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Alexandroff Manifolds and Homogeneous Continua

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Abstract. We prove the following result announced by the second and third authors: Any homogeneous, metric ANR-continuum is a V^n_G-continuum provided $\dim_G X = n \geq 1$ and $\check{H}^n(X;G) \neq 0$, where G is a principal ideal domain. This implies that any homogeneous n-dimensional metric ANR-continuum is a V^n-continuum in the sense of Alexandroff. We also prove that any finite-dimensional cyclic in dimension n homogeneous metric continuum X, satisfying $\check{H}^n(X;G) \neq 0$ for some group G and $n \geq 1$, cannot be separated by a compactum K with $\check{H}^{n-1}(K;G) = 0$ and $\dim_G K \leq n-1$. This provides a partial answer to a question of Kallipoliti–Papasoglu as to whether a two-dimensional homogeneous Peano continuum can be separated by arcs.

1 Introduction

Cantor manifolds and stronger versions of Cantor manifolds were introduced to describe some properties of Euclidean manifolds. According to the Bing–Borsuk conjecture [2] that any homogeneous metric ANR-compactum of dimension n is an n-manifold, finite-dimensional homogeneous metric ANR-continua are supposed to share some properties with Euclidean manifolds. One of the first results in that direction established by Krupski [14] is that any homogeneous metric continuum of dimension n is a Cantor n-manifold. Recall that a space X is a *Cantor n-manifold* if any partition of X is of dimension at least $n-1$ [19] (a partition of X is a closed set $P \subset X$ such that $X \backslash P$ is the union of two open disjoint sets). In other words, X cannot be the union of two proper closed sets whose intersection is of covering dimension at most $n-2$. Stronger versions of Cantor manifolds were considered by Hadžiivanov [9] and Hadžiivanov and Todorov [10]. But the strongest specification of Cantor manifolds is the notion of V^n-continua introduced by Alexandroff [1]: a compactum X is a *V^n-continuum* if for every two closed disjoint massive subsets X_0, X_1 of X there exists an open cover ω of X such that there is no partition P in X between X_0 and X_1 admitting an ω-map into a space Y with $\dim Y \leq n-2$ ($f\colon P \to Y$ is said to be an ω-map if there exists an open cover γ of Y such that $f^{-1}(\gamma)$ refines ω). Recall that a massive subset of X is a set with non-empty interior in X.

More general concepts of the above notions were considered in [12]. In particular, we are going to use the following one, where $\mathcal{C}$ is a class of topological spaces.

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Definition 1.1 A space X is an *Alexandroff manifold with respect to* $\mathcal{C}$ (briefly, *Alexandroff* $\mathcal{C}$*-manifold*) if for every two closed, disjoint, massive subsets X_0, X_1 of X there exists an open cover ω of X such that there is no partition P in X between X_0 and X_1 admitting an ω-map onto a space $Y \in \mathcal{C}$.

In this paper we continue investigating to what extent homogeneous continua have common properties with Euclidean manifolds. One of the main questions in this direction is whether any homogeneous n-dimensional metric ANR-compactum X is a V^n-continuum; see [18]. A partial answer to this question, when the Čech cohomology group $\check{H}^n(X)$ is non-trivial, was announced in [18]. One of the aims of the paper is to provide the proof of this fact; see Section 3. Our proof is based on the properties of (n, G)-bubbles and V^n_G-continua investigated in Section 2. We also provide a partial answer to a question of Kallipoliti–Papasoglu [11].

2 (n, G)-bubbles and V^n_G-continua

In this section we investigate the connection between (n, G)-bubbles and V^n_G-continua.

For every abelian group G let $\dim_G X$ be the cohomological dimension of X with respect to G, and let $\check{H}^n(X; G)$ denote the reduced n-th Čech cohomology group of X with coefficients in G.

Reformulating the original definition of Kuperberg [15], Yokoi [20] provided the following definition (see also [3] and [13]).

Definition 2.1 If G is an abelian group and $n \geq 0$, a compactum X is called an (n, G)*-bubble* if $\check{H}^n(X; G) \neq 0$ and $\check{H}^n(A; G) = 0$ for every proper closed subset A of X. Following [17] we say that a compactum X is a *generalized* (n, G)*-bubble* provided there exists a surjective map $f\colon X \to Y$ such that the homomorphism $f^*\colon \check{H}^n(Y; G) \to \check{H}^n(X; G)$ is nontrivial, but $f_A^*(\check{H}^n(Y; G)) = 0$ for any proper closed subset A of X, where f_A is the restriction of f over A.

We also need the following notion.

Definition 2.2 A compactum X is said to be a V^n_G*-continuum* [16] if for every two closed, disjoint, massive subsets X_0, X_1 of X there exists an open cover ω of X such that any partition P in X between X_0 and X_1 does not admit an ω-map g onto a space Y with $g^*\colon \check{H}^{n-1}(Y; G) \to \check{H}^{n-1}(P; G)$ being trivial.

Since $\check{H}^{n-1}(Y; G) = 0$ for any compactum Y with $\dim_G Y \leq n-2$, V^n_G-continua are Alexandroff manifolds with respect to the class D^{n-2}_G of all spaces of dimension $\dim_G \leq n-2$. Moreover, if $X \in V^n_G$, then for every partition C of X we have $\check{H}^{n-1}(C; G) \neq 0$. The last observation implies that $\dim_G X \geq n$ provided X is a metric V^n_G-compactum such that either $X \in ANR$ or $\dim X < \infty$ and G is countable. Indeed, if $\dim_G X \leq n-1$, then each $x \in X$ has a local base of open sets U whose boundaries are of dimension $\dim_G \leq n-2$; see [6]. Hence, any such a boundary Γ is a partition of X with $\check{H}^{n-1}(\Gamma; G) = 0$.

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The following theorem was established in [16, Theorem 3] for finite-dimensional metric (n, G)-bubbles. Let us note that, according to [20], the examples of Dranishnikov [4] and Dydak–Walsh [7] show the existence of an infinite-dimensional $(n, \mathbb{Z})$-bubble with $n \geq 5$.

Theorem 2.3 *Any generalized (n, G)-bubble X is a V^n_G-continuum.*

Proof Let $f\colon X \to Y$ be a map such that $f^*(\check{H}^n(Y; G)) \neq 0$ and $f^*_A(\check{H}^n(Y; G)) = 0$ for any proper closed set $A \subset X$. If ω is a finite open cover of a closed set $Z \subset X$, we denote by $|\omega|$ and p_ω, respectively, the nerve of ω and a map from Z onto $|\omega|$ generated by a partition of unity subordinated to ω. Furthermore, if $C \subset Z$ and $\omega(C) = \{W \in \omega : W \cap C \neq \varnothing\}$, then $p_{\omega(C)}\colon C \to |\omega(C)|$ is the restriction $p_\omega|C$. Recall also that p_ω generates maps $p^*_\omega\colon \check{H}^k(|\omega|; G) \to \check{H}^k(Z; G)$, $k \geq 0$. Moreover, if $q_\omega\colon Z \to |\omega|$ is a map generating by (another) partition of unity subordinated to $|\omega|$, then p_ω and q_ω are homotopic. So, $p^*_\omega = q^*_\omega$.

Claim 1 For every pair of non-empty open sets U_1 and U_2 in X with $\overline{U}_1 \cap \overline{U}_2 = \varnothing$ there exist an open cover ω of $X \backslash (U_1 \cup U_2)$, a map $p_\omega\colon X \backslash (U_1 \cup U_2) \to |\omega|$ and an element $e \in \check{H}^{n-1}(|\omega|; G)$ such that $p^*_{\omega(C)}(i^*_C(\mathrm{e})) \neq 0$ for every partition C of X between $\overline{U}_1$ and $\overline{U}_2$, where i_C is the inclusion $|\omega(C)| \hookrightarrow |\omega|$.

Proof of Claim 1 To prove this claim we follow the arguments from the proof of [17, Theorem]. Let U_1 and U_2 be non-empty open subsets of X with disjoint closures, and $i_k\colon F_k \hookrightarrow X$ be the inclusion of $F_k = X \backslash U_k$ into X, $k = 1, 2$. Consider the Mayer–Vietoris exact sequence

$$\check{H}^{n-1}(F_1 \cap F_2; G) \xrightarrow{\ \delta\ } \check{H}^n(X; G) \xrightarrow{\ j\ } \check{H}^n(F_1; G) \oplus \check{H}^n(F_2; G)$$

with $j = (i^*_1, i^*_2)$, and choose a non-zero element $\mathrm{e}_1 \in f^*(\check{H}^n(Y; G)) \subset \check{H}^n(X; G)$. For each $k = 1, 2$ we have the commutative diagram, where δ_k is the inclusion of $f(F_k)$ into Y:

$$\begin{array}{ccc} \check{H}^n(Y; G) & \xrightarrow{\ f^*\ } & \check{H}^n(X; G) \\ \Big\downarrow \delta^*_k & & \Big\downarrow i^*_k \\ \check{H}^n(f(F_k); G) & \xrightarrow{\ f^*_{F_k}\ } & \check{H}^n(F_k; G). \end{array}$$

So $i^*_k(\mathrm{e}_1) = 0$, $k = 1, 2$, which yields $\mathrm{e}_1 = \delta(\mathrm{e}_2)$ for some non-zero element $\mathrm{e}_2 \in \check{H}^{n-1}(F_1 \cap F_2; G)$. Then there exist an open cover ω of $F_1 \cap F_2 = X \backslash (U_1 \cup U_2)$, a map $p_\omega\colon F_1 \cap F_2 \to |\omega|$, and $\mathrm{e} \in \check{H}^{n-1}(|\omega|; G)$ with $p^*_\omega(\mathrm{e}) = \mathrm{e}_2$.

Let C be a partition of X between $\overline{U}_1$ and $\overline{U}_2$. So $X = P_1 \cup P_2$ and $C = P_1 \cap P_2$, where each P_k is a closed subset of X containing $\overline{U}_k$, $k = 1, 2$. Denote by $i\colon C \hookrightarrow F_1 \cap F_2$, $in_1\colon P_1 \hookrightarrow F_2$ and $in_2\colon P_2 \hookrightarrow F_1$ the corresponding inclusions. Then we have

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the following commutative diagram, whose rows are Mayer–Vietoris sequences:

$$\begin{array}{ccccc}
\check{H}^{n-1}(F_1\cap F_2;G) & \xrightarrow{\ \delta\ } & \check{H}^{n}(X;G) & \xrightarrow{\ j\ } & \check{H}^{n}(F_2;G)\oplus \check{H}^{n}(F_1;G)\\
\Big\downarrow{\scriptstyle i^*} & & \Big\downarrow{\scriptstyle id} & & \Big\downarrow{\scriptstyle in_1^*\oplus in_2^*}\\
\check{H}^{n-1}(C;G) & \xrightarrow{\ \delta_1\ } & \check{H}^{n}(X;G) & \xrightarrow{\ j_1\ } & \check{H}^{n}(P_1;G)\oplus \check{H}^{n}(P_2;G).
\end{array}$$

Obviously,

$$\delta_1\big(i^*(\mathrm{e}_2)\big) = \mathrm{id}\big(\delta(\mathrm{e}_2)\big) = \mathrm{e}_1 \neq 0. \tag{2.1}$$

On the other hand, the commutativity of the diagram

$$\begin{array}{ccc}
\check{H}^{n-1}(|\omega|;G) & \xrightarrow{\ p_\omega^*\ } & \check{H}^{n-1}(F_1\cap F_2;G)\\
\Big\downarrow{\scriptstyle i_C^*} & & \Big\downarrow{\scriptstyle i^*}\\
\check{H}^{n-1}(|\omega(C)|;G) & \xrightarrow{\ p_{\omega(C)}^*\ } & \check{H}^{n-1}(C;G)
\end{array}$$

implies that $p^*_{\omega(C)}(i^*_C(\mathrm{e})) = i^*(p^*_\omega(\mathrm{e})) = i^*(\mathrm{e}_2)$. Therefore, according to (2.1), $p^*_{\omega(C)}(i^*_C(\mathrm{e})) \neq 0$. This completes the proof of Claim 1. ∎

Now, we can show that $X \in V^n_G$. Let U_1 and U_2 be non-empty open subsets of X with disjoint closures. Then there exists a finite open cover ω of $X\backslash(U_1\cup U_2)$, a map $p_\omega\colon X\backslash(U_1\cap U_2)\to|\omega|$ and an element $\mathrm{e}\in\check{H}^{n-1}(|\omega|;G)$ satisfying the conditions from Claim 1. For each $W\in\omega$ let $h(W)$ be an open subset of X extending W. So, $\gamma=\{h(W) : W\in\omega\}\cup\{U_1,U_2\}$ is a finite open cover of X whose restriction on $X\backslash(U_1\cup U_2)$ is ω.

Suppose there exists a partition C of X between $\overline{U}_1$ and $\overline{U}_2$ admitting a γ-map g onto a space T with $g^*(\check{H}^{n-1}(T;G)) = 0$. Thus, we can find a finite open cover α of T such that $\beta = g^{-1}(\alpha)$ refines ω. Let $p_\beta\colon C\to|\beta|$ be a map onto the nerve of β generated by a partition of unity subordinated to β. Obviously, the function $V\in\alpha\to g^{-1}(V)\in\beta$ generates a a simplicial homeomorphism $g^\alpha_\beta\colon|\alpha|\to|\beta|$. Then the maps p_β and $g_\alpha = g^\alpha_\beta\circ\pi_\alpha\circ g$, where $\pi_\alpha\colon T\to|\alpha|$ is a map generated by a partition of unity subordinated to $|\alpha|$, are homotopic. Hence, $p^*_\beta = g^*\circ\pi^*_\alpha\circ(g^\alpha_\beta)^*$. Because $g^*\colon\check{H}^{n-1}(T;G)\to\check{H}^{n-1}(C;G)$ is a trivial map, the last equality implies that so is the map $p^*_\beta\colon\check{H}^{n-1}(|\beta|;G)\to\check{H}^{n-1}(C;G)$. On the other hand, since β refines ω, we can find a map $\varphi_\beta\colon|\beta|\to|\omega(C)|$ such that $p_{\omega(C)}$ and $\varphi_\beta\circ p_\beta$ are homotopic. Therefore, $p^*_{\omega(C)} = p^*_\beta\circ\varphi^*_\beta$. According to Claim 1, $p^*_{\omega(C)}(\mathrm{e}_C)\neq 0$, where e_C is the element $i^*_C(\mathrm{e})\in\check{H}^{n-1}(|\omega(C)|;G)$. So, $p^*_\beta(\varphi^*_\beta(\mathrm{e}_C))\neq 0$, which contradicts the triviality of p^*_β. ∎

We can extend the definition of V^n_G-continua as follows.

Definition 2.4 A compactum X is said to be a V^n_G*-continuum with respect to a given class* $\mathcal{A}$ if for every two closed, disjoint, massive subsets X_0, X_1 of X there exists an open cover ω of X such that any partition P in X between X_0 and X_1 does not admit an ω-map g onto a space $Y \in \mathcal{A}$ with $g^*\colon \check{H}^{n-1}(Y;G) \to \check{H}^{n-1}(P;G)$ being trivial.

Recall that a metric space X is *strongly n-universal* if any map $g\colon K \to X$, where K is a metric compactum of dimension $\dim K \le n$, can be approximated by embeddings.

Theorem 2.5 *Let X be a metric compactum containing a strongly n-universal dense subspace M such that M is an absolute extensor for n-dimensional compacta with* $n \ge 1$. *Then X is a* V^n_G*-continuum with respect to the class* D^{n-1}_G *of all spaces of dimension* $\dim_G \le n-1$. *In particular, X is an Alexandroff manifold with respect to the class* D^{n-2}_G.

Proof Suppose that X is not a V^n_G-continuum with respect to the class D^{n-1}_G. So we can find open sets U and V in X with disjoint closures such that for every $\epsilon > 0$ there exists a partition C_ϵ between $\overline{U}$ and $\overline{V}$ admitting an ϵ-map g_ϵ onto a space $Y_\epsilon \in D^{n-1}_G$ such that $g^*_\epsilon\colon \check{H}^{n-1}(Y_\epsilon;G) \to \check{H}^{n-1}(C_\epsilon;G)$ is trivial.

Consider two different points a, b from the n-sphere $\mathbb{S}^n$ and a map $f\colon \mathbb{S}^n \to M$ with $f(a) \in U \cap M$ and $f(b) \in V \cap M$ (such a map exists because M is an absolute extensor for n-dimensional compacta). Since M is strongly n-universal, we can approximate f by a homeomorphism $h\colon \mathbb{S}^n \to M$ such that $h(a) \in U$ and $h(b) \in V$. Therefore, $K_\epsilon = C_\epsilon \cap h(\mathbb{S}^n)$ is a partition of $h(\mathbb{S}^n)$ between $h(\mathbb{S}^n) \cap \overline{U}$ and $h(\mathbb{S}^n) \cap \overline{V}$. Then $Z = g_\epsilon(K_\epsilon)$ is a closed subset of Y_ϵ, and since $\dim_G Y_\epsilon \le n-1$, $i^*_Z\colon \check{H}^{n-1}(Y_\epsilon;G) \to \check{H}^{n-1}(Z;G)$ is a surjective map, where $i_Z\colon Z \hookrightarrow Y_\epsilon$ is the inclusion. So, we have the following commutative diagram with $g_{K_\epsilon} = g|K_\epsilon$ and $i_{K_\epsilon}\colon K_\epsilon \hookrightarrow C_\epsilon$:

$$\begin{array}{ccc} \check{H}^{n-1}(Y_\epsilon;G) & \xrightarrow{g^*_\epsilon} & \check{H}^{n-1}(C_\epsilon;G) \\ \Big\downarrow i^*_Z & & \Big\downarrow i^*_{K_\epsilon} \\ \check{H}^{n-1}(Z;G) & \xrightarrow{g^*_{K_\epsilon}} & \check{H}^{n-1}(K_\epsilon;G). \end{array}$$

Because g^*_ϵ is trivial and i^*_Z is surjective, $g^*_{K_\epsilon}$ is also trivial. Hence, for every $\epsilon > 0$ there exists a partition K_ϵ between $h(\mathbb{S}^n) \cap \overline{U}$ and $h(\mathbb{S}^n) \cap \overline{V}$ admitting an ϵ-map g_{K_ϵ} onto a space Z such that $g^*_{K_\epsilon}\colon \check{H}^{n-1}(Z;G) \to \check{H}^{n-1}(K_\epsilon;G)$ is trivial. This means that $\mathbb{S}^n$ is not a V^n_G-continuum. On the other hand, $\mathbb{S}^n$ is an (n, G)-bubble for all G. So by Theorem 2.3 , $\mathbb{S}^n$ is a V^n_G-continuum, a contradiction. ∎

Corollary 2.6 *Let X be either the universal Menger compactum* μ^n *or X be a metric compactification of the universal Nöbeling space* ν^n. *Then X is a* V^n_G*-continuum with respect to the class* D^{n-1}_G *for any G. Moreover,* μ^n *is not a* V^n_G*-continuum.*

Proof Since both μ^n and ν^n are strongly n-universal absolute extensors for n-dimensional compacta, it follows from Theorem 2.5 that X is a V^n_G-continuum with respect to the class D_G^{n-1}. To show that μ^n is not a V_G^n-continuum, it suffices to find a partition E of μ^n with trivial $\check{H}^{n-1}(E;G)$. One can show the existence of such partitions using the geometric construction of the Menger compactum. We provide a proof of this fact using Dranishnikov's results from [5]. Indeed, by [5, Theorem 2], there exists a map $g\colon \mu^n \to \mathbb{I}^\infty$ such that $g^{-1}(P)$ is homeomorphic to μ^n for any AR-space $P \subset \mathbb{I}^\infty$. If $P \in AR$ is a partition of $\mathbb{I}^\infty$, then $g^{-1}(P)$ is a partition of μ^n homeomorphic to μ^n. Hence, $\check{H}^{n-1}(g^{-1}(P);G) = 0$. ■

3 Homogeneous Continua

In this section we prove that some homogeneous continua are V_G^n-continua. Recall that a space X is said to be *homogeneous* if for every two points $x, y \in X$ there exists a homeomorphism $h\colon X \to X$ with $h(x) = y$. Krupski has conjectured that any n-dimensional, homogeneous metric ANR-continuum is a V^n-continuum.[1] The next result provides a partial solution to Krupski's conjecture and a partial answer to [18, Question 2.4].

Theorem 3.1 *Let X be a homogeneous, metric ANR-continuum such that $\dim_G X = n \geq 1$ and $\check{H}^n(X;G) \neq 0$, where G is a principal ideal domain. Then X is a V_G^n-continuum.*

Proof According to [20, Theorem 3.3], any space X satisfying the conditions from this theorem is an (n, G)-bubble. Hence, by Theorem 2.3, $X \in V_G^n$. ■

Bing and Borsuk [2] raised the question whether no compact acyclic in dimension $n-1$ subset of X separates X, where X is a metric n-dimensional homogeneous ANR-continuum. Yokoi [20, Corollary 3.4] provided a partial positive answer to this question in the case where X is a homogeneous metric n-dimensional ANR-continuum such that $\check{H}^n(X;\mathbb{Z}) \neq 0$. The next proposition is a version of Yokoi's result when X is not necessarily an ANR.

Proposition 3.2 *Let X be a finite-dimensional homogeneous metric continuum with $\check{H}^n(X;G) \neq 0$. Then $\check{H}^{n-1}(C;G) \neq 0$ for any partition C of X such that $\dim_G C \leq n-1$.*

Proof Suppose there exists a partition C of X such that

$$\check{H}^{n-1}(C;G) = 0 \quad \text{and} \quad \dim_G C \leq n-1.$$

The last inequality implies that the inclusion homomorphism

$$\check{H}^{n-1}(C;G) \to \check{H}^{n-1}(A;G)$$

is an epimorphism for every closed set $A \subset C$. So, $\check{H}^{n-1}(A;G) = 0$ for all closed subsets of C. Therefore, we may assume that C does not have any interior points. Since $\check{H}^n(X;G) \neq 0$, according to [16, Theorem 2], there exists a compact subset

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$K \subset X$ with $K \in V_G^n$. Since X is homogeneous, we may also assume that $K \cap C \neq \varnothing$. Observe that $z \in K \backslash C$ for some z. Indeed, the inclusion $K \subset C$ would imply that $\check{H}^{n-1}(P; G) = 0$ for every partition P of K. Let $X \backslash C = U \cup V$ and $z \in V$, where U and V are nonempty, open ,and disjoint sets in X. Then the Effros theorem [8] allows us to push K towards U by a small homeomorphism $h\colon X \to X$ so that the image $h(K)$ meets both U and V (see the proof of [14, Lemma 2] for a similar application of Effros' theorem). To do this, we let ϵ be the distance from z to the boundary of V and choose δ so that the pair (ϵ, δ) satisfies the Effros property. Further, we pick points $x \in K$ and $y \in U$ such that $\operatorname{dist}(x, y) < \delta$ and choose a homeomorphism h such that $h(x) = h(y)$ and h is ϵ-close to the identity. Therefore, $S = h(K) \cap C$ is a partition of $h(K)$ and $\check{H}^{n-1}(S; G) = 0$, because $S \subset C$, a contradiction. ■

Proposition 3.2 provides a partial answer to Kallipoliti–Papasoglu's question [11] as to whether homogeneous, two-dimensional, metric, locally connected continua can be separated by arcs.

Corollary 3.3 *No finite-dimensional, metric, homogeneous continuum X having $\check{H}^2(X; G) \neq 0$ can be separated by any one-dimensional compactum C with $\check{H}^1(C; G) = 0$.*

4 Some Remarks and Problems

The class of (n, G)-bubbles is stable in the sense of the following proposition.

Proposition 4.1 *Let X be a metric compactum admitting an ϵ-map onto an (n, G)-bubble for any $\epsilon > 0$. Then X is also an (n, G)-bubble.*

Proof First, let us show that $\check{H}^n(X; G) \neq 0$. Take any open cover ω of X and let ϵ be the Lebesgue number of ω. There exists a surjective ϵ-map $f\colon X \to Y_\epsilon$ with Y_ϵ being an (n, G)-bubble. Since $\check{H}^n(Y_\epsilon; G) \neq 0$, we can find an open cover α of Y_ϵ such that $\check{H}^n(|\alpha|; G) \neq 0$ (we use the notations from the proof of Theorem 2.3). The $\beta = f^{-1}(\alpha)$ is an open cover of X refining ω such that $|\beta|$ is homeomorphic to $|\alpha|$. So, $\check{H}^n(|\beta|; G) \neq 0$, which implies that $\check{H}^n(X; G) \neq 0$.

Suppose now that A is a proper closed subset of X and γ is an open (in A) cover of A. Extend each $U \in \gamma$ to an open set $V(U)$ in X and let $W = \cup\{V(U) : U \in \gamma\}$. We can suppose that $W \neq X$. Choose a surjective η-map $g\colon X \to Y_\eta$ such that Y_η is an (n, G)-bubble with η being a positive number smaller than both $\operatorname{dist}(A, X \backslash W)$ and the Lebesgue number of γ. Then $B = g(A)$ is a proper closed subset of Y_η such that $g^{-1}(B) \subset W$. There exists an open cover θ of B such that the family $\delta = \{g^{-1}(G) \cap A : G \in \theta\}$ is an open cover of A refining γ. Obviously, $|\delta|$ is homeomorphic to $|\theta|$. Since $\check{H}^n(B; G) = 0$, we have $\check{H}^n(|\theta|; G) = \check{H}^n(|\delta|; G) = 0$. Hence $\check{H}^n(A; G) = 0$, which completes the proof. ■

Now, we are going to discuss some problems. The main question suggested by the results of this paper is whether any of the conditions for X can be removed in Theorem 3.1. Since, according to Theorem 2.3, μ^n is not a V_G^n-continuum for any G, the condition X to be an *ANR* cannot be removed. So, we have the following question.

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Problem 4.2 *Let X be a homogeneous metric ANR-continuum X with $\dim_G X = n$, where G is any abelian group. Is X a V^n_G-continuum?*

Since any V^n_G-continuum with respect to the class D^{n-1}_G is V^n, the next question is still interesting.

Problem 4.3 *Let X be a homogeneous metric continuum X with $\dim_G X = n$. Is X a V^n_G-continuum with respect to the class D^{n-1}_G? What if $\check{H}^n(X;G) \neq 0$?*

Another question is whether finite-dimensionality can be removed from the result of Stefanov [16], which was applied above.

Problem 4.4 *Let X be a metrizable compactum with $\check{H}^n(X;G) \neq 0$ for some group G and $n \geq 1$. Does X contain a V^n_G-continuum?*

We can show that any finite simplicial complex is a generalized (n, G)-bubble if and only if it is an (n, G)-bubble. So, our last question is whether this remains true for all metric compacta.

Problem 4.5 *Is there any metric compactum X that is a generalized (n, G)-bubble but not an (n, G)-bubble?*

5 Added in Proof

Recently, Problem 4.3 has been solved in the positive in a forthcoming paper by the third author.

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