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A new criterion for the existence of multiple solutions in cones

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We provide new sufficient conditions for the existence of multiple fixed points for a map between ordered Banach spaces. An interesting feature of this approach is that we require conditions not on two boundaries, but rather on one boundary and a point with some extra information on the monotonicity of the nonlinearity on a certain set. We apply our results to prove the existence of at least two positive solutions for a nonlinear boundary-value problem that models a thermostat.

1. Introduction

The cone compression–expansion theorem of Krasnosel′skiĭ (see theorem 2.2) and the classical fixed-point index in cones (see, for example, [1]) are two tools that are commonly used for proving the existence of multiple non-trivial solutions of certain boundary-value problems. The use of these tools relies on the compactness of an associated integral operator combined with the behaviour of this operator on the boundary of certain shells. Our approach will be of a different flavour, and will exploit ideas from the recent papers by Persson [14], Cabada and Cid [2] and Cid *et al*. [3].

Using the properties of the topological degree as his main tool, Persson [14, theorem 5] presented sufficient conditions for the existence of a non-negative fixed point for monotone maps in finite-dimensional spaces. This result was extended to non-decreasing and completely continuous operators between infinitely dimensional ordered Banach spaces by Cabada and Cid [2, theorem 2.1], assuming certain conditions on the set of supersolutions associated to the operator and using a combination of the monotone iterative method (see theorem 2.1) with the cone compression–expansion theorem of Krasnosel′skiĭ. Cid and co-authors [3, theorem 2.3] improved the results of [2] by relaxing the monotonicity condition, with the aim of giving new existence conditions for the solutions of a fourth-order problem that models the stationary states of the deflection of an elastic beam with both ends hinged.

Here we give a refinement of the results of [3], valid for the existence of one non-trivial solution, by relaxing the assumptions on the set of supersolutions. An interesting feature of this approach is that, unlike the classical result of Krasnosel′skiĭ, we shall require conditions not on two boundaries, but rather on a point and a boundary, jointly with some extra information on the monotonicity in a certain set. This will be an essential ingredient for our new multiplicity result. We note that Persson showed [14, proposition 7] that it is possible to employ his existence result, combined with Tarski's fixed-point theorem, to prove the existence of more than one fixed point in the finite-dimensional setting. Persson's proof also uses the boundedness of the set of supersolutions. As far as we are aware, this is the first time that the multiplicity has been treated in the infinite-dimensional setting via an extension of the ideas of Persson.

In the last section we illustrate how our result can be applied to prove the existence of at least two positive solutions of the non-local boundary-value problem (BVP)

$$u''(t) + \lambda g(t) f(u(t)) = 0, \quad t \in (0,1),$$

with the non-local boundary conditions (BCs)

$$u'(0) = 0, \qquad \sigma u'(1) + u(\eta) = 0, \quad \eta \in [0,1].$$

This BVP arises in the study of the steady states of a heated bar of length 1, where the left end of the bar is isolated and a controller in the right end adds or removes heat according to the temperatures detected by a sensor placed at another point of the bar. These types of heat-flow problem have been studied by Infante and Webb [8], who were motivated by some earlier work of Guidotti and Merino [5], and further studied in [6, 7, 9, 11, 13, 15, 16].

2. Some preliminary material

We begin with some notation, definitions and some classical results that we use in the remainder of the paper.

A subset K of a real Banach space N is a *cone* if it is closed, $K + K \subset K$, $\lambda K \subset K$ for all $\lambda \geqslant 0$ and $K \cap (-K) = \{\theta\}$. A cone K defines the partial ordering in N given by $x \leqslant y$ if and only if $y - x \in K$. The cone K is *c-normal* with normal constant $c \geqslant 1$ if $\|x\| \leqslant c\|y\|$ for all $x, y \in N$ with $x \leqslant y$. Whenever $\operatorname{int}(K) \neq \emptyset$, the symbol $x \ll y$ means $y - x \in \operatorname{int}(K)$ and the cone is said to be solid. We denote by ∂K the boundary of K and by $d(x, \partial K)$ the distance from x to the boundary of K.

If $T\colon K \to K$ satisfies the conditions

$$Tx \nleq x \qquad \text{for all } x \text{ with } \|x\| = R$$

and

$$x \nleq Tx \quad \text{for all } x \text{ with } \|x\| = \bar{R},$$

then it is called a cone compression when $0 < R < \bar{R}$ and a cone expansion when $0 < \bar{R} < R$. On the other hand, if for each $u, v \in M \subset N$ with $v \leqslant u$ we have $Tv \leqslant Tv$, then T is called non-decreasing in M.

We denote the closed ball of centre $x_0 \in N$ and radius $r > 0$ by

$$B[x_0, r] = \{x \in N \colon \|x - x_0\| \leqslant r\},$$

and for $x, y \in N$, with $x \leqslant y$, the interval

$$[x, y] = \{z \in N \colon x \leqslant z \leqslant y\}.$$

Now we recall two classical fixed-point results. The first one is known as the monotone iterative method (see, for example, [17, theorem 7.A]) and the second one, which is widely used for the existence of fixed points in cones, is due to Krasnosel′skiĭ (see, for example, [17, theorem 13.D]).

Theorem 2.1. *Let N be a real Banach space with normal order cone K. Suppose that there exist $\alpha \leqslant \beta$ such that $T \colon [\alpha, \beta] \subset N \to N$ is a compact monotone non-decreasing operator with $\alpha \leqslant T\alpha$ and $T\beta \leqslant \beta$. Then T has a fixed point and the iterative sequence $\alpha_{n+1} = T\alpha_n$, with $\alpha_0 = \alpha$, converges to the greatest fixed point of T in $[\alpha, \beta]$, and the sequence $\beta_{n+1} = T\beta_n$, with $\beta_0 = \beta$, converges to the smallest fixed point of T in $[\alpha, \beta]$.*

Theorem 2.2. *Let N be a real Banach space with order cone K. Suppose that the operator $T \colon K \to K$ is completely continuous and either a cone compression or expansion. Then T has a fixed point x on K and*

$$\min\{R, \bar{R}\} < \|x\| < \max\{R, \bar{R}\}.$$

3. One or more non-zero fixed points

We can now formulate a result regarding the existence of one non-trivial fixed point.

Theorem 3.1. *Let N be a real Banach space, let K be a normal cone with normal constant $c \geqslant 1$ and non-empty interior (i.e. solid) and let $T \colon K \to K$ be a completely continuous operator.*

Assume that

(i) *there exist $\beta_1 \in K$, with $T\beta_1 \leqslant \beta_1$ and $R_1 > 0$ such that $B[\beta_1, R_1] \subset K$,*

(ii) *the map T is non-decreasing in the set*

$$K_1 = \left\{x \in K \colon \frac{R_1}{c} \leqslant \|x\| \leqslant c\|\beta_1\|\right\},$$

(iii) *there exists $r_1 > 0$, with $r_1 \neq R_1$, such that $Tx \nleqslant x$ for all $x \in K$ with $\|x\| = r_1$.*

Then the map T has at least one non-zero fixed point x_1 in K that either belongs to K_1 or is such that

$$\min\{r_1, R_1\} < \|x_1\| < \max\{r_1, R_1\}.$$

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Proof. Since $B[\beta_1, R_1] \subset K$ we have that if $x \in K$ with $\|x\| = R_1$, then $x \leqslant \beta_1$.

Suppose first that we can choose $\alpha_1 \in K$ with $\|\alpha_1\| = R_1$ and $T\alpha_1 \geqslant \alpha_1$. Since $\alpha_1 \leqslant \beta_1$ and due to the normality of the cone K we have that $[\alpha_1, \beta_1] \subset K_1$, which implies that T is non-decreasing on $[\alpha_1, \beta_1]$. Then we can apply theorem 2.1 to ensure the existence of extremal fixed points of T on $[\alpha_1, \beta_1]$, which, in particular, are non-trivial fixed points.

Now suppose that such an α_1 does not exist. Thus, $Tx \ngeqslant x$ for all $x \in K$ with $\|x\| = R_1$. Since $Tx \nleqslant x$ for all $x \in K$ with $\|x\| = r_1$, we obtain by theorem 2.2 the existence of a non-trivial fixed point x_1. □

Remark 3.2. The main novelty of this result with respect to [3, theorem 2.3] is that we do not require the set $S = \{x \in K : Tx \leqslant x\}$ to be either bounded or bounded away from 0.

Remark 3.3. We note that the compactness assumption in theorem 3.1 can be relaxed. It is well known that theorems 2.1 and 2.2 are valid for condensing maps (see, for example, [17]), so we can replace the assumption that T is completely continuous with the weaker condition of T being a condensing map.

We can now use a nesting argument similar to those used, for example, in [9, 12], where classical fixed-point index techniques were used, and in [4, 10], where theorem 2.2 was used, to prove a new result regarding the existence of multiple fixed points. This is a consequence of theorem 3.1; here we include a direct proof for completeness. The special case of $n = 2$ is illustrated in detail in the next section.

Theorem 3.4. *Let N be a real Banach space ordered by a c-normal solid cone K and let $T : K \to K$ a completely continuous map. Assume that there exist $\beta_1, \dots, \beta_n \in \operatorname{int}(K)$, real numbers $R_n > \cdots > R_1 > 0$ and real numbers $r_n \geqslant \cdots \geqslant r_1 > 0$ such that*

(i) *$T\beta_i \leqslant \beta_i$, $R_i < d(\beta_i, \partial K)$ and $R_i \neq r_i$ for $i = 1, \dots, n$,*

(ii) *T is non-decreasing in any order interval contained in the set*

$$\bigcup_{i=1,\dots,n} \{x \in K : R_i \leqslant c\|x\| \leqslant c^2\|\beta_i\|\},$$

(iii) *$Tx \nleqslant x$ for all $x \in K$ with $\|x\| \in \{r_1, \dots, r_n\}$,*

(iv) *$\max\{cr_i, c^2\|\beta_i\|\} < R_{i+1}$ and $c\|\beta_i\| < r_{i+1}$ for $i = 1, \dots, n-1$.*

Then, T has at least n non-zero fixed points.

Proof. Define the following conical shells for $i = 1, \dots, n$:

$$A_i = \left\{x \in K : \frac{R_i}{c} \leqslant \|x\| \leqslant c\|\beta_i\|\right\}$$

and

$$B_i = \{x \in K : \min\{r_i, R_i\} < \|x\| < \max\{r_i, R_i\}\}.$$

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Clearly, the sets A_i and B_i are not empty. Moreover, it is easy to verify that, for $i \neq j$, condition (iv) guarantees

$$A_i \cap A_j = B_i \cap B_j = B_i \cap A_j = \emptyset.$$

Therefore, if we prove that there exists at least one fixed point in $A_i \cup B_i$ for each $i = 1, \dots, n$, then we are done.

On the one hand, suppose that we can choose $\alpha_i \in K$ with $\|\alpha_i\| = R_i$ and $\alpha_i \leqslant T\alpha_i$. Since $\alpha_i \leqslant \beta_i$ and due to the normality of the cone K, we have that

$$[\alpha_i, \beta_i] \subset \bigcup_{i=1,\dots,n} \{x \in K\colon R_i \leqslant c\|x\| \leqslant c^2\|\beta_i\|\},$$

which, by hypothesis (ii), implies that T is non-decreasing in $[\alpha_i, \beta_i]$. Then we are in a position to apply theorem 2.1 to ensure the existence of extremal fixed points of T on $[\alpha_i, \beta_i]$ which are contained in the conical shell A_i.

On the other hand, suppose that such α_i does not exist. Thus, $x \not\leqslant Tx$ for all $x \in K$ with $\|x\| = R_i$. Using condition (iii) and theorem 2.2, we obtain the existence of a fixed point in the conical shell B_i. □

Remark 3.5. The choice $R_i = d(\beta_i, \partial K)$ in theorems 3.1 and 3.4 weakens the assumptions on the monotonicity of the operator T.

Remark 3.6. We also note that when $n > 1$ theorem 3.4 allows $r_i = r_{i+1}$ for some i. However, if that happens, the rest of conditions in the result force $r_{i+2} > r_{i+1} = r_i$. In consequence, if n is even we need at least $\frac{1}{2}n$ different radii r_i and if n is odd we need at least $\frac{1}{2}(n+1)$ radii r_i.

4. An application

We now discuss the existence of positive solutions of the BVP

$$u''(t) + \lambda g(t) f(u(t)) = 0, \quad t \in (0,1), \tag{4.1}$$

with the non-local BCs

$$u'(0) = 0, \qquad \sigma u'(1) + u(\eta) = 0, \quad \eta \in [0,1], \tag{4.2}$$

which models a thermostat. Here $g \in L^1[0,1]$, $g \geqslant 0$ almost everywhere, $f\colon [0,\infty) \to [0,\infty)$ is continuous. We focus on the case $\sigma + \eta > 1$ that leads to the existence of (strictly) positive solutions.

This BVP (see, for example, [8]), can be rewritten as a Hammerstein integral equation of the form

$$u(t) = \int_0^1 k(t,s)\lambda g(s) f(u(s))\,\mathrm{d}s := Tu(t), \tag{4.3}$$

where

$$k(t,s) = \sigma + \begin{cases} \eta - s, & s \leqslant \eta \\ 0, & s > \eta \end{cases} - \begin{cases} t - s, & s \leqslant t, \\ 0, & s > t. \end{cases}$$

It is known [8] that for every $(t, s) \in [0,1] \times [0,1]$ we have

$$\hat{c}k(0, s) \leqslant k(t, s) \leqslant k(0, s),$$

where

$$\hat{c} = 1 - \frac{1}{\sigma + \eta}.$$

With the above conditions, it is routine to prove that $T\colon C[0,1] \to C[0,1]$ leaves invariant the cone

$$K = \Bigl\{u \in C[0,1]\colon \min_{t\in[0,1]} u(t) \geqslant \hat{c}\|u\|\Bigr\},$$

where in $C[0,1]$ we are considering the supremum norm $\|u\| = \sup\{u(t)\colon t \in [0,1]\}$.

It is also known that K is a normal solid cone with constant $c = 1$.

The advantage of considering this cone is that it provides a lower bound for the functions belonging to A_i and, as we shall see, this will be useful for the monotonicity argument.

We make use of the numbers

$$\gamma_* = \inf_{t\in[0,1]} \int_0^1 k(t, s)g(s)\,\mathrm{d}s, \qquad \gamma^* = \sup_{t\in[0,1]} \int_0^1 k(t, s)g(s)\,\mathrm{d}s,$$

note that $\gamma_* = 1/M$ and $\gamma^* = 1/m$, in the notation of [8].

THEOREM 4.1. *Let $\beta_1, \beta_2, R_1, R_2 \in (0, +\infty)$ be such that $\beta_1 < \hat{c}R_2$ and $\beta_i \geqslant R_i(2(\sigma + \eta) - 1)$ for every i. Assume g satisfies $\gamma_* > 0$, and f is non-decreasing on $[\hat{c}R_i, \beta_i]$ for every i and*

$$\lim_{u\to 0^+} \frac{f(u)}{u} = \lim_{u\to+\infty} \frac{f(u)}{u} = +\infty.$$

Then the BVP (4.1), (4.2) has at least two (strictly) positive solutions for any λ such that

$$0 < \lambda < \min\left\{\frac{\beta_1}{\gamma^* f(\beta_1)}, \frac{\beta_2}{\gamma^* f(\beta_2)}\right\}. \tag{4.4}$$

Proof. We shall show that the following conditions, which clearly guarantee those assumed in theorem 3.4 for $n = 2$, hold:

(1′) there exist $\beta_1, \beta_2 \in K$ and $R_1, R_2 > 0$ such that $c^2\|\beta_1\| < R_2$ and $T\beta_i \leqslant \beta_i$ and $B[\beta_i, R_i] \subset K$ for every $i = 1, 2$,

(2′) the map T is non-decreasing in the sets

$$K_i = \left\{x \in K\colon \frac{R_i}{c} \leqslant \|x\| \leqslant c\|\beta_i\|\right\}, \quad i = 1, 2,$$

(3′) there exist r_1, r_2, with $0 < cr_1 < R_2$, $\max\{r_1, c\|\beta_1\|\} < r_2$, such that $r_i \neq R_i$ and $Tx \nleq x$ for all $x \in K$ with $\|x\| = r_i$ for every $i = 1, 2$.

Since $\beta_1, \beta_2 > 0$, then clearly (with an abuse of notation) $\beta_i \in K$. Take $u \in B[\beta_i, R_i]$. Since $\beta_i \geqslant R_i(2(\sigma+\eta)-1)$, it follows that $u \in K$. The hypothesis $(1')$ is satisfied since

$$T\beta_i = \int_0^1 k(t,s)g(s)\lambda f(\beta_i)\,\mathrm{d}s \leqslant \lambda\gamma_i^* f(\beta_i) \leqslant \beta_i \quad \text{for every } i.$$

Since f is non-decreasing in $[\hat{c}R_i, \beta_i]$, if we take $u_i, v_i \in K_i$ with $u_i \leqslant v_i$, we have, for every $t \in [0,1]$,

$$Tv_i(t) - Tu_i(t) = \int_0^1 k(t,s)\lambda g(s) f(v_i(s))\,\mathrm{d}s - \int_0^1 k(t,s)\lambda g(s) f(u_i(s))\,\mathrm{d}s \geqslant 0.$$

Furthermore, for $t, r \in [0,1]$, we have

$$\begin{aligned} Tv_i(t) - Tu_i(t) &= \int_0^1 k(t,s)\lambda g(s)[f(v_i(s)) - f(u_i(s))]\,\mathrm{d}s \\ &\geqslant \int_0^1 \hat{c}k(0,s)\lambda g(s)[f(v_i(s)) - f(u_i(s))]\,\mathrm{d}s \\ &\geqslant \hat{c}\int_0^1 k(r,s)\lambda g(s)[f(v_i(s)) - f(u_i(s))]\,\mathrm{d}s \\ &= \hat{c}[Tv_i(r) - Tu_i(r)]. \end{aligned}$$

Therefore, $\min_{t\in[0,1]}[Tv_i(t) - Tu_i(t)] \geqslant \hat{c}\|Tv_i - Tu_i\|$. Thus, T is non-decreasing on K_i and $(2')$ is satisfied.

The behaviour of the nonlinearity f ensures that $(3')$ is satisfied. We prove that $Tx \nleq x$ on a small sphere; a similar result holds on a large sphere.

For a fixed λ satisfying (4.4), choose $L > 0$ large enough such that $\lambda\gamma_* L\hat{c} > 1$ and $r > 0$ (small), satisfying $f(s) \geqslant Ls$ provided that $s \leqslant r$. For $u \in K$ with $\|u\| = r$, we have

$$Tu(t) = \int_0^1 k(t,s)g(s)\lambda f(u(s))\,\mathrm{d}s \geqslant \lambda\gamma_* L\hat{c}\|u\| > \|u\|,$$

which implies that $Tu \nleq u$. □

We illustrate the above theorem in the following example.

Example 4.2. We consider the BVP

$$u'' + \lambda(\sqrt{u} + u(u-6)^2) = 0, \qquad u'(0) = 0, \qquad \tfrac{3}{4}u'(1) + u(\tfrac{2}{3}) = 0.$$

In this case $\hat{c} = \frac{5}{17}$, and a direct calculation gives

$$\gamma^* = \sigma + \tfrac{1}{2}\eta^2 = \tfrac{35}{36} \quad \text{and} \quad \gamma_* = \tfrac{1}{2}(2\sigma - 1 + \eta^2) = \tfrac{17}{36}.$$

Now choose $\beta_1 = 2$, $R_1 = \frac{12}{11}$, $R_2 = \frac{102}{5}$ and $\beta_2 = \frac{187}{5}$; clearly f is non-decreasing on the intervals $[\frac{60}{187}, 2]$ and $[6, \frac{187}{5}]$.

Theorem 4.1 provides the existence of two positive solutions (the last number is rounded to the third decimal place) for every

$$0 < \lambda < \min\left\{\frac{72}{35(\sqrt{2}+32)}, \frac{33\,660}{7(25\sqrt{187}\sqrt{5} + 4\,609\,363)}\right\} = 0.001.$$

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