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On a family of solutions of the second Painlevé equation related to superconductivity

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In a recent paper devoted to the study of the superheating field attached to a semi-infinite superconductor, Chapman [1] constructs a family of approximate solutions of the Ginzburg–Landau system. This construction, based on a matching procedure, implicitly uses the existence of a family of solutions depending on a parameter $c \in \mathbb{R}$ of the Painlevé equation in a semi-infinite interval $(0, +\infty)$

$$u''(t) + (t - c)u(t) - u(t)^3 = 0,$$

with a Neumann condition at 0

$$u'(0) = 0,$$

and having a prescribed behaviour at $+\infty$

$$u(t) \sim \sqrt{t}.$$

In this paper we prove the existence of such a family of solutions and investigate its properties. Moreover, we prove that the second coefficient in Chapman's expansion of the superheating field is finite.

1 Introduction

In a recent paper devoted to the study of the superheating field attached to a semi-infinite superconductor, Chapman [1] constructed a family of approximate solutions of the Ginzburg–Landau system using a procedure of matching so-called inner and outer solutions. The details of this construction will be presented later, but let us just emphasize that the inner solutions, up to rescaling, turn out to be solutions of the Painlevé equation in a semi-infinite interval $(0, +\infty)$

$$u''(t) + (t - c)u(t) - u(t)^3 = 0, \tag{1.1}$$

(depending on $c \in \mathbb{R}$) with a Neumann condition

$$u'(0) = 0, \tag{1.2}$$

and having a prescribed behaviour at $+\infty$

$$u(t) \sim \sqrt{t}. \tag{1.3}$$

Although a sizable literature has been devoted to the study of the Painlevé equation [2, 3], we understand that the solution of this problem is not explicitly known. The only case of the above problem which has been rigorously analysed is in fact the limit case, corresponding to the construction by Hastings and McLeod [4] of a unique strictly positive solution of $u''(t) + tu(t) - u(t)^3 = 0$ in $I\!R$ such that $\lim_{t\to-\infty} u(t) = 0$, and such that $u(t) \sim \sqrt{t}$ at $+\infty$.

We prove here, with similar techniques, the existence of a unique solution satisfying (1.2) and (1.3). This will be the object of § 2. In § 3, we investigate the dependence of these solutions on the parameter c. In particular, we analyse a certain integral (3.5) related to these solutions. Also, we show that the Hastings–McLeod solution is indeed the limit as c goes to infinity of the solutions constructed in this paper. In § 4, we discuss the origin of the problem and explain how the present work relates to Chapman's interesting contribution. The main achievement of our analysis is to prove that the coefficient of the second term of Chapman's formal expansion for the superheating field is well defined. (See Remark 2.17, Proposition 3.2 and formulas (4.19) and (4.20) below.)

2 Shooting method for the Painlevé equation

We consider the following initial value problem:

$$\begin{cases} u''(t) + (t-c)u(t) - u(t)^3 = 0, \\ u(0) = \alpha > 0, \\ u'(0) = 0, \end{cases} \tag{2.1}$$

where c is an arbitrary but fixed real number. For each $\alpha > 0$, there exists a unique maximal solution $u : [0, T^*(\alpha)) \mapsto I\!R$. This solution u is of class C^2 on $[0, T^*(\alpha))$, and in fact C^∞ since $u \mapsto u^3$ is C^∞. Also, the existence time of the maximal solution is characterized by the fact that if $T^*(\alpha) < \infty$, then

$$\lim_{t\to T^*(\alpha)} [|u(t)| + |u'(t)|] = \infty.$$

To show the dependence of the solution on α, we sometimes write $u(t, \alpha)$. Our goal is to prove the following theorem.

Theorem 2.1 *For each $c \in I\!R$, there exists a unique $\alpha > 0$ such that $u = u(\cdot, \alpha)$ has the following properties:*

(1) $u(t) > 0$ *for all* $t \in [0, T^*(\alpha))$,

(2) $T^*(\alpha) = \infty$,

(3) $lim_{t\to+\infty}(u(t) - \sqrt{t}) = 0$.

We prove the theorem using a shooting argument. Before describing the details of this argument, we make a few preliminary observations concerning the solutions to the problem (2.1). Throughout this section, $f : [0, \infty) \mapsto I\!R$ denotes the function given by

$$f(t) = \begin{cases} 0, & \text{if } t \geqslant 0 \text{ and } t < c, \\ \sqrt{(t-c)}, & \text{if } t \geqslant 0 \text{ and } t \geqslant c. \end{cases}$$

Clearly, if $c \leqslant 0$, the first condition is vacuous. Note that the differential equation in (2.1) can be rewritten as

$$u''(t) = u(t)[u(t)^2 - (t-c)]. \tag{2.2}$$

The following lemma is therefore immediate.

Lemma 2.2 *Let $t \in [0, T^*(\alpha))$ and $u(t) > 0$.*

(1) *If $u(t) > f(t)$, then u is strictly convex in a neighbourhood of t.*

(2) *If $u(t) < f(t)$, then u is strictly concave in a neighbourhood of t.*

The following lemma is also easy to prove, and will play a central role in the analysis.

Lemma 2.3 *Suppose there exists $t_0 \in [0, T^*(\alpha))$ such that*

(1) $t_0 > c$,

(2) $u(t_0) \geqslant f(t_0) = \sqrt{(t_0 - c)}$,

(3) $u'(t_0) \geqslant f'(t_0) = (2\sqrt{(t_0-c)})^{-1}$.

It follows that $u(t) > f(t)$ for all $t \in (t_0\,, T^*(\alpha))$. In particular, $u(t)$ is strictly convex on $(t_0\,, T^*(\alpha))$.

Proof Since $u(t_0) \geqslant \sqrt{(t_0-c)}$, it follows from (2.2) that $u''(t_0) \geqslant 0$. On the other hand, $f''(t_0) < 0$, so $f''(t_0) < u''(t_0)$. It follows that $f(t) < u(t)$ for t in some small interval $(t_0\,, t_0+\epsilon)$, where $\epsilon > 0$.

Suppose there exists $t_1 \in (t_0, T^*(\alpha))$ such that $f(t) < u(t)$ for all $t \in (t_0, t_1)$ and $f(t_1) = u(t_1)$. On the interval (t_0, t_1), u is strictly convex and f is strictly concave. Since $u'(t_0) \geqslant f'(t_0)$, it follows that $u'(t) > f'(t)$ on $(t_0\,, t_1)$, i.e. $u(t) - f(t)$ is an increasing, positive function. Thus, it is impossible that $f(t_1) = u(t_1)$. This proves that $f(t) < u(t)$ on $(t_0\,, T^*(\alpha))$. □

Corollary 2.4 *If $u(t) > 0$ for all $t \in [0, T^*(\alpha))$, then the set*

$$\{t \in [0, T^*(\alpha)) : u(t) \leqslant f(t)\}$$

is an interval.

Proof This an immediate consequence of Lemma 2.3. □

Lemma 2.5 *If $T^*(\alpha) < \infty$ and $u(t) > 0$ for all $t \in [0, T^*(\alpha))$, then $\lim_{t\to T^*(\alpha)} u(t) = \infty$.*

Proof We first show that $\sup_{[0,T^*(\alpha))} u(t) = \infty$. Suppose not. Multiplying the equation in (2.1) by $u'(t)$, we see that

$$E'(t) = \frac{u(t)^2}{2}\,, \tag{2.3}$$

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where

$$E(t) = \frac{u'(t)^2}{2} + \frac{(t-c)u(t)^2}{2} - \frac{u(t)^4}{4}. \tag{2.4}$$

If $\sup_{[0,T^*(\alpha))} u(t) < \infty$, then $E'(t)$ is bounded on $[0, T^*(\alpha))$. Since $T^*(\alpha) < \infty$, it follows that $E(t)$ is bounded on $[0, T^*(\alpha))$. This in turn implies that $u'(t)^2$ is bounded on $[0, T^*(\alpha))$, which contradicts the basic property of $T^*(\alpha)$. This shows that $\sup_{t\in[0,T^*(\alpha))} u(t) = \infty$.

It therefore follows from Corollary 2.4 and the hypothesis $T^*(\alpha) < \infty$ that there exists $t_0 \in [0, T^*(\alpha))$ such that $u(t) > f(t)$ for all $t \in (t_0, T^*(\alpha))$. Since $\sup_{[0,T^*(\alpha))} u(t) = \infty$ and u is convex on $(t_0, T^*(\alpha))$, it follows that $\lim_{t\to T^*(\alpha)} u(t) = \infty$. □

Lemma 2.6 *Let* $0 < \alpha < \beta$, *and let* $u(t) = u(t, \alpha)$ *and* $v(t) = u(t, \beta)$. *Suppose* $T > 0$ *is such that*

(1) $T < T^*(\alpha)$ *and* $T < T^*(\beta)$,

(2) $u(t) > 0$ *and* $v(t) > 0$ *on* $[0, T]$.

It follows that $u(t) < v(t)$ *on* $[0, T]$. *In other words, two different solutions of (2.1) cannot cross as long as they both remain positive.*

Proof Let $W(t) = v'(t)u(t) - v(t)u'(t)$. Then

$$W'(t) = v''(t)u(t) - v(t)u''(t) = u(t)v(t)[v(t)^2 - u(t)^2]. \tag{2.5}$$

We note that $W(0) = 0$, since $u'(0) = v'(0) = 0$. Also, since $\alpha < \beta$, $u(t) < v(t)$ for $t > 0$ in some interval $t \in (0, \epsilon)$.

We argue by contradiction, and so we suppose there exists $t_1 \in [0, T]$ such that

$$u(t) < v(t), \quad t \in [0, t_1),$$
$$u(t_1) = v(t_1).$$

It follows that $u'(t_1) \geqslant v'(t_1)$, and so

$$W(t_1) = v'(t_1)u(t_1) - v(t_1)u'(t_1) = [v'(t_1) - u'(t_1)]v(t_1) \leqslant 0.$$

On the other hand, it follows from (2.5) that $W'(t) > 0$ on $[0, t_1)$, and so $W(t_1) > 0$. This contradiction establishes the lemma. □

We now begin the shooting argument. We consider the following two sets:

$$A = \{\alpha > 0 : \text{ there exists } t_0 \in (0, T^*(\alpha)) \text{ such that } u(t,\alpha) > 0 \text{ for all } t \in [0, t_0) \text{ and } u(t_0, \alpha) = 0\}.$$
$$B = \{\alpha > 0 : u(t,\alpha) > 0 \text{ for all } t \in [0, T^*(\alpha)) \text{ and there exists } t_0 \in (0, T^*(\alpha)) \text{ such that } u(t,\alpha) > f(t) \text{ on } (t_0, T^*(\alpha))\}.$$

It is clear that A and B are disjoint. We first turn our attention to the set A.

Proposition 2.7 *A is open.*

Proof This is an immediate consequence of continuous dependence of $u(t, \alpha)$ on α. Let $\alpha \in A$ and let t_0 be as in the definition of A. It follows that $u'(t_0) < 0$, for if $u'(t_0) = 0$, then $u(t)$ would be identically zero by uniqueness of solutions to (2.1). Thus, $u(t) < 0$ for t in some interval $(t_0, t_0 + \epsilon)$. By continuous dependence on the initial value, it follows that $u(t, \beta)$ must take on some negative values for β close enough to α. Evidently, such values of β belong to A. □

Proposition 2.8 *A in an interval.*

Proof This is an immediate consequence of Lemmas 2.5 and 2.6. □

Proposition 2.9 *A is non-empty. More precisely, A includes all sufficiently small* $\alpha > 0$.

Proof We first consider the solution to the following linear initial value problem:

$$\left.\begin{array}{l} w''(t) + (t - c)w(t) = 0, \\ w(0) = 1, \\ w'(0) = 0. \end{array}\right\} \tag{2.6}$$

Multiplying the above equation by $\sin(t - \tau)$ for large τ and integrating by parts over the interval $[\tau, \tau + \pi]$ one sees that the solution of (2.6) cannot remain positive for all $t > 0$. Let t_0 be the smallest positive zero of w. Clearly, then, $w'(t_0) < 0$, and so $w(t) < 0$ for t in some interval $(t_0, t_0 + \epsilon)$.

Now set $v(t, \alpha) = \alpha^{-1} u(t, \alpha)$. Then $v = v(\cdot, \alpha)$ satisfies

$$\left.\begin{array}{l} v''(t) + (t - c)v(t) - \alpha^2 v(t)^3 = 0, \\ v(0) = 1, \\ v'(0) = 0. \end{array}\right\} \tag{2.7}$$

By the continuous dependence on the parameter α, it follows that if α^2 is sufficiently small, then the solution of (2.7) must have a zero. Therefore, if $\alpha > 0$ is sufficiently small, then $u(t, \alpha)$ has a zero, i.e. $\alpha \in A$. □

Next, we give a characterization of A in terms of the whether or not $u(t, \alpha)$ has a local maximum.

Proposition 2.10

(1) *If* $c \geqslant 0$, *then* $A = \{\alpha > 0 \ : \ u(\cdot, \alpha)$ *has at least one positive local maximum at some* $t_0 > 0\}$.

(2) *If* $c < 0$, *then* $A = (0, \sqrt{-c}\,] \cup \{\alpha > \sqrt{-c} \ : \ u(\cdot, \alpha)$ *has at least one positive local maximum at some* $t_0 > 0\}$.

Proof We note first that if $c < 0$ and $\alpha < \sqrt{-c}$, then $u''(0) < 0$. Thus, u is strictly concave and decreasing on some interval $(0, \epsilon)$, where $\epsilon > 0$. Since $u(0) < f(0)$ and $f(t)$ is increasing, an easy argument shows that $u(t)$ remains strictly concave and decreasing as long as it is positive. Therefore, it must have a positive zero.

If $c < 0$ and $\alpha = \sqrt{-c}$, then $u(t)$ is the limit of the concave, decreasing functions $u(\cdot, \beta)$ as $\beta \to \alpha^-$, at least on some interval $(0, \epsilon)$. Thus, u is concave and non-increasing on some

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interval $(0,\epsilon)$. Since $u(0) = f(0)$ and $f(t)$ is increasing, one can show as above that $u(\cdot,\alpha)$ must have a zero. Thus, if $c < 0$, then $(0,\sqrt{-c}\,]$ is a subset of A.

Let c now be arbitrary, and suppose that $\alpha > \sqrt{-c}$ in case $c < 0$. It follows that $u''(0) > 0$, and so u is strictly convex and increasing on some interval $(0,\epsilon)$. Thus, if $\alpha \in A$, i.e. if $u(\cdot\,,\alpha)$ has a zero, it must first have a positive local maximum.

Finally, suppose that $u(\cdot\,,\alpha)$ has a positive local maximum at some $t_0 > 0$. Clearly $u(t_0) \leqslant f(t_0)$, for if not $u''(t_0) > 0$. We claim in fact that $u(t_0) < f(t_0)$. Suppose to the contrary that $u(t_0) = f(t_0)$. Since $f(t_0) = u(t_0) > 0$, we must have $t > c$. Since $u'(t_0) = 0$ and $f'(t_0) > 0$, it follows that $u(t) > f(t)$ for t in some interval $(t_0 - \epsilon, t_0)$. It follows that u is strictly convex on the interval $(t_0 - \epsilon, t_0)$, which makes it impossible for u to have a maximum at t_0.

We conclude that $u(t_0) < f(t_0)$. Thus, u is strictly concave in a neighbourhood of t_0. Since $f(t)$ is increasing, $u(t)$ is strictly concave and decreasing for $t > t_0$ as long as it remains positive. It must therefore have a zero. □

We now turn our attention to the set B. It is not immediately clear that B is open. However, we first show that B is non-empty.

Proposition 2.11 *B is non-empty. In fact, if $\alpha > 0$ satisfies*

$$\alpha(\alpha^2 + c) > 1, \tag{2.8}$$

then $u(t) > f(t)$ for all $t \in [0, T^*(\alpha))$.

Proof Suppose this condition is satisfied, and so in particular $\alpha^2 + c > 0$. We claim that $u(t)$ is strictly convex on the interval $[0, \min(T^*(\alpha), \alpha^2 + c))$. Indeed,

$$u''(0) = u(0)[u(0)^2 + c] = \alpha(\alpha^2 + c) > 0$$

and $u'(0) = 0$, so u is strictly convex and increasing on some interval $(0,\epsilon)$. Since $f(t) < \alpha$ for all $t \in [0, \alpha^2 + c)$, it follows that $u(t)$ remains strictly convex and increasing on $[0, T^*(\alpha)) \cap [0, \alpha^2 + c)$. In particular, $u(t) \geqslant u(0) = \alpha$ on this interval.

Observing that the lemma is proved in the case when $T^*(\alpha) \leqslant \alpha^2 + c$, we may now suppose that $T^*(\alpha) > \alpha^2 + c$. Therefore, for all $t \in [0, \alpha^2 + c]$,

$$u''(t) = u(t)[u(t)^2 - (t - c)] \geqslant \alpha\,(\alpha^2 - t + c) \geqslant 0.$$

It follows that $u'(t) \geqslant \alpha((\alpha^2 + c)t - t^2/2)$ on this interval, and in particular that $u'(\alpha^2 + c) \geqslant \frac{\alpha}{2}(\alpha^2 + c)^2$. On the other hand, $f'(\alpha^2 + c) = \frac{1}{2\alpha}$. The condition (2.8) gives exactly that $f'(\alpha^2 + c) < u'(\alpha^2 + c)$. By Lemma 2.3, it follows that $u(t) > f(t)$ for all $t \in (\alpha^2 + c, T^*(\alpha))$. □

Proposition 2.12 *B is an interval.*

Proof Let $\alpha \in B$ and suppose $\beta > \alpha$. We will show that $\beta \in B$. Let $u(t) = u(t,\alpha)$ and $v(t) = u(t,\beta)$. Lemma 2.6 implies that $u(t) < v(t)$ for all $t < \min[T^*(\alpha), T^*(\beta)]$. It follows that $T^*(\beta) \leqslant T^*(\alpha)$. Indeed, if $T^*(\alpha) < T^*(\beta)$, then $T^*(\alpha) < \infty$ and so

$\lim_{t\to T^*(\alpha)} u(t) = \infty$ by Lemma 2.5. It then follows that $\lim_{t\to T^*(\alpha)} v(t) = \infty$, which is impossible if $T^*(\beta) > T^*(\alpha)$.

Thus, $T^*(\beta) \leqslant T^*(\alpha)$. Since $\alpha \in B$, there exists $t_0 \in (0, T^*(\alpha))$ such that $u(t) > f(t)$ for all $t \in (t_0, T^*(\alpha))$. If $t_0 < T^*(\beta)$, then $v(t) > u(t) > f(t)$ for all $t \in (t_0, T^*(\beta))$, and so $\beta \in B$. On the other hand, if $T^*(\beta) \leqslant t_0$, then $T^*(\beta) < \infty$, and so $\lim_{t\to T^*(\beta)} v(t) = \infty$ by Lemma 2.5. Thus, $v(t) > f(t)$ for t sufficiently close to $T^*(\beta)$, i.e. $\beta \in B$. □

Lemma 2.13 *If $\alpha \in B$, then there exists $t_1 \in (0, T^*(\alpha))$ such that $u(\cdot, \alpha)$ is strictly convex and increasing on the interval $(t_1, T^*(\alpha))$.*

Proof By the definition of B, there exists $t_0 \in (0, T^*(\alpha))$ such that $u(\cdot, \alpha)$ is strictly convex on the interval $(t_0, T^*(\alpha))$. To prove the lemma, it suffices to show that $u(t)$ cannot be decreasing on $(t_0, T^*(\alpha))$. If $u(t)$ is decreasing on $(t_0, T^*(\alpha))$, then Lemma 2.5 implies that $T^*(\alpha) = \infty$. In this case, the fact that $u(t)$ is decreasing on (t_0, ∞) implies that $u(t) < f(t)$ for large t. This contradicts the hypothesis that $\alpha \in B$. □

Proposition 2.14 *B is open.*

Proof Let $\alpha \in B$, and let $u(t) = u(t, \alpha)$. We need to prove that if β is in a small enough neighbourhood of α, then $\beta \in B$.

We distinguish two cases, $T^*(\alpha) > c$ and $T^*(\alpha) \leqslant c$, and we suppose first that $T^*(\alpha) > c$. Since $\alpha \in B$, there exists $t_0 > c$ such that $u(t) > f(t)$ for all $t \in (t_0, T^*(\alpha))$. We claim that there exists $t_1 \in (t_0, T^*(\alpha))$ such that $u'(t_1) > f'(t_1)$. If not, then

$$u'(t) \leqslant (2\sqrt{(t-c)})^{-1}$$

for all $(t_0, T^*(\alpha))$. Integrating this inequality, we see that for some $K > 0$,

$$u(t) \leqslant \sqrt{(t-c)} + K \tag{2.9}$$

for all $(t_0, T^*(\alpha))$. By Lemma 2.5, this implies that $T^*(\alpha) = \infty$. In this case, formula (2.9) is incompatible with the fact that $u(t)$ is strictly convex and increasing for large t. This proves that there exists $t_1 > t_0$ such that $u'(t_1) > f'(t_1)$.

Thus, the number $t_1 > c$ is such that $u(t_1) > f(t_1)$ and $u'(t_1) > f'(t_1)$. By continuous dependence, if β is close enough to α, and if $v(t)$ denotes the solution of (2.1) with initial value β, then $v(t_1) > f(t_1)$ and $v'(t_1) > f'(t_1)$. Lemma 2.3 then implies that $\beta \in B$.

Next suppose that $T^*(\alpha) \leqslant c$ (and in particular that $c > 0$.) By Lemma 2.5, we know that $u(t) \to \infty$ as $t \to T^*(\alpha)$. Since $T^*(\alpha) < \infty$, it follows also that $u'(t) \to \infty$ as $t \to T^*(\alpha)$. Indeed, $u'(t)$ must be unbounded on $[0, T^*(\alpha))$, for otherwise $u(t)$ would be bounded on that interval. Since $u'(t)$ is increasing on $[0, T^*(\alpha))$, it follows that $u'(t) \to \infty$ as $t \to T^*(\alpha)$.

Fix $b > 0$ such that $(t-c) + b > f(t)$ for all $t > c$. Choose $t_1 \in [0, T^*(\alpha))$ such that $u'(t_1) > 1$ and $u(t_1) > b$. By continuous dependence on the initial value, for β close enough to α, we have that $v'(t_1) > 1$ and $v(t_1) > b$. Since $t_1 < c$, $v(t_1) > f(t_1)$, and so $v(t)$ is convex

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in a neighbourhood of t_1. Thus, as long as $v(t)$ remains convex for $t \geqslant t_1$, it follows that

$$\begin{aligned} v(t) &\geqslant v'(t_1)(t - t_1) + v(t_1) \\ &\geqslant (t - t_1) + b \\ &\geqslant t - c + b \\ &> f(t). \end{aligned}$$

It follows that $v(t)$ remains convex throughout the interval $[0, T^*(\beta))$, and so $\beta \in B$. □

To summarize what has been proved, A and B are two open sets of the form

$$A = (0, a), \quad B = (b, \infty),$$

where $0 < a < b$, and (if $c < 0$) $a > \sqrt{-c}$.

We now need to do two things:

- examine the asymptotic behaviour of $u(t, \alpha)$ for $\alpha > 0$ belonging to neither A nor B, and
- show that there is only one such $\alpha > 0$.

Proposition 2.15 *Let $\alpha > 0$ be such that $\alpha \notin A$ and $\alpha \notin B$, and let $u = u(\cdot\,, \alpha)$. It follows that*

(1) $u(t) > 0$ *for all* $t \in [0, T^*(\alpha))$,
(2) $u'(t) > 0$ *for all* $t \in (0, T^*(\alpha))$,
(3) *there exists* t_0 *such that* $u(t) < f(t)$ *for all* $t \in (t_0, T^*(\alpha))$,
(4) $T^*(\alpha) = \infty$,
(5) $\lim_{t\to+\infty} u'(t) = 0$,
(6) $\lim_{t\to+\infty} u(t) = \infty$,
(7) $0 < \alpha(\alpha^2 + c) \leqslant 1$.

Proof Statement (1) is a consequence of the definition of the set A. Statement (2) is a consequence of Proposition 2.10. Indeed, $u(t) > f(t)$ for t in some small interval $(0, \epsilon)$, and so u is strictly convex and increasing on $(0, \epsilon)$. If statement (2) is false, let t_1 be the first positive zero of u'. Thus $u'(t) > 0$ on $(0, t_1)$. It follows that u cannot be strictly convex on $(0, t_1)$; and so (by Lemma 2.3), there exists $t_0 \in (0, t_1)$ such that $u(t) < f(t)$ on (t_0, t_1). Moreover, $u(t_1) < f(t_1)$, since $u'(t_1) = 0 < f'(t_1)$ and $u(t) < f(t)$ on (t_0, t_1). Thus, u has a strict local maximum at t_1, which is impossible by Proposition 2.10 since $\alpha \notin A$. This proves (2).

Statement (3) follows from Lemma 2.3 (and Corollary 2.4) since $\alpha \notin B$. Statement (4) is a consequence of statement (3) and Lemma 2.5.

To prove statement (5), we note that by statements (2), (3) and (4), u' is a positive function, decreasing for large values of t. Thus, $\lim_{t\to+\infty} u'(t) = L \geqslant 0$ exists. If $L > 0$, then u is bounded below by some linear function of t, for large t, contradicting statement (3). Thus, $L = 0$, which proves statement (5).

To prove statement (6) we observe that u is a positive, increasing function, and so $\lim_{t\to+\infty} u(t) = L > 0$ exists, where L might be infinite. Suppose $L < \infty$. It follows from the equation in (2.1) that $\lim_{t\to+\infty} u''(t) = -\infty$, which is impossible for a positive function defined for all $t > 0$.

Finally, (7) is an immediate consequence of Propositions 2.10 and 2.11. □

Proposition 2.16 *Let $\alpha > 0$ be such that $\alpha \notin A$ and $\alpha \notin B$, and let $u = u(\cdot, \alpha)$. It follows that* $\lim_{t\to+\infty}(f(t) - u(t)) = 0$.

Proof From the definition of $f(t)$ and (2.2), it follows that if $t > 0$ and $t > c$, then

$$u''(t) = u(t)[u(t)^2 - f(t)^2]. \tag{2.10}$$

Integrating from τ to ρ, where $\tau < \rho$ and $u(t) < f(t)$ for all $t > \tau$, we obtain

$$u'(\rho) - u'(\tau) = \int_\tau^\rho u(t)[u(t)^2 - f(t)^2]\mathrm{d}t. \tag{2.11}$$

Since, $\lim_{\rho\to\infty} u'(\rho) = 0$, and since $u(t)[u(t)^2 - f(t)^2] < 0$ for all $t > \tau$, it follows that there exists a sequence $t_k \to \infty$ such that

$$\lim_{k\to\infty} (f(t_k) - u(t_k)) = 0.$$

Suppose it is false that $\lim_{t\to+\infty} (f(t) - u(t)) = 0$. Then there exists another sequence $y_k \to \infty$ such that

$$\left.\begin{aligned} \inf_k[f(y_k) - u(y_k)] > 0, \\ f'(y_k) - u'(y_k) = 0, \\ f''(y_k) - u''(y_k) \leqslant 0. \end{aligned}\right\} \tag{2.12}$$

In other words, the y_k form a sequence of local maxima of $f(t) - u(t)$ out to infinity. This last relation says simply that

$$u''(y_k) \geqslant \frac{-1}{4(y_k - c)^{3/2}} = \frac{-1}{4f(y_k)^3},$$

Substituting this inequality into (2.10), we see that

$$u(y_k)[f(y_k)^2 - u(y_k)^2] \leqslant \frac{1}{4f(y_k)^3},$$

which clearly implies that $\lim_{k\to\infty}(f(y_k) - u(y_k)) = 0$, contradicting (2.12). This proves the proposition. (The last part of our proof was adapted from Hastings & McLeod [4, pp. 36–37], but without using the change of variables.) □

Remark 2.17 *We deduce in particular from* (2.11) *the property that the ultimately positive function*

$$t \mapsto f(t)^2 - u(t)^2 \text{ belongs to } L^1((0, +\infty)). \tag{2.13}$$

This property is also needed in the construction of Chapman [1] *as we shall see in* § *4.*

Proposition 2.18 *There is only one positive real number α belonging to neither A nor B.*

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Proof Suppose $0 < \alpha < \beta$ and that $\alpha \notin A$, $\alpha \notin B$ and $\beta \notin A$, $\beta \notin B$. Let $u(t) = u(t, \alpha)$ and $v(t) = u(t, \beta)$. By Lemma 2.6, $u(t) < v(t)$ for all $t > 0$. Furthermore, by formula (2.5), it follows that $W'(t) > 0$ for all $t > 0$, where $W(t) = v'(t)u(t) - v(t)u'(t)$. Since $W(0) = 0$, it follows that

$$\lim_{t\to+\infty} W(t) > 0. \tag{2.14}$$

Also,

$$f(t)[v'(t) - u'(t)] = W(t) + v'(t)[f(t) - u(t)] - [f(t) - v(t)]u'(t). \tag{2.15}$$

Since

$$\begin{aligned}\lim_{t\to+\infty} v'(t) &= \lim_{t\to+\infty} u'(t)\\ &= \lim_{t\to+\infty} (f(t) - u(t))\\ &= \lim_{t\to+\infty} (f(t) - v(t)) = 0,\end{aligned}$$

formulas (2.14) and (2.15) imply that

$$\lim_{t\to+\infty} f(t)[v'(t) - u'(t)] > 0.$$

In other words, there exists $K > 0$ such that

$$v'(t) - u'(t) \geqslant \frac{K}{\sqrt{t-c}}$$

for all large t. Integrating this inequality, we see that there exist $K_1 > 0$ and K_2 such that

$$v(t) - u(t) \geqslant K_1\sqrt{t-c} + K_2,$$

for all large t. On the other hand,

$$\lim_{t\to+\infty} [v(t) - u(t)] = \lim_{t\to+\infty} [v(t) - f(t) - (u(t) - f(t))] = 0.$$

This contradiction proves that there is only one $\alpha > 0$ not in A or B, and concludes the proof of the theorem. □

3 On the asymptotic behaviour of the solutions with respect to t or c

3.1 Presentation

We denote by $u = u_c(t)$ the solution of (1.1)–(1.3) which was constructed in § 2. Recall that (see point (5) in Proposition 2.15 and Proposition 2.16)

$$u_c(t) - \sqrt{t} \to 0 \quad \text{as} \quad t \to +\infty, \tag{3.1}$$

and that also

$$u_c'(t) \to 0 \quad \text{as} \quad t \to +\infty. \tag{3.2}$$

It is an interesting problem in itself to improve these asymptotic formulae. Also, we need to improve these estimates in order to analyse $\int_0^\infty (u_c(t)^2 - (t-c))\mathrm{d}t$ and to compare the various solutions discussed in this paper. For this purpose, it is better to consider the function

$$t \mapsto v_c(t) = u_c(t+c) \quad \text{for} \quad t \in (-c, +\infty). \tag{3.3}$$

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Of course, this family of solutions is not defined on a fixed interval, but this translation has the advantage that all the functions v_c are solutions of the same equation

$$v'' + tv - v^3 = 0. \tag{3.4}$$

As we shall see, this will make the comparison of the solutions for different values of c easier (see § 3.4), and we shall prove that all the expansions at ∞ are the same for all solutions. Section 3.3 is devoted to a careful study of the function

$$I\!R \ni c \mapsto \eta(c) := \int_0^{+\infty} (u_c(t)^2 - (t-c))\mathrm{d}t \; - \frac{c^2}{2}, \tag{3.5}$$

whose physical interpretation is explained in § 4.

3.2 Sharper asymptotics of the solution as $t \to +\infty$

It is 'well known' [2][1] that

$$v_c(t) \sim \sqrt{t} + \gamma_5 t^{-\frac{5}{2}} + \sum_{j>5} \gamma_j t^{-\frac{j}{2}}, \tag{3.6}$$

where the γ_j are independent of c. Since we have no references for the proof of this 'standard' result, we present here a proof of a slightly weaker result, inspired partially by an L^2-proof suggested by Pierre Bolley on the basis of Bolley & Camus [5] or Bolley *et al.* [6], but combined with an argument used in a similar context by Hervé & Hervé [7]. More precisely, we prove the following proposition:

Proposition 3.1 *The solution $v_c(t)$ has, for any c, the following behaviour at $+\infty$*

$$\sqrt{t} - v_c(t) = \mathcal{O}\left(t^{-\frac{5}{2}}\right). \tag{3.7}$$

Similarly,

$$2^{-\frac{1}{2}} t^{-\frac{1}{2}} - v_c'(t) = \mathcal{O}\left(t^{-2}\right). \tag{3.8}$$

Proof We analyse the equation satisfied by $w(t) := \sqrt{t} - v_c(t)$, i.e.

$$\left(\frac{d^2}{\mathrm{d}t^2} - 2t\right) w = -\frac{1}{4} t^{-\frac{3}{2}} - 3t^{\frac{1}{2}} w^2 + w^3, \tag{3.9}$$

in $[T, +\infty)$. Note that the left-hand side corresponds to Airy's equation.

Our starting point is that

$$\lim_{t \to +\infty} w(t) = 0\,, \; \lim_{s \to +\infty} w'(t) = 0.$$

The change of variables $s = t^{\frac{3}{2}}$ leads to the following equation for $y(s) = w(s^{\frac{2}{3}})$:

$$Ay := \left(\frac{9}{4}\frac{d^2}{ds^2} - 2\right) y = h, \tag{3.10}$$

with

$$h(s) = -\frac{3}{4} s^{-1} y'(s) - \frac{1}{4} s^{-\frac{5}{3}} - 3s^{-\frac{1}{3}} y^2 + s^{-\frac{2}{3}} y^3.$$

[1] As also confirmed by N. Joshi, this behaviour at ∞ is well known to the specialists in the Painlevé equation.

We observe for future use that

$$y'(s) = \frac{2}{3} s^{-\frac{1}{3}} w'(s^{\frac{2}{3}}).$$

The lemma of Hervé & Hervé [7, p. 435], which is a consequence of the explicit expression for the solution using the variation of constants formula, says simply that if y tends to 0 and if $h = Ay$ in (3.10) is either $o(s^k)$ or $\mathcal{O}(s^k)$ for some negative k, then y and y' have the same property.

Consequently, we know right away that $y(s) = o(1)$, $y'(s) = o(1)$, and consequently $h(s) = o(s^{-\frac{1}{3}})$ as $s \to +\infty$. The above-mentioned lemma of Hervé and Hervé gives that $y(s) = o(s^{-\frac{1}{3}})$ and $y'(s) = o(s^{-\frac{1}{3}})$. This implies first that $h(s) = o(s^{-1})$, and applying the lemma again, that $y(s) = o(s^{-1})$ and $y'(s) = o(s^{-1})$. We now get that $h(s) = \mathcal{O}(s^{-\frac{5}{3}})$ and the lemma gives $y(s) = \mathcal{O}(s^{-\frac{5}{3}})$ and $y'(s) = \mathcal{O}(s^{-\frac{5}{3}})$. To complete the proof of the proposition it suffices to translate these estimates in terms of the original variables. □

Similar arguments can probably be used to prove the existence of the complete expansion (3.6) announced above.

3.3 On the variation of the energy

We wish to analyse $\eta(c)$, defined by (3.5). This expression corresponds to $\beta(c) - c^2/\sqrt{2}$ in §4. In terms of the function v_c, given by (3.3), the formula for $\eta(c)$ becomes

$$\eta(c) = \int_{-c}^{+\infty} (v_c(t)^2 - t)\mathrm{d}t - \frac{c^2}{2}. \tag{3.11}$$

The local energy E of u was introduced in (2.4) and can be expressed (with $\mathcal{E}(t; c)) = E(t + c; c))$ as

$$\mathcal{E}(t; c) = \frac{v_c'(t)^2}{2} + \frac{t v_c(t)^2}{2} - \frac{v_c(t)^4}{4}. \tag{3.12}$$

The corresponding derivative (see (2.3)) is

$$\mathcal{E}'(t; c) = (\partial_t \mathcal{E})(t; c) = \frac{v_c(t)^2}{2}. \tag{3.13}$$

Let us now give the formula for $\eta(c)$:

$$\begin{aligned} \eta(c) &= \lim_{A \to +\infty} \int_{-c}^{A} (v_c(t)^2 - t)\mathrm{d}t - c^2/2 \\ &= -2\mathcal{E}(-c; c) + 2 \lim_{A \to +\infty} (\mathcal{E}(A; c) - A^2/4). \end{aligned}$$

The function $\eta(c)$ thereby appears as (twice) the renormalized variation of the energy $E(t; c)$ between 0 and $+\infty$.

By the definition of $\mathcal{E}$, we have that

$$-2\mathcal{E}(-c; c) = \frac{1}{2} v_c(-c)^2 (2c + v_c(-c)^2).$$

We now study the asymptotic behaviour of $\mathcal{E}(A; c)$ as $A \to +\infty$. The asymptotic analysis

(Proposition 3.1) of $v_c(t)$ and $v_c'(t)$ as $t \to +\infty$ gives

$$\mathscr{E}(A;c) = \frac{A^2}{4} + \mathcal{O}(A^{-1}).$$

We finally obtain the following simple formula:

$$\eta(c) = \frac{1}{2}v_c(-c)^2(2c + v_c(-c)^2) = \frac{1}{2}u_c(0)^2(2c + u_c(0)^2) = \frac{1}{2}\alpha_c^2(2c + \alpha_c^2), \tag{3.14}$$

where α_c is defined by

$$\alpha_c = u_c(0) = v_c(-c).$$

We have seen in Propositions 2.10 and 2.15 that

$$0 < \alpha_c(\alpha_c^2 + c) \leqslant 1. \tag{3.15}$$

When $c > 0$, this immediately implies that

$$0 \leqslant c\,\alpha_c \leqslant 1\ ,$$

and

$$\lim_{c\to+\infty} \eta(c) = 0.$$

When $c < 0$, we get, from (3.15) and the fact that $\alpha_c \geqslant \sqrt{-c}$, the inequality

$$0 \leqslant \alpha_c^2 + c \leqslant \frac{1}{\alpha_c} \leqslant \frac{1}{\sqrt{-c}}.$$

We observe also that

$$\eta(c) + \frac{c^2}{2} = \frac{1}{2}(\alpha_c^2 + c)^2,$$

and so

$$\eta(c) + \frac{c^2}{2} = \mathcal{O}((-c)^{-1}), \quad \text{as} \quad c \to -\infty. \tag{3.16}$$

We have now almost completely proved the following result.

Proposition 3.2 *The function $c \mapsto \eta(c)$ introduced in* (3.5) *is a continuous function and has the following behaviour:*

$$\lim_{c\to+\infty} \eta(c) = 0, \tag{3.17}$$

$$\lim_{c\to-\infty} \eta(c) = -\infty, \tag{3.18}$$

and

$$\eta(c) > 0, \quad \forall c \geqslant 0. \tag{3.19}$$

In particular, $\eta(c)$ has a positive global maximum.

Proof It remains only to prove the continuity of $c \mapsto \alpha_c$. Coming back to the proofs in § 2, we indicate the dependence of the sets A and B on the parameter c by writing $A = A(c)$ and $B = B(c)$. Recall that $\alpha_c = \sup A(c) = \inf B(c)$. Let

$$\mathscr{A} = \{(\alpha, c) \in (0, +\infty) \times \mathbb{R} : \alpha \in A(c)\},$$

$$\mathscr{B} = \{(\alpha, c) \in (0, +\infty) \times \mathbb{R} : \alpha \in B(c)\}.$$

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Obvious modifications of the proofs of Propositions 2.7 and 2.14 show that $\mathscr{A}$ and $\mathscr{B}$ are open in $(0, +\infty) \times I\!R$. Consequently,

$$\{(\alpha_c, c) : c \in I\!R\} = (0, \infty) \times I\!R \setminus (\mathscr{A} \cup \mathscr{B}),$$

is closed in $(0, \infty) \times I\!R$.

Let c_k be a sequence tending to c as $k \to +\infty$. We have to show that $\alpha_{c_k} \to \alpha_c$.

If β is a limit point of α_{c_k} with $0 < \beta < \infty$, then by closure we have $(\beta, c) \in (0, +\infty) \times I\!R \setminus (A \cup \mathscr{B})$. Thus $\beta = \alpha_c$.

To complete the proof, we need to show that α_{c_k} cannot have a subsequence tending to $+\infty$ or a subsequence tending to 0. However, (3.15) shows that α_{c_k} cannot have a subsequence tending to $+\infty$. Also, the proof of Proposition 2.9, where the continuous dependence of the solution of (2.7) on α and c is used, shows that α_{c_k} cannot having a subsequence tending to 0. □

3.4 Comparison with the solution of Hastings–McLeod

The goal of this subsection is to compare the family v_c introduced in the preceding subsection with the solution constructed by Hastings & McLeod [4].

Let us first recall the result obtained by these authors:

Theorem 3.3 *There exists a unique solution u of the equation*

$$u''(t) + tu - u^3 = 0, \tag{3.20}$$

in $I\!R$ with

$$u(t) \sim \sqrt{t} \quad as \quad t \to +\infty, \tag{3.21}$$

and

$$\lim_{t \to -\infty} u(t) = 0. \tag{3.22}$$

Moreover, this solution has the following properties:

$$u(t) > 0, \tag{3.23}$$

and

$$u'(t) > 0. \tag{3.24}$$

We shall denote this solution by v_∞. In fact, as $t \to +\infty$, v_∞ has the same asymptotic properties as do the functions v_c.

Proposition 3.4 *There exists t_0 such that*

$$v_\infty(t) \leqslant \sqrt{t} \quad for \quad t \in [t_0, +\infty). \tag{3.25}$$

Moreover, v_∞ has the following behaviour, as $t \to +\infty$,

$$v_\infty(t) - \sqrt{t} = \mathscr{O}(t^{-\frac{5}{2}}), \tag{3.26}$$

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and

$$v'_\infty(t) - \frac{1}{2}\frac{1}{\sqrt{t}} = \mathcal{O}(t^{-2}). \tag{3.27}$$

Proof Starting with Theorem 3.3, in particular that $v_\infty(t) \sim \sqrt{t}$ as $t \to +\infty$, we see that v_∞ cannot be ultimately strictly convex. To prove that this solution satisfies $v(t) \leqslant \sqrt{t}$ for t large enough and $\lim_{t\to+\infty} v'(t) = 0$, we argue essentially as in the proof of Proposition 2.15 ((3) and (5)) observing that (1), (2) and (4) were established through Theorem 3.3. The asymptotic estimates can now be obtained by the same arguments as in the proof of Proposition 3.1. □

Our main result in this subsection is the following:

Theorem 3.5 *The family $c \mapsto v_c(t)$ is a monotonic decreasing sequence of functions converging uniformly on any compact set to v_∞ as c goes to $+\infty$.*

The proof of the theorem is based on two lemmas which are quite similar in spirit to Lemma 2.6.

Lemma 3.6 *Let $c_1 < c_2$ and v_{c_1} and v_{c_2} the corresponding solutions respectively defined on $[-c_1, +\infty)$ and $[-c_2, +\infty)$. Then*

$$v_{c_2}(t) < v_{c_1}(t), \quad \forall t \in [-c_1, +\infty).$$

Proof This comparison is based on the study of the Wronskian

$$W(t) := v'_{c_1}(t)v_{c_2}(t) - v_{c_1}(t)v'_{c_2}(t). \tag{3.28}$$

We first observe that, as a consequence of Proposition 3.1, we have

$$\lim_{t\to+\infty} W(t) = 0. \tag{3.29}$$

We recall also that

$$W'(t) = v_{c_1}(t)v_{c_2}(t)\left(v_{c_1}(t)^2 - v_{c_2}(t)^2\right). \tag{3.30}$$

An immediate computation gives

$$W(-c_1) = -v_{c_1}(-c_1)v'_{c_2}(-c_1),$$

which implies

$$W(-c_1) < 0. \tag{3.31}$$

Step 1 We first prove that

$$v_{c_2}(-c_1) < v_{c_1}(-c_1). \tag{3.32}$$

Suppose not. Since $v'_{c_2}(-c_1) > 0$, there exists $\epsilon > 0$ such that $v_{c_2}(t) > v_{c_1}(t)$, for $t \in (-c_1, -c_1 + \epsilon)$. In this interval, we have $W'(t) < 0$. We first observe that it is impossible that $\epsilon = +\infty$ (contradiction between (3.29), (3.31) and $W'(t) < 0$ on $(-c_1, +\infty)$).

Let t_0 the smallest $t > -c_1$ such that $v_{c_1}(t_0) = v_{c_2}(t_0)$. At this point, we would have

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$v'_{c_2}(t_0) \leqslant v'_{c_1}(t_0)$ and this would give $W(t_0) \geqslant 0$. This is again a contradiction with (3.29) and $W'(t) \leqslant 0$ on $(-c_1, t_0)$. This proves (3.32).

Step 2 We now prove that

$$v_{c_2}(t) < v_{c_1}(t), \quad \forall t \in (-c_1, +\infty). \tag{3.33}$$

Suppose not. Then there exists $t_1 > -c_1$ such that

$$\begin{cases} v_{c_2}(t_1) = v_{c_1}(t_1), \\ v_{c_2}(t) \ < v_{c_1}(t)\ , \qquad \forall t \in (-c_1, t_1), \\ v'_{c_2}(t_1) \geqslant v'_{c_1}(t_1). \end{cases}$$

Moreover, by the Cauchy uniqueness, we have actually

$$v'_{c_2}(t_1) > v'_{c_1}(t_1).$$

But we then have $W(t_1) < 0$, and the argument we have used in Step 1 still works if we replace $-c_1$ by t_1. □

Lemma 3.7 *For any $c_1 \in I\!R$, we have*

$$v_\infty(t) < v_{c_1}(t), \quad \forall t \in [-c_1, +\infty).$$

Proof The proof of this lemma is essentially the same as the proof of the previous lemma, using the asymptotic properties of v_∞ proved in Proposition 3.4. □

Proof of Theorem 3.5 We first observe that, for any $t \in I\!R$, $v_c(t)$ is a decreasing family $(c > -t)$ bounded from below by $v_\infty(t)$ and consequently converges to some limit $w_\infty(t)$ satisfying $w_\infty(t) \geqslant v_\infty(t)$. The function $t \mapsto w_\infty(t)$ is measurable, monotonically increasing and locally in L^∞.

Let us also observe the identity

$$\begin{aligned} v_c(x+2a) - 2v_c(x+a) + v_c(x) &= \int_x^{x+a} \int_s^{s+a} v''_c(t)\mathrm{d}t\mathrm{d}s \\ &= \int_x^{x+a} \int_s^{s+a} [v_c(t)^3 - tv_c(t)]\mathrm{d}t\mathrm{d}s, \end{aligned}$$

which is valid for any c, x, and a such that $x > -c$ and $x + 2a > -c$.

By the Dominated Convergence Theorem, we obtain, taking the limit $c \to +\infty$,

$$w_\infty(x+2a) - 2w_\infty(x+a) + w_\infty(x) = \int_x^{x+a} \int_s^{s+a} [w_\infty(t)^3 - tw_\infty(t)]\mathrm{d}t\mathrm{d}s. \tag{3.34}$$

We note that the right-hand side in (3.34) tends to 0 as $a \to 0$ (for fixed x). It follows, since w_∞ is a monotone function, that w_∞ is in fact continuous. Indeed, (3.34) implies easily that

$$\lim_{a \to 0^\pm} w_\infty(x+a) = w_\infty(x),$$

since the two limits (corresponding to $\pm$) are known to exist.

By Dini's theorem (which can be applied because the limit is continuous), the decreasing family $v_c(t)$ converges uniformly on any compact set to the limit $w_\infty(t)$. But for $-c < a$,

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all the functions v_c are solutions of the same equation in $(a, +\infty)$. It is then clear that the function w_∞ satisfies in the distributional sense the equation

$$u'' + tu - u^3 = 0$$

in $\mathbb{R}$, and using the ellipticity of the operator $u \mapsto u''$ and a bootstrap argument, we get that w_∞ is a C^∞ solution of the same equation.

Moreover, the function w_∞ satisfies

$$v_\infty(t) \leqslant w_\infty(t) \leqslant v_0(t).$$

According to the properties of v_∞ and v_0, it is then clear that

$$w_\infty \sim \sqrt{t},$$

as $t \to +\infty$.

We now observe that, for any $t \in \mathbb{R}$

$$w_\infty(t) \leqslant v_{-t}(t),$$

and that according to Eq. (3.15) (and its consequence for $c > 0$), $v_{-t}(t) \to 0$ as $t \to -\infty$. We have therefore shown that

$$\lim_{t \to -\infty} w_\infty(t) = 0.$$

By uniqueness, it follows that $w_\infty = v_\infty$ and the theorem is proved.

4 Connection with superconductivity

4.1 The Ginzburg–Landau functional

As mentioned in the introduction, our problem concerning the Painlevé equation comes from superconductivity. The starting point is the study of the so-called Ginzburg–Landau functional defined in a suitable domain Ω. In the case when the domain is a film, a standard (heuristic) reduction leads to the study of a reduced Ginzburg–Landau functional corresponding to one-dimensional problems in an interval. One then obtains, when considering the limiting case of the infinite interval and when restricting the analysis to symmetric solutions, the following variational problem.

Consider the local minima of the following functional:

$$\varepsilon_\infty(f, A; h) = \int_0^{+\infty} \left[\tfrac{1}{2}(f^2 - 1)^2 + \kappa^{-2} f'^2 + f^2 A^2 + A'^2\right] \mathrm{d}x + 2h\, A(0), \tag{4.1}$$

which is defined on the set of the pairs (f, A) such that $(1 - f) \in H^1(\mathbb{R}^+)$ and $A \in H^1(\mathbb{R}^+)$. The corresponding Ginzburg–Landau equations, which are simply the Euler–Lagrange equations for this functional, are then:

$$-\kappa^{-2} f'' - f + f^3 + A^2 f = 0 \quad \text{on} \quad (0, +\infty) \tag{4.2}$$

$$-A'' + f^2 A = 0 \quad \text{on} \quad (0, +\infty) \tag{4.3}$$

$$f'(0) = 0, \quad \lim_{x \to +\infty} f(x) = 1, \tag{4.4}$$

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$$A'(0) = h, \quad \lim_{x \to +\infty} A(x) = 0 \tag{4.5}$$

with $A \in H^2(]0, +\infty[)$ and $(1 - f) \in H^2(]0, +\infty[)$.

The superheating field $h^{sh}(\kappa)$ is defined as follows. Let $\mathscr{H}^{sh}(\kappa)$ be the set of $h \geqslant 0$ such that there exists a solution of the Ginzburg–Landau equations corresponding to local minima of the functional. The superheating field is then defined as the supremum of $\mathscr{H}^{sh}(\kappa)$. While it is well-known (see de Gennes, 1966) that

$$\lim_{\kappa \to +\infty} h^{sh}(\kappa) = \frac{1}{\sqrt{2}}, \tag{4.6}$$

to our knowledge, the first rigorous proof was given by Bolley & Helffer [8]. Chapman [1] proposes a much more precise formula:

$$h^{sh}(\kappa) = \frac{1}{\sqrt{2}} + C\kappa^{-\frac{4}{3}} + o(\kappa^{-\frac{4}{3}}) \tag{4.7}$$

with an inspired heuristic proof.

The mathematical proof of this formula, however, is open. The goal of this section is to analyse what is missing in this direction, and to show how the results of the present article fill in some of the gaps.

To this end, we note the following:

- Chapman constructs formal solutions to the Ginzburg–Landau equations via an expansion in κ^{-1}. However, the construction of solutions can only give a lower bound on the superheating field $h^{sh}(\kappa)$. We recall that upper bounds are available by *a priori* estimates and give (see Bolley & Helffer [8]) for the moment the weaker result
$$h^{sh}(\kappa) \leqslant \frac{1}{\sqrt{2}} + \mathcal{O}(\kappa^{-\frac{2}{3}}). \tag{4.8}$$
- Once a 'formal' solution, which is at best an approximate solution, is constructed, one would like to prove that there exists a real solution near the formal solution. In some cases, this can be done by suitable constructions of subsolutions (see, for example, Bolley & Helffer [8]).
- The formal construction by Chapman assumes the existence of a family of solutions of the Painlevé equation.
- Once the solutions to the Painlevé equation are proved to exist, it is still not clear that the value of C in (4.7) proposed by Chapman is well-defined.

As mentioned in the introduction, our contribution concerns the latter two points.

4.2 On formal computations of Chapman

In this section we recall the basic elements of Chapman's formal construction of the superheating field, and show how the results of § 2 and § 3 give rigorous meaning to the second term of the expansion. Chapman uses a technique of matching so-called outer solutions and inner solutions. An 'inner' solution corresponds to a solution near 0, and an 'outer' solution corresponds to a solution at ∞. The matching procedure leads to a family

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of 'approximate' pairs (f, A) of the Ginzburg–Landau system and the maximal value of $A'(0)$ over the family will give a candidate for the superheating field.

Outer Solution

One formally expands solutions in powers of $\epsilon = 1/\kappa$, giving

$$\begin{aligned} f_o &= f_o^{(0)} + \epsilon f_o^{(1)} + \cdots , \\ A_o &= A_o^{(0)} + \epsilon A_o^{(1)} + \cdots , \\ H_o &= H_o^{(0)} + \epsilon H_o^{(1)} + \cdots , \end{aligned}$$

where $H_o = A_o'$ and $H(0) = h$. Substituting these expansions into the Ginzburg–Landau equations, one obtains, after easy computations, the a-dependent family defined for $-1 - \sqrt{2} \leqslant a \leqslant -1$ by

$$A_o^{(0)}(x) = \frac{2\sqrt{2}a \exp x}{1 + a^2 \exp 2x}. \tag{4.9}$$

The function $f_o^{(0)}$ is then defined by the equation

$$f_o^{(0)}(x)^2 + A_o^{(0)}(x)^2 = 1. \tag{4.10}$$

So

$$H_o^{(0)}(x) = \frac{2\sqrt{2}a \exp x(1 - a^2 \exp 2x)}{(1 + a^2 \exp 2x)^2}. \tag{4.11}$$

This gives the leading outer solution, which corresponds to the formal limiting case $\kappa = +\infty$. One verifies that

$$H_o^{(0)}(0) = \frac{2\sqrt{2}\, a\, (1 - a^2)}{(1 + a^2)^2}. \tag{4.12}$$

As a result of Chapman's analysis, the first term in (4.7) is given by the maximal value of $H_o^{(0)}(0)$ over a in the appropriate range. This value is precisely $1/\sqrt{2}$, with $a = -(1 + \sqrt{2})$.

Inner Solution

To carry out the matching procedure for the outer solution with $h = 1/\sqrt{2}$, Chapman first remarks that as $x \to 0$, we have the expansion

$$f_o^{(0)}(x) \sim 2^{\frac{1}{4}} x^{\frac{1}{2}}. \tag{4.13}$$

He then defines the new variable $x = \epsilon^{\frac{2}{3}} t$ and introduces the functions $f_i = \epsilon^{\frac{1}{3}} \hat{f}_i$ and $A_i = -1 + \epsilon^{\frac{2}{3}} \hat{A}_i$. In these new coordinates the Ginzburg–Landau system is written as:

$$\left.\begin{aligned} &(a) \quad \hat{f}_i'' = \hat{f}_i^3 - 2\hat{f}_i \hat{A}_i + \epsilon^{\frac{2}{3}} \hat{f}_i \hat{A}_i^2, \\ &(b) \quad \hat{A}_i'' = -\epsilon^{\frac{4}{3}} \hat{f}_i^2 + \epsilon^2 \hat{f}_i^2 \hat{A}_i, \\ &(c) \quad H_i = \hat{A}_i'. \end{aligned}\right\} \tag{4.14}$$

We now expand $\hat{f}_i$, $\hat{A}_i$ and H_i in powers of $\epsilon^{\frac{2}{3}}$:

$$\left.\begin{aligned} &(a) \quad \hat{f}_i = \hat{f}_i^{(0)} + \epsilon^{\frac{2}{3}} \hat{f}_i^{(1)} + \epsilon^{\frac{4}{3}} \hat{f}_i^{(2)} + \cdots , \\ &(b) \quad \hat{A}_i = \hat{A}_i^{(0)} + \epsilon^{\frac{2}{3}} \hat{A}_i^{(1)} + \epsilon^{\frac{4}{3}} \hat{A}_i^{(2)} + \cdots , \\ &(c) \quad H_i = H_i^{(0)} + \epsilon^{\frac{2}{3}} H_i^{(1)} + \epsilon^{\frac{4}{3}} H_i^{(2)} + \cdots . \end{aligned}\right\} \tag{4.15}$$

Substituting these expansions into the above system and equating powers of $\epsilon^{\frac{2}{3}}$ yields at leading order

$$\left.\begin{array}{ll} (a) & (\hat{f}_i^{(0)})'' = (\hat{f}_i^{(0)})^3 - 2\hat{f}_i^{(0)}(\hat{A}_i^{(0)}), \\ (b) & (\hat{A}_i^{(0)})'' = 0, \\ (c) & H_i^{(0)} = (\hat{A}_i^{(0)})'. \end{array}\right\} \tag{4.16}$$

Since Chapman starts with $H_i^{(0)} = 1/\sqrt{2}$, it follows that

$$\hat{A}_i^{(0)} = \frac{t-c}{\sqrt{2}}. \tag{4.17}$$

The equation for $\hat{f}_i^{(0)}$ is now the equation for the second Painlevé transcendent (see Hastings & McLeod [4] and Levi & Winternitz [3]). We observe indeed that up to the scaling $\hat{f}_i^{(0)}(t) = 2^{\frac{1}{6}}u(2^{\frac{1}{6}}t)$, this equation is the same as (1.1). The Neumann condition at 0, i.e. (1.2), corresponds to the first part of (4.4).

To match with the outer solution, Chapman requires (according to (4.13)) the following condition:

$$\hat{f}_i^{(0)} \sim 2^{\frac{1}{4}}t^{\frac{1}{2}}\ , \text{ as } t \to +\infty\ , \tag{4.18}$$

which gives the boundary condition (1.3) under the same change of variables given just above. It is now clear that the family of solutions constructed in § 2 are precisely those required by Chapman (up to a change of variables).

As a result of a rather complicated 'matching' procedure based on the formal 'Van Dyke' rule [9], he then gets the following candidate for the superheating field:

$$h^{sh}(\kappa) = \frac{1}{\sqrt{2}} + \kappa^{-\frac{4}{3}} \sup_c \left(\beta(c) - \frac{c^2}{\sqrt{2}}\right) + o(\kappa^{-\frac{4}{3}}), \tag{4.19}$$

where $\beta(c)$ is defined by

$$\beta(c) = \int_0^\infty [(\hat{f}_i^{(0)})^2 - \sqrt{2}(t-c)]\mathrm{d}t. \tag{4.20}$$

By the same rescaling as above, one can check that

$$\beta(c) - \frac{c^2}{\sqrt{2}} = 2^{\frac{1}{6}}\eta(2^{\frac{1}{6}}c)\ ,$$

where η was defined by (3.5). Proposition 3.2 therefore shows that the supremum in formula (4.19) is finite and positive, and is achieved at a finite value of c.

Numerical computations give

$$\sup_c \left(\beta(c) - \frac{c^2}{\sqrt{2}}\right) \sim 0.326. \tag{4.21}$$

(See Chapman [1] or Dolgert *et al.* [10], who refer to an unpublished computation by A. Dolgert and S. J. Di Bartolo.) Chapman also observes that numerically

$$\lim_{c\to+\infty} \left(\beta(c) - \frac{c^2}{\sqrt{2}}\right) = 0, \tag{4.22}$$

$$\lim_{c\to-\infty} \left(\beta(c) - \frac{c^2}{\sqrt{2}}\right) = -\infty, \tag{4.23}$$

and that the maximum of $\beta(c) - \frac{c^2}{\sqrt{2}}$ is obtained for

$$c \sim 0.3. \tag{4.24}$$

We recall that Eqs. (4.22) and (4.23) are proved rigorously in § 3.3. The mathematical proof that $c \mapsto (\beta(c) - c^2/\sqrt{2})$ has a unique maximum is still open.

5 Conclusion

Chapman's formal contruction of solutions to the Ginzburg–Landau equations and the resulting asymptotic expansion he obtains for the superheating field leave open a number of interesting mathematical questions. In this paper, we have focused on two such issues: the existence of 'inner' solutions, and the finiteness of the coefficient of the second term in the asymptotic expansion. A next logical step in the analysis would be to provide good estimates of this coefficient. In any case, it seems that there is considerable work left to do to rigorously justify Chapman's calculations.

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