We can note that the point (p, q) is invariant under this transformation, and so is any point on the line $M = 0$. Fixing just one more point fixes the whole plane.

There are many ways that this investigation could be extended — we could, for example, work in three dimensions. Take a point (p, q, r) that is NOT on the plane $ax + by + cz + d = 0$, and so on; in this case the central matrix that emerges has eigenvalues $3pqr$ and $-pqr$ (three times). A simple question accessible to A Level students is this one:

1. Suppose the point (p, q) is INSIDE the circle $(x - a)^2 + (y - b)^2 = r^2$.
2. The point (p, q) is ON the three circles $(x - a')^2 + (y - b)^2 = r^2$, $(x - a)^2 + (y - b')^2 = r^2$ and $(x - a)^2 + (y - b)^2 = r'^2$, where $a' < a$, $b' < b, r' < r$.
3. Show that (p, q) is OUTSIDE the circle $(x - a')^2 + (y - b')^2 = r'^2$.

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99.25 The work done by friction

It is common in A Level Mechanics to teach that the work done on a particle by gravity is $-mg(y_2 - y_1)$, where y_1 and y_2 are respectively the initial and final heights of the particle. The work done is independent of the path taken, and may be positive or negative. This is used alongside the fact that the gain in kinetic energy is equal to the total work done by all the forces on the particle.

We recently noticed that the work done by friction can be treated in a similar way. Provided that $\dot{x} \geqslant 0$ throughout, the work done by friction on a particle is $-\mu mg(x_2 - x_1)$, where x_1 and x_2 are respectively the initial and final horizontal positions of the particle. Here μ is the coefficient of (dynamic) friction in Coulomb's standard model for friction. The work done is independent of the path taken and is necessarily negative.

In this note, first we derive the result then we apply it to some particular situations. This method is certainly not new, but it deserves to be more widely known.

Deriving the result

We consider a particle of mass m sliding down a slope which at this instant makes a slope of angle θ with the horizontal, as in Figure 1.

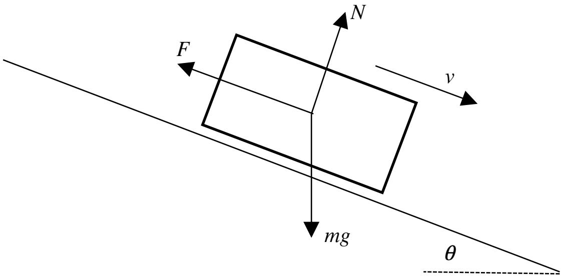

FIGURE 1

Equilibrium perpendicular to the slope ensures that the normal contact force is given by $N = mg\cos\theta$. Since the particle is moving, the friction is given by $F = \mu N = \mu mg\cos\theta$. Friction does work at a rate of $-Fv = -\mu mgv\cos\theta = -\mu mg\dot{x}$, provided that $\dot{x} \geqslant 0$. Integrating with respect to time, shows that the work done by friction is $-\mu mg(x_2 - x_1)$ as claimed.

Similarly, if $\dot{x} \leqslant 0$ throughout, the work done by friction on a particle is $\mu mg(x_2 - x_1)$. We may cover both cases by the expression $-\mu mg|x_2 - x_1|$, but we must be careful not to combine rightward and leftward movements.

If the only forces doing work on a particle are gravity and friction, there is a line of zero total work done, given by $-mg(y - y_1) - \mu mg(x - x_1) = 0$ for $x \geqslant x_1$, where (x_1, y_1) is the initial position. This simplifies to $y - y_1 = -\mu(x - x_1)$, and the corresponding equation for $x \leqslant x_1$ is $y - y_1 = \mu(x - x_1)$. The path of the particle must stay below these lines. When the path reaches the line, the particle will reverse direction if the magnitude of the gradient exceeds μ; otherwise the particle will come to rest at that point.

Straight slides

The problem from [1] which originally inspired this article is as follows. A waterslide consists of two straight sections, with equal coefficients of friction μ. A swimmer accelerates for 10 metres down a section at 40 degrees to the horizontal and then slides at constant speed for 20 metres down a section at 11 degrees to the horizontal. Calculate μ and the speed on the second section, assuming no loss of speed at the join. See Figure 2.

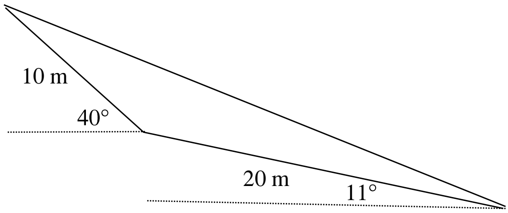

FIGURE 2

The interest lies in the final part of the question. What is the final speed if the slide is replaced by a single straight slide starting and ending in the same positions? The answer can be obtained tediously by much Trigonometry and Mechanics, but it turns out to be the same as the original speed.

The work done on the first section is $10mg(\sin 40° - \mu \cos 40°)$. The work done on the second section is $20mg(\sin 11° - \mu \cos 11°) = 0$, leading to $\mu = \tan 11°$. But the work done on the alternative slide must equal the sum of these expressions, since the overall horizontal and vertical displacements are the same.

A parabolic surface

Consider the surface given by rotating the curve $y = x^2$ about the y-axis, as shown in Figure 3. If a particle starts at position (a, a^2) with $a < 0$, it may slide to a position (b, b^2) provided that $b^2 - a^2 = -\mu(b - a)$, so that $b = -a - \mu$. But for this to be valid we require $b \geqslant a$ or $a \leqslant -\frac{\mu}{2}$. Similarly if a particle starts at position (b, b^2) with $b > 0$, it may slide to a position (c, c^2) provided that $c^2 - b^2 = \mu(c - b)$, so that $c = -b + \mu$. But for this to be valid we require $c \leqslant b$ or $b \geqslant \frac{\mu}{2}$.

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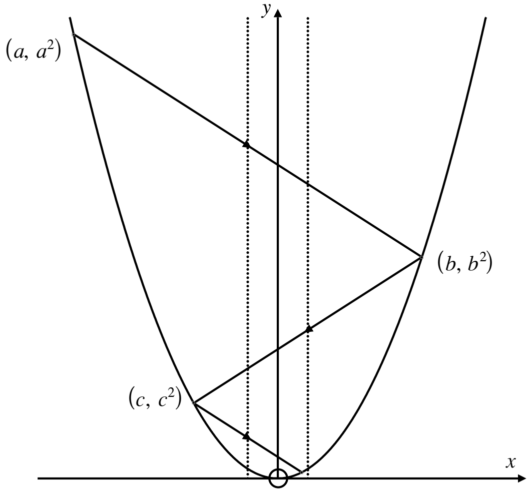

FIGURE 3

The particle will continue to slide back and forth until it comes to rest between the dotted lines shown at $x = \pm\frac{\mu}{2}$, which are also the points where the magnitude of the gradient is equal to μ.

Inside a hemisphere

Consider a hemisphere of radius r, as shown in Figure 4. Here $\mu = \tan\theta$, so that θ is the well-known angle of friction. If a particle starts at angle α below the horizontal from the centre, it may slide to an angle β below the horizontal on the other side. A simple angle chase shows that $\beta = \alpha + 2\theta$. But for this to be valid, we require $\alpha + \beta \leqslant 180°$ or $\alpha \leqslant 90° - \theta$.

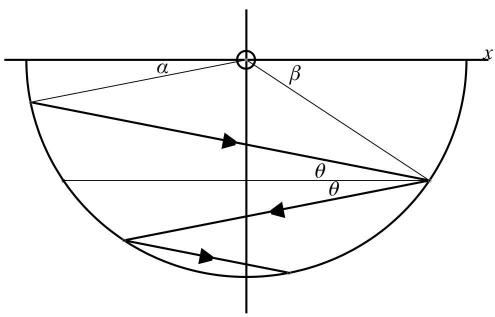

FIGURE 4

The particle will continue to slide back and forth until it comes to rest in the region where $90° - \theta \leqslant \alpha \leqslant 90° + \theta$. Again these limits correspond to the points where the magnitude of the gradient is equal to μ.

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Outside a hemisphere

A classic problem in circular motion involves a particle sliding down the outside of a *smooth* hemisphere. For variations on this theme, see [2]. With a *rough* hemisphere, the particle will not slide if it is placed at the top, but it will start to slide if it is placed at angle $\alpha \geqslant \theta$ where $\mu = \tan\theta$.

Consider a hemisphere of radius r, as shown in Figure 5. Point A is where the particle of mass m starts from rest and point B is a general point on its journey. Considering kinetic energy and work done we have

$$\tfrac{1}{2}mv^2 = mgr\left(\cos\alpha - \cos\beta\right) - \mu mgr\left(\sin\beta - \sin\alpha\right).$$

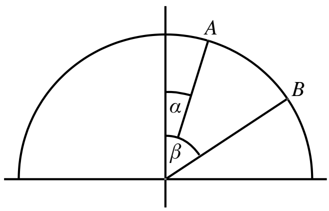

FIGURE 5

If N is the normal contact force, the radial equation for motion is

$$mg\cos\beta - N = \frac{mv^2}{r}.$$

The particle leaves the hemisphere when $N = 0$, leading to

$$\cos\beta = \frac{v^2}{gr} = 2(\cos\alpha - \cos\beta) - 2\mu(\sin\beta - \sin\alpha).$$

This simplifies to

$$3\cos\beta + 2\mu\sin\beta = 2\cos\alpha + 2\mu\sin\alpha.$$

This can always be solved by expressing the left-hand side as $r\cos(\beta - \gamma)$ where $r = \sqrt{3^2 + 4\mu^2}$ and $\gamma = \arctan\left(\tfrac{2}{3}\mu\right)$, but if $\mu > 0$ this does not seem to lead to answers as nice as the previous problems.

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