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A NEW DP-MINIMAL EXPANSION OF THE INTEGERS

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Abstract. We consider the structure $(\mathbb{Z}, +, 0, |_{p_1}, \dots, |_{p_n})$, where $x|_p y$ means $v_p(x) \leq v_p(y)$ and v_p is the p-adic valuation. We prove that this structure has quantifier elimination in a natural expansion of the language of abelian groups, and that it has dp-rank n. In addition, we prove that a first order structure with universe $\mathbb{Z}$ which is an expansion of $(\mathbb{Z}, +, 0)$ and a reduct of $(\mathbb{Z}, +, 0, |_p)$ must be interdefinable with one of them. We also give an alternative proof for Conant's analogous result about $(\mathbb{Z}, +, 0, <)$.

§1. Introduction. The study of "well-behaved" expansions of $(\mathbb{Z}, +, 0)$ is a recent subject. Until not long ago, no examples of such structures were studied, other than $(\mathbb{Z}, +, 0, <)$. The first stable examples were given independently by Palacín and Sklinos [13] and by Poizat [15]. Specifically, they both proved, using different methods, that for any integer $q \geq 2$ the structure $(\mathbb{Z}, +, 0, \prod_q)$ is superstable of U-rank ω, where $\prod_q = \{q^n : n \in \mathbb{N}\}$. Palacín and Sklinos also showed the same result for other examples, such as $(\mathbb{Z}, +, 0, \mathrm{Fac})$, where $\mathrm{Fac} = \{n! : n \in \mathbb{N}\}$. Conant [4] and Lambotte and Point [10] independently generalized these results. For a subset $A \subseteq \mathbb{Z}$ with either an upper bound or a lower bound, they give some sparsity conditions on A which are sufficient for the structure $(\mathbb{Z}, +, 0, A)$ to be superstable of U-rank ω. Conant also gives sparsity conditions which are necessary for the structure $(\mathbb{Z}, +, 0, A)$ to be stable.

A different kind of example was given recently by Kaplan and Shelah in [9]. They proved that for $\mathrm{Pr} = \{p \in \mathbb{Z} : |p| \text{ is prime}\}$, the structure $(\mathbb{Z}, +, 0, \mathrm{Pr})$ has the independence property (and even the n-independence property for all n) hence it is unstable. On the other hand, assuming Dickson's Conjecture,[1] it is supersimple of U-rank 1.

In contrast to the above, $(\mathbb{Z}, +, 0, <)$ remained the only known unstable dp-minimal expansion of $(\mathbb{Z}, +, 0)$. In [1, Question 5.32], Aschenbrenner, Dolich, Haskell, Macpherson, and Starchenko ask $(\star)$ whether every dp-minimal expansion of $(\mathbb{Z}, +, 0)$ is a reduct of $(\mathbb{Z}, +, 0, <)$. In [2] the same authors prove that $(\mathbb{Z}, +, 0, <)$ has no proper dp-minimal expansions. This was later strengthened by Dolich and Goodrick, who proved in [6] that $(\mathbb{Z}, +, 0, <)$ has no proper strong expansions. Together with a result of Conant which we describe below (Fact 1.8),

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[1]A strong number-theoretic conjecture about primes in arithmetic progressions, which generalizes Dirichlet's theorem on prime numbers.

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this means that any other unstable dp-minimal expansion of $(\mathbb{Z}, +, 0)$, if exists, is not a reduct, nor an expansion of $(\mathbb{Z}, +, 0, <)$.

In the first part of this article we introduce a new family of dp-minimal expansions of $(\mathbb{Z}, +, 0)$, thus giving a negative answer to the question $(\star)$ above. More generally, for every $n \in \mathbb{N} \cup \{\omega\}$ we introduce a family of expansions of $(\mathbb{Z}, +, 0)$ having dp-rank n. For a prime number p, let $v_p : \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$ be the p-adic valuation, namely, $v_p(a) = \sup\{k \in \mathbb{N} : p^k | a\}$. Let $\emptyset \neq P \subseteq \mathbb{N}$ be a (possibly infinite) set of primes, and let L_P be the language $\{+, 0\} \cup \{|_p : p \in P\}$, where each $|_p$ is a binary relation. We expand $(\mathbb{Z}, +, 0)$ to an L_P-structure $\mathcal{Z}_P$ by interpreting $a|_p b$ as $v_p(a) \leq v_p(b)$ for each $p \in P$. We denote $T_P := Th(\mathcal{Z}_P)$. For convenience, we enumerate P by $P = \{p_\alpha : \alpha < |P|\}$, and p without a subscript usually denotes some $p \in P$. If $P = \{p\}$ we write T_p instead of $T_{\{p\}}$, etc.

We first prove that T_P eliminates quantifiers in a natural definitional expansion. Let $L_P^E = L_P \cup \{-, 1\} \cup \{D_n : n \geq 1\}$, where $-$ and 1 are interpreted in the obvious way, and for each $n \geq 1$, D_n is an unary relation symbol interpreted as $\{na : a \in \mathbb{Z}\}$.

THEOREM 1.1. *For every nonempty set P of primes, the theory T_P eliminates quantifiers in the language L_P^E.*

After proving this we were informed that a similar result has been proved independently by François Guignot [8], and again by Nathanaël Mariaule [11, Corollary 2.11].

Using quantifier elimination, we are able to determine the dp-rank of T_P.

THEOREM 1.2. *For every nonempty set P of primes, dp-rank$(T_P) = |P|$.*

In particular, for a single prime p we have that T_p is dp-minimal, i.e., dp-rank$(T_p) = 1$.

We now move to our second result. We first give some context and history.

DEFINITION 1.3. Let L_1 and L_2 be two first-order languages, and let $\mathcal{M}_1$ be an L_1-structure and $\mathcal{M}_2$ an L_2-structure, both with the same underlying universe M. Let $A \subseteq M$ be a set of parameters.

(1) We say that $\mathcal{M}_1$ is an *A-reduct* of $\mathcal{M}_2$, and $\mathcal{M}_2$ is an *A-expansion* of $\mathcal{M}_1$, if for every $n \geq 1$, every subset of M^n which is L_1-definable over $\emptyset$ (equivalently, over A) is also L_2-definable over A. When $A = M$ we just say that $\mathcal{M}_1$ is a *reduct* of $\mathcal{M}_2$, and $\mathcal{M}_2$ is an *expansion* of $\mathcal{M}_1$. We will mostly use this with either $A = \emptyset$ or $A = M$.

(2) We say that $\mathcal{M}_1$ and $\mathcal{M}_2$ are *A-interdefinable* if $\mathcal{M}_1$ is an A-reduct of $\mathcal{M}_2$ and $\mathcal{M}_2$ is an A-reduct of $\mathcal{M}_1$. When $A = M$ we just say that $\mathcal{M}_1$ and $\mathcal{M}_2$ are *interdefinable.*

(3) Let $A \subseteq B \subseteq M$ be another set of parameters. We say that $\mathcal{M}_1$ is a *B-proper A-reduct* of $\mathcal{M}_2$, and $\mathcal{M}_2$ is a *B-proper A-expansion* of $\mathcal{M}_1$, if $\mathcal{M}_1$ is an A-reduct of $\mathcal{M}_2$, but $\mathcal{M}_2$ is not a B-reduct of $\mathcal{M}_1$. When $B = M$ we just say *proper* instead of B-proper. We will mostly use this with either $B = M$ or $B = \emptyset$.

Let $\mathcal{M}_1$ be an L_1-structure and $\mathcal{M}_2$ an L_2-structure, both with the same underlying universe M, and suppose that $\mathcal{M}_1$ is a $\emptyset$-reduct of $\mathcal{M}_2$. Then we can replace L_2 by $L_2 \cup L_1$, interpreting each L_1-symbol in $\mathcal{M}_2$ as it is interpreted in $\mathcal{M}_1$. As

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we have not added new $\emptyset$-definable sets, this new structure is $\emptyset$-interdefinable with the original $\mathcal{M}_2$. Therefore we may always assume for simplicity of notation that $L_1 \subseteq L_2$ and $\mathcal{M}_1 = \mathcal{M}_2|_{L_1}$.

A-reducts are preserved by elementary extensions and elementary substructures containing A, in the following sense:

OBSERVATION 1.4. *Let $\mathcal{M} \prec \mathcal{N}$ be two L-structures with universes M and N respectively. Let $A \subseteq M$ and let $\mathcal{N}'$ be an A-reduct of $\mathcal{N}$ with language L'. Let $\mathcal{M}'$ be the structure obtained by restricting the relations and functions of $\mathcal{N}'$ to M. Then*:

(1) *$\mathcal{M}'$ is well defined, it is an A-reduct of $\mathcal{M}$, and $\mathcal{M}' \prec \mathcal{N}'$.*
(2) *$\mathcal{N}'$ is an A-proper A-reduct of $\mathcal{N}$ if and only if $\mathcal{M}'$ is an A-proper A-reduct of $\mathcal{M}$.*
(3) *$\mathcal{N}'$ is a proper A-reduct of $\mathcal{N}$ if and only if $\mathcal{M}'$ is a proper A-reduct of $\mathcal{M}$.*

REMARK 1.5. Observation 1.4 is not necessarily true if $A \not\subseteq M$. If $\mathcal{N}'$ contains a constant $c \notin M$, or a n-ary function f such that $f(M^n) \not\subseteq M$, then $\mathcal{M}'$ is not well-defined. Even when it is well-defined, the rest is still not necessarily true. For example, let $\mathcal{M} = (\mathbb{Z}, +, 0, 1, <)$, and let $\mathcal{N} = (N, +, 0, 1, <)$ be a nontrivial elementary extension of $\mathcal{M}$. Let $b \in N$ be a positive infinite element, and let $\mathcal{N}' = (N, +, 0, 1, [0, b])$. Then $\mathcal{M}' = (\mathbb{Z}, +, 0, 1, \mathbb{N}) \not\prec \mathcal{N}'$ (as $[0, b]$ contains an element $x = b$ such that $x \in [0, b]$ but $x + 1 \notin [0, b]$). Also, $\mathcal{M}'$ is interdefinable with $\mathcal{M}$, but we will see that $\mathcal{N}'$ is a proper reduct of $\mathcal{N}$.

DEFINITION 1.6. Let $\mathcal{F}$ be a family of first-order structures, and let $\mathcal{M} \in \mathcal{F}$. We say that $\mathcal{M}$ is A-*minimal* in $\mathcal{F}$ if there are no A-proper A-reducts of $\mathcal{M}$ in $\mathcal{F}$. We say that $\mathcal{M}$ is A-*maximal* in $\mathcal{F}$ if there are no A-proper A-expansions of $\mathcal{M}$ in $\mathcal{F}$. When $A = M$ we just say that $\mathcal{M}$ is *minimal* or *maximal*, respectively.

An example of this phenomenon was given by Pillay and Steinhorn, who proved in [14] that $(\mathbb{N}, <)$ has no proper o-minimal expansions, i.e., it is a maximal o-minimal structure. Another example was given by Marker, who proved in [12] that if $\mathcal{N}$ is a $\emptyset$-expansion of $(\mathbb{C}, +, \cdot, 0, 1)$ and a reduct of $(\mathbb{C}, +, \cdot, 0, 1, \mathbb{R})$, then $\mathcal{N}$ is interdefinable with either $(\mathbb{C}, +, \cdot, 0, 1)$ or $(\mathbb{C}, +, \cdot, 0, 1, \mathbb{R})$, i.e., $(\mathbb{C}, +, \cdot, 0, 1, \mathbb{R})$ is minimal among the proper expansions of $(\mathbb{C}, +, \cdot, 0, 1)$. A much more recent example, given by Dolich and Goodrick in [6], was already mentioned above: $(\mathbb{Z}, +, 0, <)$ has no proper strong expansions, i.e., it is maximal among the strong structures.[2]

A concrete example to an even stronger phenomenon was recently given. Based on a result by Palacín and Sklinos [13], Conant and Pillay proved in [5] the following:

FACT 1.7 ([5, Theorem 1.2]). $(\mathbb{Z}, +, 0, 1)$ *has no proper stable expansions of finite dp-rank*.

In other words, $(\mathbb{Z}, +, 0, 1)$ is maximal among the stable structures of finite dp-rank. This theorem is no longer true if we replace $(\mathbb{Z}, +, 0, 1)$ by an elementarily

[2]For a more general example, by Zorn's Lemma, every stable structure $\mathcal{M}$ has an expansion which is maximal among the stable expansions of $\mathcal{M}$. And as stability is preserved under nonproper expansions, this maximal expansion may be chosen to be a $\emptyset$-expansion. Similarly, for every $n \geq 1$, by Zorn's Lemma, every stable structure $\mathcal{M}$ of dp-rank n has an expansion which is maximal among the stable expansions of $\mathcal{M}$ of dp-rank n.

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equivalent structure $(N, +, 0, 1)$. Let $(N, +, 0, 1, |_p)$ be a nontrivial elementary extension of $(\mathbb{Z}, +, 0, 1, |_p)$, let $b \in N$ be such that $\gamma := v_p(b)$ is nonstandard, and let $B = \{a \in N : b|_p a\} = \{a \in N : v_p(a) \geq \gamma\}$. Then $(N, +, 0, 1, B)$ is a proper expansion of $(N, +, 0, 1)$ of dp-rank 1, and in Proposition 6.1 we show that it is also stable.

As $(\mathbb{Z}, +, 0, <)$ is dp-minimal, an immediate consequence of the above is that there are no stable structures which are both proper expansions of $(\mathbb{Z}, +, 0)$ and proper reducts of $(\mathbb{Z}, +, 0, <)$. In [3] Conant strengthened this result by proving that there are no structures *at all* which are both proper expansions of $(\mathbb{Z}, +, 0)$ and proper reducts of $(\mathbb{Z}, +, 0, <)$. Again, by interdefinability, we may replace $(\mathbb{Z}, +, 0)$ by $(\mathbb{Z}, +, 0, 1)$ and $(\mathbb{Z}, +, 0, <)$ by $(\mathbb{Z}, +, 0, 1, <)$. So we have:

Fact 1.8 ([3, Theorem 1.1]). $(\mathbb{Z}, +, 0, 1, <)$ *is minimal among the proper expansions of* $(\mathbb{Z}, +, 0, 1)$.

Again, this is no longer true if we replace $(\mathbb{Z}, +, 0, 1, <)$ by an elementarily equivalent structure. In private communication, Conant mentioned the following possible counterexample: Let $(N, +, 0, 1, <)$ be a nontrivial elementary extension of $(\mathbb{Z}, +, 0, 1, <)$, let $b \in N$ be a positive nonstandard element, and let $B = [0, b]$. Then $(N, +, 0, 1, B)$ is a proper expansion of $(N, +, 0, 1)$, and in Proposition 6.3 we show that it is indeed also a proper reduct of $(N, +, 0, 1, <)$. Note that the formula $y - x \in B$ defines the ordering on B, so this structure is unstable. We will see (Remark 5.17) that every structure which is a proper expansion of $(N, +, 0, 1)$ and a reduct of $(N, +, 0, 1, <)$, and which has a definable one-dimensional set which is not definable in $(N, +, 0, 1)$, defines a set of the form $[0, b]$ for a positive nonstandard b. Hence a stable intermediate structure between $(N, +, 0, 1, <)$ and $(N, +, 0, 1)$, if such exists, cannot contain new definable sets of dimension one.

Nevertheless, a weaker version of Fact 1.8 does hold as well for elementarily equivalent structures. As $(\mathbb{Z}, +, 0, 1, <)$ is a $\emptyset$-expansion of $(\mathbb{Z}, +, 0, 1)$, by Fact 1.8 it is obviously minimal among the proper $\emptyset$-expansions of $(\mathbb{Z}, +, 0, 1)$. In $(\mathbb{Z}, +, 0, 1)$, every element is $\emptyset$-definable, so a proper $\emptyset$-expansion of $(\mathbb{Z}, +, 0, 1)$ is the same as a $\emptyset$-proper $\emptyset$-expansion of $(\mathbb{Z}, +, 0, 1)$. Now if $\mathcal{N}$ is a $\emptyset$-proper $\emptyset$-reduct of $(\mathbb{Z}, +, 0, 1, <)$, and a $\emptyset$-proper $\emptyset$-expansion of $(\mathbb{Z}, +, 0, 1)$, then also in $\mathcal{N}$ every element is $\emptyset$-definable, so $\mathcal{N}$ is a proper reduct of $(\mathbb{Z}, +, 0, 1, <)$. Hence $(\mathbb{Z}, +, 0, 1, <)$ is $\emptyset$-minimal among the $\emptyset$-proper $\emptyset$-expansions of $(\mathbb{Z}, +, 0, 1)$. By Observation 1.4, we get:

Corollary 1.9. *Let* $(N, +, 0, 1, <)$ *be an elementary extension of* $(\mathbb{Z}, +, 0, 1, <)$. *Then* $(N, +, 0, 1, <)$ *is* $\emptyset$*-minimal among the* $\emptyset$*-proper* $\emptyset$*-expansions of* $(N, +, 0, 1)$.

Conant's proof of Fact 1.8 is very elementary from a model-theoretic point of view. In particular, it does not use Fact 1.7. On the other hand, it is somewhat complicated, involving detailed analysis of definable sets in arbitrary dimension. Conant asked whether this theorem can be proved using model theoretic methods which incorporate Fact 1.7. Here we give such a proof. Utilizing a basic property of (un)stability, we were able to prove minimality among unstable expansions by reducing the problem to the one-dimensional case (in an elementary extension), which is much easier.

Using the same reduction to dimension 1, and additional technical lemmas, we prove:

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THEOREM 1.10. *Let $(N, +, 0, 1, |_p)$ be an elementary extension of $(\mathbb{Z}, +, 0, 1, |_p)$. Then $(N, +, 0, 1, |_p)$ is $\emptyset$-minimal among the* <u>*unstable*</u> *$\emptyset$-proper $\emptyset$-expansions of $(N, +, 0, 1)$.*

Combined with Fact 1.7 and Theorem 1.2, we obtain:

THEOREM 1.11. *Let $(N, +, 0, 1, |_p)$ be an elementary extension of $(\mathbb{Z}, +, 0, 1, |_p)$. Then $(N, +, 0, 1, |_p)$ is $\emptyset$-minimal among the $\emptyset$-proper $\emptyset$-expansions of $(N, +, 0, 1)$.*

In particular:

COROLLARY 1.12. *$(\mathbb{Z}, +, 0, 1, |_p)$ is minimal among the proper expansions of $(\mathbb{Z}, +, 0, 1)$.*

Again, Corollary 1.12 fails for elementary extensions, see Proposition 6.2.

§2. Axioms and basic sentences of T_P.

In this section, we present a set of axioms for a subtheory $T'_P \subseteq T_P$, and use them to prove a number of (families of) sentences of T'_P. In Section 3 we will use these sentences to prove quantifier elimination for T'_P, from which it will also follow that in fact $T'_P = T_P$.

For convenience, we will work with the valuation functions v_p instead of the relations $|_p$. Let us define a multisorted language L_P^M for the valuations v_p on $(\mathbb{Z}, +, 0)$ for $p \in P$ as follows: let Z be the main sort with a function symbol $+$ and a constant symbol 0, interpreted as in $(\mathbb{Z}, +, 0)$. For each $p \in P$ we add a distinct sort Γ_p together with the symbols $<_p$, 0_p, S_p and ∞_p, interpreted as a distinct copy of $(\mathbb{N} \cup \{\infty\}, <, 0, S, \infty)$ where S is the successor function. Finally, we add a function symbol $v_p : Z \to \Gamma_p$, interpreted as the p-adic valuation.[3] When confusion is possible, we denote by $\mathbf{v}_p$ the usual valuation in the metatheory, to distinguish it from the function symbol v_p. We omit the subscript p in $<_p$, 0_p, S_p, ∞_p and Γ_p when no confusion is possible.

We use the following standard notation. Let $k \in \mathbb{N}$ be a nonnegative integer.

- In the Z sort, $\underline{k}$ denotes $\underbrace{1 + 1 + \cdots + 1}_{k \text{ times}}$ if $k > 0$ and 0 if $k = 0$. Also, $\underline{-k}$ denotes $-\underline{k}$.
- For an element a from Z, ka denotes $\underbrace{a + a + \cdots + a}_{k \text{ times}}$ if $k > 0$ and 0 if $k = 0$, $(-k)a$ denotes $-(ka)$, similarly for a variable x in place of a.
- For an element γ from Γ_p, $\gamma + \underline{k}$ denotes $\underbrace{S(S(\dots(\gamma)\dots))}_{k \text{ times}}$, similarly for a variable u in place of γ, and $\underline{k}$ is an abbreviation for $0 + \underline{k}$.

The group $(\mathbb{Z}, +, 0)$ with valuations v_p for $p \in P$ can be seen as an L_P-structure and an L_P^M-structure which are interdefinable (with imaginaries) so they essentially define the same sets. We will therefore not distinguish between the L_P-structure and the L_P^M-structure on $(\mathbb{Z}, +, 0)$, except when dealing with dp-rank, where we always refer to the one-sorted language L_P.

[3]It could be interesting to consider the language with just one sort $(N, <, 0, S, \infty)$ for valuation, instead of one for each $p \in P$. Since different valuations are allowed to interact with each other, the resulting structures might be much more complicated.

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For quantifier elimination we define $L_P^{M,E} = L_P^M \cup \{-,1\} \cup \{D_n : n \geq 1\}$ as before. In the L_P^E-structure on $\mathbb{Z}$, every atomic formula without parameters is definable by a quantifier-free formula without parameters and with variables in the Z sort in the $L_P^{M,E}$-structure on $\mathbb{Z}$, and vice-versa. Hence quantifier elimination in L_P^E follows from quantifier elimination in $L_P^{M,E}$. We will therefore prove quantifier elimination for the $L_P^{M,E}$-structure on $\mathbb{Z}$.

For $a \in \mathbb{Z}$ and $p \in P$, let $(a_i)_{i\in\mathbb{N}}$ be the p-adic representation of a, i.e., $a = \sum_{i\in\mathbb{N}} a_i p^i$ and each a_i is in $\{0,\dots,p-1\}$. For $\gamma \in \mathbb{N}$, the *prefix of a of length γ* is the sequence $(a_i)_{i<\gamma}$. The *ball* of radius γ and center a is the set of all integers with same prefix of length γ as a.

Proposition 2.1. *The following sentences are true in $\mathcal{Z}_P$ and therefore are in T_P:*

(1) *Any axiomatization for $Th(\mathbb{Z},+,-,0,1,\{D_n\}_{n\geq 1})$ in the Z sort.*
(2) *For each p, any axiomatization of $Th(\mathbb{N}\cup\{\infty\},<,0,S,\infty)$ in the sort $(\Gamma_p,<_p, 0_p,S_p,\infty_p)$.*
(3) *For each $p: \forall x(v_p(x) \geq 0 \wedge (v_p(x)=\infty \leftrightarrow x=0))$.*
(4) *For each $p: \forall x,y(v_p(x+y) \geq min(v_p(x),v_p(y)))$.*
(5) *For each $p: \forall x,y(v_p(x) \neq v_p(y) \rightarrow v_p(x+y) = min(v_p(x),v_p(y)))$.*
(6) *For each p and $0 \neq n \in \mathbb{Z}: \forall x(v_p(nx) = v_p(x) + \underline{\mathbf{v}_p(n)})$.*
(7) *For each $p: v_p(\underline{p}) = 1$.*
(8) *For each p and $k \in \mathbb{N}$: Every ball in v_p of radius γ consists of exactly p^k disjoint balls of radius $\gamma+k$.*

Proof. (1)–(7) are obvious. For (8), let $a \in \mathbb{Z}$ and $\gamma \in \mathbb{N}$. The ball in v_p of radius γ around a is the set of integers such that, in p-adic representation, their prefix of length γ is the same as the prefix of a of length γ. There are p possibilities for each digit, so p^k possibilities for the k digits with indices $\gamma,\dots,\gamma+k-1$, which exactly correspond to the balls of radius $\gamma+k$ contained in the original ball. ⊣

Let T_P' be the theory implied by the axioms (1)–(8). All of the following propositions are first order, and we prove them using only T_P'. Let $\mathcal{M}$ be some fixed model of T_P', with $\mathcal{Z}$ the Z-sort and Γ_p the Γ_p-sort.

Lemma 2.2. *For each p:*

(1) $\forall x,y(v_p(x-y) \geq min(v_p(x),v_p(y)))$.
(2) $\forall u\forall y\exists x(v_p(x-y)=u)$. *In particular, v_p is surjective.*
(3) *For each $n \neq 0$, $v_p(\underline{n}) = \underline{\mathbf{v}_p(n)}$.*
(4) *For each $k \geq 1: \forall x(v_p(x) \geq \underline{k} \leftrightarrow D_{p^k}(x))$.*

Proof. We only prove item (2), the others are easy to check. By Axiom (8) with $k=1$, there are x_1,x_2 such that $v_p(x_1-y) \geq u$, $v_p(x_2-y) \geq u$, and $v_p(x_1-x_2) < u+\underline{1}$. Hence by (1) above, $u+\underline{1} > v_p(x_1-x_2) = v_p((x_1-y)-(x_2-y)) \geq \min(v_p(x_1-y),v_p(x_2-y)) \geq u$. So either $v_p(x_1-y)=u$ or $v_p(x_2-y)=u$. ⊣

The following lemmas are left as an exercise.

Lemma 2.3. (1) *Let $n_1,\dots,n_l \in \mathbb{N}$, and let $N \in \mathbb{N}$ be such that $n_i|N$ for all $1 \leq i \leq l$. Let $b_1,\dots,b_n$ be element of $\mathcal{Z}$. Then every boolean combination of formulas of the form $D_{n_i}(k_i x - b_i)$ is equivalent to a disjunction (possibly*

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empty, i.e., a contradiction) of formulas of the form $D_N(x-\underline{r}_j)$, *where for each* j, $r_j \in \{0,1,\dots,N-1\}$.

(2) *Let* $m \in \mathbb{N}$ *and let* $m', k \in \mathbb{N}$ *be such that* $m = p^k \cdot m'$ *and* $gcd(m', p) = 1$. *Let* $r \in \mathbb{Z}$, *and let* $r_1 = r \bmod m'$, $r_2 = r \bmod p^k$. *Then the formula* $D_m(x-\underline{r})$ *is equivalent to* $D_{m'}(x-\underline{r}_1) \wedge (v_p(x-\underline{r}_2) \geq k)$.

LEMMA 2.4. *For* a_1 *and* a_2 *in* $\mathcal{Z}$.

(1) *For every* $k \geq 1$, *the formula* $v_p(x-a_1) < v_p(x-a_2)+\underline{k}$ *is equivalent to*

$$v_p(x-a_2) < v_p(a_2-a_1) \vee v_p(x-a_2) > v_p(a_2-a_1) \vee v_p(x-a_1) < v_p(a_2-a_1)+\underline{k}.$$

(2) *For every* $k \geq 0$, *the formula* $v_p(x-a_1)+\underline{k} < v_p(x-a_2)$ *is equivalent to* $v_p(x-a_2) > v_p(a_2-a_1)+\underline{k}$.

LEMMA 2.5. *For a fixed* $p \in P$, a_0, a_1 *in* $\mathcal{Z}$ *and* $\gamma_0, \gamma_1 \in \Gamma_p$.

(1) *Every formula of the form* $v_p(x-a_0) \geq \gamma_0 \wedge v_p(x-a_1) < \gamma_1$ *where* $\gamma_0 \geq \gamma_1$, *is either inconsistent (if* $v_p(a_0-a_1) \geq \gamma_1$*) or equivalent to* $v_p(x-a_0) \geq \gamma_0$ *(if* $v_p(a_0-a_1) < \gamma_1$*).*

(2) *Every formula of the form* $v_p(x-a_0) \geq \gamma_0 \wedge v_p(x-a_1) < \gamma_1$ *where* $\gamma_0 < \gamma_1$ *and* $v_p(a_0-a_1) < \gamma_0$ *is equivalent to just* $v_p(x-a_0) \geq \gamma_0$.

LEMMA 2.6. *Every two balls in* Γ_p *are either disjoint, or one is contained in the other. More generally, for* $(a_i)_i \in \mathcal{Z}$, $(\gamma_i)_i \in \Gamma_p$, *every conjunction of formulas of the form* $v_p(x-a_i) \geq \gamma_i$ *is either inconsistent, or equivalent to a single formula* $v_p(x-a_{i_0}) \geq \gamma_{i_0}$, *where* $\gamma_{i_0} = max\{\gamma_i\}$.

DEFINITION 2.7. For $a, b \in Z$, $\gamma, \delta \in \Gamma_p$, define $(a,\gamma) \leq_p (b,\delta)$ if $\gamma \leq \delta$ and $v_p(a-b) \geq \gamma$. Define $(a,\gamma) \sim_p (b,\delta)$ if $(a,\gamma) \leq_p (b,\delta)$ and $(a,\gamma) \geq_p (b,\delta)$.

$(a,\gamma) \leq_p (b,\delta)$ means that $\gamma \leq \delta$ and, in p-adic representation, the prefix of a of length γ is contained in the prefix of b of length δ. This is equivalent to saying that the ball of radius γ around a (namely, $\{x : v_p(x-a) \geq \gamma\}$) contains the ball of radius δ around b.

Note that $\leq_p$ and $\sim_p$ are defined by quantifier-free formulas, and so do not depend on the model containing the elements under consideration.

LEMMA 2.8. *The parameters* a_i *are in* $\mathcal{Z}$ *and* γ_i *are in* Γ_p *for some* $p \in P$.

(1) *Every formula of the form* $v_p(x-a_0) \geq \gamma_0 \wedge \bigwedge_{m=1}^{n} v_p(x-a_m) < \gamma_m$ *is equivalent to the formula* $v_p(x-a_0) \geq \gamma_0 \wedge \bigwedge_{m\in C} v_p(x-a_m) < \gamma_m$, *for every* $C \subseteq \{1,\dots,n\}$ *such that* $\{(a_m,\gamma_m) : m \in C\}$ *contains at least one element from each* $\sim_p$*-equivalence class of* $\leq_p$*-minimal elements among* $\{(a_m,\gamma_m) : 1 \leq m \leq n\}$ *(i.e., representatives for all the maximal balls). In particular, this is true for* C *consisting of one element from each such class, i.e., for* C *an antichain.*

(2) *Assume that* $(a_0,\gamma_0),\dots,(a_n,\gamma_n)$ *are such that for all* $1 \leq m \leq n$ *we have* $\gamma_m > \gamma_0$, $v_p(a_m-a_0) \geq \gamma_0$, *and* $k_m := \gamma_m - \gamma_0$ *is a standard integer. Assume further that* $\{(a_m,\gamma_m) : 1 \leq m \leq n\}$ *is an antichain with respect to* $\leq_p$. *Then every formula of the form* $v_p(x-a_0) \geq \gamma_0 \wedge \bigwedge_{m=1}^{n} v_p(x-a_m) < \gamma_m$ *is equivalent to a formula of the form* $\bigvee_{i=1}^{l} v_p(x-b_i) \geq \gamma_N$ *with* N *such that* $\gamma_N = \max\{\gamma_m : 1 \leq m \leq n\}$, *where for all* i, $v_p(b_i-a_0) \geq \gamma_0$ *and for* $i \neq j$,

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$v_p(b_i - b_j) < \gamma_N$, and where $l = p^{k_N} - \sum_m p^{k_N - k_m} \geq 0$ (it may be that $l = 0$, i.e., a contradiction). In particular, l does not depend on the model $\mathcal{M}$ of T_P' containing the a_i's and γ_i's.

PROOF. We prove (1). Let C be such. For each $1 \leq m \leq n$ there is an m' such that $(a_{m'}, \gamma_{m'}) \leq (a_m, \gamma_m)$ and $(a_{m'}, \gamma_{m'})$ is minimal among the (a_i, γ_i)'s. So $\forall x (v_p(x - a_{m'}) < \gamma_{m'} \rightarrow v_p(x - a_m) < \gamma_m)$. As $\{(a_i, \gamma_i) : i \in C\}$ contains one element from each $\sim$-equivalence class of $\leq$-minimal elements, we may assume $m' \in C$.

We prove (2). Assume without loss of generality that $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n$. Let $b_0, \ldots, b_{p^{k_n}-1}$ be the $x_0, \ldots, x_{p^k-1}$ from Axiom 8 for k_n, γ_0, a_0. Then $v_p(x - a_0) \geq \gamma_0$ is equivalent to $\bigvee_{i=0}^{p^{k_n}-1} (v_p(x - b_i) \geq \gamma_n)$. For every $m \geq 1$, let $c_{m,0}, \ldots, c_{m,p^{k_n-k_m}-1}$ be the $x_0, \ldots, x_{p^k-1}$ from Axiom 8 for $k_n - k_m$, γ_m, a_m. Then $v_p(x - a_m) \geq \gamma_m$ is equivalent to $\bigvee_{i=0}^{p^{k_n-k_m}-1} (v_p(x - c_{m,i}) \geq \gamma_n)$. For every m, $v_p(a_0 - a_m) \geq \gamma_0$, so for every $0 \leq i \leq p^{k_n-k_m} - 1$, $v_p(c_{m,i} - a_0) \geq \gamma_0$. Hence by the choice of $\{b_j\}_j$, there is a unique $s_{m,i} < p^{k_n}$ such that $v_p(c_{m,i} - b_{s_{m,i}}) \geq \gamma_n$. So $v_p(x - a_m) \geq \gamma_m$ is equivalent to $\bigvee_{i=0}^{p^{k_n-k_m}-1} (v_p(x - b_{s_{m,i}}) \geq \gamma_n)$.

By the choice of $\{c_{m,i}\}_i$, $\bigwedge_{i \neq j} (v_p(c_{m,i} - c_{m,j}) < \gamma_n)$, so also $\bigwedge_{i \neq j} (v_p(b_{s_{m,i}} - b_{s_{m,j}}) < \gamma_n)$. In particular, $i \mapsto s_{m,i}$ is injective for a fixed m, hence $F_m := \{s_{m,i} : 0 \leq i \leq p^{k_n-k_m} - 1\}$ is of size $p^{k_n-k_m}$.

The sets $\{F_m\}_{m=1}^n$ must be mutually disjoint. Otherwise, there are $m_1 < m_2$ and i, j such that $s_{m_1,i} = s_{m_2,j}$. Since $v_p(c_{m_1,i} - b_{s_{m_1,i}}) \geq \gamma_n$ and $v_p(c_{m_2,j} - b_{s_{m_2,j}}) \geq \gamma_n$ we get $v_p(c_{m_1,i} - c_{m_2,j}) \geq \gamma_n \geq \gamma_{m_1}$. Since $v_p(c_{m_1,i} - a_{m_1}) \geq \gamma_{m_1}$ and $v_p(c_{m_2,j} - a_{m_2}) \geq \gamma_{m_2} \geq \gamma_{m_1}$, we get $v_p(a_{m_1} - a_{m_2}) \geq \gamma_{m_1}$, a contradiction to the antichain assumption.

Let $F := \bigcup_{m=1}^n F_m$. By the above, $|F| = \sum_m p^{k_n-k_m}$ and

$$\forall x \Big(\big(v_p(x - a_0) \geq \gamma_0 \wedge \bigwedge_{m=1}^{n} v_p(x - a_m) < \gamma_m\big) \leftrightarrow \big(\bigvee_{i \notin F} v_p(x - b_i) \geq \gamma_n\big) \Big). \qquad \dashv$$

LEMMA 2.9. *For all elements $a_i, a_{i,j}$ in $\mathcal{Z}$ and γ_i in Γ_p for some $p \in P$, we have the following.*

1. *If b is a solution to $v_p(x - a_0) \geq \gamma_0 \wedge \bigwedge_{i=1}^n v_p(x - a_i) < \gamma_i$ and $v_p(b' - b) \geq \gamma := max\{\gamma_0, \ldots, \gamma_n\}$ then b' is also a solution.*
2. *Every formula of the form $v_p(x - a_0) \geq \gamma_0 \wedge \bigwedge_{m=1}^n v_p(x - a_m) < \gamma_m$ where for each $1 \leq m \leq n$, $\gamma_m \geq \gamma_0 + \underline{n}$, has a solution.*
3. *If $p_1, \ldots, p_l \in P$ are different primes not dividing m and $\gamma_i \in \Gamma_{p_i}$, then every formula of the form $(\bigwedge_{k=1}^l v_{p_k}(x - a_k) \geq \gamma_k) \wedge D_m(x - r)$ has an infinite number of solutions.*
4. *If $p_1, \ldots, p_l \in P$ are different primes not dividing m and $\gamma_{k,j} \in \Gamma_{p_k}$, then every formula of the form*

$$\bigwedge_{k=1}^{l} \left(v_{p_k}(x - a_{k,0}) \geq \gamma_{k,0} \wedge \bigwedge_{i=1}^{n_k} v_{p_k}(x - a_{k,i}) < \gamma_{k,i} \right) \wedge D_m(x - r)$$

where for each $1 \leq k \leq l$ and $1 \leq i \leq n_k$, $\gamma_{k,i} \geq \gamma_{k,0} + \underline{n}_k$, has an infinite number of solutions. In particular, this holds if each $\gamma_{k,i} - \gamma_{k,0}$ is a nonstandard integer.

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PROOF. The proofs of (1) and (3) are left as an easy exercice. We prove (2). By Axiom 8 for $k = n$, there are $b_0, \dots, b_{p^n-1}$ such that for all i, $v_p(b_i - a_0) \geq \gamma_0$, and for all $i \neq j$, $v_p(b_i - b_j) < \gamma_0 + \underline{n}$. Then some b_i must satisfy $\bigwedge_{m=1}^{n} v_p(x - a_m) < \gamma_m$, otherwise, since $p^n > n$, by the Pigeonhole Principle there are $i \neq j$ and m such that $v_p(b_i - a_m) \geq \gamma_m$ and $v_p(b_j - a_m) \geq \gamma_m$, and therefore also $v_p(b_i - b_j) \geq \gamma_m \geq \gamma_0 + \underline{n}$, a contradiction.

We prove (4). For each $1 \leq k \leq l$, by (2) the formula $v_{p_k}(x - a_{k,0}) \geq \gamma_{k,0} \wedge (\bigwedge_{i=1}^{n_k} v_{p_k}(x - a_{k,i}) < \gamma_{k,i})$ has a solution b_k. Let $\gamma_k := \max\{\gamma_{k,0}, \dots, \gamma_{k,n_k}\}$. By (3) the formula $(\bigwedge_{k=1}^{l} v_{p_k}(x - b_k) \geq \gamma_k) \wedge D_m(x - r)$ has an infinite number of solutions $\{b'_j\}_{j\geq 1}$. By (1), every b'_j is a solution to

$$\bigwedge_{k=1}^{l} \left(v_{p_k}(x - a_{k,0}) \geq \gamma_{k,0} \wedge \bigwedge_{i=1}^{n_k} v_{p_k}(x - a_{k,i}) < \gamma_{k,i} \right) \wedge D_m(x - r).$$

⊣

§3. Quantifier elimination.

PROOF OF THEOREM 1.1. As mentioned previously, we will in fact prove quantifier elimination for $T'_P \subseteq T_P$. It is enough to prove that for all models $\mathcal{M}_1$ and $\mathcal{M}_2$ of T'_P, with a common substructure A, and for all formulas $\phi(x)$ in a single variable x over A which are a conjunction of atomic or negated atomic formulas, we have $\mathcal{M}_1 \vDash \exists x \phi(x) \Rightarrow \mathcal{M}_2 \vDash \exists x \phi(x)$. Let $\mathcal{M}_1$, $\mathcal{M}_2$, A, and $\phi(x)$ be such, and let $b \in \mathcal{M}_1$ be such that $\mathcal{M}_1 \vDash \phi(b)$.

As v_p is surjective for all $p \in P$, we may assume that x is of the Z sort. Since ϕ contains only finitely many symbols from L_P, we may assume for simplicity of notation that P is finite. So $\phi(x)$ is equivalent[4] to a conjunction of formulas of the forms:

(1) $n_i x = a_i$, for some $n_i \neq 0$.
(2) $n_i x \neq a_i$, for some $n_i \neq 0$.
(3) $D_{m_i}(n_i x - a_i)$, for some $n_i \neq 0$.
(4) $\neg D_{m_i}(n_i x - a_i)$, for some $n_i \neq 0$.
(5) $v_{p_\alpha}(n_{i,1}x - a_{i,1}) < v_{p_\alpha}(n_{i,2}x - a_{i,2}) + \underline{k_i}$, for some $p_\alpha \in P$, $n_{i,1} \neq 0$ or $n_{i,2} \neq 0$, and $k_i \in \mathbb{N}$.
(6) $v_{p_\alpha}(n_{i,1}x - a_{i,1}) + \underline{k_i} < v_{p_\alpha}(n_{i,2}x - a_{i,2})$, for some $p_\alpha \in P$, $n_{i,1} \neq 0$ or $n_{i,2} \neq 0$, and $k_i \in \mathbb{N}$.
(7) $v_{p_\alpha}(n_i x - a_i) \geq \gamma_i$, for some $p_\alpha \in P$ and $n_i \neq 0$.
(8) $v_{p_\alpha}(n_i x - a_i) < \gamma_i$, for some $p_\alpha \in P$ and $n_i \neq 0$.

By multiplicativity of the valuations we may assume that for all formulas of forms (5) or (6), either $n_{i,1} = n_{i,2}$, $n_{i,1} = 0$, or $n_{i,2} = 0$. Therefore, by Lemma 2.4, we may assume that every formula of form (5) or (6) is equivalent to a formula of form (7) or (8).

[4]The negation of a formula of form (5) is $v_{p_\alpha}(n_{i,1}x - a_{i,1}) \geq v_{p_\alpha}(n_{i,2}x - a_{i,2}) + \underline{k}$, which is equivalent to $v_{p_\alpha}(n_{i,2}x - a_{i,2}) + \underline{k-1} < v_{p_\alpha}(n_{i,1}x - a_{i,1})$ if $k > 0$, which is of form (6), and to $v_{p_\alpha}(n_{i,2}x - a_{i,2}) < v_{p_\alpha}(n_{i,1}x - a_{i,1}) + 1$ if $k = 0$, which is of form (5). Similarly for the negation of a formula of form (6). Also, (7) and (8) are in essence special cases of (5) or (6), but they are required because in A the valuation may be not surjective.

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By Lemma 2.3, the conjunction of all the formulas of the forms (3) or (4) is equivalent to a formula of the form

$$\bigvee_j \left(D_{m_j}(x - r_j) \wedge \bigwedge_{\alpha<|P|} v_{p_\alpha}(x - s_{j,\alpha}) \geq \underline{k}_{j,\alpha} \right)$$

where for all j and α, $\gcd(m_j, p_\alpha) = 1$. As $\mathcal{M}_1 \models \phi(b)$, this disjunction is not empty. Let $D_m(x - r) \wedge \bigwedge_{\alpha<|P|} v_{p_\alpha}(x - s_\alpha) \geq \underline{k}_\alpha$ be one of the disjuncts which are satisfied by b. It is enough to find $b' \in \mathcal{M}_2$ which satisfies this disjunct, along with all the formulas of other forms. Note that $v_{p_\alpha}(x - s_\alpha) \geq \underline{k}_\alpha$ is of form (7), so altogether we want to find $b' \in \mathcal{M}_2$ which satisfies a conjunction of formulas of the forms:

(1) $n_i x = a_i$, $n_i \neq 0$.
(2) $n_i x \neq a_i$, $n_i \neq 0$.
(3) $D_m(x-r)$, where for all $\alpha < |P|$, $\gcd(m, p_\alpha) = 1$ (only a single such formula).
(4) $v_{p_\alpha}(n_i x - a_i) \geq \gamma_i$, $\alpha < |P|$, $n_i \neq 0$.
(5) $v_{p_\alpha}(n_i x - a_i) < \gamma_i$, $\alpha < |P|$, $n_i \neq 0$.

By a standard argument, we may assume that the conjunction does not contain formulas of form (1). For each formula of form (2), there is at most one element which does not satisfy it. So it is enough to prove that there are infinitely many elements in $\mathcal{M}_2$ which satisfy all the formulas of forms (3), (4), or (5).

Let $n := \prod_i n_i$. By multiplicativity of the valuations, the conjunction of formulas of forms (3), (4), or (5) is equivalent to the conjunction of:

(1) $v_{p_\alpha}(nx - \frac{n}{n_i} a_i) \geq \gamma_i + \mathbf{v}_{p_\alpha}(\frac{n}{n_i})$.
(2) $v_{p_\alpha}(nx - \frac{n}{n_i} a_i) < \gamma_i + \mathbf{v}_{p_\alpha}(\frac{n}{n_i})$.
(3) $D_{nm}(nx - nr)$.

By substituting $y = nx$, it is equivalent to satisfy:

(1) $v_{p_\alpha}(y - \frac{n}{n_i} a_i) \geq \gamma_i + \mathbf{v}_{p_\alpha}(\frac{n}{n_i})$.
(2) $v_{p_\alpha}(y - \frac{n}{n_i} a_i) < \gamma_i + \mathbf{v}_{p_\alpha}(\frac{n}{n_i})$.
(3) $D_{nm}(y - nr)$.
(4) $D_n(y)$.

Notice that formula (4) is already implied by formula (3). Again by Lemma 2.3, we may exchange $D_{nm}(y - nr)$ by a formula $D_{m'}(y - r')$, where for all $\alpha < |P|$, $\gcd(m', p_\alpha) = 1$. Also, by Lemma 2.6 we may assume that for each $\alpha < |P|$, there is only one formula of form (1). Altogether, it is enough to prove that in $\mathcal{M}_2$ there are infinitely many elements which satisfy the conjunction of the following formulas:

(1) $v_{p_\alpha}(x - a_{\alpha,0}) \geq \gamma_{\alpha,0}$ for all $\alpha < |P|$.
(2) $v_{p_\alpha}(x - a_{\alpha,i}) < \gamma_{\alpha,i}$ for all $\alpha < |P|$, $1 \leq i \leq n_\alpha$. $\circledast$
(3) $D_m(x-r)$, where for all $\alpha < |P|$, $\gcd(m, p_\alpha) = 1$ (only a single such formula).

By Lemma 2.5 (and since this formula is consistent in $\mathcal{M}_1$) we may assume that for all $\alpha < |P|$, $1 \leq i \leq n_\alpha$ we have $\gamma_{\alpha,0} < \gamma_{\alpha,i}$ and $v_{p_\alpha}(a_{\alpha,0} - a_{\alpha,i}) \geq \gamma_{\alpha,0}$. By Lemma 2.8(1), we may assume that for each $\alpha < |P|$, the set

$$\{(a_{\alpha,i}, \gamma_{\alpha,i}) \,:\, 1 \leq i \leq n_\alpha \,,\, \gamma_{\alpha,i} - \gamma_{\alpha,0} \text{ is a standard integer}\}$$

is an antichain with respect to $\leq_{p_\alpha}$ (Definition 2.7).

For each $\alpha < |P|$, let $S_\alpha = \{0 \le i \le n_\alpha : \gamma_{\alpha,i} - \gamma_{\alpha,0}$ is a standard integer$\}$ and $\gamma'_{\alpha,0} = max\{\gamma_{\alpha,i} : i \in S_\alpha\}$. For $s = 1, 2$ and for each $\alpha < |P|$, by Lemma 2.8(2) the conjunction $v_{p_\alpha}(x - a_{\alpha,0}) \ge \gamma_{\alpha,0} \wedge \bigwedge_{i \in S_\alpha} v_{p_\alpha}(x - a_{\alpha,i}) < \gamma_{\alpha,i}$ is equivalent in $\mathcal{M}_s$ to a formula of the form $\bigvee_{i=1}^{l_\alpha} v_{p_\alpha}(x - a^s_{\alpha,0,i}) \ge \gamma'_{\alpha,0}$, where for all i, $a^s_{\alpha,0,i} \in \mathcal{M}_s$ and l_α does not depend on s. Note that $a^s_{\alpha,0,i}$ may not be in A. Furthermore, by Lemma 2.8(2), $v_{p_\alpha}(a^s_{\alpha,0,i} - a_{\alpha,0}) \ge \gamma_{\alpha,0}$ and for $i \ne j$, $v_{p_\alpha}(a^s_{\alpha,0,i} - a^s_{\alpha,0,j}) < \gamma'_{\alpha,0}$.

Together, the conjunction of the formulas in $\circledast$ is equivalent in $\mathcal{M}_s$ to the disjunction $\psi_s = \bigvee_{k=1}^{l} \psi_{s,k}$, where for each k, $\psi_{s,k}$ is the conjunction of the following formulas:

(1) $v_{p_\alpha}(x - a^s_{\alpha,0,k}) \ge \gamma'_{\alpha,0}$, for all $\alpha < |P|$.
(2) $v_{p_\alpha}(x - a_{\alpha,i}) < \gamma_{\alpha,i}$, for all $\alpha < |P|$, $i \notin S_\alpha$ (so $\gamma_{\alpha,0} < \gamma_{\alpha,i}$ and $\gamma_{\alpha,i} - \gamma_{\alpha,0}$ is not a standard integer).
(3) $D_m(x-r)$, where for all $\alpha < |P|$, $\gcd(m, p_\alpha) = 1$ (only a single such formula).

Furthermore, $l = \prod_{\alpha<|P|} l_\alpha$ does not depend on s.

Since ψ_1 is consistent in $\mathcal{M}_1$ (satisfied by nb), the disjunction for $s = 1$ is not empty, i.e., $l \ge 1$. And since l does not depend on s, the disjunction for $s = 2$ is also not empty. Consider one such disjunct, $\psi_{2,k}$. By Lemma 2.9(4), it has an infinite number of solutions. This completes the proof. ⊣

COROLLARY 3.1. *T'_P is a complete theory. Hence $T'_P = T_P$.*

PROOF. By quantifier elimination, it is enough to show that T'_P decides every atomic sentence. These are just the sentences equivalent to one of the forms: $\underline{n}_1 = \underline{n}_2$ in any sort, $\underline{k}_1 <_p \underline{k}_2$ in Γ_p, $D_m(\underline{n})$ in the Z sort and $v_p(\underline{n}_1) < v_p(\underline{n}_2)$ in the Z sort, all of which are clearly decided by T'_P. ⊣

REMARK 3.2. Suppose $\mathcal{M} \models T_P$ and $\phi(x)$ is a consistent formula in a single variable with parameters from $\mathcal{M}$. Then by quantifier elimination and Lemmas 2.3 and 2.4, $\phi(x)$ is equivalent to a disjunction of formulas, which are either of the form $x = a$ or of the form

$$D_m(x-r) \wedge \bigwedge_j nx \ne a_j \wedge \bigwedge_{p \in F} \left(v_p(n_p x - a_{p,0}) \ge \gamma_{p,0} \wedge \bigwedge_{i=1}^{l_p} v_p(n_p x - a_{p,i}) < \gamma_{p,i} \right),$$

where $F \subseteq P$ is finite and $\gcd(m, p) = 1$ for all $p \in F$. Moreover, one may assume $\gcd(n_p, p) = 1$ for each $p \in F$.

For p a single prime number and $\mathcal{M} \models T_p$, the following lemma says that the definable subgroups of $(\mathcal{M}, +)$ are only those of the form $m\mathcal{M} \cap \{a \in \mathcal{M} : v(a) \ge \gamma\}$, for $m \in \mathbb{Z}$ and $\gamma \in \Gamma$ and for each such defining formula, there are only finitely many possible m's when varying the parameters of the formula.

LEMMA 3.3. *For a single prime p, let $\phi(x, y)$ be any L_p^M-formula, and let $\theta(y)$ be the formula for "$(\phi(x, y), +)$ is a subgroup". Then there are $n_1, \dots, n_k \ge 1$, having $gcd(n_i, p) = 1$ for each i, such that the following sentence is true in T_p:*

$$\forall y \left(\theta(y) \to \bigvee_{i=1}^{k} \exists w \forall x (\phi(x, y) \leftrightarrow (D_{n_i}(x) \wedge (v_p(x) \ge v_p(w))) \right).$$

PROOF. It is enough to work in $\mathbb{Z}$.

By quantifier elimination (and Lemma 2.3(2)), $\phi(x, y)$ is equivalent to a formula of the form $\bigvee_i \bigwedge_j \phi_{i,j}(x, y)$, where for each i, j, $\phi_{i,j}(x, y)$ is one of the following:

(1) $t_{i,j}(x, y) = 0$, where $t_{i,j}(x, y)$ is a $\{+, -, 1\}$-term, i.e., of the form $k_{i,j}x + l_{i,j}y + \underline{r}_{i,j}$ for $k_{i,j}, l_{i,j}, r_{i,j} \in \mathbb{Z}$.
(2) $t_{i,j}(x, y) \neq 0$, where $t_{i,j}(x, y)$ is a $\{+, -, 1\}$-term.
(3) $v(t_{i,j}(x, y)) \geq v(s_{i,j}(x, y))$, where $t_{i,j}(x, y)$, $s_{i,j}(x, y)$ are $\{+, -, 1\}$-terms. ($v(t_{i,j}(x, y)) < v(s_{i,j}(x, y))$ is equivalent to $v(p \cdot t_{i,j}(x, y)) \leq v(s_{i,j}(x, y))$, which is of the same form).
(4) $D_{m_{i,j}}(t_{i,j}(x, y))$, where $t_{i,j}(x, y)$ is a $\{+, -, 1\}$-term and $\gcd(m_{i,j}, p) = 1$.

For each i, let $J_i = \{j : \phi_{i,j}(x, y)$ is of the form $D_{m_{i,j}}(t_{i,j}(x, y))\}$, and let $m_i = \prod_{j \in J_i} m_{i,j}$. As in the proof of Lemma 2.3(1), the satisfaction of the formula $D_{m_{i,j}}(t_{i,j}(x, y))$ depends only on the remainders of x and y mod $m_{i,j}$, which are determined by the remainders of x and y mod m_i. So there is a set $R_i \subseteq \{0, 1, \ldots, m_i - 1\}^2$ such that $\bigwedge_{j \in J_i} \phi_{i,j}(x, y)$ is equivalent to $\bigvee_{(r,s) \in R_i}(D_{m_i}(x - \underline{r}) \wedge D_{m_i}(y - \underline{s}))$. Therefore, $\phi(x, y)$ is equivalent to a formula of the form $\bigvee_i(D_{m_i}(x - \underline{r}_i) \wedge D_{m_i}(y - \underline{s}_i) \wedge \bigwedge_j \phi_{i,j}(x, y))$, where $\gcd(m_i, p) = 1$ and for each i, j, $\phi_{i,j}(x, y)$ is one of the following:

(1) $t_{i,j}(x, y) = 0$, where $t_{i,j}(x, y)$ is a $\{+, -, 1\}$-term.
(2) $t_{i,j}(x, y) \neq 0$, where $t_{i,j}(x, y)$ is a $\{+, -, 1\}$-term.
(3) $v(t_{i,j}(x, y)) \geq v(s_{i,j}(x, y))$, where $t_{i,j}(x, y)$, $s_{i,j}(x, y)$ are $\{+, -, 1\}$-terms.

For each i, let $\phi_i(x, y)$ be the i'th disjunct, i.e., the formula $D_{m_i}(x - \underline{r}_i) \wedge D_{m_i}(y - \underline{s}_i) \wedge \bigwedge_j \phi_{i,j}(x, y)$.

Let $b \in \mathbb{Z}$ be such that $\phi(\mathbb{Z}, b)$ is a subgroup. If $\phi(\mathbb{Z}, b)$ is finite, it must be $\{0\}$. To account for this case, we may take $n_1 = 1$, and for $w = 0$ we have that $\phi(x, b)$ is equivalent to $D_{n_1}(x) \wedge (v_p(x) \geq v_p(0))$. If $\phi(\mathbb{Z}, b)$ is infinite, then $\phi(\mathbb{Z}, b) = n\mathbb{Z}$ for some $n \geq 1$. Moreover, there must be an i_0 such that $\phi_{i_0}(\mathbb{Z}, b)$ is infinite. So $D_{m_{i_0}}(b - \underline{s}_{i_0})$ holds, hence $\phi_{i_0}(x, b)$ is equivalent to just $D_{m_{i_0}}(x - \underline{r}_{i_0}) \wedge \bigwedge_j \phi_{i_0,j}(x, b)$. As $\phi(\mathbb{Z}, b)$ is infinite, it is clear that no formula $\phi_{i_0,j}(x, y)$ is of the form (1), hence $\phi_{i_0}(x, b)$ is equivalent to $D_{m_{i_0}}(x - \underline{r}_{i_0}) \wedge \bigwedge_j \phi_{i_0,j}(x, b)$, where for each j, $\phi_{i_0,j}(x, b)$ is one of the following:

(1) $k_{i_0,j}x \neq c_{i_0,j}$.
(2) $v(k'_{i_0,j}x - c'_{i_0,j}) \geq v(k''_{i_0,j}x - c''_{i_0,j})$.

Applying Lemma 2.4 to formulas as in (2), we may assume that $\phi_{i_0}(x, b)$ is equivalent to $D_{m_{i_0}}(x - \underline{r}_{i_0}) \wedge \bigwedge_j \phi_{i_0,j}(x, b)$, where for each j, $\phi_{i_0,j}(x, b)$ is one of the following:

(1) $k_{i_0,j}x \neq c_{i_0,j}$.
(2) $v(k_{i_0,j}x - c_{i_0,j}) \geq \gamma_{i_0,j}$.
(3) $v(k_{i_0,j}x - c_{i_0,j}) < \gamma_{i_0,j}$.

The formula $v(k_{i_0,j}x - c_{i_0,j}) \geq \gamma_{i_0,j}$ defines a coset of $p^{\gamma_{i_0,j}}\mathbb{Z}$, and the formula $v(k_{i_0,j}x - c_{i_0,j}) < \gamma_{i_0,j}$ defines a finite union of cosets of $p^{\gamma_{i_0,j}}\mathbb{Z}$. Let $J = \{j : \phi_{i_0,j}(x, b)$ is of form 2 or 3$\}$, and let $\delta = \max\{\gamma_{i_0,j} : j \in J\}$. Then for every $j \in J$, every coset of $p^{\gamma_{i_0,j}}\mathbb{Z}$ is a finite union of cosets of $p^{\delta}\mathbb{Z}$. So $\bigcap_{j \in J} \phi_{i_0,j}(\mathbb{Z}, b)$ is a finite intersection of finite unions of cosets of $p^{\delta}\mathbb{Z}$, and hence is itself just a finite union of cosets of $p^{\delta}\mathbb{Z}$ (since every two cosets are either equal or disjoint).

Therefore, $\phi_{i_0}(\mathbb{Z}, b)$ is a set of the form $U \backslash F$, where F is a finite set (the set of points excluded by the inequalities $k_{i_0,j}x \neq c_{i_0,j}$), and U is a finite union of the form $\bigcup_{j=1}^{N}((m_{i_0}\mathbb{Z} + r_{i_0}) \cap (p^\delta\mathbb{Z} + c_j))$. For each j, $(m_{i_0}\mathbb{Z} + r_{i_0}) \cap (p^\delta\mathbb{Z} + c_j)$ is a coset of $m_{i_0}p^\delta\mathbb{Z}$ (it is not empty, since $\gcd(m_{i_0}, p) = 1$), so U is of the form $\bigcup_{j=1}^{N}(m_{i_0}p^\delta\mathbb{Z} + d_j)$. As $\phi_{i_0}(\mathbb{Z}, b)$ is infinite, this union is not empty.

Now, $(m_{i_0}p^\delta\mathbb{Z} + d_1)\backslash F \subseteq U\backslash F = \phi_{i_0}(\mathbb{Z}, b) \subseteq \phi(\mathbb{Z}, b) = n\mathbb{Z}$, so n divides $m_{i_0}p^\delta$ since F is finite. Write $n = n'p^\gamma$ with $\gcd(n', p) = 1$. Then $n'|m_{i_0}$, and in particular, $n' \leq m_{i_0}$. So $\phi(x, b)$ is equivalent to $D_n(x)$, which is equivalent to $D_{n'}(x) \wedge v(x) \geq \gamma$, and $n' \leq m_{i_0}$. Recall that i_0 depends on b, but there are only finitely many i's, so $m = \max\{m_i\}$ exists, and hence, for any b such that $\phi(x, b)$ is a subgroup, there is an $n' \leq m$ with $\gcd(n', p) = 1$, and there is a γ such that $\phi(x, b)$ is equivalent to $D_{n'}(x) \wedge v(x) \geq \gamma$, and we are done. ⊣

§4. dp-rank of T_P.

Quantifier elimination now enables us to determine the dp-rank of T_P. We first review two equivalent definitions of dp-rank. More details about dp-rank can be found, e.g., in [17]. We work in a monster model $\mathbb{M}$ of some complete L-theory T, for some langage L.

Definition 4.1. Let $\phi(x, b)$ be an L-formula, with parameters b from $\mathbb{M}$, and let κ be a (finite or infinite) cardinal. We say dp-rank$(\phi(x, b)) < \kappa$ if for every family $(I_t : t < \kappa)$ of mutually indiscernible sequences over b and $a \models \phi(x, b)$, there is $t < \kappa$ such that I_t is indiscernible over ab.

We say that dp-rank$(\phi(x, b)) = \kappa$ if dp-rank$(\phi(x, b)) < \kappa^+$ but not dp-rank$(\phi(x, b)) < \kappa$. We say that dp-rank$(\phi(x, b)) \leq \kappa$ if dp-rank$(\phi(x, b)) < \kappa$ or dp-rank$(\phi(x, b)) = \kappa$. Note that if κ is a limit cardinal, it may happen that dp-rank$(\phi(x, b)) < \kappa$ but dp-rank$(\phi(x, b)) \geq \lambda$ for all $\lambda < \kappa$.

For a theory T we denote dp-rank(T) = dp-rank$(x = x)$ where $|x| = 1$. If dp-rank$(T) = 1$ we say that T is *dp-minimal*.

Definition 4.2. Let κ be a cardinal. An *ict-pattern of length* κ consists of:

- a collection of formulas $(\phi_\alpha(x; y_\alpha) : \alpha < \kappa)$, with $|x| = 1$,
- an array $(b_i^\alpha : i < \omega, \alpha < \kappa)$ of tuples, with $|b_i^\alpha| = |y_\alpha|$

such that for every $\eta : \kappa \to \omega$ there exists an element $a_\eta \in \mathbb{M}$ such that

$$\models \phi_\alpha(a_\eta; b_i^\alpha) \iff \eta(\alpha) = i.$$

We define κ_{ict} as the minimal κ such that there does not exist an ict-pattern of length κ.

Fact 4.3 ([17, Proposition 4.22]). *For any cardinal κ, we have dp-rank$(T) < \kappa$ if and only if $\kappa_{ict} \leq \kappa$.*

Proposition 4.4. *For any prime p, T_p is dp-minimal (in the one-sorted language).*

Proof. Denote $L = L_p^E$ and $T = T_p$. Let L^- contain the symbols of L, except for the divisibility relations $\{D_n\}_{n\geq 1}$. Let $\mathcal{Z}^-$ be the reduct of $\mathcal{Z}_p$ to L^-. Let $\mathbb{Q}_p^-$ be $\mathbb{Q}_p$ as an L^--structure. It is a reduct of the structure $(\mathbb{Q}_p, +, -, \cdot, 0, 1, |_p)$, which is dp-minimal (see [7, Theorem 6.6]), and therefore is also dp-minimal. Note that $\mathcal{Z}^-$ is a substructure of $\mathbb{Q}_p^-$.

Let $L' = L \cup \{Z\}$. Interpret Z in $\mathbb{Q}_p$ as $\mathbb{Z}$, and interpret each D_n such that $D_n \cap \mathbb{Z}$ is the usual divisibility relation and $D_n \cap (\mathbb{Q}_p \backslash \mathbb{Z}) = \emptyset$, thus making it an L'-structure $\mathbb{Q}'_p$. Let $\mathcal{M}$ be an ω_1-saturated model of $Th(\mathbb{Q}'_p)$, and let $A = Z(\mathcal{M})$ be the interpretation of Z in it. Then A is an ω_1-saturated model of T.

Suppose that T is not dp-minimal. Then there are formulas $\phi(x, y)$, $\psi(x, z)$ in L with $|x| = 1$, and elements $(b_i : i < \omega)$, $(c_j : j < \omega)$, $(a_{i,j} : i, j < \omega)$ in A such that $\phi(a_{i,j}, b_{i'})$ if and only if $i = i'$ and $\psi(a_{i,j}, c_{j'})$ iff $j = j'$. By Theorem 1.1 we may assume that ϕ, ψ are quantifier-free and in disjunctive normal form. Let N be the largest n such that D_n appears in ϕ or ψ. Color each pair (i, j) such that $i > j$ by $a_{i,j}$ mod $N!$ By Ramsey Theorem, we may assume that all the elements $a_{i,j}$ with $i > j$ have the same residue modulo $N!$ and so modulo all $n \leq N$.

Write ϕ as $\bigvee_k \bigwedge_l (\phi'_{k,l} \wedge \phi''_{k,l})$ and ψ as $\bigvee_k \bigwedge_l (\psi'_{k,l} \wedge \psi''_{k,l})$, where $\phi'_{k,l}$, $\psi'_{k,l}$ are atomic or negated atomic L^--formulas and $\phi''_{k,l}$, $\psi''_{k,l}$ are atomic or negated atomic formulas containing no relations other than $\{D_n\}_{n\geq 1}$. For each k, denote by ϕ_k, ψ_k the formulas $\bigwedge_l (\phi'_{k,l} \wedge \phi''_{k,l})$ and $\bigwedge_l (\psi'_{k,l} \wedge \psi''_{k,l})$ respectively.

For every $i > j$ we have $\phi(a_{i,j}, b_i)$, so there is a $k_{i,j}$ such that $\phi_{k_{i,j}}(a_{i,j}, b_i)$. Again by Ramsey Theorem, we may assume that all the $k_{i,j}$'s are equal to some k_0, so for every $i > j$ we have $\phi_{k_0}(a_{i,j}, b_i)$. For every $i' \neq i$ we have $\neg\phi(a_{i',j}, b_i)$, so in particular $\neg\phi_{k_0}(a_{i',j}, b_i)$. Similarly, we may assume that for some k_1, for every $i > j$ we have $\psi_{k_1}(a_{i,j'}, c_j)$ iff $j = j'$.

Let ϕ'_k, ψ'_k be the formulas obtained from ϕ_k, ψ_k respectively, by deleting all the formulas $\phi''_{k,l}$, $\psi''_{k,l}$. So ϕ'_k, ψ'_k are L^--formulas.

For every $m \in \mathbb{N}$, let $I_m = \{m + 1, \ldots, 2m\}$, $J_m = \{1, \ldots, m\}$. For every $(i, j) \in I_m \times J_m$, we have $\phi_{k_0}(a_{i,j}, b_i)$ and therefore also $\phi'_{k_0}(a_{i,j}, b_i)$. Let $i \neq i' \in I_m$, and suppose for a contradiction that $\phi'_{k_0}(a_{i',j}, b_i)$, i.e., $\bigwedge_l (\phi'_{k_0,l}(a_{i',j}, b_i))$. But we know that $\neg\phi_{k_0}(a_{i',j}, b_i)$, so for some l_0 we have $\neg\phi'_{k_0,l_0}(a_{i',j}, b_i) \vee \neg\phi''_{k_0,l_0}(a_{i',j}, b_i)$. Therefore, we get $\neg\phi''_{k_0,l_0}(a_{i',j}, b_i)$. But from $\phi_{k_0}(a_{i,j}, b_i)$ we also get $\phi''_{k_0,l_0}(a_{i,j}, b_i)$. Together, this contradicts the fact that all the elements $a_{i,j}$ with $i > j$ have the same residue modulo all $n \leq N$.

Altogether, in A, for every $(i, j) \in I_m \times J_m$ we have $\phi'_{k_0}(a_{i,j}, b_{i'})$ if and only if $i = i'$, and similarly also $\psi'_{k_1}(a_{i,j}, c_{j'})$ iff $j = j'$. Since ϕ'_{k_0}, ψ'_{k_1} are quantifier-free, and A is a substructure of $\mathcal{M}$, this holds also in $\mathcal{M}$. As m is arbitrary, this contradicts the dp-minimality of $Th(\mathbb{Q}_p^-)$. ⊣

Lemma 4.5. *Let $L = \bigcup_{\alpha<\kappa} L_\alpha$ be a language such that every atomic formula in L is in L_α for some α. Let T be an L-theory that eliminates quantifiers, and for $\alpha < \kappa$ let T_α be its reduction to L_α. Let μ_α be cardinals such that dp-rank$(T_\alpha) \leq \mu_\alpha$. Then dp-rank$(T) \leq \sum_{\alpha<\kappa} \mu_\alpha$, where $\sum$ is the cardinal sum.*

Proof. Suppose not. Let $\lambda := \sum_{\alpha<k} \mu_\alpha$. Then there is a family $(\mathcal{I}_t : t < \lambda^+)$ of mutually indiscernible sequences over $\emptyset$, $\mathcal{I}_t = (a_{t,i} : i \in I_t)$, and a singleton b, such that for all t, $\mathcal{I}_t$ is not indiscernible over b. For every $t < \lambda^+$, let $\phi_t(\bar{x}) = \phi_t(\bar{x}, b)$ be a formula over b and let $\bar{c}_{t,1}$ and $\bar{c}_{t,2}$ be two finite tuples of elements of $\mathcal{I}_t$ of length $|\bar{x}|$ such that $\phi_t(\bar{c}_{t,1})$ and $\neg\phi_t(\bar{c}_{t,2})$, i.e., witnessing the nonindiscernibility of $\mathcal{I}_t$ over b. By quantifier elimination in T, we may assume that ϕ_t is quantifier-free. Hence there must be an atomic formula $\psi_t(\bar{x}) = \psi_t(\bar{x}, b)$ such that $\psi_t(\bar{c}_{t,1})$ and $\neg\psi_t(\bar{c}_{t,2})$. By the assumption on L, there is an $\alpha_t < \kappa$ such that $\psi_t(\bar{x}, y)$ is in L_{α_t}. Therefore,

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there must be an $\alpha < \kappa$ such that $|\{t < \lambda^+ : \alpha_t = \alpha\}| > \mu_\alpha$, as otherwise we get

$$\lambda^+ = \left|\bigcup_{\alpha<\kappa} \{t < \lambda^+ : \alpha_t = \alpha\}\right| \leq \sum_{\alpha<\kappa} \left|\{t < \lambda^+ : \alpha_t = \alpha\}\right| \leq \sum_{\alpha<\kappa} \mu_\alpha = \lambda,$$

a contradiction. But then $(\mathcal{I}_t : t < \lambda^+, \alpha_t = \alpha)$ is a family of more than μ_α mutually indiscernible sequences over $\emptyset$ with respect to L_α, and for all t such that $\alpha_t = \alpha$, $\mathcal{I}_t$ is not indiscernible over b with respect to L_α, a contradiction to dp-rank$(T_\alpha) \leq \mu_\alpha$. ⊣

Now Theorem 1.2 follows:

Proof of Theorem 1.2. dp-rank$(T_P) \leq |P|$ follows from Proposition 4.4 and Lemma 4.5 for $L_P^E = \bigcup_{\alpha<|P|} L_{p_\alpha}^E$. For $\alpha < |P|$ let $\phi_\alpha(x, y)$ be the formula $x|_{p_\alpha} y \wedge y|_{p_\alpha} x$ (i.e., $v_{p_\alpha}(x) = v_{p_\alpha}(y)$), and for $\alpha < |P|$, $i \in \mathbb{N}$ let $a_{\alpha,i}$ be such that $v_{p_\alpha}(a_{\alpha,i}) = i$. Let $F \subseteq |P|$ be finite. By Lemma 2.9(4), for every $\eta : F \to \mathbb{N}$ there is a b_η such that for every $\alpha \in F$, $v_{p_\alpha}(b_\eta) = v_{p_\alpha}(a_{\alpha,\eta(\alpha)})$. If P is finite, just take $F = |P|$. Otherwise, by compactness, there are such b_η for $F = |P|$ as well. These $\phi_\alpha(x, y)$, $a_{\alpha,i}$, and b_η form an ict-pattern of length $|P|$, so dp-rank$(T_P) \geq |P|$. ⊣

§5. There are no intermediate structures between $(\mathbb{Z}, +, 0)$ and $(\mathbb{Z}, +, 0, |_p)$.

In this section we focus on a single valuation. Let p be any prime. Unless stated otherwise, we work in a monster model $\mathcal{M} = (M, +, 0, |_p)$ of T_p, and denote its value set by Γ. We may omit the subscript p when it is clear from the context. Recall that Γ is an elementary extension of $(\mathbb{N}, <, 0, S)$.

5.1. Preliminaries. For $a \in M$, $\gamma \in \Gamma$, we denote by $B(a, \gamma)$ the definable set $\{x : v(x - a) \geq \gamma\}$ and call it the *ball* of *radius* γ around a. If $\gamma = \infty$ then $B(a, \gamma)$ is just $\{a\}$, and we call such balls *trivial*. Unless stated otherwise, balls are assumed to be nontrivial. Of course, $a \in B(a, \gamma)$, and if $b \in B(a, \gamma)$ then $B(b, \gamma) = B(a, \gamma)$. Also, by Lemma 2.2(2), if $\delta \neq \gamma$ then $B(a, \delta) \neq B(a, \gamma)$. So the radius of a ball is well defined. We denote the radius of a ball B by $rad(B)$.

We call a *swiss cheese* any nonempty set F that can be written as $F = B_0 \setminus \bigcup_{i=1}^n B_i$, where $\{B_i\}_{i=0}^n$ are balls. Note that this representation is not unique. As the intersection of any two balls is either empty or equals one of them, we may always assume that $\{B_i\}_{i=1}^n$ are nonempty, pairwise disjoint and contained in B_0.

Remark 5.1. Rephrasing Lemma 2.9(2), if $B_0, B_1, \dots, B_n$ are balls such that for all $i \geq 1$, $rad(B_i) \geq rad(B_0) + \underline{n}$, then $B_0 \setminus \bigcup_{i=1}^n B_i \neq \emptyset$. In particular, this holds if $|rad(B_i) - rad(B_0)| \notin \mathbb{N}$.

Proposition 5.2. *Let $\emptyset \neq F = B_0 \setminus \bigcup_{i=1}^n B_i$ be a swiss cheese. Then there exists a unique ball B_0' such that $F \subseteq B_0'$ and B_0' is minimal with respect to this property. This B_0' satisfies $B_0' \subseteq B_0$, $|rad(B_0') - rad(B_0)| \in \mathbb{N}$, and it is also the unique ball $B \subseteq B_0$ such that there are at least two distinct balls B_1'' and B_2'', satisfying $rad(B_j'') = rad(B_0') + 1$ and $B_j'' \cap F \neq \emptyset$ for $j = 1, 2$.*

Proof. Let $I_1 = \{1 \leq i \leq n : |rad(B_i) - rad(B_0)| \in \mathbb{N}\}$, $I_2 = \{1, \dots, n\} \setminus I_1$. By applying Lemma 2.8(2) to $B_0 \setminus \bigcup_{i \in I_1} B_i \neq \emptyset$, we see that $B_0 \setminus \bigcup_{i \in I_1} B_i = \bigsqcup_{j=1}^l B_j''$, where $l \geq 1$ and for all j, $B_j'' \subseteq B_0$ and $rad(B_j'') = \max\{rad(B_i) : i \in I_1\}$. So

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$F = \bigsqcup_{j=1}^{l}(B_j''\setminus\bigcup_{i\in I_2} B_i)$. By Remark 5.1, for each j, $B_j''\setminus\bigcup_{i\in I_2} B_i \neq \emptyset$. If C is a ball such that $F \subseteq C$, then for each j, $B_j''\setminus\bigcup_{i\in I_2} B_i \subseteq C$, and we claim that in fact $B_j'' \subseteq C$. Indeed, by Axiom 8, $B_j'' = \bigsqcup_{t=1}^{p} B_{j,t}''$ with $rad(B_{j,t}'') = rad(B_j'') + 1$, and again by Remark 5.1, for each t, $B_{j,t}''\setminus\bigcup_{i\in I_2} B_i \neq \emptyset$. So $C \cap B_{j,t}'' \neq \emptyset$ but $C \not\subseteq B_{j,t}''$ (as also for $s \neq t$, $C \cap B_{j,s}'' \neq \emptyset$), therefore $B_{j,t}'' \subseteq C$. This holds for all t, hence $B_j'' \subseteq C$. In particular, $B_1'' \subseteq C$. As $|rad(B_1'') - rad(B_0)| \in \mathbb{N}$, there are only finitely many balls B such that $B_1'' \subseteq B \subseteq B_0$, so we may choose B_0' to be a minimal one (with respect to inclusion) among those that also satisfy $F \subseteq B$ (exists, since B_0 satisfies this). By this choice, $B_0' \subseteq B_0$ and $|rad(B_0') - rad(B_0)| \in \mathbb{N}$. If B is another ball such that $F \subseteq B$, then $F \subseteq B \cap B_0'$, and $B \cap B_0' \neq \emptyset$ is also a ball. Also, as we have shown, $B_1'' \subseteq B$, so $B_1'' \subseteq B \cap B_0' \subseteq B_0$. Hence by the choice of B_0', $B_0' = B \cap B_0' \subseteq B$. This shows that B_0' is the unique minimal ball containing F. Finally, let D be a ball and assume $F \subseteq D$. By Axiom 8 write $D = \bigsqcup_{t=1}^{p} D_t''$ with $rad(D_t'') = rad(D) + 1$. Then D is minimal if and only if for all t, $F \not\subseteq D_t''$, iff there are $t \neq s$ such that $F \cap D_t'' \neq \emptyset$ and $F \cap D_s'' \neq \emptyset$. ⊣

Let F be a swiss cheese. By Proposition 5.2 we may write $F = B_0\setminus\bigcup_{i=1}^{n} B_i$ where B_0 is the unique minimal ball containing F. We may also assume that $\{B_i\}_{i=1}^{n}$ are nonempty, pairwise disjoint and contained in B_0. Unless stated otherwise, all representations are assumed to satisfy these conditions. We call B_0 the *outer ball* of F, and define the *radius* of F to be $rad(F) := rad(B_0)$. We also call $\{B_i\}_{i=1}^{n}$ the *holes* of F. Note that this representation is still not unique (unless there are no holes at all), as each hole may always be split into p smaller holes, and sometimes there are sets of p holes which may each be combined into a single hole. There is a canonical representation for F, namely, the one with the minimal number of holes. But we will not use it. Rather, when dealing with holes without mentioning a specific representation, either the intended representation is clear from the context (e.g., when using Remark 5.3(2) or (3) to split a swiss cheese with a given representation), or we may choose any representation and stick with it.

We say that B_i is a *proper hole* of F if $|rad(B_i) - rad(B_0)| \notin \mathbb{N}$. We call F a *proper cheese* if all of its holes are proper. Note that by Remark 5.1, being a proper cheese does not depend on the representation of the holes.

REMARK 5.3. (1) If $B_0, B_1, \dots, B_n$ are balls such that for all $i \geq 1$, $B_i \subseteq B_0$ and $|rad(B_i) - rad(B_0)| \notin \mathbb{N}$, then B_0 is the outer ball of the swiss cheese $F = B_0\setminus\bigcup_{i=1}^{n} B_i$, which is therefore proper.

(2) Let F be a swiss cheese, and let $k \geq 1$. Then F may be written as a disjoint union $F = \bigsqcup_{i=1}^{l} F_i$, where $1 \leq l \leq p^k$, and for each i, F_i is a swiss cheese such that $rad(F_i) \geq rad(F) + k$ and $|rad(F_i) - rad(F)| \in \mathbb{N}$. Each hole of F_i is already a hole of F, and each hole of F is a hole of at most one of the $\{F_i\}_i$.
If F is proper, then $l = p^k$ and each F_i is a proper cheese of radius $rad(F_i) = rad(F) + k$. In this case, each hole of F is a hole of exactly one of the $\{F_i\}_i$.

(3) Let $F = B_0\setminus\bigcup_{i=1}^{n} B_i$ be a swiss cheese, let $I_1 = \{1 \leq i \leq n : |rad(B_i) - rad(B_0)| \in \mathbb{N}\}$, and let $k_0 = \max\{rad(B_i) - rad(B_0) : i \in I_1\} \in \mathbb{N}$. Then for each $k \geq k_0$, F may be written as a disjoint union $F = \bigsqcup_{i=1}^{l} F_i$, where $1 \leq l \leq p^k$, and for each i, F_i is a *proper* swiss cheese of radius

$rad(F_i) = rad(F) + k$. Each hole of F_i is already a proper hole of F, and each proper hole of F is a hole of exactly one of the $\{F_i\}_i$.

(4) Let F', F'' be two swiss cheeses of radiuses γ', γ'' respectively, and let $\gamma = \max\{\gamma', \gamma''\}$. Then $F' \cap F''$ is either empty, or also a swiss cheese of radius $rad(F' \cap F'') \geq \gamma$ such that $|rad(F' \cap F'') - \gamma| \in \mathbb{N}$.

(5) If both F', F'' are proper and $\gamma' = \gamma''$, and if $F' \cap F''$ is not empty, then F', F'' have the same outer ball, and $F' \cap F''$ is also a proper cheese of the same outer ball.

LEMMA 5.4. *Let F, F' be two swiss cheeses of radiuses $\gamma \leq \gamma'$ respectively. If $F \cap F' \neq \emptyset$, then $F \cup F'$ is also a swiss cheese, of radius exactly γ. The set of holes of $F \cup F'$ is a subset of the union of the set of holes of F and the set of holes of F'.*

PROOF. Write $F = B_0 \setminus \bigcup_{i=1}^n B_i$, $F' = B'_0 \setminus \bigcup_{j=1}^m B'_j$. If $F \cap F' \neq \emptyset$ then $B_0 \cap B'_0 \neq \emptyset$, hence $B_0 \supseteq B'_0$. Therefore,

$$F' \setminus F = F' \setminus \left(B_0 \setminus \bigcup_{i=1}^n B_i \right) = F' \setminus B_0 \cup \left(F' \cap \bigcup_{i=1}^n B_i \right) = \bigcup_{i=1}^n F' \cap B_i.$$

For each i: if $B'_0 \cap B_i = \emptyset$ then $F' \cap B_i = \emptyset$. Otherwise, as $B_0 \supseteq B'_0$, we also get $B_i \subseteq B'_0$ ($B_i \supseteq B'_0$ is impossible, as it implies $F \cap F' = \emptyset$), and in this case, $F' \cap B_i = B_i \setminus \bigcup_{j=1}^m (B_i \cap B'_j)$. Together, we get

$$F \cup F' = F \cup (F' \setminus F) = B_0 \setminus \left(\bigcup_{i \in I_1} B_i \cup \bigcup_{i \in I_2} \bigcup_{j=1}^m (B_i \cap B'_j) \right)$$

where I_1 is the set of i such that $B'_0 \cap B_i = \emptyset$ and I_2 is the set of i such that $B_i \subseteq B'_0$. This is a swiss cheese, and as $F \subseteq F \cup F' \subseteq B_0$ and $rad(F) = rad(B_0) = \gamma$, also $rad(F \cup F') = \gamma$ and B_0 is its outer ball. For each i such that $B_i \subseteq B'_0$ and each j, either $B_i \cap B'_j = \emptyset$ (in which case $B_i \cap B'_j$ does not appear as a hole of $F \cup F'$), or $B_i \cap B'_j = B_i$ or $B_i \cap B'_j = B'_j$, so the last part holds. ⊣

Sometimes we want disjoint swiss cheeses to also have disjoint outer balls, but unfortunately, that is not always possible. An example for this is a union of two swiss cheeses, $F_1 \cup F_2$, with $F_2 \subseteq B$ where B is one of the holes of F_1. If $|rad(B) - rad(F_1)| \in \mathbb{N}$, we may rewrite F_1 as a union of swiss cheeses of radius $rad(B)$, and, together with F_2, we have a union of swiss cheeses with disjoint outer balls. But if $|rad(B) - rad(F_1)| \notin \mathbb{N}$, we cannot do such a thing.

DEFINITION 5.5. A *pseudo swiss cheese* is a definable set P such that there is a swiss cheese F with outer ball B such that $F \subseteq P \subseteq B$. By the following remark, we may call B the *outer ball* of P, and define the *radius* of P to be $rad(P) := rad(B)$. We also call P *pseudo proper cheese* if there is a proper cheese F with outer ball B such that $F \subseteq P \subseteq B$.

REMARK 5.6. (1) In the previous definition, B is uniquely determined by P. Indeed, suppose F_1, F_2 are two swiss cheeses with outer balls B_1, B_2, respectively, such that $F_1 \subseteq P \subseteq B_1$ and $F_2 \subseteq P \subseteq B_2$. Then $rad(B_1) = rad(F_1) \geq rad(B_2)$ and $rad(B_2) = rad(F_2) \geq rad(B_1)$, so $rad(B_1) = rad(B_2)$. Also, $P \subseteq B_1 \cap B_2 \neq \emptyset$, so we must have $B_1 = B_2$.

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(2) For every $k \geq 1$, every proper pseudo swiss cheese of radius γ can be written as a union of exactly p^k proper pseudo cheeses with disjoint outer balls of radius exactly $\gamma + k$.

(3) Note that the analogue to Remark 5.3(2) is not true for pseudo swiss cheeses. For example, let B be a ball of radius γ, let $\{B_i\}_{i=0}^{p-1}$ be all the balls of radius $\gamma + 1$ contained in B, let $\{B_{i,j}\}_{j=0}^{p-1}$ be all the balls of radius $\gamma + 2$ contained in B_i, and let $C \subseteq B_{0,1}$ be a ball of radius $\delta > \gamma$ such that $|\delta - \gamma| \notin \mathbb{N}$. Then $P = C \sqcup \bigsqcup_{i=0}^{p-1} B_{i,0}$ is a pseudo swiss cheese of radius γ, but cannot be written as $\leq p$ pseudo swiss cheeses of radius $\geq \gamma + 1$, because $P \cap B_0$ is not a pseudo swiss cheese. Also note that the intersection of two pseudo swiss cheeses is not necessarily a single pseudo swiss cheese. For example, take $P \cap B_0$ from above.

LEMMA 5.7. (1) *Let P_1,P_2 be two pseudo swiss cheeses with outer balls B_1,B_2, respectively, such that $rad(B_1) \geq rad(B_2)$. If $B_1 \cap B_2 \neq \emptyset$ then $P_1 \cup P_2$ is also a pseudo swiss cheese, with outer ball B_2. If P_2 is proper, then $P_1 \cup P_2$ is also proper.*

(2) *Any finite union of pseudo swiss cheeses may be written as a union of pseudo swiss cheeses having disjoint outer balls. Also, any finite union of pseudo proper cheeses may be written as a union of pseudo proper cheeses having disjoint outer balls.*

PROOF. We prove (1). $B_1 \cap B_2 \neq \emptyset$ and $rad(B_1) \geq rad(B_2)$, so $B_1 \subseteq B_2$ and therefore also $P_1 \subseteq B_2$. Let F_2 be a swiss cheese with outer ball B_2 such that $F_2 \subseteq P_2 \subseteq B_2$. Then $F_2 \subseteq P_1 \cup P_2 \subseteq B_2$. If P_2 is proper, then we may take F_2 to be proper, and so $P_1 \cup P_2$ is also proper.
We prove (2). Let $A = \bigcup_{i=1}^n P_i$ such that for each i, P_i is a pseudo swiss cheese with outer ball B_i. Let $\{B'_j\}_{j=1}^m$ be the set of all the maximal balls (with respect to inclusion) among $\{B_i\}_{i=1}^n$. Then $\{B'_j\}_{j=1}^m$ are pairwise disjoint. For each $1 \leq j \leq m$, let $I_j = \{i \,:\, B_i \cap B'_j \neq \emptyset\}$ and $P'_j = \bigcup_{i \in I_j} P_i$. So $\{1, \ldots, n\} = \bigsqcup_{j=1}^m I_j$ and therefore $A = \bigcup_{j=1}^m P'_j$. By (1), P'_j is a pseudo swiss cheese with outer ball B'_j. If for each i, P_i is proper, then by (1), for each j, P'_j is also proper. ⊣

REMARK 5.8. The valuation v_p induces a topology on $\mathcal{M}$, generated by the balls. By Lemma 2.9(3), if $\gcd(m, p) = 1$, then the sets defined by $D_m(x - r)$ are dense in $\mathcal{M}$.

LEMMA 5.9. *Let P be a pseudo swiss cheese with outer ball B and radius α, and assume $0 \in B$. Let G be a dense subgroup of $\mathcal{M}$, and let $A = P \cap G$. Then there exists $N \in \mathbb{N}$ and $a_1, \ldots, a_N \in B \cap G$ such that $\bigcup_{i=1}^N (A + a_i) = B \cap G$.*

PROOF. Observe that B is a subgroup of $\mathcal{M}$ since $0 \in B$. Let F be a swiss cheese with outer ball B such that $F \subseteq P \subseteq B$. By Remark 5.3(3), for some finite k we may find a proper cheese $F' \subseteq F$ of radius $\alpha + k$. Let s be the number of holes in F'. By Remark 5.3(2), we may write F' as a union of exactly p^s proper cheeses of radius $\alpha + k + s$. As $p^s > s$, at least one of these proper cheeses must have no holes, i.e., must be a ball, say D. Let $x \in D$ and $D_0 = D - x$. Then D_0 is a subgroup of B of index $N := p^{k+s}$. Let $x_1, \ldots, x_N$ be representatives of the cosets, so $B = \bigcup_{i=1}^N x_i + D_0$. For each i, let $a_i \in x_i + D_0 \cap G$. As $a_i \in B \cap G$ and $A \subseteq B \cap G$,

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we have $(A+a_i) \subseteq B \cap G$, and therefore $\bigcup_{i=1}^{N}(A+a_i) \subseteq B \cap G$. On the other hand, as $A \supseteq D \cap G$, we also have $\bigcup_{i=1}^{N}(A+a_i) \supseteq B \cap G$. ⊣

LEMMA 5.10. *Let $A = G \cap \bigsqcup_{i=1}^{n} F_i$ where G is a dense subgroup of $\mathcal{M}$ and $\{F_i\}_{i=1}^{n}$ are disjoint proper cheeses with nonstandard radiuses. Then there are $N, m \in \mathbb{N}$ and $c_1, \dots, c_N \in G$ such that $\bigcap_{i=1}^{N}(A-c_i) = G \cap \bigsqcup_{i=1}^{m} P_i$ with P_i pseudo proper cheeses with disjoint outer balls, all of the same nonstandard radius, and $0 \in P_1$.*

PROOF. It is of course enough to prove the lemma without the requirement $0 \in P_1$, as we may then arrange that by shifting by some $c \in G \cap P_1$.

Preparation step. By Remark 5.3(2), if F is a proper cheese of infinite radius γ then, for all $k \geq 0$, F can be written as a disjoint union of proper cheeses of radius $\gamma + k$. So there exists $\gamma_1, \dots, \gamma_n$, in distinct archimedean classes of Γ, such that we can write

$$\bigsqcup_{i=1}^{n} F_i = \bigsqcup_{i=1}^{m} \bigsqcup_{j=1}^{s_i} F_j^i,$$

where $s_1, \dots, s_m \geq 1$ and for all $1 \leq i \leq m$ and $1 \leq j \leq s_i$, $rad(F_j^i) = \gamma_i$ and F_j^i has a swiss cheese representation in which the radiuses of all the holes are in

$$R := \{\alpha \in \Gamma : \text{for all } 1 \leq k \leq m, \text{ if } |\alpha - \gamma_k| \in \mathbb{N} \text{ then } \alpha \leq \gamma_k\}.$$

We call this representation of A a *good representation of A with respect to* $\{\gamma_i\}_{i=1}^{m}$.

If $m = 1$, we already have what we want, so we may assume that $m > 1$. For each i, j, let B_j^i be the outer ball of F_j^i. There are two cases:

CASE 1. For every $1 < l \leq m$ and every $1 \leq u \leq s_l$ there is some $1 \leq v \leq s_1$ such that $B_v^1 \cap B_u^l \neq \emptyset$.

This means that $\{B_j^1\}_{j=1}^{s_1}$ is the set of all the maximal balls with respect to inclusion among $\{B_j^i : 1 \leq i \leq m, 1 \leq j \leq s_i\}$. It follows that $\{B_j^1\}_{j=1}^{s_1}$ are outer balls of pseudo proper cheese containing all the F_j^i. Indeed, by the proof of Lemma 5.7(2), we may write

$$\bigsqcup_{i=1}^{m} \bigsqcup_{j=1}^{s_i} F_j^i = \bigsqcup_{j=1}^{s_1} P_j,$$

where for each j, P_j is a pseudo proper cheese such that $F_j^1 \subseteq P_j \subseteq B_j^1$. So these are pseudo proper cheeses with disjoint outer balls, all of the same radius γ_1. So in this case we are done.

CASE 2. There are $1 < l \leq m$ and $1 \leq v \leq s_l$ such that for every $1 \leq j \leq s_1$, $B_j^1 \cap B_v^l = \emptyset$.

Let $a \in F_1^1 \cap G$ and $b \in F_v^l \cap G$ and set $A' = (A-a) \cap (A-b)$. Then $0 \in A' \neq \emptyset$. We show that A' has a good representation with respect to a subset of $\{\gamma_i\}_{i=1}^{m}$, of the form

$$A' = G \cap \bigsqcup_{i=1}^{m'} \bigsqcup_{j=1}^{s_i'} \tilde{F}_j^i$$

such that either there are no more proper cheeses of radius γ_1, or the number s_1' of proper cheeses of radius γ_1 is strictly less than s_1. By reiterating this process, it will terminate either to the case in which every proper cheese is of the same radius or to Case 1, which proves the Lemma.

Write $A' = G \cap (\bigsqcup_{i=1}^{m} \bigsqcup_{j=1}^{s_i} \bigsqcup_{q=1}^{m} \bigsqcup_{r=1}^{s_i} (F_j^i - a) \cap (F_r^q - b))$. By the good representation, for each i, j we write $F_j^i = B_j^i \setminus \bigsqcup_t B_{j,t}^i$ with $rad(B_{j,t}^i) \in R$.

For every i and j, k, if $B_j^i - a \neq B_k^i - b$, then $(F_j^i - a) \cap (F_k^i - b) = \emptyset$, and if $B_j^i - a = B_k^i - b$, then $(F_j^i - a) \cap (F_k^i - b)$ is a proper cheese of radius $\gamma_i \geq \gamma_1$ such that all its holes can be written with radiuses in R.

For every $i < i'$ and j, k, if $(B_j^i - a) \cap (B_k^{i'} - b) = \emptyset$, then also $(F_j^i - a) \cap (F_k^{i'} - b) = \emptyset$. Otherwise, $(B_j^i - a) \supseteq (B_k^{i'} - b)$ and

$$(F_j^i - a) \cap (F_k^{i'} - b) = ((B_k^{i'} - b) \setminus \bigsqcup_{t'} (B_{k,t'}^{i'} - b)) \setminus \bigsqcup_t (B_{j,t}^i - a).$$

For each t such that $(B_{j,t}^i - a) \cap (B_k^{i'} - b) \neq \emptyset$ there are three cases:

(1) $rad(B_k^{i'} - b) > rad(B_{j,t}^i - a)$. Then $(B_k^{i'} - b)$ is included in the hole $(B_{j,t}^i - a)$ hence $(F_j^i - a) \cap (F_k^{i'} - b) = \emptyset$.

(2) $rad(B_k^{i'} - b) \leq rad(B_{j,t}^i - a)$ and $rad(B_{j,t}^i - a)$ is at finite distance from $\gamma_{i'}$. As $rad(B_{j,t}^i - a) = rad(B_{j,t}^i) \in R$, we get

$$rad(B_k^{i'} - b) = rad(B_k^{i'}) = \gamma_{i'} \geq rad(B_{j,t}^i - a).$$

So $rad(B_k^{i'} - b) = rad(B_{j,t}^i - a)$, and so $(B_k^{i'} - b) = (B_{j,t}^i - a)$ and therefore $(F_j^i - a) \cap (F_k^{i'} - b) = \emptyset$.

(3) $rad(B_k^{i'} - b) \leq rad(B_{j,t}^i - a)$ and $rad(B_{j,t}^i - a)$ is not at finite distance from $\gamma_{i'}$. Then $B_{j,t}^i - a$ is a proper hole of $(F_j^i - a) \cap (F_k^{i'} - b)$.

Therefore $(F_j^i - a) \cap (F_k^{i'} - b)$ is either empty or a proper cheese of radius $\gamma_{i'} > \gamma_i \geq \gamma_1$ such that all its holes can be written with radiuses in R.

So A' has a good representation that is the intersection of G with a (nonempty) disjoint union of proper cheeses, with radiuses among $\{\gamma_i\}_{i=1}^m$, such that all their holes have radiuses in R. Now either $s_1 = 1$, hence F_1^1 is the only cheese of radius γ_1 in the good representation of A and hence in the good representation of A' there are no more proper cheeses of radius γ_1. Otherwise we have a good representation with respect to a subset of $\{\gamma_i\}_{i=1}^m$ of the form

$$A' = G \cap \bigsqcup_{i=1}^{m'} \bigsqcup_{j=1}^{s_i'} \tilde{F}_j^i$$

where $s_1', \ldots, s_{m'}' \geq 1$, and s_1' is the number of cheese of radius γ_1. For every $1 \leq l \leq s_1'$, there must be j, k such that $\tilde{F}_l^1 = (F_j^1 - a) \cap (F_k^1 - b)$. As $(F_j^1 - a) \cap (F_k^1 - b) \neq \emptyset \iff B_j^1 - a = B_k^1 - b$, for every j there is at most one k such that $(F_j^1 - a) \cap (F_k^1 - b) \neq \emptyset$, therefore $s_1' \leq s_1$. Suppose towards contradiction that $s_1' = s_1$. Then for every j there is exactly one k such that $(F_j^1 - a) \cap (F_k^1 - b) \neq \emptyset$, in particular, for $j = 1$ there is exactly one l such that $(F_1^1 - a) \cap (F_l^1 - b) \neq \emptyset$, and so also $B_1^1 - a = B_l^1 - b$. By the choice of a, b, we have $0 \in (B_1^1 - a) \cap (B_v^l - b) = (B_l^1 - b) \cap (B_v^l - b)$, so $b \in B_l^1 \cap B_v^l \neq \emptyset$, a contradiction. Therefore $s_1' < s_1$. ⊣

Lemma 5.11. *Let $A = G \cap \bigsqcup_{i=1}^n P_i$ where G is a dense subgroup of $\mathcal{M}$ and $\{P_i\}_{i=1}^n$ are pseudo proper cheeses with disjoint outer balls, all of the same nonstandard* *radius* α*, such that* $0 \in P_1$*. Then there exists* $N \in \mathbb{N}$ *and* $c_1, \dots, c_N \in G$ *such that* $\bigcap_{i=1}^N (A - c_i) = G \cap P$ *for some pseudo proper cheese* P *of nonstandard radius such that* $0 \in P$.

Proof. It is of course enough to prove the lemma without the requirement $0 \in P$. We proceed by induction on n. For $n = 1$ we have nothing to prove. Suppose that the lemma holds for all $n' < n$. For each $1 \leq i \leq n$ let B_i be the outer ball of P_i, and let F_i be a proper cheese with outer ball B_i such that $F_i \subseteq P_i \subseteq B_i$. Let S be the set of all the balls of radius α, and let $S' = \{B_i : 1 \leq i \leq n\}$. Observe that $(S, +)$ is an infinite group with neutral element B_1 (since $0 \in P_1 \subseteq B_1$), and in particular, $S' \subsetneq S$. Let $C := \bigcup S' = \bigsqcup_{i=1}^n B_i$.

Claim 1. *If for every* $1 \leq i \leq n$ *there is* $a \in B_i$ *such that* $S' - a = S'$*, then* S' *is a subgroup of* S.

Proof of claim. If $B, B' \in S$ then $rad(B) = rad(B')$, hence $(B - a) \cap B' \neq \emptyset \Rightarrow B - a = B'$. Also, for all $B'' \in S$ and $a, a' \in B''$, $a - a' \in B_1$ and therefore $B - a' = (B - a) + (a - a') = B - a$. From this and the hypothesis of the claim it follows that for each $1 \leq i \leq n$, $S' - B_i := \{B - B_i : B \in S'\} = S'$, which implies that S' is a subgroup of S. ⊣

There are two cases:

Case 1. S' is a subgroup of S. Then $(C, +)$ is a subgroup of $(M, +)$, and S' is the quotient group C/B_1. As $(C, +)$ is definable, by Lemma 3.3 it must be of the form $C = B(0, \beta)$ (as $B_1 \not\subseteq mM$ for every $m > 1$ with $\gcd(m, p) = 1$). In fact, since $|S'| = n$, it must be that $\beta = \alpha - k$, where k satisfies $n = p^k$. In particular, β is nonstandard. For each i, let H_i be (any choice for) the set of holes of F_i, and let $H = \bigcup_i H_i$. Then we can rewrite $\bigsqcup_{i=1}^n F_i$ as $F = B(0, \beta) \setminus \bigcup H$, which is a single proper cheese, with outer ball $B(0, \beta)$. Let $P = \bigsqcup_{i=1}^n P_i$. Then $F \subseteq P \subseteq B(0, \beta)$, so P is a pseudo proper cheese, and we are done.

Case 2. S' is not a subgroup of S. Then by the claim, there is some $1 \leq i_0 \leq n$ such that for all $a \in B_{i_0}$, $S' - a \neq S'$ (in fact $1 < i_0$). Let $a \in G \cap P_{i_0} \subseteq B_{i_0}$ (which exists because G is dense), and let $A' = A \cap (A - a)$. Then $0 \in A' \neq \emptyset$.

Write $A' = G \cap (\bigsqcup_{i=1}^n \bigsqcup_{j=1}^n P_i \cap (P_j - a))$. Then

$$G \cap \bigsqcup_{i=1}^n \bigsqcup_{j=1}^n F_i \cap (F_j - a) \subseteq A' \subseteq G \cap \bigsqcup_{i=1}^n \bigsqcup_{j=1}^n B_i \cap (B_j - a).$$

For all $1 \leq i, j \leq n$, $rad(B_i) = rad(B_j) = \alpha$ and therefore, as in Lemma 5.10, $B_i \cap (B_j - a) \neq \emptyset \iff B_i = B_j - a \iff F_i \cap (F_j - a) \neq \emptyset$, and in this case, $F_i \cap (F_j - a)$ is a proper cheese with outer ball B_i. We also have that $F_i \cap (F_j - a) \subseteq P_i \cap (P_j - a) \subseteq B_i \cap (B_j - a)$, so $P_i \cap (P_j - a) \neq \emptyset \iff B_i \cap (B_j - a) \neq \emptyset$, and in this case, $P_i \cap (P_j - a)$ is a pseudo proper cheese with outer ball B_i. Therefore, $G \cap (\bigsqcup_{i=1}^n \bigsqcup_{j=1}^n B_i \cap (B_j - a)) = G \cap (\bigsqcup_{i=1}^{n'} B'_i)$, $G \cap (\bigsqcup_{i=1}^n \bigsqcup_{j=1}^n F_i \cap (F_j - a)) = G \cap (\bigsqcup_{i=1}^{n'} F'_i)$, and $A' = G \cap (\bigsqcup_{i=1}^{n'} P'_i)$, where for each i, $B'_i \in S'$, F'_i is a proper swiss cheese with outer ball B'_i, and P'_i is a pseudo proper cheese such that $F'_i \subseteq P'_i \subseteq B'_i$.

Moreover, for every i there is at most one j such that $B_i \cap (B_j - a) \neq \emptyset$, therefore $n' \leq n$. But by the choice of a, $S' - a \neq S'$, so there is an $1 \leq i \leq n$ such that $B_i \neq B_j - a$ for all $1 \leq j \leq n$. Therefore $n' < n$, and by the induction hypothesis we are done. ⊣

5.2. Proof of the theorem. To prove Theorem 1.10 we first prove a lemma that enables us to reduce the problem to single variable formulas. Recall the following:

FACT 5.12 ([16, Theorem 2.13]). *A theory T is stable if and only if all formulas $\phi(x, y)$ over $\emptyset$ with $|x| = 1$ are stable.*

Using this, we can prove:

LEMMA 5.13. *Let L be any language and let T be an unstable L-theory with monster model $\mathcal{M}$. Let $L^- \subseteq L$ be such that $T|_{L^-}$ is stable. Then there exists an L-formula $\phi(x, y)$ over $\emptyset$ with $|x| = 1$ and $b \in \mathcal{M}$ such that $\phi(x, b)$ is not L^--definable with parameters in $\mathcal{M}$.*

PROOF. By 5.12 there is an unstable L-formula $\phi(x, y)$ over $\emptyset$ with $|x| = 1$. Let $(a_i)_{i\in\mathbb{Z}}$, $(b_i)_{i\in\mathbb{Z}}$ be two indiscernible sequences in $\mathcal{M}$ witnessing the instability of $\phi(x, y)$, i.e., $\phi(a_i, b_j)$ if and only if $i < j$. Assume towards contradiction that $\phi(x, b_0)$ is definable by an L^--formula $\psi(x, c_0)$ with parameters c_0 in $\mathcal{M}$. For each $k \in \mathbb{Z}\backslash\{0\}$, as $tp(b_k/\emptyset) = tp(b_0/\emptyset)$ there is an automorphism of L-structures $\sigma_k \in Aut(\mathcal{M}/\emptyset)$ such that $\sigma_k(b_0) = b_k$. Let $c_k = \sigma_k(c_0)$. Then $\phi(x, b_k)$ is equivalent to $\psi(x, c_k)$, and hence $\psi(a_i, c_j)$ if and only if $i < j$, a contradiction to the stability of $T|_{L^-}$. ⊣

Lemma 5.13 allows us to give a simple proof for the unstable case of Corollary 1.9:

THEOREM 5.14 (Conant, unstable case of Corollary 1.9). *Let $(N, +, 0, 1, <)$ be an elementary extension of $(\mathbb{Z}, +, 0, 1, <)$. Then $(N, +, 0, 1, <)$ is $\emptyset$-minimal among the* <u>*unstable*</u> *$\emptyset$-proper $\emptyset$-expansions of $(N, +, 0, 1)$.*

PROOF. Let $\mathcal{N}$ be any unstable structure with universe N, which is a $\emptyset$-proper $\emptyset$-expansion of $(N, +, 0, 1)$ and a $\emptyset$-reduct of $(N, +, 0, 1, <)$. We show that $\mathcal{N}$ is $\emptyset$-interdefinable with $(N, +, 0, 1, <)$. It is enough to show that $x \geq 0$ is definable over $\emptyset$ in $\mathcal{N}$. Let L be the language of $\mathcal{N}$, $L^- = \{+, 0, 1\}$ and $L_< = \{+, 0, 1, <\}$. We may expand all these languages by adding the symbols $\{-\} \cup \{D_n : n \geq 1\}$, as all of them are already definable over $\emptyset$ in all three languages. As $\mathcal{N}$ is a $\emptyset$-expansion of $(N, +, 0, 1)$ and a $\emptyset$-reduct of $(N, +, 0, 1, <)$, we may replace L with $L \cup L^-$ and $L_<$ with $L_< \cup L \cup L^-$ without adding new $\emptyset$-definable sets to any structure. So we may assume that $L^- \subseteq L \subseteq L_<$.

Let $\mathcal{M}$ be a monster model for $Th(\mathbb{Z}, +, 0, 1, <)$, so $\mathcal{M}|_L$ is a monster for $Th(\mathcal{N})$. As $(N, +, 0, 1)$ is stable but $\mathcal{N}$ is not, by Lemma 5.13 there exist an L-formula $\phi(x, y)$ over $\emptyset$ with $|x| = 1$ and $b \in \mathcal{M}$ such that $\phi(x, b)$ is not L^--definable with parameters in $\mathcal{M}$. By quantifier elimination in $Th(\mathbb{Z}, +, 0, 1, <)$ and Lemma 2.3(1) (which is a theorem of $Th(\mathbb{Z}, +, 0, 1)$), $\phi(x, b)$ is equivalent to a formula of the form

$$\bigvee_i (D_{m_i}(x - k_i) \wedge x \in [c_i, c_i'])$$

where $c_i, c_i' \in M \cup \{-\infty, +\infty\}$ and $[c_i, c_i']$ denotes the closed interval except if one of the bounds is infinite, in which case it is open on the infinite side. Let $m = \prod_i m_i$.

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As each formula of the form $D_{m_i}(x-k)$ is equivalent to a disjunction of formulas of the form $D_m(x-k')$, we can rewrite this as

$$\bigvee_i (D_m(x-k_i) \wedge x \in [c_i, c_i'])$$

(with possibly different k_i's and numbering). By grouping together disjuncts with the same k_i, we can rewrite this as

$$\bigvee_i (D_m(x-k_i) \wedge \bigvee_j x \in [c_{i,j}, c_{i,j}'])$$

where for $i_1 \neq i_2$, $k_{i_1} \not\equiv k_{i_2} \mod m$. As this formula is equivalent to $\phi(x,b)$, which is not L^--definable with parameters in $\mathcal{M}$, there must be an i_0 such that $D_m(x - k_{i_0}) \wedge \bigvee_j x \in [c_{i_0,j}, c_{i_0,j}']$ is not L^--definable with parameters in $\mathcal{M}$. This latter formula, which we denote by $\phi_{i_0}(x,b)$, is equivalent to $\phi(x,b) \wedge D_m(x-k_{i_0})$, and so is L-definable. Let $\psi(x,b)$ be the formula $\phi_{i_0}(mx+k_{i_0}, b)$. Then $\psi(x,b)$ is L-definable and equivalent to just $\bigvee_j mx + k_{i_0} \in [c_{i_0,j}, c_{i_0,j}']$. This substitution is reversible as $\phi_{i_0}(x,b)$ is equivalent to $D_m(x-k_{i_0}) \wedge \psi(\frac{x-k_{i_0}}{m}, b)$, therefore also $\psi(x,b)$ is not L^--definable with parameters in $\mathcal{M}$. Each formula of the form $mx+k \in [c,c']$ is equivalent to the formula $x \in [\lceil \frac{c-k}{m} \rceil, \lfloor \frac{c'-k}{m} \rfloor]$, so we can rewrite $\psi(x,b)$ as $\bigvee_{i=1}^n x \in [c_i, c_i']$. By reordering and combining intersecting intervals, we may assume that the intervals are disjoint and increasing, i.e., for all $i < n$, $c_i' < c_{i+1}$.

Now we show how from $\psi(x,b)$ we can get an L-definable formula equivalent to $[0,a]$, for a a positive nonstandard integer in $\mathcal{M}$. For each i, if $[c_i, c_i']$ defines in $\mathcal{M}$ a finite set then it is L^--definable, and so $\psi(x,b) \wedge x \notin [c_i, c_i']$ is also L-definable but not L^--definable (since $(\psi(x,b) \wedge x \notin [c_i, c_i']) \vee x \in [c_i, c_i']$ is again equivalent to $\psi(x,b)$). So we may assume that for all i, $[c_i, c_i']$ is infinite. Note that as $\psi(x,b)$ is not L^--definable, it cannot be empty.

We want $\psi(x,b)$ to have a lower bound, i.e., $-\infty < c_1$. If $c_1 = -\infty$ but $c_n' \neq +\infty$, then we can just replace $\psi(x,b)$ with $\psi(-x,b)$. If both $c_1 = -\infty$ and $c_n' = +\infty$, we can replace $\psi(x,b)$ with $\neg\psi(x,b)$ and again remove all finite intervals. In both cases, $\psi(x,b)$ is still L-definable but not L^--definable, so it is still a nonempty disjunction of infinite disjoint intervals.

By replacing $\psi(x,b)$ with $\psi(x+c_1,b)$ we may assume that $c_1 = 0$, so the leftmost interval is $[0, c_1']$. If $c_1' \neq +\infty$ let $a' = c_1'$, otherwise let $a' \in \mathcal{M}$ be any positive nonstandard integer. Let $\theta(x,b')$ denote the formula $\psi(x,b) \wedge \psi(a'-x,b)$. Then $\theta(x,b')$ is L-definable and equivalent to the infinite interval $[0,a']$. The proof of the following claim is an obvious consequence of quantifier elimination for Presburger arithmetic and is left to the reader.

CLAIM 5.15. *For every $c \geq 0$ there exist $a > c$ and b such that $\theta(x,b)$ is equivalent to the interval* $[0,a]$.

In particular, as N is a small subset of $\mathcal{M}$, there exists $c \in \mathcal{M}$ bigger than all elements of N. By the claim, there exist $\tilde{a} > c$ and $\tilde{b}$ such that $\theta(x,\tilde{b})$ is equivalent to the interval $[0,\tilde{a}]$, and so $\theta(N,\tilde{b}) = \{s \in N : s \geq 0\}$.

Let $\chi(y,z)$ be the formula $\chi_1(y,z) \wedge \chi_2(y,z) \wedge \chi_3(y,z)$ where:

- $\chi_1(y,z)$ is the formula $\theta(0,z) \wedge \theta(y,z) \wedge \neg\theta(-1,z) \wedge \neg\theta(y+1,z) \wedge \neg\theta(2y,z)$.
- $\chi_2(y,z)$ is the formula $\forall w((w \neq 0 \wedge \theta(w,z)) \rightarrow \theta(w-1,z))$.

- $\chi_3(y,z)$ is the formula $\forall w((w \neq y \wedge \theta(w,z)) \to \theta(w+1,z))$.

So $\chi(y,z)$ is L-definable over $\emptyset$.

Claim 5.16. *For every $a, b \in \mathcal{M}$, $\mathcal{M} \vDash \chi(a,b)$ if and only if $a > 0$ and $\theta(\mathcal{M}, b) = [0,a]$.*

Proof. This can be formulated as a first order sentence in $L_<$ without parameters:

$$\mathcal{M} \vDash \forall y, z(\chi(y,z) \leftrightarrow (y > 0 \wedge \forall x(\theta(x,z) \leftrightarrow 0 \leq x \leq y))),$$

so it is enough to prove this for $\mathbb{Z}$. Let $a, b \in \mathbb{Z}$. If $a > 0$ and $\theta(\mathbb{Z}, b) = [0,a]$, then clearly $\mathbb{Z} \vDash \chi(a,b)$. Suppose $\mathbb{Z} \vDash \chi(a,b)$, and denote $A := \theta(\mathbb{Z}, b)$. By χ_1, $0, a \in A$ and $-1, a+1, 2a \notin A$. Suppose towards contradiction that $a < 0$. Then from χ_2 it follows by induction that $(-\infty, a] \subseteq A$. But then $2a \in A$, a contradiction. So $a \geq 0$. If $a = 0$ then again $2a \in A$ is a contradiction. So $a > 0$. From χ_2 it follows by induction that $[0,a] \subseteq A$. Also, from $a+1 \notin A$ and χ_2 it follows by induction that $[a+1, \infty) \cap A = \emptyset$, and from $-1 \notin A$ and χ_3 it follows by induction that $(-\infty, -1] \cap A = \emptyset$. So $A = [0,a]$. ⊣

Now, let $\delta(x)$ be the formula

$$\exists y, z(\chi(y,z) \wedge \theta(x,z)).$$

Then $\delta(x)$ is L-definable over $\emptyset$, and we claim that it defines $x \geq 0$ in $\mathcal{N}$: For $s \in N$, if $\mathcal{N} \vDash \delta(s)$ then there are $a, b \in N$ such that $\mathcal{N} \vDash \chi(a,b) \wedge \theta(s,b)$, so by Claim 5.16, $s \in [0,a]$ hence $s \geq 0$. On the other hand, suppose $s \geq 0$. By the choice of $\tilde{a}, \tilde{b}$, $\mathcal{M} \vDash \chi(\tilde{a}, \tilde{b}) \wedge \theta(s, \tilde{b})$, so $\mathcal{M} \vDash \delta(s)$, and by elementarity, $\mathcal{N} \vDash \delta(s)$. Therefore, $x \geq 0$ is definable over $\emptyset$ in $\mathcal{N}$. ⊣

Remark 5.17. The part in the proof where we start with an L-formula $\phi(x,y)$ over $\emptyset$ with $|x| = 1$ and $b \in \mathcal{M}$ such that $\phi(x,b)$ is not L^--definable with parameters in $\mathcal{M}$, and show that there exists a formula $\theta(x,b')$ which is L-definable and equivalent to the infinite interval $[0,a']$, works the same for any structure $\mathcal{N}$ which is a proper expansion of $(N,+,0,1)$ and a reduct of $(N,+,0,1,<)$. $\mathcal{N}$ does not have to be a $\emptyset$-expansion of $(N,+,0,1)$ or a $\emptyset$-reduct of $(N,+,0,1,<)$, nor unstable, as long as such $\phi(x,y)$ and b exist (being a $\emptyset$-reduct is needed in the proof for $\phi(x,y)$ to also be $\emptyset$-definable in $L_<$). So in any structure $\mathcal{N}$ which is a proper expansion of $(N,+,0,1)$ and a reduct of $(N,+,0,1,<)$, and which has a definable one-dimensional set which is not definable in $(N,+,0,1)$, there exists a definable infinite interval, and hence it is unstable.

Combined with Fact 1.7, we recover Corollary 1.9 and Fact 1.8:

Proof of Corollary 1.9. Suppose for a contradiction that there exists a structure $\mathcal{N}$ with universe N, which is a $\emptyset$-proper $\emptyset$-expansion of $(N,+,0,1)$ and a $\emptyset$-proper $\emptyset$-reduct of $(N,+,0,1,<)$. So $\mathcal{N}$ is dp-minimal, and by Theorem 5.14, it must also be stable. By Observation 1.4, relativization to $\mathbb{Z}$ gives us a structure $\mathcal{Z} \prec \mathcal{N}$ which is a $\emptyset$-proper $\emptyset$-expansion of $(\mathbb{Z},+,0,1)$ and a $\emptyset$-proper $\emptyset$-reduct of $(\mathbb{Z},+,0,1,<)$. As every element of $(\mathbb{Z},+,0,1)$ is $\emptyset$-definable, a reduct of $(\mathbb{Z},+,0,1)$ is in fact a $\emptyset$-reduct, and so a $\emptyset$-proper $\emptyset$-expansion of $(\mathbb{Z},+,0,1)$ is in fact a proper $\emptyset$-expansion of $(\mathbb{Z},+,0,1)$, which is of course a proper expansion. So $\mathcal{Z}$ is a stable dp-minimal proper expansion of $(\mathbb{Z},+,0,1)$, a contradiction to Fact 1.7. ⊣

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Proof of Fact 1.8. Suppose for a contradiction that there exists a structure $\mathcal{Z}$ with universe $\mathbb{Z}$, which is a proper expansion of $(\mathbb{Z}, +, 0, 1)$ and a proper reduct of $(\mathbb{Z}, +, 0, 1, <)$. In $\mathcal{Z}$, $+$, 0, and 1 are definable, but not necessarily $\emptyset$-definable. We expand $\mathcal{Z}$ to a structure $\mathcal{Z}'$ by adding $+$, 0, and 1 to the language. So $\mathcal{Z}'$ is a proper $\emptyset$-expansion of $(\mathbb{Z}, +, 0, 1)$, and still a proper reduct of $(\mathbb{Z}, +, 0, 1, <)$. As every element of $(\mathbb{Z}, +, 0, 1, <)$ is $\emptyset$-definable, a reduct of $(\mathbb{Z}, +, 0, 1, <)$ is in fact a $\emptyset$-reduct. So $\mathcal{Z}'$ is a proper $\emptyset$-expansion of $(\mathbb{Z}, +, 0, 1)$, and a proper $\emptyset$-reduct of $(\mathbb{Z}, +, 0, 1, <)$. As a proper $\emptyset$-expansion/reduct is obviously a $\emptyset$-proper $\emptyset$-expansion/reduct, this contradicts Corollary 1.9. ⊣

The proof of Theorem 1.10 is similar, but more involved and relies on Section 5.1.

Proof of Theorem 1.10. Let $\mathcal{N}$ be any unstable structure with universe N, which is a $\emptyset$-proper $\emptyset$-expansion of $(N, +, 0, 1)$ and a $\emptyset$-reduct of $(N, +, 0, 1, |_p)$. We show that $\mathcal{N}$ is $\emptyset$-interdefinable with $(N, +, 0, 1, |_p)$. It is enough to show that $x|_p y$ is definable over $\emptyset$ in $\mathcal{N}$. Let L be the language of $\mathcal{N}$ and $L^- = \{+, 0, 1\}$. As in the proof of Theorem 5.14, we may assume that all languages contain $\{-\} \cup \{D_n : n \geq 1\}$, and (by being a $\emptyset$-reduct and $\emptyset$-expansion) that $L^- \subseteq L \subseteq L_p^E$.

Let $\mathcal{M}$ be a monster model for T_p, so $\mathcal{M}|_L$ is a monster for $Th(\mathcal{N})$. As $(N, +, 0, 1)$ is stable but $\mathcal{N}$ is not, by Lemma 5.13 there exist an L-formula $\phi(x, y)$ over $\emptyset$ with $|x| = 1$ and $b \in \mathcal{M}$ such that $\phi(x, b)$ is not L^--definable with parameters in $\mathcal{M}$. By Theorem 1.1 (quantifier elimination) and Remark 3.2, $\phi(x, b)$ is equivalent to a formula of the form

$$\bigvee_i \left(D_m(x - r_i) \wedge kx \in F_i \wedge \bigwedge_j k'x \neq a_{i,j} \right) \vee \bigvee_{i'} x = c_{i'}$$

where $m, k, k', r_i \in \mathbb{Z}$, $\gcd(m, p) = \gcd(k, p) = 1$, $k' = p^l k$ for some $l \geq 0$, $a_{i,j}, c_{i'} \in \mathcal{M}$ and each F_i is a swiss cheese in $\mathcal{M}$.

The first step of the proof is to show the existence of an L-definable formula which is equivalent to a formula of the form $D_m(x) \wedge x \in B(0, \gamma)$, i.e., $D_m(x) \wedge v(x) \geq \gamma$, for some nonstandard $\gamma \in \Gamma$ and integer m such that $\gcd(m, p) = 1$. Let $\phi'(x, b)$ be the formula

$$\bigvee_i (D_m(x - r_i) \wedge kx \in F_i).$$

The symmetric difference $\phi(x, b) \triangle \phi'(x, b)$ is finite, hence L^--definable, and therefore $\phi'(x, b)$ is also L-definable but not L^--definable. So we may replace $\phi(x, b)$ by $\phi'(x, b)$. For each i, the formula $D_m(x - r_i)$ is equivalent to $D_{km}(kx - kr_i)$, so $\phi(x, b)$ is equivalent to the formula

$$\bigvee_i (D_{km}(kx - kr_i) \wedge kx \in F_i).$$

Let $\phi'(x, b)$ be the formula $D_k(x) \wedge \phi(\frac{x}{k}, b)$. Then $\phi'(x, b)$ is L-definable and equivalent to the formula

$$\bigvee_i (D_{m'}(x - r_i') \wedge x \in F_i)$$

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where $m' = km$ and $r_i' = kr_i$. This substitution is reversible as $\phi(x, b)$ is equivalent to $\phi'(kx, b)$, therefore also $\phi'(x, b)$ is not L^--definable. So again we may replace $\phi(x, b)$ by $\phi'(x, b)$.

We want each F_i to have a nonstandard radiuses. For each i, choose a representation for F_i as a swiss cheese $F_i = B_{i,0} \setminus \bigcup_{j=1}^{n_i} B_{i,j}$, where $B_{i,j} = B(a_{i,j}, \gamma_{i,j})$. Let $J_i = \{1 \leq j \leq n_i : \gamma_{i,j} \notin \mathbb{N}\}$, i.e., the set of indices of the infinite holes, and let

$$B_{i,0}' = \begin{cases} B(0,0) & \gamma_{i,0} \in \mathbb{N}, \\ B_{i,0} & \gamma_{i,0} \notin \mathbb{N}, \end{cases} \text{ and } B_{i,0}'' = \begin{cases} B_{i,0} & \gamma_{i,0} \in \mathbb{N}, \\ B(0,0) & \gamma_{i,0} \notin \mathbb{N}, \end{cases}$$

(note that $B(0, 0) = M$). Let $F_i' = B_{i,0}' \setminus \bigcup_{j \in J_i} B_{i,j}$, and let $F_i'' = B_{i,0}'' \setminus \bigcup_{j \notin J_i} B_{i,j}$. Then $F_i = F_i' \cap F_i''$, and so $\phi(x, b)$ is equivalent to

$$\bigvee_i (D_{m'}(x - r_i') \wedge x \in F_i'' \wedge x \in F_i').$$

Each hole of F_i' has nonstandard radius, and its outer ball either has an nonstandard radius or has radius 0. On the other hand, both the outer ball and all the holes of F_i'' have finite radiuses. In general, if $B(a, \gamma)$ has finite radius, then the formula $x \in B(a, \gamma)$ is equivalent to $D_{p^\gamma}(x - a)$. So $x \in F_i''$ is equivalent to a boolean combination of such formulas, and therefore, by Lemma 2.3(1) (choosing the same m'' for all the i's and rearranging the disjunction), $\phi(x, b)$ is equivalent to a formula of the form

$$\bigvee_i (D_{m''}(x - r_i') \wedge x \in F_i')$$

where each hole of F_i' has a nonstandard radius, and its outer ball either has an nonstandard radius or has radius 0. Note that now it may be that $p|m''$. By grouping together disjuncts with the same r_i', we can rewrite this as

$$\bigvee_i (D_{m''}(x - r_i') \wedge \bigvee_j x \in F_{i,j}')$$

where for $i_1 \neq i_2$, $r_{i_1}' \not\equiv r_{i_2}' \bmod m''$. As this formula is equivalent to $\phi(x, b)$, which is not L^--definable with parameters in $\mathcal{M}$, there must be an i_0 such that $D_{m''}(x - r_{i_0}') \wedge \bigvee_j x \in F_{i_0,j}'$ is not L^--definable with parameters in $\mathcal{M}$. This latter formula, which we denote by $\phi_{i_0}(x, b)$, is equivalent to $\phi(x, b) \wedge D_{m''}(x - r_{i_0}')$, and so is L-definable. So we may replace $\phi(x, b)$ by $\phi_{i_0}(x, b)$. For simplicity of notation we rewrite this as

$$D_m(x - r) \wedge \bigvee_i x \in F_i.$$

By Lemma 5.4 we may assume that $\{F_i\}_i$ are pairwise disjoint, and still have that for each i, all the holes of F_i have nonstandard radiuses and its outer ball either has a nonstandard radius or has radius 0. By Remark 5.1 two proper cheeses having the same outer ball must intersect. Applying this to all the F_i's having radius 0 (which are all proper, as all the holes are of nonstandard radius), we see that there can be at most one i such that F_i has radius 0.

We want all proper cheeses to have nonstandard radius. If there is i_0 such that the proper cheese F_{i_0} has radius 0, let $\phi'(x, b)$ be the formula $D_m(x - r) \wedge \neg\phi(x, b)$.

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Then $\phi'(x, b)$ is L-definable and, as $\phi(x, b)$ is equivalent to $D_m(x - r) \wedge \neg\phi'(x, b)$, it is also not L^--definable. The formula $\phi'(x, b)$ is equivalent to

$$D_m(x - r) \wedge \bigwedge_i x \in F_i^c.$$

We may write $F_{i_0} = B(0, 0) \setminus \bigcup_{j=1}^n B_j$, where for each j, $rad(B_j)$ is nonstandard. So $F_{i_0}^c = \bigcup_{j=1}^n B_j$, and $\phi'(x, b)$ is equivalent to

$$D_m(x - r) \wedge \bigvee_{j=1}^n (x \in B_j \wedge \bigwedge_{i \neq i_0} x \in F_i^c).$$

For each $i \neq i_0$, F_i^c is a finite union of swiss cheeses (specifically, a union of a single swiss cheese of radius 0 and a finite number of balls). Therefore, by Remark 5.3(4), for each j, $B_j \cap \bigcap_{i \neq i_0} F_i^c$ is a finite union of swiss cheeses, each of radius at least $rad(B_j)$, so nonstandard. So $\phi'(x, b)$ is equivalent to a formula of the form

$$D_m(x - r) \wedge \bigvee_i x \in F_i'$$

where each F_i' is a swiss cheese of nonstandard radius. Again by Lemma 5.4, we may assume in addition that $\{F_i'\}_i$ are pairwise disjoint. As $\phi'(x, b)$ is not L^--definable, the disjunction cannot be empty. So we may replace $\phi(x, b)$ by $\phi'(x, b)$ and rename F_i' as F_i.

We may assume that for each i, $D_m(x - r) \wedge x \in F_i$ defines a nonempty set, as otherwise we may just drop the i'th disjunct. Write $m = p^k m'$ with $\gcd(m', p) = 1$. Then $D_m(x - r)$ is equivalent to $D_{m'}(x - r_1) \wedge (v_p(x - r_2) \geq k)$, where $r_1 = r \bmod m'$ and $r_2 = r \bmod p^k$. So $\phi(x, b)$ is equivalent to

$$D_{m'}(x - r_1) \wedge \bigvee_i (v_p(x - r_2 \geq k) \wedge x \in F_i).$$

The formula $v_p(x - r_2) \geq k$ defines the ball $B(r_2, k)$, of finite radius k, and for each i, the outer ball of F_i has a nonstandard radius. As $D_m(x - r) \wedge x \in F_i$ defines a nonempty set, so too does $v_p(x - r_2) \geq k \wedge x \in F_i$, and hence the outer ball of F_i is contained in $B(r_2, k)$. Therefore $v_p(x - r_2) \geq k \wedge x \in F_i$ is equivalent to just $x \in F_i$, and so $\phi(x, b)$ is equivalent to

$$D_{m'}(x - r_1) \wedge \bigvee_i x \in F_i.$$

By Remark 5.3(3) we may assume that each F_i is a proper cheese. We replace $\phi(x, b)$ by $\phi(x + r_1, b)$, and rename m' as m and each $F_i - r_1$ as F_i. Altogether, $\phi(x, b)$ is equivalent to a formula of the form

$$D_m(x) \wedge \bigvee_i x \in F_i$$

where $\gcd(m, p) = 1$, and $\{F_i\}_i$ are disjoint proper cheeses having nonstandard radiuses. As $\phi(x, b)$ is not L^--definable, the disjunction cannot be empty.

By Remark 5.8, $D_m(x)$ defines a dense subgroup of $\mathcal{M}$. By successively applying Lemmas 5.10, 5.11, and 5.9, we get an L-definable formula of the form

$$D_m(x) \wedge x \in B(0, \gamma) \tag{$\star$}$$

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with γ nonstandard and $\gcd(m, p) = 1$. We will now assume that $\phi(x, b)$ is of this form.

To finish, we need the following:

CLAIM 5.18. *Let $\psi(x, z)$ be any L_p-formula with $|x| = 1$.*

(1) *Suppose there exists $a \in \mathcal{M}$ with $v(a)$ nonstandard, for which there exists b such that $\psi(x, b)$ is equivalent to $v(x) \geq v(a)$. Then for any c such that $v(c)$ is nonstandard there is $b' \in \mathcal{M}$ such that $tp(b'/\emptyset) = tp(b/\emptyset)$ (in L_p) and $\psi(x, b')$ is equivalent to $v(x) \geq v(c)$.*

(2) *Let $\theta(z)$ be an L_p-formula. Then there exists $K \in \mathbb{N}$ such that for any $a \in \mathcal{M}$ with $v(a) \geq K$, if there exists b such that $\theta(b)$ holds and $\psi(x, b)$ is equivalent to $v(x) \geq v(a)$, then for any c such that $v(c) \geq K$ there is $b' \in \mathcal{M}$ such that $\theta(b')$ and $\psi(x, b')$ is equivalent to $v(x) \geq v(c)$. That is, let $\alpha(w)$ be the formula defined by*

$$\exists z(\theta(z) \wedge \forall x(\psi(x, z) \leftrightarrow v(x) \geq v(w)))$$

and let $\chi(w)$ be the formula defined by

$$\alpha(w) \rightarrow \forall w'(v(w') \geq K \rightarrow \alpha(w')).$$

Then $\chi(w)$ is satisfied by any a such that $v(a) \geq K$.

PROOF OF CLAIM. Proof of (1). We show that we can find $a' \in \mathcal{M}$ such that $tp(a'/\emptyset) = tp(a/\emptyset)$ and $v(a') = v(c)$. Indeed, let $\Sigma(x)$ be the partial type $tp(a/\emptyset) \cup \{v(x) = v(c)\}$. We show that it is consistent. Let $F \subseteq \Sigma(x)$ be a finite subset. As $v(a)$ is nonstandard, we may assume that F is of the form

$$\{x \neq j \; : \; -n \leq j \leq n\} \cup \{D_{m_k}(x - r_k) \; : \; 1 \leq k \leq s\} \cup \{v(x) = v(c)\}.$$

Let $m = \prod_k m_k$, and write $m = p^l m'$ with $\gcd(m', p) = 1$. By Lemma 2.9(4), there exists $\tilde{a} \in \mathcal{M}$ satisfying the formula $D_{m'}(x - a) \wedge (v(x) = v(c))$. So $v(\tilde{a}) = v(c)$ is nonstandard. As $v(a)$ is also nonstandard, $\tilde{a}$ also satisfies $D_{p^l}(x - a)$, so it satisfies $D_m(x - a)$, and therefore it satisfies $\{D_{m_k}(x - r_k) \; : \; 1 \leq k \leq s\}$. Also, as $v(\tilde{a})$ is nonstandard, $\tilde{a} \notin \mathbb{Z}$. Together we have that $\tilde{a}$ satisfies F.

So $\Sigma(x)$ is consistent. Let $a' \in \mathcal{M}$ be a realization of $\Sigma(x)$. As $tp(a'/\emptyset) = tp(a/\emptyset)$, there is an automorphism of L_p-structures $\sigma \in Aut(\mathcal{M}/\emptyset)$ such that $\sigma(a) = a'$. Let $b' = \sigma(b)$. So $tp(b'/\emptyset) = tp(b/\emptyset)$ and $\psi(x, b')$ is equivalent to $v(x) \geq v(a')$. As $v(a') = v(c)$, we have what we want.

Proof of (2). Let $\xi(w, w')$ be the formula defined by $\alpha(w) \rightarrow \alpha(w')$. By (1), $\xi(a, c)$ holds for any a, c such that $v(a)$ and $v(c)$ are nonstandard, so the result follows by compactness. ⊣

Now, let $\theta(z)$ be the formula expressing that $(\phi(x, z), +)$ is a subgroup. By Lemma 3.3 there are $n_1, \ldots, n_k$, having $\gcd(n_i, p) = 1$ for each i, such that for all $c \in \mathcal{M}$ for which $\theta(c)$ holds, $\phi(x, c)$ is equivalent to a formula of the form $D_{n_i}(x) \wedge v(x) \geq v(d)$ for some i and some $d \in \mathcal{M}$. As $(N, +, 0, |_p)$ is an elementary substructure, if $c \in N$ then there exists such $d \in N$. Let $n = \prod_i n_i$, and let $\psi(x, z)$ be the formula $\phi(nx, z)$. Then for all $c \in \mathcal{M}$ for which $\theta(c)$ holds, $\psi(x, c)$ is equivalent to $v(x) \geq v(d)$, for the same d corresponding to $\phi(x, c)$ (as $v(n) = 0$).

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Let $K \in \mathbb{N}$ be as given by the claim for $\psi(x,z)$ and $\theta(z)$, and let $\alpha(w)$ and $\chi(w)$ be as in the claim. We have that $\psi(x,b)$ is equivalent to $v(x) \geq \gamma$. In particular, the formula $\rho(z)$ defined by

$$\theta(z) \wedge \exists w(v(w) \geq K \wedge \forall x(\psi(x,z) \leftrightarrow v(x) \geq v(w)))$$

is satisfied by b. Since $\rho(z)$ contains no parameters, there exists $c \in N$ such that $(N,+,0,|_p) \vDash \rho(c)$. So $\theta(c)$ holds and there exists $d \in N$ such that $v(d) \geq K$ and $\psi(x,c)$ is equivalent to $v(x) \geq v(d)$. So $(N,+,0,|_p) \vDash \alpha(d)$. As $v(d) \geq K$, by the claim we have $\mathcal{M} \vDash \chi(d)$. Since $\chi(w)$ contains no parameters, also $(N,+,0,|_p) \vDash \chi(d)$. Hence, as v_p is surjective, for every $\gamma \in \Gamma(N)$ such that $\gamma \geq K$ there exists $c_\gamma \in N$ such that $\theta(c_\gamma)$ holds and $\psi(x,c_\gamma)$ is equivalent to $v(x) \geq \gamma$.

Let $\delta(x,y)$ be the formula

$$\bigwedge_{k=1}^{K-1} (D_{p^k}(x) \rightarrow D_{p^k}(y)) \wedge \forall z(\theta(z) \rightarrow (\psi(x,z) \rightarrow \psi(y,z))).$$

Then $\delta(x,y)$ is L-definable over $\emptyset$, and we claim that it defines $v(x) \leq v(y)$ in $\mathcal{N}$: Let $a_1, a_2 \in N$, and suppose $v(a_1) \leq v(a_2)$. Then of course $\bigwedge_{k=1}^{K-1}(D_{p^k}(a_1) \rightarrow D_{p^k}(a_2))$. Let $c \in N$ such that $\theta(c)$. Then there exists $d \in N$ such that $\psi(x,c)$ is equivalent to $v(x) \geq v(d)$, and therefore also $\psi(a_1,c) \rightarrow \psi(a_2,c)$. So we have $\delta(a_1,a_2)$. On the other hand, suppose $\delta(a_1,a_2)$. If $v(a_1) \leq K-1$, then by $\bigwedge_{k=1}^{K-1}(D_{p^k}(a_1) \rightarrow D_{p^k}(a_2))$ we get $v(a_1) \leq v(a_2)$. Otherwise, we have that $\gamma := v(a_1) \geq K$ and hence $\psi(a_1,c_\gamma)$. From $\forall z(\theta(z) \rightarrow (\psi(a_1,z) \rightarrow \psi(a_2,z)))$, as $\theta(c_\gamma)$ holds, we get in particular $\psi(a_1,c_\gamma) \rightarrow \psi(a_2,c_\gamma)$, and therefore we get $\psi(a_2,c_\gamma)$, which means $v(a_2) \geq \gamma = v(a_1)$. Therefore, $v(x) \leq v(y)$ is definable over $\emptyset$ in $\mathcal{N}$. ⊣

Combined with Fact 1.7 and Theorem 1.2, we obtain Theorem 1.11 and Corollary 1.12:

Proof of Theorem 1.11. Identical to the proof of Corollary 1.9 from Theorem 5.14. ⊣

Proof of Corollary 1.12. Identical to the proof of Fact 1.8 from Corollary 1.9. ⊣

§6. Intermediate structures in elementary extensions. In this section, we show that Fact 1.7, Fact 1.8 and Corollary 1.12 are no longer true if we replace $\mathbb{Z}$ by an elementarily equivalent structure. In the case of Corollary 1.12, there are both stable and unstable counterexamples. For Fact 1.8 there are unstable counterexamples, but we do not know whether there are stable ones.

For each of the above we give a family of counterexamples.

Proposition 6.1. *Let $(N,+,0,1,|_p)$ be a nontrivial elementary extension of $(\mathbb{Z},+,0,1,|_p)$, let $b \in N$ be such that $\gamma := v_p(b)$ is nonstandard, and let $B = \{a \in N : v_p(a) \geq \gamma\}$. Then $(N,+,0,1,B)$ is a stable proper expansion of $(N,+,0,1)$ of dp-rank 1. In particular, it is a proper reduct of $(N,+,0,1,|_p)$.*

Proof. It is clear that $(N,+,0,1,B)$ is a proper expansion of $(N,+,0,1)$, and, as a reduct of $(N,+,0,1,|_p)$, by Theorem 1.2 it is of dp-rank 1. It remains to show stability. In [18, Example 0.3.1 and Theorem 4.2.8], Wagner defines an abelian

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structure to be an abelian group together with some predicates for subgroups of powers of this group. Every module is an abelian structure. Wagner proves that, as with modules, in an abelian structure every definable set is equal to a boolean combination of cosets of $acl(\emptyset)$-definable subgroups. As a consequence, every abelian structure is stable. Under the assumptions of Proposition 6.1, B is a subgroup of N, so $(N, +, 0, 1, B)$ is an abelian structure, hence stable. ⊣

Let $(N, +, 0, 1, |_p)$ be a nontrivial elementary extension of $(\mathbb{Z}, +, 0, 1, |_p)$. For $\gamma \in \Gamma$ we define

$$C_\gamma = \left\{(a,b) \in N^2 : v_p(a) \leq \gamma \wedge v_p(b) \leq \gamma \wedge v_p(a) \leq v_p(b)\right\}.$$

Proposition 6.2. *There is a nontrivial elementary extension* $(N, +, 0, 1, |_p)$ *of* $(\mathbb{Z}, +, 0, 1, |_p)$ *and a nonstandard* $\gamma \in \Gamma$ *such that* $(N, +, 0, 1, C_\gamma)$ *is an unstable expansion of* $(N, +, 0, 1)$ *and a proper reduct of* $(N, +, 0, 1, |_p)$.

Proof. For each $m \in \mathbb{N}$, let

$$\begin{aligned} C_m &= \left\{(a,b) \in \mathbb{Z}^2 : a|_p p^m \wedge b|_p p^m \wedge a|_p b\right\} \\ &= \{(a,b) \in \mathbb{Z}^2 : \neg D_{p^{m+1}}(a) \wedge \neg D_{p^{m+1}}(b) \wedge \bigwedge_{i=1}^{m} (D_{p^i}(a) \to D_{p^i}(b))\}. \end{aligned}$$

Let $\mathcal{Z}_m = (\mathbb{Z}, +, 0, 1, |_p, C_m)$. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$, and let $\mathcal{N} = \prod_{\mathcal{U}} \mathcal{Z}_m = (N, +, 0, 1, |_p, C)$. Let $\psi(z)$ be the formula $\forall x, y(C(x,y) \leftrightarrow x|_p z \wedge y|_p z \wedge x|_p y)$. For any $m \geq k \geq 0$, $\mathcal{Z}_m \models \exists z\psi(z) \wedge \forall z(\psi(z) \to p^k|_p z)$, and therefore also $\mathcal{N} \models \exists z\psi(z) \wedge \forall z(\psi(z) \to p^k|_p z)$. Hence there exists $c \in N$ such that $\gamma := v_p(c)$ is nonstandard and $C = C_\gamma$.

Suppose for a contradiction that $|_p$ is definable in $(N, +, 0, 1, C)$. Then there is a formula $\phi(x, y, z)$ in the language of $(N, +, 0, 1, C)$ with $|x| = |y| = 1$, and there is $d \in N$, such that $\mathcal{N} \models \forall x, y(x|_p y \leftrightarrow \phi(x, y, d))$. Let $(d_m)_{m\in\mathbb{N}}$ be a representative for d mod $\mathcal{U}$. Then there exists $m \in \mathbb{N}$ such that $\mathcal{Z}_m \models \forall x, y(x|_p y \leftrightarrow \phi(x, y, d_m))$. Hence $|_p$ is definable in $(\mathbb{Z}, +, 0, 1, C_m)$. But C_m is definable in $(\mathbb{Z}, +, 0, 1)$, a contradiction.

It is clear that $(N, +, 0, 1, C)$ is an unstable proper expansion of $(N, +, 0, 1)$. ⊣

Proposition 6.3. *There is a nontrivial elementary extension* $(N, +, 0, 1, <)$ *of* $(\mathbb{Z}, +, 0, 1, <)$, *and a positive nonstandard* $b \in N$, *such that* $(N, +, 0, 1, [0, b])$ *is an unstable expansion of* $(N, +, 0, 1)$ *and a proper reduct of* $(N, +, 0, 1, <)$.

Proof. For each $m \in \mathbb{N}$, let $B_m = [0, m] = \{0, 1, \ldots, m\}$, and let $\mathcal{Z}_m = (\mathbb{Z}, +, 0, 1, <, B_m)$. Let $\mathcal{N} = \prod_{\mathcal{U}} \mathcal{Z}_m = (N, +, 0, 1, <, B)$ be the ultraproduct of $\{\mathcal{Z}_m\}_m$ with respect to some nonprincipal ultrafilter $\mathcal{U}$ over $\mathbb{N}$. For any $m \geq k \geq 0$, $\mathcal{Z}_m \models \exists! x(\forall y(B_m(y) \leftrightarrow 0 \leq y \leq x) \wedge x \geq \underline{k})$ and therefore also $\mathcal{N} \models \exists! x(\forall y(B(y) \leftrightarrow 0 \leq y \leq x) \wedge x \geq \underline{k})$. Hence there exists a positive nonstandard element $b \in N$ such that $B = [0, b]$. Suppose for a contradiction that $<$ is definable in $(N, +, 0, 1, B)$. Then there is a formula $\phi(x, y, z)$ in the language of $(N, +, 0, 1, B)$ with $|x| = |y| = 1$, and there is $c \in N$, such that $\mathcal{N} \models \forall x, y(x < y \leftrightarrow \phi(x, y, c))$. Let $(c_m)_{m\in\mathbb{N}}$ be a representative for c mod $\mathcal{U}$. Then there exists $m \in \mathbb{N}$ such that $\mathcal{Z}_m \models \forall x, y(x < y \leftrightarrow \phi(x, y, c_m))$. Hence $<$ is definable in $(\mathbb{Z}, +, 0, 1, B_m)$, a contradiction. It is clear that $(N, +, 0, 1, B)$ is a proper expansion of $(N, +, 0, 1)$. The formula $B(y - x)$ defines the ordering on B, so this structure is unstable. ⊣

REMARK 6.4. The conclusions of Propositions 6.2 and 6.3 in fact hold for *any* nontrivial elementary extension and nonstandard $\gamma \in \Gamma$ or positive nonstandard $b \in N$, respectively. In both cases, this can be proved by showing that any structure of this form is sufficiently elementarily equivalent to the specific examples in Propositions 6.2 and 6.3. We leave this as an exercice.

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