ON THE ADJOINT GROUP OF SOME RADICAL RINGS

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1. Introduction. A ring R is called *radical* if it coincides with its Jacobson radical, which means that R forms a group under the operation $a \circ b = a + b + ab$ for all a and b in R. This group is called the *adjoint group* R° of R. The relation between the adjoint group R° and the additive group R^+ of a radical ring R is an interesting topic to study. It has been shown in [**1**] that the finiteness conditions "minimax", "finite Prüfer rank", "finite abelian subgroup rank" and "finite torsionfree rank" carry over from the adjoint group to the additive group of a radical ring. The converse is true for the minimax condition, while it fails for all the other above finiteness conditions by an example due to Sysak [**6**] (see also [**2**, Theorem 6.1.2]). However, we will show that the converse holds if we restrict to the class of *nil rings*, i.e. the rings R such that for any $a \in R$ there exists an $n = n(a)$ with $a^n = 0$.

Recall that a group G is called a *minimax group* if it has a series of finite length whose factors satisfy the minimum or maximum condition on subgroups. The group G has *finite torsion-free rank* if it has a finite series whose factors are either periodic or infinite cyclic. The number of infinite cyclic factors in any such series is an invariant of G denoted by $r_0(G)$. The group G has *finite abelian subgroup rank* if each abelian subgroup of G has finite torsion-free rank and each abelian p-subgroup of G has finite Prüfer rank for every prime p. Here a group G is said to have *finite Prüfer rank* $r = r(G)$ if every finitely generated subgroup of G can be generated by r elements, and r is the least positive integer with this property. For the relation between these finiteness conditions see Chapter 6.3 of [**4**].

THEOREM A. *Let R be a nil ring. Then the following hold.*

(a) *If R^+ has finite torsion-free rank n, then also $r_0(R^\circ) = n$.*

(b) *If R^+ has finite abelian subgroup rank, then so does R°.*

(c) *If R^+ has finite Prüfer rank, then so does R°, and $r(R^\circ) \leq 3 \,.\, r(R^+)$. If R^+ contains no elements of order* 2 *then even $r(R^\circ) \leq 2 \,.\, r(R^+)$.*

The situation for the class of radical rings with a periodic additive group is similar, as the following result shows.

THEOREM B. *Let the additive group R^+ of the radical ring R be periodic. Then the following hold.*

(a) *If R^+ has finite abelian subgroup rank, then so does R°.*

(b) *If R^+ has finite Prüfer rank, then so does R°, and $r(R^\circ) \leq 3 \,.\, r(R^+)$. If R^+ contains no elements of order* 2 *then even $r(R^\circ) \leq 2 \,.\, r(R^+)$.*

At the end of Section 2, an example of a radical ring R with R^+ being an elementary abelian p-group shows that in the situation of Theorem B, the adjoint group R° may have infinite torsion-free rank. The rank inequalities in part (c) of Theorem A and part (b) of Theorem B depend on the following proposition.

Proposition. *Let R be a nil ring and n a positive integer. If $S = nR$, then $nS = (S^\circ)^{\{n\}} = (S^\circ)^n$.*

Here $G^{\{n\}}$ denotes the set of all nth powers of the elements of a group G and G^n the subgroup of G generated by this set.

It seems to be unknown whether the bounds in part (c) of Theorem A and part (b) of Theorem B are best possible. This question will be discussed in more detail at the end of Section 3.

The notation is standard and can for instance be found in [**4**] and [**5**]. Note that the adjoint inverse of an element a of a radical ring will be denoted by a'.

2. Proof of the proposition and Theorem B. The following is a technical lemma on formal power series.

Lemma 2.1. *Let $\mathbb{Z}[\![x]\!]$ be the ring of formal power series in the variable x over the ring $\mathbb{Z}$ of integers. If n is a positive integer, then $1 + n^2x$ can be written as $(1 + n\,.\,f)^n$ for some $f \in x\mathbb{Z}[\![x]\!]$.*

Proof. Considering the binomial series for $(1 + n^2x)^{1/n}$, we obtain that

$$f = \sum_{m=1}^{\infty} n^{2m-1}\binom{1/n}{m}x^m \in x\mathbb{R}[\![x]\!]$$

satisfies $1 + n^2x = (1 + nf)^n$. Thus it suffices to show that

$$n^{2m-1}\binom{1/n}{m} = n^{m-1}\frac{1(1-n)(1-2n)\ldots(1-(m-1)n)}{m!}$$

is an integer for $n, m \geq 1$. If p is a prime, then the number of times that p divides $m!$ is

$$v_m = \sum_{i \geq 1}\left[\frac{m}{p^i}\right],$$

where $[x]$ denotes the greatest integer not exceeding the real number x. Hence we only need to show that p divides $n^{m-1}1(1-n)(1-2n)\ldots(1-(m-1)n)$ at least v_m times. As

$$v_m < \frac{m}{p}\sum_{i=0}^{\infty}\left(\frac{1}{p}\right)^i = \frac{m}{p-1} \leq m,$$

this is clear if p divides n. Therefore we may suppose that p does not divide n. Then at least $\left[\frac{m}{p^i}\right]$ of the factors

$$1, 1-n, 1-2n, \ldots, 1-(m-1)n$$

are divisible by p^i for every $i \geq 1$, from which it follows that p divides the product $1(1-n)(1-2n)\ldots(1-(m-1)n)$ at least v_m times. This completes the proof of the lemma.

Proof of the proposition. As nS is an ideal of R, it forms a subgroup of the adjoint group R°. Thus we only have to show that $nS = (S^\circ)^{\{n\}}$.

Let $t \in nS$, i.e. $t = ns$ for some $s \in S$. Then $s = nr$ for a suitable $r \in R$. By Lemma 2.1 there exists a formal power series $f \in x\mathbb{Z}[\![x]\!]$ such that $(1+nf)^n = 1 + n^2x$. Putting $a = n \,.\, f(r) \in nR = S$ and using a formal identity 1, we obtain $(1+a)^n = 1 + n^2r = 1 + t$. Note that the substitution of r into f is possible, since R is nil. It follows that t is the adjoint nth power of $a \in S$, which implies $nS \subseteq (S^\circ)^{\{n\}}$.

Now let p be a prime dividing n. If $s \in S$, then $s = pr$ for some $r \in R$. It follows that

$$(1+s)^p = 1 + \sum_{i=1}^{p-1} \binom{p}{i} s^i + s^p,$$

where

$$\sum_{i=1}^{p-1} \binom{p}{i} s^i = ps \,.\, \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} s^{i-1} \in pS$$

and

$$s^p = p^p r^p = p^{p-2} r^{p-1} \,.\, ps \in pS.$$

Hence we have $(1+S)^{\{p\}} \subseteq 1 + pS$. Writing $n = p_1 \ldots p_k$ as a product of primes, it now follows by induction on k that $(1+S)^{\{n\}} \subseteq 1 + nS$. Thus $(S^\circ)^{\{n\}} \subseteq nS$. The proposition is proved.

To apply the proposition for radical p-rings recall that a finite p-group G is called *powerful* if either $p = 2$ and $G' \leq G^4$ or p is an odd prime and $G' \leq G^p$. Writing $d(G)$ for the minimal number of elements from G necessary to generate G, we have the following facts, which can for instance be found in [**3**].

LEMMA 2.2. *Let G be a finite p-group.*

(a) (*Burnside Basis Theorem.*) *If $\Phi(G)$ denotes the Frattini subgroup of G, then* $p^{d(G)} = |G/\Phi(G)|$.

(b) *If G is powerful, then $r(G) = d(G)$ and $\Phi(G) = G^p$.*

LEMMA 2.3. *Let R be a finite nilpotent p-ring.*

(a) *If $p = 2$, then $r((4R)^\circ) = r((4R)^+)$.*

(b) *If p is an odd prime, then $r((pR)^\circ) = r((pR)^+)$.*

Proof. Let $S = nR$, where $n = 4$ if $p = 2$, and $n = p$ if p is odd. For all $x, y \in S$, the adjoint commutator $x' \circ y' \circ x \circ y = (1+x')(1+y')(xy - yx)$ lies in $S^2 = n^2R^2 \subseteq n^2R = nS$. Hence it follows from the proposition that $(S^\circ)' \subseteq (S^\circ)^n$. Thus S° is powerful. Now Lemma 2.2 and again the proposition yield

$$p^{r(S^\circ)} = p^{d(S^\circ)} = |S^\circ/\Phi(S^\circ)| = |S^\circ/(S^\circ)^p| = |S^+/(pS)^+| = p^{r(S^+)}.$$

This proves the lemma.

As a consequence, part (b) of Theorem B follows for radical p-rings.

LEMMA 2.4. *Let R be a radical p-ring whose additive group R^+ has finite Prüfer rank. Then R° has likewise finite Prüfer rank and the following hold.*

(a) *If $p = 2$, then $r(R^\circ) \leq 3 \,.\, r(R^+)$.*

(b) *If p is an odd prime, then* $r(R^\circ) \leq 2 \,.\, r(R^+)$.

Proof. Consider first the case that R is finite. Then R is nilpotent; see [**5**, Theorem 2.5.16]. For any subgroup U of $(R/pR)^\circ$, it follows by the Burnside Basis Theorem that

$$p^{d(U)} = |U/\Phi(U)| \leq |R/pR| = p^{r((R/pR)^+)}.$$

Hence

$$r((R/pR)^\circ) = \max_{U \leq (R/pR)^\circ} d(U) \leq r((R/pR)^+) \leq r(R^+).$$

In case (b), Lemma 2.3 yields

$$r((pR)^\circ) = r((pR)^+),$$

from which it follows that

$$r(R^\circ) \leq r(R^\circ/(pR)^\circ) + r((pR)^\circ) = r((R/pR)^\circ) + r((pR)^\circ) \leq 2 \,.\, r(R^+).$$

Case (a) is treated in the same way by considering the chain

$$0 \leq 4R \leq 2R \leq R$$

and observing that the ring $2R/4R$ has trivial multiplication, so that its additive and adjoint groups coincide.

Consider now the general case of an arbitrary radical p-ring. For all $n \geq 0$ let R_n be the ideal $\{r \in R \mid p^n r = 0\}$ of R. As R is a p-ring, we have

$$R = \bigcup_{n \geq 0} R_n.$$

Let U be a finitely generated subgroup of R_n^+. Then U is an r-generated abelian group of exponent dividing p^n, where $r = r(R^+)$. Thus $|U| \leq (p^n)^r = p^{nr}$. Hence each R_n is finite. Let $c = 3$ for $p = 2$ and $c = 2$ for $p \neq 2$. By the finite case we have

$$r(R_n^\circ) \leq c \,.\, r(R_n^+) \leq c \,.\, r(R^+)$$

for all $n \geq 0$. Since R° is the union of the R_n°, we obtain

$$r(R^\circ) \leq c \,.\, r(R^+).$$

The lemma is proved.

To complete the proofs of both of the theorems we will need the following result.

LEMMA 2.5 ([**1**, Lemma 2.4]). *If R is a nil ring and p a prime, then the following hold.*
(a) *R^+ is a p-group if and only if R° is a p-group.*
(b) *R^+ is torsion-free if and only if R° is torsion-free.*

Proof of Theorem B. To prove part (a), suppose that R^+ is periodic with finite abelian subgroup rank. For each prime p the p-component of R^+ forms an ideal T_p of the ring R, and $R = \bigoplus_p T_p$. By [**4**, Vol. 2, p. 38, Corollary 1], each T_p^+ is a Chernikov-group. Hence by Theorem A of [**1**], each T_p° is a Chernikov-group and each of the ideals T_p of R is nilpotent. In particular, each T_p° is a nilpotent group. As R° is the direct product of the T_p°, it follows that R° is locally nilpotent (i.e. each of its finitely generated subgroups is nilpotent). By the nilpotency of the rings T_p and Lemma 2.5 each T_p° is a

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Chernikov p-group. Thus $R^\circ = \bigotimes_p T_p^\circ$ has finite abelian subgroup rank by [**4**, Vol. 2, p. 38, Corollary 1].

Part (b) is proved in the same way, using [**4**, Vol. 2, p. 38, Corollary 2]. Here, by Lemma 2.4, the bound $r = r(R^+)$ for the Prüfer ranks of the T_p^+ carries over to the required bound for the Prüfer ranks of the T_p°.

We finish this section with the example mentioned in the introduction.

Example (see [**6**, p. 28]). Let p be a prime and $K = \mathrm{GF}(p)$ the field with p elements. If $K[\![x]\!]$ denotes the ring of formal power series over K then $R = xK[\![x]\!]$ is a radical ring with an elementary abelian additive group R^+. But obviously the element x of R° has infinite order in R°. (In fact, it can easily be shown that R° is torsion-free.) Assume $r_0(R^\circ) < \infty$. Then Theorem B of [**1**] implies that $r_0(R^\circ) = r_0(R^+) = 0$, contradicting the fact that R° contains elements of infinite order. Hence $r_0(R^\circ) = \infty$.

3. Proof of Theorem A. A ring R is called *locally nilpotent* if each of its finitely generated subrings is nilpotent.

Lemma 3.1 ([**1**, Lemma 2.1]). *Let R be a nilpotent ring and $\mathfrak{X}$ a class of groups which is closed under the forming of subgroups, epimorphic images and extensions. Then the adjoint group R° of R is an $\mathfrak{X}$-group if and only if the additive group R^+ of R is an $\mathfrak{X}$-group.*

Lemma 3.2 ([**1**, Lemma 3.1]). *If R is a locally nilpotent ring whose additive group R^+ is torsion-free with finite torsion-free rank n, then $R^{n+1} = 0$.*

Lemma 3.3 (Special case of [**7**, Theorem 6]). *Let R be an arbitrary ring and S a nilpotent proper subring of R. Then S is properly contained in its idealizer $\mathrm{Id}_R(S) = \{r \in R \mid rS + Sr \subseteq S\}$.*

Lemma 3.4. *Let G be a locally nilpotent torsion-free group with finite torsion-free rank. Then $r(G) \le r_0(G) < \infty$.*

Proof. We may assume that G is finitely generated and hence nilpotent. Let

$$1 = Z_1 \le Z_2 \le \ldots \le Z_n = G$$

be the upper central series of G. As Z_1 is torsion-free, each of the factors Z_{i+1}/Z_i for $i < n$ is torsion-free abelian; see [**4**, Vol. 1, Theorem 2.25]. Thus

$$r(G) \le \sum_{i=1}^{n-1} r(Z_{i+1}/Z_i) = \sum_{i=1}^{n-1} r_0(Z_{i+1}/Z_i) = r_0(G).$$

The lemma is proved.

Proof of Theorem A. The torsion subgroup of R^+ forms an ideal T of R. If the ideals T_p of R are defined as in the proof of Theorem B, then $T = \bigoplus T_p$. By Lemma 2.5, each T_p° is a p-group and $(R/T)^\circ$ is torsion-free.

To prove (a), note that $T^\circ = \bigotimes T_p{}^\circ$ is periodic, so that we may assume $T = 0$. Hence R^+ is torsion-free. By Zorn's Lemma there exists a maximal locally nilpotent subring S of R, which is even nilpotent by Lemma 3.2. Assume now that $S \ne R$. Then by Lemma 3.3, S is properly contained in its idealizer $I = \mathrm{Id}_R(S)$. Hence there exists an element a in the subring I of R which is not in S. The subring $\hat{S}$ generated by $S \cup \{a\}$ is contained in the idealizer I of S, and therefore S is an ideal of $\hat{S}$. The quotient ring $\hat{S}/S$ is generated by $a+S$. As R is nil, it follows that $\hat{S}/S$ is nilpotent. Thus $\hat{S}$ is a nilpotent subring of R containing S properly. This contradiction shows that $R=S$ is nilpotent. Now Lemma 3.1 yields $r_0(R^\circ)<\infty$, so that Theorem B of [**1**] implies $r_0(R^\circ)=r_0(R^+)$. This proves part (a).

To prove (b), let R^+ have finite abelian subgroup rank. Then T^+ and hence by Theorem B also T° have finite abelian subgroup rank. Moreover, T is a locally nilpotent ring, since we have seen in the proof of Theorem B, that T is the direct sum of the nilpotent ideals T_p. As $r_0((R/T)^+)<\infty$, it follows as in the proof of (a) that R/T is a nilpotent ring. Its additive group $(R/T)^+$ is torsion-free with finite abelian subgroup rank and hence has finite Prüfer rank by [**4**, Vol. 2, p. 38, Corollary 1]. Thus Lemma 3.1 implies $r((R/T)^\circ)<\infty$ and, in particular, $(R/T)^\circ$ has finite abelian subgroup rank. As R/T is nilpotent and T is locally nilpotent, the ring R is locally nilpotent. Thus R° is a locally nilpotent extension of T° by $(R/T)^\circ$, which both have finite abelian subgroup rank. Hence R° has finite abelian subgroup rank. This proves (b).

To prove (c), suppose that R^+ has finite Prüfer rank. Then it follows as in the proof of (b) that R is a locally nilpotent ring and that R° is a locally nilpotent group. Moreover,

$$r((R/T)^\circ)\le r_0((R/T)^\circ)=r_0((R/T)^+) \tag{1}$$

by Lemma 3.4 and part (a). On the other hand, Theorem B yields

$$r(T^\circ)\le c\,.\,r(T^+)<\infty, \tag{2}$$

where $c=2$ if R contains no elements of order 2 and otherwise $c=3$. Combining equations (1) and (2), we obtain

$$\begin{aligned} r(R^\circ) &\le r(R^\circ/T^\circ)+r(T^\circ)=r((R/T)^\circ)+r(T^\circ)\\ &\le r_0((R/T)^+)+c\,.\,r(T^+)\\ &\le c\,.\,(r_0(R^+/T^+)+r(T^+))\\ &=c\,.\,r(R^+). \end{aligned}$$

This completes the proof of Theorem A.

REMARKS. (a) Note that our main results together with Theorem B of [**1**] imply that the rings R considered in the theorems with R^+ having finite abelian subgroup rank are two-sided T-nilpotent, i.e. each non-trivial epimorphic image of R has a non-trivial two-sided annihilator. It is easy to see that such rings are locally nilpotent.

(b) In both theorems, the inequality $r(R^\circ)\le c\,.\,r(R^+)$ for the Prüfer ranks is given, where $c=2$ if R^+ contains no elements of order 2 and otherwise $c=3$. It remains open whether these bounds are best possible. For rings with elements of additive order 2, the worst case known to the author is the ring $R=2\mathbb{Z}/8\mathbb{Z}$ with $r(R^+)=1$ and $r(R^\circ)=2$, while in the special case in which R^+ contains no elements of order 2, no example R with $r(R^\circ)>r(R^+)$ seems to be known. Hence it can be conjectured that the constant c can be decreased by 1 in either case.

(c) A slight modification of our proofs leads to the following minor improvement of the inequality just discussed:

$$r(R^\circ)\le r_0(R^+)+\max\{3\,.\,r(T_2^+),\,2\,.\,r(T_p^+)\mid p\ne 2\},$$

where the T_p are defined as in the proof of Theorem B.

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