Canad. J. Math. Vol. 72 (3), 2020 pp. 774–804
http://dx.doi.org/10.4153/S0008414X19000087

Lipschitz-free Spaces on Finite Metric Spaces

Stephen J. Dilworth, Denka Kutzarova, and Mikhail I. Ostrovskii

Abstract. Main results of the paper are as follows:
(1) For any finite metric space M the Lipschitz-free space on M contains a large well-complemented subspace that is close to ℓ_1^n.
(2) Lipschitz-free spaces on large classes of recursively defined sequences of graphs are not uniformly isomorphic to ℓ_1^n of the corresponding dimensions. These classes contain well-known families of diamond graphs and Laakso graphs.
Interesting features of our approach are: (a) We consider averages over groups of cycle-preserving bijections of edge sets of graphs that are not necessarily graph automorphisms. (b) In the case of such recursive families of graphs as Laakso graphs, we use the well-known approach of Grünbaum (1960) and Rudin (1962) for estimating projection constants in the case where invariant projections are not unique.

1 Introduction

1.1 Definitions and Basic Properties of Lipschitz-free Spaces

Basic facts about Lipschitz-free spaces can be found in [49, Chapter 10] and [59, Chapter 3] (in [59] Lipschitz-free spaces are called Arens–Eells spaces).

Definition 1.1 Let X be a metric space. A *molecule* of X is a function $m\colon X \to \mathbb{R}$ that is supported on a finite set and satisfies $\sum_{p\in X} m(p) = 0$. For $p, q \in X$, define the molecule m_{pq} by $m_{pq} = \mathbf{1}_p - \mathbf{1}_q$, where $\mathbf{1}_p$ and $\mathbf{1}_q$ are indicator functions of singleton sets $\{p\}$ and $\{q\}$. We endow the space of molecules with the seminorm

$$\|m\|_{\mathrm{LF}} = \inf\Big\{\sum_{i=1}^{n} |a_i| d_X(p_i, q_i) : \ m = \sum_{i=1}^{n} a_i m_{p_i q_i}\Big\}.$$

It is not difficult to see that this is actually a norm. The *Lipschitz-free space* over X is defined as the completion of the space of all molecules with respect to the norm $\|\cdot\|_{\mathrm{LF}}$. We denote the Lipschitz-free space over X by $\mathrm{LF}(X)$.

Received by the editors July 10, 2018; revised February 2, 2019.
Published online on Cambridge Core February 13, 2019.
Author S. D. was supported by the National Science Foundation under Grant Number DMS–1361461. Authors S. D. and D. K. were supported by the Workshop in Analysis and Probability at Texas A&M University in 2017. Author M. O. was supported by the National Science Foundation under Grant Number DMS–1700176.
AMS subject classification: **52A21**, 30L05, 42C10, 46B07, 46B20, 46B85.
Keywords: Arens–Eells space, diamond graph, earth mover distance, Kantorovich–Rubinstein distance, Laakso graph, Lipschitz-free space, recursive family of graphs, transportation cost, Wasserstein distance.

By a *pointed metric space*, we mean a metric space with a distinguished point, denoted O. By $\mathrm{Lip}_0(X)$ we denote the space of all Lipschitz functions $f\colon X \to \mathbb{R}$ satisfying $f(O) = 0$, where O is the distinguished point of a pointed metric space X. It is not difficult to check that $\mathrm{Lip}_0(X)$ is a Banach space with respect to the norm $\|f\| = \mathrm{Lip}(f)$. As is well known [49, 59], the duality

$$\mathrm{LF}(X)^* = \mathrm{Lip}_0(X) \tag{1.1}$$

holds with respect to the pairing $\langle f, m\rangle = \sum_{x\in X} f(x)m(x)$ defined for $f \in \mathrm{Lip}_0(X)$ and a molecule m.

We also need the following description of $\mathrm{LF}(X)$ in the case where X is a vertex set of an unweighted graph with its graph distance. Let $G = (V(G), E(G)) = (V, E)$ be a finite graph. Let $\ell_1(E)$ be the space of real-valued functions on E with the norm $\|f\| = \sum_{e\in E} |f(e)|$. We consider some orientation on E, so each edge of E is a directed edge. For a directed cycle C in E (we mean that the cycle can be "walked around" following the direction, which is not related to the orientation of E), we introduce the *signed indicator function* of C by

$$\chi_C(e) = \begin{cases} 1 & \text{if } e \in C \text{ and its orientations in } C \text{ and } G \text{ are the same,} \\ -1 & \text{if } e \in C \text{ but its orientations in } C \text{ and } G \text{ are different,} \\ 0 & \text{if } e \notin C. \end{cases} \tag{1.2}$$

The *cycle space* $Z(G)$ of G is the subspace of $\ell_1(E)$ spanned by the signed indicator functions of all cycles in G. We will use the fact that $\mathrm{LF}(G)$ for unweighted graphs G ([49, Proposition 10.10]) is isometrically isomorphic to the quotient of $\ell_1(E)$ over $Z(G)$:

$$\mathrm{LF}(G) = \ell_1(E)/Z(G). \tag{1.3}$$

We use the standard terminology of Banach space theory [6], graph theory [8, 16], and the theory of metric embeddings [49].

1.2 Historical and Terminological Remarks

The Lipschitz-free spaces are studied by several groups of researchers, for different reasons and under different names. Some authors use the term *Arens–Eells space* (see [34, 59]), which reflects the contribution of Arens and Eells [5]. The norm of this space and a more general space of measures (see [57–59]) is called the *Kantorovich–Rubinstein distance (or norm)* to acknowledge the contribution of Kantorovich and Rubinstein [35, 36], or *Wasserstein distance (or norm)*, (see [3, 46]) to acknowledge the contribution of Wasserstein [56] (whose name is transliterated from Russian as Vasershtein); see the paper [20], where the term Wasserstein distance was introduced. The term *Wasserstein norm* is also used for the p-analogue of the distance. The term *Lipschitz-free space* is commonly used (especially in the Banach space theory) after the publication of the paper [26]. The names used for this distance in computer science are *earth mover distance* and *transportation cost* (see [1, 2, 37, 47]). All of the above-mentioned notions are equivalent for finite metric spaces that we consider in this paper. For this reason we decided not to attach any of the mentioned names to the objects of our study and to use the neutral name *Lipschitz-free space* (which only reflects the connection of this notion with the notion of a Lipschitz function).

Lipschitz-free spaces are of significant interest for computer science (see [31]), functional analysis ([25,34,59]), metric geometry ([3], [46, p. 134], [49]), and optimal transportation ([57,58]).

1.3 Overview of the Paper

Our interest in Lipschitz-free spaces is inspired by the theory of metric embeddings (see [49]): we are interested in studying properties of Banach spaces admitting an isometric embedding of a given metric space. We are going to focus on finite metric spaces.

Our main results and observations are as follows.

(a) We show that for any finite metric space M the space $\mathrm{LF}(M)$ contains a half-dimensional well-complemented subspace that is close to ℓ_1^n, see Section 3.

(b) We prove that the Lipschitz-free spaces on large classes of recursively defined sequences of graphs (see Section 1.4 for definitions) are not uniformly isomorphic to ℓ_1^n of the corresponding dimensions (Section 4). These classes contain well-known families of diamond graphs and Laakso graphs; see Section 1.4 for definitions and Section 5 for proofs. The case of diamond graphs can also be handled using classical theory of orthogonal series. Since this approach has its advantages and leads to more precise results, we include the corresponding argument in Section 6.

Interesting features of our approach are: (1) We consider averages over groups of cycle-preserving bijections of edge sets of graphs that are not necessarily graph automorphisms (see Section 4.3); (2) In the case of such recursive families of graphs as Laakso graphs, we use the well-known approach of Grünbaum [28] and Rudin [52] for estimating projection constants in the case where invariant projections are not unique (see Sections 4.4, 4.6, 4.7, and 5.3).

(c) We observe (Section 2) that the known fact (see [13, 15]) that Lipschitz-free spaces on finite ultrametrics are close to ℓ_1 in the Banach–Mazur distance immediately follows from the result of Gupta [29] on Steiner points and the well-known result on isometric embeddability of ultrametrics into weighted trees.

(d) We finish this section by observing that the result of Erdős and Pósa [22] on edge-disjoint cycles implies that the cycle space (considered as a subspace of $\ell_1(E)$) always contains a "large" 1-complemented in $\ell_1(E)$ subspace isometric to ℓ_1^n.

Observe that the subspace in $Z(G)$ spanned by the signed indicator functions of a family of edge-disjoint cycles is isometric to ℓ_1^n of the corresponding dimension and is 1-complemented in $\ell_1(E(G))$, and so in $Z(G)$.

This makes us interested in the estimates of the amount of edge-disjoint cycles in terms of the dimension of the cycle space. Such estimates, sharp up to the constants involved in them, were obtained by Erdős and Pósa [22]. Denote by $\mu(G)$ the dimension of the cycle space of G. It is well known (see [7, Proposition 2.1]) that for connected graphs, $\mu(G) = |E(G)| - |V(G)| + 1$. Let $\nu(G)$ be the maximal number of edge-disjoint cycles in G.

Theorem 1.2 (Erdős and Pósa [22, Theorem 4]) *The relation*

$$\nu(G) = \Omega\Big(\frac{\mu(G)}{\log(\mu(G))}\Big)$$

holds for any family of graphs, and, for some family of graphs,

$$\nu(G) = O\Big(\frac{\mu(G)}{\log(\mu(G))}\Big).$$

Remark 1.3 It is worth mentioning that Erdős and Pósa state their result slightly differently. They do not require graphs to be simple or connected and denote by $g(k)$ the smallest integer such that for any $n \in \mathbb{N}$ a graph with n vertices and $n + g(k)$ edges contains at least k edge-disjoint cycles. Theorem 4 in [22] states that

$$g(k) = \Theta(k \log k).$$

It is easy to see that Theorem 1.2 follows from this result.

1.4 Recursive Families of Graphs, Diamond Graphs, and Laakso Graphs

We are going to use the general definition of recursive sequences of graphs introduced by Lee and Raghavendra [42].

Definition 1.4 Let H and G be two finite connected directed graphs having distinguished vertices, which we call *top* and *bottom*, respectively. The *composition* $H \oslash G$ is obtained by replacing each edge $\overrightarrow{uv} \in E(H)$ by a copy of G; the vertex u is identified with the bottom of G, and the vertex v is identified with the top of G. Directions of edges in $H \oslash G$ are inherited from G. The *top* and *bottom* of the obtained graph are defined as the top and bottom of H, respectively.

When we consider these graphs as metric spaces we use the graph distances of the underlying undirected graphs (that is, we ignore the directions of edges).

It is straightforward to verify the following lemma.

Lemma 1.5 (Associativity of $\oslash$) *For any three graphs F, G, H, the sides of*

$$(F \oslash G) \oslash H = F \oslash (G \oslash H),$$

are equal both as directed graphs and as metric spaces.

Let B be a connected unweighted finite simple directed graph having two distinguished vertices, which we call *top* and *bottom*, respectively. We use B to construct a recursive family of graphs as follows:

Definition 1.6 We say that the graphs $\{B_n\}_{n=0}^{\infty}$ are defined by *recursive composition* or that $\{B_n\}_{n=0}^{\infty}$ is a *recursive sequence* or *recursive family* of graphs if

- the graph B_0 consists of one directed edge with *bottom* being the initial vertex and *top* being the terminal vertex;
- $B_n = B_{n-1} \oslash B$.

https://doi.org/10.4153/S0008414X19000087 Published online by Cambridge University Press

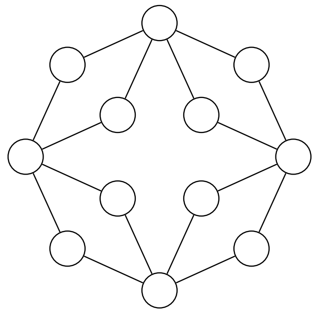

Figure 1: Diamond D_2.

Observe that Lemma 1.5 implies that for every $k \in \{0, 1, \ldots, n\}$, we have

$$B_n = B_{n-k} \oslash B_k, \tag{1.4}$$

and $B_1 = B$. The authors of [42] use the notation $B_n = B^{\oslash n}$.

Observe that in the case where the graph B has an automorphism that maps its bottom to top and top to bottom, the choice of directions on edges will not affect the isomorphic structure of the underlying undirected graphs. For this reason to define recursive families in such cases we do not not need to assign directions to edges.

Interesting and important examples of recursive families of graphs have been extensively studied in the literature. One of the most well known families, and one important for the theory of metric embeddings, was introduced in [30] (the conference version was published in 1999). This family (which turned out to be very useful in the theory of metric characterizations of classes of Banach spaces [32], see also [49, Section 9.3.2]) corresponds to the special case of Definition 1.6, where B is a square and one pair of its opposite vertices is chosen to play roles of the top and the bottom. The usual definition of diamond graphs is the following.

Definition 1.7 (Diamond graphs) Diamond graphs $\{D_n\}_{n=0}^{\infty}$ are defined recursively: The *diamond graph* of level 0 has two vertices joined by an edge of length 1 and is denoted by D_0. The *diamond graph* D_n is obtained from D_{n-1} in the following way. Given an edge $uv \in E(D_{n-1})$, it is replaced by a quadrilateral u, a, v, b, with edges ua, av, vb, bu. (See Figure 1.)

Let us count some parameters associated with graphs D_n. Denote by $V(D_n)$ and $E(D_n)$ the vertex set and edge set of D_n, respectively. We need the following simple observations about cardinalities of these sets:

(A) $|E(D_n)| = 4^n$.

(B) $|V(D_{n+1})| = |V(D_n)| + 2|E(D_n)|$.

Hence, $|V(D_n)| = 2(1 + \sum_{i=0}^{n-1} 4^i)$.

The next special case of the general Definition 1.6, whose metric geometry was studied in [42,50], corresponds to the case where $B = K_{2,n}$, and the vertices in the part

https://doi.org/10.4153/S0008414X19000087 Published online by Cambridge University Press

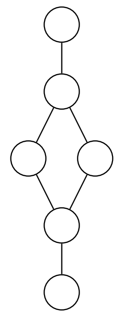

Figure 2: Laakso graph $\mathcal{L}_1$.

containing two vertices play the roles of the top and the bottom. The usual definition is the following.

Definition 1.8 (Multibranching diamonds) For any integer $k \geq 2$, we define $D_{0,k}$ to be the graph consisting of two vertices joined by one edge. For any $n \in \mathbb{N}$, if the graph $D_{n-1,k}$ is already defined, the graph $D_{n,k}$ is defined as the graph obtained from $D_{n-1,k}$ by replacing each edge uv in $D_{n-1,k}$ by a set of k independent paths of length 2 joining u and v. We endow $D_{n,k}$ with the shortest path distance. We call $\{D_{n,k}\}_{n=0}^{\infty}$ *diamond graphs of branching* k, or *diamonds of branching* k.

The last special case of the general Definition 1.6 that we consider in this paper goes back to Laakso [40]. The corresponding recursive family of graphs was introduced by Lang and Plaut [41]. In [48] it was shown that these graphs are incomparable with diamond graphs in the following sense: elements of none of these families admit bilipschitz embeddings into the other family with uniformly bounded distortions. Laakso graphs correspond to the case where the graph B is the graph shown in Figure 2 with the natural choice for the top and the bottom.

Definition 1.9 Laakso graphs $\{\mathcal{L}_n\}_{n=0}^{\infty}$ are defined recursively. The *Laakso graph* of level 0 has two vertices joined by an edge of length 1 and is denoted $\mathcal{L}_0$. The *Laakso graph* $\mathcal{L}_n$ is obtained from $\mathcal{L}_{n-1}$ according to the following procedure. Each edge $uv \in E(\mathcal{L}_{n-1})$ is replaced by the graph $\mathcal{L}_1$ exhibited in Figure 2, the vertices u and v are identified with the vertices of degree 1 of $\mathcal{L}_1$.

2 Lipschitz-free Spaces Close to ℓ_1^n

Our first proposition is known (see [24, Corollary 3.3]) we give a direct proof of it for convenience of the reader.

Proposition 2.1 *Let T be a finite weighted tree. Then* $\mathrm{LF}(T)$ *is isometric to* ℓ_1^k, *where k is the number of edges in the tree.*

https://doi.org/10.4153/S0008414X19000087 Published online by Cambridge University Press

Proof Let $f \mapsto e_f$ be a bijection between the edge set of T and the unit vector basis in ℓ_1^k. We denote the weight of f by $w(f)$. We consider the following map F of the set of molecules on T into ℓ_1^k.

For each edge $f = \{u, v\}$ we let $F(\mathbf{1}_u - \mathbf{1}_v) = w(f)e_f$. It is clear that each molecule in $\mathrm{LF}(T)$ can be (uniquely) written as a linear combination of molecules $\{\mathbf{1}_u - \mathbf{1}_v\}_{\{u,v\}\in E(T)}$. We define F to be the linear extension of the defined map to $\mathrm{LF}(T)$; it is clear from this definition that F is a surjective map onto ℓ_1^k.

By the duality (1.1), to show that F is an isometry of $\mathrm{LF}(T)$ onto ℓ_1^k, it is enough to find a 1-Lipschitz function $L \in \mathrm{Lip}_0(T)$ (the base point O is chosen arbitrarily) such that

$$L\Big(\sum_{\{u,v\}\in E(T)} a_{uv}(\mathbf{1}_u - \mathbf{1}_v)\Big) = \sum_{\{u,v\}\in E(T)} |a_{uv}| \cdot w(uv),$$

where $a_{uv} \in \mathbb{R}$.

Construction of such 1-Lipschitz function L is quite straightforward. We let $L(O) = 0$. If the function is already defined on one end u of an edge $\{u, v\}$, we set $L(v) = L(u) \pm w(uv)$, where we choose + if the coefficient of $\mathbf{1}_v - \mathbf{1}_u$ in m is nonnegative, and − if the coefficient of $\mathbf{1}_v - \mathbf{1}_u$ in m is negative. It is clear that L is 1-Lipschitz and $L(m) = \sum_{\{u,v\}\in E(T)} |a_{uv}| \cdot w(uv)$. ∎

The following result is very useful in the current context.

Theorem 2.2 ([29]) *Let T be a weighted tree and let M be a subset of $V(T)$. Then there is a weighted tree $\widetilde{T}$ with the vertex set M such that the distances induced by T and $\widetilde{T}$ on M are* 8*-equivalent.*

Corollary 2.3 *Let T be a weighted tree and let M be a subset of $V(T)$. Then the Banach–Mazur distance between* $\mathrm{LF}(M)$ *(where M is endowed with the metric induced from T) and ℓ_1^k of the corresponding dimension does not exceed* 8.

Remark 2.4 Gupta [29] did not show that the constant 8 is sharp; his lower estimate for the constant is 4. It is not clear what the optimal constant is in Corollary 2.3.

Since it is well known that ultrametrics can be isometrically embedded into weighted trees (see, for example, [12, Theorem 9], and also [21, Section 3]), we get also the following finite version of results of [13, 15]:

Corollary 2.5 *Let M be a finite ultrametric space. Then* $\mathrm{LF}(M)$ *is* 8*-isomorphic to ℓ_1^k, where $k = |M| - 1$.*

To see that there are metric spaces of different nature whose Lipschitz-free spaces are also close to ℓ_1^k of the corresponding dimension, we use (1.3). This equality implies that if we consider a graph G which contains a small amount of cycles, or all cycles in it are disjoint, then $\mathrm{LF}(G)$ is close to ℓ_1^n of the corresponding dimension.

The space $\mathrm{LF}(G)$ remains close to ℓ_1^n for metric spaces that are bilipschitz equivalent to graphs having properties described in the previous paragraph. One of the ways of getting such metric spaces is deletion of edges forming short cycles; see [51] on results related to this construction, especially [51, Section 17.2]. It is worth mentioning that bilipschitz equivalent metric spaces can have quite different structure of cycle spaces. Consider, for example, K_n (complete graph on n vertices) and the graph $K_{1,n-1}$ consisting of n vertices in which the first vertex is adjacent to all other vertices, and there are no other edges. Any bijection between these metric spaces has distortion 2; the cycle space $Z(K_n)$ is a large space, whereas $Z(K_{1,n-1})$ is trivial.

Problem 2.6 *It would be very interesting to find a condition on a finite metric space M that is equivalent to the condition that the space* $\mathrm{LF}(M)$ *is Banach–Mazur close to ℓ_1^n of the corresponding dimension. It is not clear whether it is feasible to find such a condition.*

3 Large Complemented ℓ_1^n in Finite-dimensional Lipschitz Free Spaces

The following result can be regarded as a finite-dimensional version of the result of Cúth, Doucha, and Wojtaszczyk [14] who proved that the Lipschitz-free space on an infinite metric space contains a complemented subspace isomorphic to ℓ_1.

Theorem 3.1 *For every n-point metric space M, the space* $\mathrm{LF}(M)$ *contains a* 2*-complemented* 2*-isomorphic copy of ℓ_1^k with $k = \lceil \frac{n}{2} \rceil$.*

The following lemma is a version of [14, Lemma 3.1].

Lemma 3.2 *Let (M,d) be a finite metric space and let $\{y_i\}_{i=1}^k$ be a sequence of distinct points in M such that $M\backslash\{y_i\}_{i=1}^k$ is nonempty. For each $i \in \{1,\dots,k\}$, let $x_i \in M\backslash\{y_i\}_{i=1}^k$ be such that the distance $d(x_i,y_i)$ is minimized, so $\{x_i\}_{i=1}^k$ are not necessarily distinct. Then linear combinations of the functions $f_i(x) = d(y_i,x_i)\mathbf{1}_{y_i}(x)$ satisfy the inequality*

$$\max_i |\alpha_i| \le \mathrm{Lip}\Big(\sum_{i=1}^k \alpha_i f_i\Big) \le \max\Big\{\max_{i\neq j} \frac{d(x_i,y_i)+d(x_j,y_j)}{d(y_i,y_j)}, 1\Big\} \cdot \max_i |\alpha_i|.$$

Proof The leftmost inequality is obtained by comparing the values of $\sum_{i=1}^k \alpha_i f_i$ at x_m and y_m, where $m \in \{1,\dots,k\}$ is such that $\alpha_m = \max_i |\alpha_i|$.

To prove the rightmost inequality, we perform the following analysis: consider any pair (u,v) of points in M and estimate from above the quotient

$$\Big|\sum_{i=1}^k \alpha_i f_i(u) - \sum_{i=1}^k \alpha_i f_i(v)\Big| \Big/ d(u,v).$$

If the points u and v are y_i and y_j, $i \neq j$, then the estimate from above is

$$\frac{d(x_i,y_i)+d(x_j,y_j)}{d(y_i,y_j)} \cdot \max_i |\alpha_i|.$$

https://doi.org/10.4153/S0008414X19000087 Published online by Cambridge University Press

If one of the points is y_i and the other is not in the sequence $\{y_i\}_{i=1}^k$, we get at most $\max_i |\alpha_i|$, because of the minimality property of $d(x_i, y_i)$. If both u and v are not in $\{y_i\}_{i=1}^k$, then $\sum_{i=1}^k \alpha_i f_i(u) = \sum_{i=1}^k \alpha_i f_i(v) = 0$. ∎

Proof of Theorem 3.1 Any finite metric space can be considered as a weighted graph with the weighted graph distance (we can consider elements of the metric space as vertices of a complete graph with the weight of each edge equal to the distance between its ends).

Consider the minimum weight spanning tree T in this graph constructed according to Boruvka–Kruskal procedure [39, Construction A] (see also [8, Algorithm 8.22]); that is, we list edges in the order of nondecreasing lengths; then we process this list from the beginning and pick for the spanning tree all edges which do not form cycles with the previously selected.

It is easy to see that the picked set of edges satisfies the following condition: at least one of the shortest edges incident to each of the vertices is in the spanning tree.

Any tree is a bipartite graph. Therefore, we can split M into two subsets, M_1 and M_2, such that any edge in the spanning tree T has one vertex in M_1 and the other in M_2. At least one of the sets M_1 and M_2 contains at least half of the elements of M. We assume that M_1 is such and label its vertices as $\{y_i\}_{i=1}^k$. For each i, we let x_i be the closest to y_i element of M_2 (the elements $\{x_i\}_{i=1}^k$ are not required to be distinct). The comment in the previous paragraph implies that $x_i \in M_2$ is one of the closest to y_i and different from y_i elements of M. Hence, by Lemma 3.2, the subspace of $\mathrm{Lip}_0(M)$ (we pick the base point to be any element of $M \backslash \{y_i\}_{i=1}^k$) spanned by $\{d(y_i, x_i)\mathbf{1}_{y_i}\}_{i=1}^k$ is 2-isomorphic to ℓ_∞^k and thus 2-complemented in $\mathrm{Lip}_0(M)$.

Consider the functions $u_i = (\mathbf{1}_{y_i} - \mathbf{1}_{x_i})/d(x_i, y_i)$ in $\mathrm{LF}(M)$. We claim that $\{u_i\}_{i=1}^k$ span a 2-complemented subspace in $\mathrm{LF}(M)$ that is 2-isomorphic to ℓ_1^k. It is clear that $\|u_i\| = 1$ and $f_i(u_j) = \delta_{i,j}$ (Kronecker δ). Let $\{b_i\}_{i=1}^k$ be a sequence of real numbers satisfying $\sum_{i=1}^k |b_i| = 1$, and $x = \sum_{i=1}^k b_i u_i$. We need to estimate the norm of x. Clearly, $\|x\| \le \sum_{i=1}^k |b_i| = 1$.

On the other hand, let $\alpha_i = \mathrm{sign}(b_i)$. Then, by the first part of the proof,

$$1 \le \Big\| \sum_{i=1}^k \alpha_i f_i \Big\| \le 2.$$

On the other hand,

$$\Big(\sum_{i=1}^k \alpha_i f_i \Big)(x) = \sum_{i=1}^k \alpha_i b_i = \sum_{i=1}^k |b_i|.$$

Hence, $\frac{1}{2} \le \|x\| \le 1$.

Now we show that the linear span of $\{u_i\}$ is 2-complemented. We introduce $P\colon \mathrm{LF}(M) \to \mathrm{lin}\{u_i\}$ by

$$P(u) = \sum_{i=1}^k f_i(u) u_i.$$

It is clear that P is a linear projection. Let us estimate its norm. Let $f \in \mathrm{Lip}_0(M)$ be such that $\|f\| = 1$ and $f(P(u)) = \|P(u)\|$. Then

$$\|P(u)\| = \sum_{i=1}^{k} f_i(u) f(u_i) \le \Big\| \sum_{i=1}^{k} f(u_i) f_i \Big\| \cdot \|u\| \le 2 \max_i |f(u_i)| \cdot \|u\| \le 2\|u\|.$$

It remains to recall that the construction is such that $k \ge |M|/2$. ∎

Problem 3.3 *Is the constant* 2 *in the statement "*2*-complemented* 2*-isomorphic" of Theorem 3.1 is sharp?*

It is not surprising that Theorem 3.1 can be sharpened for some classes of graphs. In Theorem 3.6 we sharpen it for the diamond graphs.

It is natural to ask: How and when can we go beyond half-dimensional subspace? It is easy to see that the following result can be proved on the same lines as Theorem 3.1.

Theorem 3.4 *Let M be a finite metric space and $\{y_i\}_{i=1}^k$ be a sequence in it such that $M \backslash \{y_i\}_{i=1}^k$ is nonempty. Let $d_i = d(y_i, (M \backslash \{y_i\}_{i=1}^k))$ and*

$$C = \max\Big\{ \max_{i \ne j} \frac{d_i + d_j}{d(y_i, y_j)}, 1 \Big\}. \tag{3.1}$$

Then $\mathrm{LF}(M)$ contains a C-complemented subspace that is C-isomorphic to ℓ_1^k and $\mathrm{Lip}_0(M)$ contains a C-complemented subspace that is C-isomorphic to ℓ_∞^k.

Corollary 3.5 *If M is a connected unweighted graph with n vertices, then for every $p \in \mathbb{N}$ with $p \le \mathrm{diam}(M) + 1$, the space $\mathrm{LF}(M)$ contains a subspace of dimension $d \ge n\left(\frac{p-1}{p}\right)$ that is $4p$-complemented and is $4p$-isomorphic to ℓ_1^d, and $\mathrm{Lip}_0(M)$ contains a $4p$-complemented subspace that is $4p$-isomorphic to ℓ_∞^d. If $p > \mathrm{diam}(M)$, we have the inequality $d_{BM}(\mathrm{LF}(M), \ell_1^{n-1}) \le 2p$ for the Banach–Mazur distance.*

Proof Let O be one of the vertices of M for which $\max_{v \in M} d_M(O, v) = \mathrm{diam}(M)$. Assume that $p \le \mathrm{diam}(M) + 1$. Consider the partition $M = \cup_{i=0}^{p-1} M_i$, where M_i is the set of vertices in M whose distance to O is $i \pmod p$. The assumption $p \le \mathrm{diam}(M) + 1$ implies that all sets M_i are nonempty. One of the sets $\{M_i\}_{i=0}^{p-1}$ has cardinality $\le \frac{n}{p}$. Let $\{y_i\}_{i=1}^k$ be the complement of this set. Its cardinality, which we denote by d, is at least $n(\frac{p-1}{p})$. On the other hand, it is clear that $d_i \le 2p$ (d_i is defined in Theorem 3.4). Thus, the constant C defined in (3.1) is $\le 4p$. The conclusion follows.

The last statement is true because $p \ge \mathrm{diam}(M)$ implies that the space M is $2p$-bilipschitz equivalent to the graph $K_{1,n-1}$ with its graph distance, and $\mathrm{LF}(K_{1,n-1})$ is isometric to ℓ_1^{n-1} by Proposition 2.1. ∎

For some graphs, the estimates of Theorems 3.1, 3.4 and Corollary 3.5 can be improved significantly. It is interesting that this can be done even in the case of diamond graphs $\{D_n\}$, while $\mathrm{LF}(D_n)$ are far from $\ell_1^{d(n)}$ of the corresponding dimension; see Corollary 3.7 and Theorem 6.5.

Theorem 3.6 LF(D_n) *contains a* 1*-complemented isometric copy of* ℓ_1^k *with* $k = 2 \cdot 4^{n-1}$.

Note that for large n, the number $2 \cdot 4^{n-1}$ is very close to $\frac{3}{4}|V(D_n)|$; see page 5.

Proof We use an argument similar to the argument of Theorem 3.1 with the following choice of $\{y_i\}_{i=1}^k$: the vertices $\{y_i\}_{i=1}^k$ are the vertices added to the graph in the last step. Formula **(B)** on page 5 implies that $k = 2 \cdot 4^{n-1}$. The vertex x_i is chosen to be one of the (two) closest to y_i vertices in D_n. In this case $d(x_i, y_i) = 1$ and $d(y_i, y_j) \geq 2$ for $i \neq j$. Hence the same argument as in Theorem 3.1 leads to a subspace isometric to ℓ_1^k and 1-complemented. ∎

We have the following corollary of Theorem 3.4 for diamonds.

Corollary 3.7 *For each* $m < n$ *the space* LF(D_n) *contains a* C*-complemented* C*-isomorphic to* ℓ_1^k *subspace with* $C = 2^{n-m}$ *and* $k = 2(1 + \sum_{i=0}^{n-1} 4^i) - 2 \cdot 4^{m-1}$.

Note that the codimension of the subspace does not exceed $\frac{3}{4^{n-m+1}}|V(D_n)|$.

Proof Consider in D_n the subset $A_{n,m}$ of vertices that were added when D_m was created. The equality **(B)** on page 5 implies that the cardinality of $A_{n,m}$ is $2 \cdot 4^{m-1}$. It is also easy to see that the distance from any other vertex to this set does not exceed 2^{n-m-1}. Define $\{y_i\}_{i=1}^k$ as $V(D_n)\backslash A_{n,m}$.

Then the constant C defined in (3.1) does not exceed 2^{n-m} and

$$k = 2\Big(1 + \sum_{i=0}^{n-1} 4^i\Big) - 2 \cdot 4^{m-1}$$ ∎

Results of this section lead us to suspect that a Lipschitz-free space of dimension n cannot be "too far" from ℓ_1^n in the Banach-Mazur distance. In this connection we ask the following question.

Problem 3.8 *Estimate the maximal possible Banach–Mazur distance between* ℓ_1^n *and a Lipschitz-free space of dimension* n.

So far all known estimates for the Banach-Mazur distance $d_{BM}(\text{LF}(M), \ell_1^n)$ (where $n = |M| - 1$) from below are at most logarithmic in n. We know two cases in which logarithmic estimates from below hold. One case is the case of diamond graphs (if we use estimates based on the theory of Haar functions); see Theorems 6.5 and 6.10.

The second case is the case where M itself has large ℓ_1-distortion. It is well known that the ℓ_1-distortion of n-vertex expanders is of order $\log n$; see [44]. Another example with $\log n$-distortion was given in [37, Corollary 1] (see also [49, Section 4.2]). Bourgain [9] proved that the ℓ_1-distortion of an n-element metric space can be estimated from above by $C \log n$. Therefore on these lines we cannot get lower estimates for $d_{BM}(\text{LF}(M), \ell_1^n)$ of higher than logarithmic order.

Observe that if M is an expander, then $d_{BM}(\text{LF}(M), \ell_1^n) \leq C \log n$, because expanders have diameter of order $\log n$, and thus are $C \log n$-bilipschitz equivalent to the tree $K_{1,n}$.

Corollary 3.5 allows us to get an estimate (Proposition 3.9) for $d_{BM}(\mathrm{LF}(M), \ell_1^n)$ from above in the case where M is an unweighted finite graph, which is slightly better than the estimate $d_{BM}(X_n, \ell_1^n)$ for a general n-dimensional Banach space X_n.

Let us recall known estimates for $\mathcal{D}_n := \max\{d_{BM}(X_n, \ell_1^n) : \dim X_n = n\}$:

$$n^{\frac{5}{9}} \log^{-C} n \le \mathcal{D}_n \le (2n)^{\frac{5}{6}}$$

for some absolute constant $0 < C < \infty$. The lower estimate is due to Tikhomirov [55]; it is an improvement of the previous estimate of [53]. The upper estimate in this form is due to Youssef [60]; it is an improvement of previous estimates of [11, 23, 54].

Proposition 3.9 *If M is an unweighted connected graph with $n+1$ vertices (endowed with its graph distance), then $d_{BM}(\mathrm{LF}(M), \ell_1^n) \le Cn^{\frac{8}{11}}$.*

Proof We will work with the dual space; that is, we will show that

$$d_{BM}(\mathrm{Lip}_0(M), \ell_\infty^n) \le Cn^{\frac{8}{11}}.$$

By Corollary 3.5, we can find elements $f_i \in \mathrm{Lip}_0(M)$ such that

$$\max_i |\alpha_i| \le \left\| \sum \alpha_i f_i \right\| \le 4p \max |\alpha_i|,$$

where the codimension of the subspace F spanned by $\{f_i\}$ is $k \le \frac{n}{p}$, provided $p \le \mathrm{diam}(M) + 1$. By an easy corollary of the Kadets–Snobar [33] theorem, every subspace of codimension m of a finite-dimensional normed space is the range of a projection of norm at most $\sqrt{m}+1$. Hence, we can find a projection P onto F of norm at most $2\sqrt{\frac{n}{p}}$. By the result of [60],

$$d_{BM}(\ker P, \ell_\infty^k) \le \left(2\frac{n}{p}\right)^{\frac{5}{6}}.$$

Therefore, we can find a sequence $\{g_i\}$ in $\ker P$ such that

$$\max_i |\beta_i| \le \left\| \sum \beta_i g_i \right\| \le \left(2\frac{n}{p}\right)^{\frac{5}{6}} \max_i |\beta_i|.$$

We have

$$\begin{aligned}\frac{1}{4\sqrt{n/p}} \max_{i,j}(|\alpha_i|, |\beta_j|) &\le \left\| \sum \alpha_i f_i + \sum \beta_j g_j \right\| \\ &\le \left(4p + (2n/p)^{\frac{5}{6}}\right) \max_{i,j}\left(|\alpha_i|, |\beta_j|\right).\end{aligned}$$

So the Banach–Mazur distance $d_{BM}(\mathrm{Lip}_0(M), \ell_\infty^n)$ can be estimated from above by

$$4\sqrt{\frac{n}{p}} \cdot \left(4p + \left(2\frac{n}{p}\right)^{\frac{5}{6}}\right).$$

Pick $p = n^{\frac{5}{11}}$. We get $d_{BM}(\mathrm{Lip}_0(M), \ell_\infty^n) \le cn^{\frac{8}{11}}$ either by applying the argument above if $n^{\frac{5}{11}} \le \mathrm{diam}(M) + 1$, or by using the final statement of Corollary 3.5 otherwise. ■

https://doi.org/10.4153/S0008414X19000087 Published online by Cambridge University Press

4 Proof for General Recursive Families

The goal of this section is to show that if the graph B satisfies the conditions listed in Section 4.1, then the Banach–Mazur distances between the Lipschitz-free spaces on B_n (see Definition 1.6) and the spaces $\ell_1^{d(n)}$ of the corresponding dimensions tend to ∞. See Theorem 4.2 for the statement of the result.

Note 4.1 It is clear that each bijection g on the edge set of a graph G induces an isometry on the space $\ell_1(E(G))$ given by

$$f \longmapsto h \Longleftrightarrow h(e) = f(g^{-1}e) \quad f, h \in \ell_1(E(G)), e \in E(G).$$

With some abuse of notation, we will keep the notation g for this isometry.

4.1 Conditions on B

The conditions below are not independent. Our goal is to list all the conditions that we use.

(a) Each edge is contained in a geodesic (a shortest path) of even length joining the bottom and the top. Each path joining the top and the bottom is geodesic.

(b) Each edge is directed to the vertex with the smaller distance to the top. The cycle space $Z(B)$ is constructed using this orientation of B. Each directed cycle in B is a union of two paths that are pieces of geodesics joining the top and the bottom. On one of these paths, the direction on the cycle coincides with the direction in B, on the other it is opposite.

(c) The (underlying) graph B has an automorphism v that interchanges top and bottom vertices. We say that v is a *vertical automorphism* of B. Here ("underlying" means that the automorphism does not respect directions of edges.)

(d) The automorphism v can be chosen in such a way that each element of $Z(B)$ is a fixed point of v.

(e) Let D be the distance between the bottom and the top in B. Consider the vector

$$\Delta = \frac{1}{DK} \sum_{p} \mathbf{1}_p \in \ell_1(E(B)), \tag{4.1}$$

where K is the number of distinct geodesics joining the bottom and the top in B, and $\mathbf{1}_p$ is the indicator function of a bottom-top geodesic, and the sum is over all distinct bottom-top geodesics.

It is easy to see that the map E_n $(n = 0, 1, 2, \dots)$ that maps the indicator function $\mathbf{1}_e$ of an edge e onto Δ in the copy of B that replaces e extends to an isometric embedding of $\ell_1(E(B_n))$ into $\ell_1(E(B_{n+1}))$, and that E_n maps $Z(B_n)$ into $Z(B_{n+1})$. We introduce the function $c(B)$ in $\ell_1(E(B)) = \ell_1(E(B_1))$ as the function whose absolute value is $E_0(\mathbf{1}_e)$ (where e is the only edge of B_0), and the signs are positive for edges that are closer to the top and negative for edges that are closer to the bottom (recall that each edge belongs to a geodesic of even length joining the top and the bottom). One of the conditions on these maps is: $v(c(B)) = -c(B)$ (see Note 4.1); this condition actually follows from other conditions. Another condition is in item (f).

https://doi.org/10.4153/S0008414X19000087 Published online by Cambridge University Press

(f) The collection $\mathcal{H}$ of all automorphisms of B for which the top and the bottom are fixed points satisfies two conditions. First, the corresponding subgroup of isometries of $\ell_1(E(B))$ has no fixed points in the cycle space $Z(B)$ except 0. Second, the function $c(B)$ is a common fixed point of all elements of $\mathcal{H}$. We call automorphisms of $\mathcal{H}$ *horizontal.*

(g) The cycle space of B is nontrivial. This is equivalent to the existence of two distinct bottom-top geodesics, and this is, in turn, equivalent to the fact that $\frac{1}{D}\mathbf{1}_p - \Delta \neq 0$ for any bottom-top geodesic p. We pick a bottom-top geodesic p for which the ℓ_1-norm $\frac{1}{D}\mathbf{1}_p - \Delta \neq 0$ is maximized; denote this difference by $d(B)$ and its norm in $\ell_1(E(B))$ by α. Observe that $d(B) \in Z(B)$.

It is worth mentioning that the graphs $\{B_n\}$, $n \geq 1$, inherit some properties of the graph $B = B_1$.

(A) Graphs B_n have properties of items (a) and (b).

Only the last condition in item (b) requires verification. This can be done using induction. We have assumed this condition for B_1. Suppose that holds for B_{n-1}. Consider a directed cycle in B_n. By (1.4), we have $B_n = B \oslash B_{n-1}$. If the cycle is contained in one of the copies of B_{n-1}, we are done by the induction hypothesis. If the cycle is not contained in any of B_{n-1}, then it can be obtained replacing each edge in the corresponding cycle in B_1 by a top-bottom path in the corresponding copy of B_{n-1} (see item (a)). The conclusion follows if we recall how edges of B_n are oriented, see Definition 1.4.

(B) The underlying graph of B_n has an automorphism v_n that interchanges top and bottom vertices.

This can be proved by induction:

- For $B_1 = B$ this is an assumption of item (c).
- Suppose that this is true for B_{n-1}, and v_{n-1} is the corresponding automorphism. By (1.4), we have $B_n = B \oslash B_{n-1}$. We consider the bijection of the edge set of B_n designed in the following way:
- If v maps an edge uw to an edge $\widehat{uw}$, with u and $\widehat{u}$ being closer to the bottom of B, we map B_{n-1} corresponding to the edge uw onto B_{n-1} corresponding to $\widehat{uw}$ "upside down", that is, using v_{n-1}.
- It is easy to see that we get an automorphism of B_n, which interchanges the top and the bottom. We denote this automorphism by v_n.

4.2 The Main Result

The following theorem is our main result on families $\{B_n\}$.

Theorem 4.2 *If the directed graph B satisfies the conditions in items (a)–(g) listed above, and $\{B_n\}_{n=0}^{\infty}$ are constructed according to Definition 1.6, then*

$$d_{BM}(\mathrm{LF}(B_n), \ell_1^{d(n)}) \geq \frac{cn}{\ln n}$$

for $n \geq 2$ and some absolute constant $c > 0$, where $d(n)$ is the dimension of $\mathrm{LF}(B_n)$.

https://doi.org/10.4153/S0008414X19000087 Published online by Cambridge University Press

To prove Theorem 4.2, we need several lemmas. The final step in the proof is presented on page 19.

Lemma 4.3 *To prove Theorem 4.2, it suffices to show that the relative projection constants of $Z(B_n)$ in $\ell_1(E(B_n))$ satisfy*

$$\lambda\big(Z(B_n), \ell_1(E(B_n))\big) \geq \frac{cn}{\ln n}$$

for $n \geq 2$ and some absolute constant $c > 0$.

Proof This lemma is a consequence of the following well-known fact. ∎

Fact 4.4 *If a quotient X/Y is such that the Banach–Mazur distance satisfies*

$$d_{BM}\big(X/Y, \ell_1(\Gamma)\big) \leq C,$$

then $\lambda(Y, X) \leq (1 + C)$.

Proof of Fact 4.4 Denote by $Q: X \to X/Y$ the quotient map. Let $T: \ell_1(\Gamma) \to X/Y$ be such that $\|T\| < C+\varepsilon$, $\|T^{-1}\| \leq 1$. By the *lifting property* of $\ell_1(\Gamma)$ (see [43, pp. 107–108]), there is a linear operator $\widehat{T}: \ell_1(\Gamma) \to X$ such that $\|\widehat{T}\| < C + \varepsilon$ and $Q\widehat{T} = T$. Then the operator $(I - \widehat{T}T^{-1}Q)$ is a projection of X onto Y, and its norm is $< (1 + C + \varepsilon)$; the conclusion follows. ∎

4.3 Cycle-preserving Bijections of B_n

For each $n \in \mathbb{N}$ we introduce G_n as the group of all *cycle-preserving bijections* of $E(B_n)$ (we consider undirected edges) satisfying the following additional condition: the edge set of any path joining the top and the bottom of B_n is mapped onto the edge set of a path joining the top and the bottom of B_n. By a *cycle-preserving bijection* we mean a bijection that maps the edge-set of any cycle to an edge-set of a cycle (we do not pay attention to directions of edges). It is clear that G_n is a finite group.

The representation (1.4) shows that for each $1 \leq k \leq n-1$, the graph B_n is a union of edge-disjoint copies of B_k. It is clear that bijections of $E(B_n)$ that leave all these copies of $E(B_k)$ invariant, and whose restrictions to them are contained in G_k, belong to G_n.

The groups G_n lead in a natural way (see Note 4.1) to subgroups of the group of isometries of $\ell_1(E(B_n))$. An important observation is that the subgroup corresponding to G_n leaves the cycle space $Z(B_n)$ invariant.

This observation can be shown as follows. By statement **(A)**, each directed cycle in B_n is a union of two pieces, C_1 and C_2, of geodesics (going up and going down). Thus, there are well-defined notions of the *top* (and *bottom*) *of the cycle*—the vertex of the cycle nearest to the top (bottom) of B_n. We join them to the top and bottom of B_n, respectively, using pieces of geodesics P_b and P_t. Then both the concatenation $P_bC_1P_t$ and $P_bC_2P_t$ are paths joining the bottom and the top of B_n. Therefore, the additional condition on cycle-preserving bijections implies that the edge sets of $P_bC_1P_t$ and $P_bC_2P_t$ are edge-sets of bottom-top paths in B_n. Also the image of the edge set of

https://doi.org/10.4153/S0008414X19000087 Published online by Cambridge University Press

the cyclic concatenation of C_1C_2 is an edge set of a cycle. It is easy to see that these conditions together imply that the images of C_1 and C_2 are parts of bottom-top geodesics. Hence the image of C is in the cycle space.

Observe that G_1 contains both $\mathcal{H}$ and the vertical automorphism v, and thus the group generated by $\mathcal{H} \cup \{v\}$.

4.4 Grünbaum–Rudin–Andrew-type Averaging

Usage of the averages of the following type for estimates of projection constants goes back at least to Grünbaum [28] and Rudin [52]. It was used in a way similar to the present context by Andrew [4].

Let P be any linear projection of $\ell_1(E(B_n))$ onto $Z(B_n)$. Since G_n is a finite group, which can be regarded as a group of isometries of $\ell_1(E(B_n))$, the following operator is well defined:

$$P_{G_n} := \frac{1}{|G_n|} \sum_{g \in G_n} g^{-1} P g. \tag{4.2}$$

This is also a projection onto $Z(B_n)$, and $\|P_{G_n}\| \leq \|P\|$. It is easy to check that P_{G_n} has the following important property:

$$\forall g \in G_n \quad P_{G_n} g = g P_{G_n} \tag{4.3}$$

We call a projection satisfying (4.3) *invariant* with respect to G_n.

The new twist in the usage of the method in our paper (see Sections 4.6 and 4.7) is that we use it in situations where the invariant projection is not unique. Namely, we observe that although in some situations that we consider, the P_{G_n} obtained by formula (4.2) is not unique (see Section 5.3), it is possible to show, see Lemma 4.8, that there is a collection of vectors in $\ell_1(E(B_n))$ that are mapped to 0 by any P_{G_n} satisfying (4.3). This allows us to show that in the cases considered, $\|P_{G_n}\|$ grows indefinitely as $n \to \infty$; see Section 4.7 for this, and to get the estimate stated in Lemma 4.3.

4.5 Bases in the Spaces $Z(B_n)$

We need to find a basis S_n in the cycle space $Z(B_n)$, $n \geq 1$. Each of the bases which we pick will satisfy the following conditions.

(i) Each element is either a fixed point of v_n, or is supported on a copy of some B_k, $1 \leq k \leq n-1$, and is an element of the corresponding S_k.

(ii) If an element is a fixed point of v_n, then its restriction to any B_k, $1 \leq k \leq n-1$, is a multiple of Δ_k, and thus is a fixed point of the corresponding G_k (see the discussion next to (4.4) below). This condition is void if $n = 1$.

Since $B_1 = B$, we let S_1 be any basis in $Z(B)$. The conditions (i) and (ii) are trivially satisfied; see item (d) in Section 4.1.

Let $e \in E(B_k)$. It is easy to verify that the function

$$E_{m+k-1}E_{m+k-2}\cdots E_k \mathbf{1}_e \in \ell_1(E(B_{m+k})),$$

https://doi.org/10.4153/S0008414X19000087 Published online by Cambridge University Press

which is supported on a copy of B_m that evolved from e, can be written (similarly to (4.1)) as

$$\Delta_m := \frac{1}{D_m K_m} \sum_p \mathbf{1}_p, \tag{4.4}$$

where K_m is the number of distinct geodesics joining the bottom and the top of the copy of B_m mentioned above, $\mathbf{1}_p$ is the indicator function of a bottom-top geodesic in B_m, and the sum is over all distinct bottom-top geodesics. It is easy to see that $\Delta_1 = \Delta$.

Observation 4.5 *Since any element of G_m maps bijectively bottom-top paths in B_m, we see that the function Δ_m is the fixed point of any element of G_m interpreted as acting on the considered copy of B_m.*

Now we pick a basis in $Z(B_n)$, assuming that we already picked a basis S_{n-1} in $Z(B_{n-1})$. The basis consists of two types of vectors.

(I) Vectors that were already picked for S_{n-1} in one of the copies on B_{n-1} in B_n. Recall that $B_n = B_1 \oslash B_{n-1}$; see (1.4).

(II) For each $f \in S_1$, we consider the following function on $B_n = B_1 \oslash B_{n-1}$: its restriction to each of the copies of B_{n-1} is a product of the corresponding Δ_{n-1} and the value of f on the edge from which the considered copy of B_{n-1} has evolved.

Observation 4.6 *Any vector of type* (II) *is a fixed point of any G_{n-1}. The same holds for any G_k, $1 \le k \le n-1$ corresponding to $B_n = B_{n-k} \oslash B_k$ and acting on one of the copies of B_k. For the second statement we need to observe that the restriction of Δ_{n-1} to B_k is a multiple of the corresponding Δ_k.*

First we need to show that conditions (i) and (ii) are satisfied. It is easy to see that the only statement requiring a proof is the fact that the function constructed in the previous paragraph is a fixed point for ν_n.

To see this we observe that the values of f corresponding to copies of B_{n-1} that are mapped one onto another by ν_n are equal, because f is a fixed point of ν and by construction of ν_n. Thus, we get the desired conclusion.

Lemma 4.7 *The set S_n is a basis of the linear space $Z(B_n)$.*

Proof We use induction. For $n = 1$ this is true according to our choice. Suppose that the statement holds for $n-1$, and show that this implies it for n. We need to show two things: completeness and linear independence.

Completeness: (1) If a cycle is contained in one of the B_{n-1}, then it is contained in the linear span of the corresponding S_{n-1} by the induction hypothesis, and we are done because S_n contains that S_{n-1}.

(2) If a cycle C is not contained in any of B_{n-1}, then, after collapsing each of B_{n-1} to the edge of B_1 from which it evolved (according to $B_n = B_1 \oslash B_{n-1}$), we get a nontrivial cycle $\widehat{C}$ in B_1. This cycle is a linear combination of cycles of S_1 (since S_1 is a basis in $Z(B_1)$), so $\widehat{C} = \sum \gamma_i s_i$ for some $\gamma_i \in \mathbb{R}$ and $s_i \in S_1$. Denote the composition $E_{n-1} \cdots E_1$

https://doi.org/10.4153/S0008414X19000087 Published online by Cambridge University Press

by $\mathcal{E}_{n-1}$. We have

$$\mathcal{E}_{n-1}\widehat{C} = \sum \gamma_i \mathcal{E}_{n-1} s_i.$$

The description of the type (II) vectors implies that vectors $\mathcal{E}_{n-1}s_i$ are elements of S_n. Therefore it remains to analyze the difference $C - \mathcal{E}_{n-1}\widehat{C}$.

For each B_{n-1} in B_n (according to $B_n = B_1 \oslash B_{n-1}$), one of the following is true:

- There are no edges of C and no edges of $\mathcal{E}_{n-1}\widehat{C}$ in B_{n-1}.
- There is a path p from the bottom to top of B_{n-1} that is contained in C, and the corresponding part of $\mathcal{E}_{n-1}\widehat{C}$ is Δ_{n-1}.

It remains to observe that that $\Delta_{n-1} - \mathbf{1}_p$ belongs to $Z(B_{n-1})$ (follows from the formula for Δ_{n-1}). Thus the difference $C - \mathcal{E}_{n-1}\widehat{C}$ can be written as a sum of elements of S_{n-1} for those B_{n-1} that contain nontrivial sub-paths of C. As a conclusion, we get that C is in the linear span of S_n.

Linear Independence It is clear that a nontrivial linear combination of vectors of type (I) cannot be equal to 0, since S_{n-1} are linearly independent and B_{n-1} are edge-disjoint.

For this reason, to prove linear independence it is enough to show that a nontrivial linear combination containing vectors of type (II) cannot be 0.

We split a linear combination as $a + b$, where a is a linear combination of vectors of type (I), and b is a linear combination of vectors of type (II). Observe that b can be obtained in the following way. We consider a non-zero vector in $Z(B_1)$ and replace each $\mathbf{1}_e$ used in this vector by the corresponding Δ_{n-1}. Because of this the restriction of b to at least one of B_{n-1} does not belong to $Z(B_{n-1})$. Hence $a + b$ restricted to that B_{n-1} is nonzero, and we are done. ■

4.6 Invariant Projections Annihilate Functions $c(B_1)$ and Their Images Under E_k

Lemma 4.8 *The projection P_{G_n} annihilates all of the functions of the form $c(B_1)$ for some B_1 in B_n, and functions that are obtained from $c(B_1)$ by repeated applications of E_k.*

Proof Let f be some function of the described form in $\ell_1(E(B_n))$ and let B_m, $m \leq n$, be a subgraph of B_n supporting f. It is easy to see that the absolute value of f is equal to the function Δ_m described in (4.4), and that f is positive on edges that are closer to the top of B_m and negative on the edges that are closer to the bottom of B_m.

Suppose, contrary to the statement of the lemma, that $P_{G_n} f = q \neq 0$. Since $q \in Z(B_n)$, it is a linear combination of vectors described in (I)–(II).

It is clear that one of the following is true:

($\leq$) One of the vectors of the basis described in (I)–(II), present in the linear combination representing q, belongs to S_k with $k \leq m$.

($>$) All vectors of the basis described in (I)–(II), present in the linear combination representing q, belong to S_k with $k > m$.

We show, that in each of these cases we get a contradiction with the invariance of P_{G_n}.

Case $(\leq)$**.** Assume that k is the smallest integer with this property. Since it is the smallest integer, all basis elements with nonzero coefficients belonging to S_k are of type (II). Therefore, they correspond to certain elements of S_1, and their linear combination μ (as it is present in the representation of q) corresponds to nonzero element τ of $Z(B_1)$. By condition (f) (on B) there exists a horizontal automorphism g of B_1 such that $g\tau \neq \tau$. Let us consider an automorphism $\widehat{g}$ of S_k *induced* by g in the following way. The automorphism g is a bijection of $E(B_1)$. In $B_k = B_1 \oslash B_{k-1}$ we consider the corresponding bijections of subgraphs B_{k-1} that evolved from those edges. It is clear that $\widehat{g} \in G_k$ and that $\widehat{g}\mu \neq \mu$.

On the other hand, it is clear that $\widehat{g}f = f$. In the case where $k = m$, this follows from the fact that $c(B)$ is a fixed point of all horizontal automorphisms (condition (f)). In the case where $k < m$, this follows from Observation (4.5). We get a contradiction with the fact that P_{G_n} is an invariant projection (see (4.3)), because

$$\begin{aligned} P_{G_n} f &= P_{G_n}\widehat{g}f = \widehat{g}P_{G_n} f = \widehat{g}q = \widehat{g}(\mu + (q-\mu)) \\ &= \widehat{g}\mu + \widehat{g}(q-\mu) = \widehat{g}(\mu) + (q-\mu) \neq \mu + q - \mu = q, \end{aligned}$$

where we used the fact that elements of the basis S_n used in the decomposition of $q - \mu$ are either edge-disjoint with the copy of B_k on which μ is supported or are proportional to Δ_k on that B_k. In either case, $(q-\mu)$ is a fixed point of $\widehat{g}$.

Case $(>)$**.** In this case, by Observation (4.5), any function used in the decomposition of q with respect to the basis S_n is a fixed point of ν_m, which was defined in **(B)**.

On the other hand, $\nu_m f = -f$, by the definitions of ν_m and f. This contradicts the fact that P_{G_n} is an invariant projection (see (4.3)), because we get

$$-P_{G_n} f = P_{G_n}\nu_m f = \nu_m P_{G_n} f = \nu_m q = q = P_{G_n} f. \qquad \blacksquare$$

4.7 Combining Everything

Proof of Theorem 4.2 Let us show, using Lemma 4.3, that in order to prove Theorem 4.2 it suffices to show that for each $r \in \mathbb{N}$, there exists $n = n(r) \in \mathbb{N}$, $C_r \in Z(B_n)$, and a linear combination A_r of vectors of the forms $c(B_1)$ and their images under $\{E_n\}$, such that

$$\|C_r + A_r\| = 1$$

and

$$\|C_r\| \geq 1 + \frac{\alpha(r-1)}{2}, \tag{4.5}$$

where $\alpha > 0$ is the number introduced in item (g) of Section 4.1, and to find a suitable estimate for the corresponding $n(r)$ in terms of r.

In fact, for every projection $P\colon \ell_1(E(B_n)) \to Z(B_n)$ we get

$$\|P\| \geq \|P_{G_n}\| \geq \|P_{G_n}(C_r + A_r)\| \overset{\text{(Lemma 4.8)}}{=} \|C_r\| \geq 1 + \frac{\alpha(r-1)}{2}.$$

This inequality, as we shall see later, leads to the estimate of the projection constant stated in Lemma 4.3.

https://doi.org/10.4153/S0008414X19000087 Published online by Cambridge University Press

Case $r = 1$. We let C_1 be any ℓ_1-normalized element of S_1 (use non-triviality); $A_1 = 0$. Everything is obvious.

Inductive step. Suppose that we have already constructed C_r and A_r in some $B_{n(r)}$.

We apply $E_{n(r)}$ to $C_r + A_r$. Observe that $E_{n(r)}$ maps the cycle space into the cycle space, and preserves the desired form of the function A_r. Observe that $C_r + A_r$, as an element of $\ell_1(E(B_{n(r)}))$ is a linear combination of edges. Therefore $E_{n(r)}(C_r + A_r)$ is of the form $\sum_{e\in E(B_{n(r)})} a_{e,1}E_{n(r)}\mathbf{1}_e$, where $a_{e,1}$ are real numbers. The functions $E_{n(r)}\mathbf{1}_e$ are of the form Δ (see (4.1)), supported on different copies of B_1, recall that

$$B_{n(r)+1} = B_{n(r)} \oslash B_1 \tag{4.6}$$

(see (1.4)). We let $C_r^1 = E_{n(r)}C_r$ and let

$$A_r^1 = E_{n(r)}A_r + \sum_{e\in E(B_{n(r)})} a_{e,1}c(B_1),$$

where $c(B_1)$ is taken on the corresponding copy of B_1, according to (4.6). It is easy to see that $\|C_r^1 + A_r^1\| = 1$, and its support is exactly half (in many respects) of the support of $E_{n(r)}(C_r + A_r)$.

We repeat the procedure for C_r^1 and A_r^1 instead of C_r and A_r. We do this t times, and get the functions which we denote C_r^t and A_r^t.

Some observations:

- The function C_r^t is an image of C_r under the composition $E_{n(r)+t-1}\cdots E_{n(r)}$.
- The function A_r^t is a linear combination of $E_{n(r)+t-1}\cdots E_{n(r)}A_r$ and images of $c(B_1)$ under some compositions of E_k.

Next, we perform a somewhat different procedure. Namely, we write $E_{n(r)+t}(C_r^t + A_r^t)$ in the form $\sum_{e\in E(B_{n(r)+t})} a_{e,t+1}E_{n(r)+t}\mathbf{1}_e$, where $a_{e,t+1}$ are real numbers. The functions $E_{n(r)+t}\mathbf{1}_e$ are multiples of Δ, supported on different copies of B_1, recall that $B_{n(r)+t+1} = B_{n(r)+t} \oslash B_1$. Now we let

$$A_{r+1} = E_{n(r)+t}(A_r^t),$$
$$C_{r+1} = E_{n(r)+t}(C_r^t) + \sum_{e\in E(B_{n(r)+t})} a_{e,t+1}d(B),$$

where $d(B)$ is the function defined in item (g) of Section 4.1 and supported on the corresponding copy of B_1.

It is clear from the definition of $d(B)$ that $\|C_{r+1} + A_{r+1}\| = 1$. It is also clear that $C_{r+1} \in Z(B_{n(r)+t+1})$, and A_{r+1} is of the desired form.

Observe that since $\|\sum a_{e,t+1}1_e\| = 1$, we have $\|\sum a_{e,t+1}d(B)\| = \alpha$ (see item (g) in Section 4.1). Our construction is such that the norm of the part of $E_{n(r)+t}\cdots E_{n(r)}C_r$ supported in the support of $\sum_{e\in E(B_{n(r)+t})} a_{e,t+1}d(B)$ is $\frac{1}{2^t}\|C_r\|$. Therefore, if we pick t in such a way that

$$\frac{1}{2^t}\|C_r\| < \frac{\alpha}{4}, \tag{4.7}$$

https://doi.org/10.4153/S0008414X19000087 Published online by Cambridge University Press

we get

$$\begin{aligned}\|C_{r+1}\| &= \Big\|E_{n(r)+t}\cdots E_{n(r)}C_r + \sum_{e\in E(B_{n(r)+t})} a_{e,t+1}d(B)\Big\| \\ &\geq \|E_{n(r)+t}\cdots E_{n(r)}C_r\| + \Big\|\sum_{e\in E(B_{n(r)+t})} a_{e,t+1}d(B)\Big\| - 2\frac{\alpha}{4} \\ &\overset{(4.5)}{\geq} 1 + \frac{\alpha(r-1)}{2} + \alpha - \frac{\alpha}{2} = 1 + \frac{\alpha r}{2}.\end{aligned}$$

It remains to find an estimate for n in terms of r. To achieve the condition (4.7) for $r \geq 2$, we need to pick $t \geq C \ln r$ for some $C > 0$.

This leads to the estimate $\lambda(Z(B_n), \ell_1(E(B_n))) \geq ck$ if $n \geq Ck \ln k$, where $c > 0$, $C < \infty$ (the constants in these statements do not have to be the same).

It is easy to see that this estimate implies

$$\lambda\big(Z(B_n), \ell_1(E(B_n))\big) \geq \frac{cn}{\ln n}. \qquad \blacksquare$$

5 Consequences for Multibranching Diamond Graphs and Laakso Graphs

Our next goal is to show that diamond graphs and Laakso graphs satisfy the conditions listed in Section 4.1.

5.1 Multibranching Diamond Graphs

Condition (a) in the case where B is $K_{2,n}$, $n \geq 2$, with the top and the bottom being the vertices of the part containing two vertices is obvious.

Condition (b) is clear.

For condition (c) we choose the automorphism in such a way that it maps each bottom-top path onto itself.

With this choice of v the condition (d) is easy to check.

Condition (e) is clearly satisfied.

In condition (f), a nonzero element of $Z(K_{2,n})$ cannot be a fixed point of $\mathcal{H}$ because (according to the directions chosen on edges) each non-zero element of $Z(K_{2,n})$ has bottom-top paths on which the value is positive and bottom-top paths on which the value is negative.

The second part of condition (f) holds because any horizontal automorphism maps edges that are closer to the top (bottom) to edges that are closer to the top (bottom).

Finally, condition (g) is satisfied, because we consider $n \geq 2$ and s_1 (element of the basis listed above) is an example of a nontrivial cycle in $Z(K_{2,n})$. The value of α is $\frac{2(n-1)}{n}$.

5.2 Laakso Graphs

Condition (a) in the case where B is $\mathcal{L}_1$ with the natural choice of the top and the bottom is obvious.

Condition (b) is clear.

https://doi.org/10.4153/S0008414X19000087 Published online by Cambridge University Press

For condition (c), we choose the automorphism ν that maps each bottom-top path onto itself.

In condition (d), there is only one cycle in $\mathcal{L}_1$, it is obviously the fixed point of the chosen automorphism of $\mathcal{L}_1$.

Condition (e) is clearly satisfied.

The first part of condition (f) holds, because, by the choice of the directions of edges, any nonzero element of $Z(\mathcal{L}_1)$ has positive value on one side and negative value on the other side, and thus is mapped onto its negative by a nontrivial element of $\mathcal{H}$.

The second part of condition (f) holds because any horizontal automorphism maps edges that are closer to the top (bottom) to edges that are closer to the top (bottom).

Condition (g) is clearly satisfied. The value of α is $\frac{1}{2}$.

5.3 Non-uniqueness of Invariant Projections of $\ell_1(E(\mathcal{L}_2))$ onto $Z(\mathcal{L}_2)$

Our main goal in this section is to show that for Laakso graphs, there is no uniqueness of invariant projections. It is clear that one of the invariant projections is the orthogonal projection onto $Z(\mathcal{L}_2)$ in $\ell_2(E(\mathcal{L}_2))$. So it is enough to construct an invariant projection that is not orthogonal.

Proposition 5.1 *There exists an invariant linear projection of $\ell_2(E(\mathcal{L}_2))$ onto $Z(\mathcal{L}_2)$ that is different from the orthogonal projection.*

Proof We consider the following projection: It is like the orthogonal projection on the top and bottom "tails" of $\mathcal{L}_2$ and is different only in the central part. In the central part there are edges that belong to the 16-cycle only and edges that belong also to 4-cycles.

We introduce the following functions in $\ell_1(E(\mathcal{L}_2))$ supported on the central part of $\mathcal{L}_2$:

(1) Indicator functions χ_C of cycles of length 4 (see (1.2)) directed counterclockwise, so they have values 1 on the right-hand sides and values -1 on the left sides.

(2) The function $F = \frac{F_1+F_2}{2}$, where F_1 is the indicator function of the directed counterclockwise "outer cycle" of length 16 and F_2 is the indicator function of the directed counterclockwise "inner cycle" of length 16.

We consider the projection that acts in the following way:

(a) It maps each edge that is in the "16-cycle only" to $\frac{\theta}{8}F$, where $\theta = 1$ on the right half and $\theta = -1$ on the left half.

(b) It maps each edge that is "both in the 16-cycle and 4-cycle" onto the $\frac{\theta}{4}\chi_C$, where C is the corresponding 4-cycle and $\theta = 1$ on the right side and $\theta = -1$ on the left-hand side.

It is straightforward to check that this projection is invariant in the sense of (4.3) and is different from the orthogonal projection. ■

https://doi.org/10.4153/S0008414X19000087 Published online by Cambridge University Press

6 Lipschitz-free Spaces on Diamond Graphs — More Precise Results using Haar Functions

In this section, we present an alternative self-contained proof of our results for the binary diamond graphs D_n. This proof uses the Haar system for $L_1[0,1]$ and makes an interesting connection with some open problems concerning the even levels of the Haar system. At the end of this section, we extend the proof to handle the multi-branching diamond graphs as well.

We begin by reformulating the definition of the binary diamond graphs in order to use the Haar system. For $n \geq 2$, we shall consider D_n as consisting of four copies of D_{n-1}, namely "top left", denoted TL_n, "bottom left", denoted BL_n, "bottom right", denoted BR_n, and "top right", denoted TR_n. In this identification, the bottom vertex of TL_n coincides with the top vertex of BL_n, etc.

We identify the edge space of D_n, denoted $\ell_1(D_n)$, with a certain subspace of $L_1[0,1]$. This identification is recursive. We identify the edge vectors of $\ell_1(D_1)$ with the functions $4 \cdot \mathbf{1}_{((i-1)/4, i/4]}$ for $1 \leq i \leq 4$, which are disjointly supported unit vectors in $L_1[0,1]$, ordering the edges $i = 1, \dots, 4$ counterclockwise from the top vertex. Now suppose that $n \geq 2$ and that $\ell_1(D_{n-1})$ has been identified with a subspace of $L_1[0,1]$. For a function $f \in L_1[0,1]$ we denote by Qf the function which is 0 in $\left(\frac{1}{4}, 1\right]$ and is given by $(Qf)(t) = 4f(4t)$ for $t \in \left[0, \frac{1}{4}\right]$. It is clear that Q is an isometric embedding of $L_1[0,1]$ into itself. Then we identify $\ell_1(TL_n)$ with $Q(\ell_1(D_{n-1}))$, and identify $\ell_1(BL_n)$, $\ell_1(BR_n)$, and $\ell_1(TR_n)$, with copies of $\ell_1(TL_n)$ translated by $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ to the right, respectively. It follows that the edge vectors of $\ell_1(D_n)$ are the functions $4^n \cdot \mathbf{1}_{((i-1)/4^n, i/4^n]}$ for $1 \leq i \leq 4^n$, which are disjointly supported unit vectors in $L_1[0,1]$.

Let us now determine the subspace of $L_1[0,1]$ that corresponds under this identification to the cycle space of D_n, denoted $Z(D_n)$. First, let us recall the definition of the Haar system $(h_i)_{i \geq 0}$. We define $h_0 := \mathbf{1}_{(0,1]}$, and for $n \geq 0$ and $0 \leq i \leq 2^n - 1$,

$$h_{2^n+i} := \mathbf{1}_{(i/2^n, (2i+1)/2^{n+1}]} - \mathbf{1}_{((2i+1)/2^{n+1}, (i+1)/2^n]}.$$

Let $H_n := \{h_i : 2^n \leq i \leq 2^{n+1} - 1\}$ be the collection of all 2^n Haar functions on the same level with support of length 2^{-n}. Let e_n be the cycle vector corresponding to the "large outer cycle" of D_n. To understand the pattern for e_n, first we calculate e_1, e_2 and e_3. Clearly,

$$e_1 = 4(\mathbf{1}_{[0,1/2]} - \mathbf{1}_{[1/2,1]}) = 4h_1,$$

and

$$\begin{aligned} e_2 &= 16(\mathbf{1}_{[0,1/8]} + \mathbf{1}_{[2/8,3/8]} - \mathbf{1}_{[5/8,6/8]} - \mathbf{1}_{[7/8,1]}) \\ &= 8(h_1 + h_4 + h_5 + h_6 + h_7) \\ &= 2e_1 + 8\Big(\sum_{h \in A_2} h\Big), \end{aligned} \tag{6.1}$$

https://doi.org/10.4153/S0008414X19000087 Published online by Cambridge University Press

where $A_2 = \{h \in H_2\colon \operatorname{supp} h \subseteq \operatorname{supp} e_1\}$. Note that

$$\begin{aligned} e_3 &= 64\big[\big(\mathbf{1}_{[0,1/32]} + \mathbf{1}_{[2/32,3/32]} + \mathbf{1}_{[8/32,9/32]} + \mathbf{1}_{[10/32,11/32]}\big) \\ &\quad - \big(\mathbf{1}_{[21/32,22/32]} + \mathbf{1}_{[23/32,24/32]} + \mathbf{1}_{[29/32,30/32]} + \mathbf{1}_{[31/32,1]}\big)\big] \\ &= 16\big[(h_1 + h_4 + h_5 + h_6 + h_7) \\ &\quad + 2(h_{16} + h_{17} + h_{20} + h_{21} + h_{26} + h_{27} + h_{30} + h_{31})\big] \\ &= 2e_2 + 32\Big(\sum_{h \in A_3} h\Big), \end{aligned} \tag{6.2}$$

where $A_3 = \{h \in H_4\colon \operatorname{supp} h \subseteq \operatorname{supp} e_2\}$. The passage from e_{n-1} to e_n in the general case is analogous to the passage from e_2 to e_3 above and is given by a procedure that we now describe. let I be a maximal dyadic subinterval of $\operatorname{supp} e_{n-1}$. Let I_1, I_2, I_3, and I_4 be the first, second, third, and fourth quarters of I ordered from left to right. To get e_n from e_{n-1}, if I is contained in the support of the *positive* part of e_{n-1}, then we replace $\mathbf{1}_I$ in the expression for e_{n-1} by $\mathbf{1}_{I_1} + \mathbf{1}_{I_3}$, and if I is contained in the support of the *negative* part of e_{n-1}, then we replace $-\mathbf{1}_I$ in the expression for e_{n-1} by $-(\mathbf{1}_{I_2} + \mathbf{1}_{I_4})$. Expressing e_n in terms of Haar functions, it follows, by analogy with (6.1) and (6.2) above, that

$$e_n = 2e_{n-1} + 2^{2n-1} \sum_{h \in A_n} h, \tag{6.3}$$

where $A_n = \{h \in H_{2n-2}\colon \operatorname{supp} h \subseteq \operatorname{supp} e_{n-1}\}$. Iterating (6.3) and recalling that $e_1 = 4h_1$, we get

$$e_n - 2^{n+1} h_1 \in \operatorname{span}\Big(\bigcup_{k=1}^{n-1} H_{2k}\Big). \tag{6.4}$$

Lemma 6.1 *For all $n \geq 1$,*

$$\ell_1(D_n) = \operatorname{span}\Big(\{h_0\} \cup \Big(\bigcup_{k=0}^{2n-1} H_k\Big)\Big),$$

$$Z(D_n) = \operatorname{span}\Big(\bigcup_{k=0}^{n-1} H_{2k}\Big). \tag{6.5}$$

Proof The description of $\ell_1(D_n)$ follows from the observation above that the edge vectors of $\ell_1(D_n)$ are the functions $4^n \mathbf{1}_{[(i-1)/4^n, i/4^n]}$ for $1 \leq i \leq 4^n$.

We prove (6.5) by induction. Note that

$$Z(D_1) = \operatorname{span}(\{h_1\}) = \operatorname{span}(H_0),$$

which verifies the base case $n = 1$. So suppose that $n \geq 2$ and that the result holds for $n - 1$. Note that

$$Z(D_n) = \operatorname{span}\big(Z(TL_n) \cup Z(BL_n) \cup Z(BR_n) \cup Z(TR_n) \cup \{e_n\}\big). \tag{6.6}$$

Recall that TL_n, BL_n, BR_n, and TR_n are translated and dilated copies of D_{n-1} on the intervals $[(i-1)/4, i/4]$ for $1 \leq i \leq 4$. Hence, $Z(TL_n)$, $Z(BL_n)$, $Z(BR_n)$, and

$Z(TR_n)$ are translated and dilated copies of $Z(D_{n-1})$ on the intervals $[(i-1)/4, i/4]$ for $1 \le i \le 4$. Applying the inductive hypothesis to $Z(D_{n-1})$, it follows that

$$\operatorname{span}\big(Z(TL_n) \cup Z(BL_n) \cup Z(BR_n) \cup Z(TR_n)\big) = \operatorname{span}\Big(\bigcup_{k=1}^{n-1} H_{2k}\Big). \tag{6.7}$$

Finally, from (6.6), (6.7), and (6.4), we get

$$\begin{aligned} Z(D_n) &= \operatorname{span}\Big(\{e_n\} \cup \Big(\bigcup_{k=1}^{n-1} H_{2k}\Big)\Big) \\ &= \operatorname{span}\Big(\{h_1\} \cup \Big(\bigcup_{k=1}^{n-1} H_{2k}\Big)\Big) = \operatorname{span}\Big(\bigcup_{k=0}^{n-1} H_{2k}\Big). \end{aligned}$$ ∎

Remark 6.2 Note that $Z(D_n)$ has dimension $\sum_{k=0}^{2n-2} 4^k = (4^{2n-1}-1)/3$. This can also be seen directly without using Lemma 6.1, since (6.6) clearly implies that $\dim Z(D_n) = 4 \dim Z(D_{n-1}) + 1$. Using this observation that the spaces have the same dimension, it suffices to show that $Z(D_n) \subseteq \operatorname{span}(\bigcup_{k=0}^{n-1} H_{2k})$, which follows from (6.4) and (6.7). Thus, the proof can be concluded slightly differently.

Our next goal is to prove that $Z(D_n)$ is not well-complemented in $\ell_1(D_n)$. This essentially follows from a result of Andrew [4]. (Note that the idea of using the average over the group of isometries to estimate norms of projections goes back at least to Grünbaum [28] and Rudin [52].) For completeness we present a slight generalization of Andrew's elegant argument. Let $X_n = \operatorname{span}(\{h_i : 0 \le i \le 2^{n+1}-1\}) = \operatorname{span}(\{h_0\} \cup (\bigcup_{k=0}^{n} H_k))$. Let $(\cdot,\cdot)$ denote the usual inner product in $L_2[0,1]$. Orthogonality will refer to this inner product.

Suppose $i \ge 1$ and that $h_i \in H_k$. Define a linear isomorphism $g_i : X_n \to X_n$ by

$$(g_i f)(t) = \begin{cases} f(t), & t \notin \operatorname{supp} h_i, \\ f(t+2^{-k-1}), & t \in h_i^{-1}(1), \\ f(t-2^{-k-1}), & t \in h_i^{-1}(-1) \end{cases}$$

for all $f \in X_n$. Suppose now that $\|\cdot\|$ is any norm on X_n with the property that each g_i acts as a linear isometry of $(X_n, \|\cdot\|)$. For our purposes, $\|\cdot\|$ will be the usual norm of $L_1[0,1]$ or of $L_\infty[0,1]$. Let G be the group of isometries generated by $(g_i)_{i\ge1}$. Note that G is finite.

In the next proposition it is convenient to set $H_{-1} := \{h_0\}$.

Lemma 6.3 *Let A be any nonempty subset of $\{-1,0,1,\dots,n\}$ and let P be any linear projection on $(X_n, \|\cdot\|)$ with range $Y := \operatorname{span}(\bigcup_{k\in A} H_k)$. Then $\|P\| \ge \|P_Y\|$, where P_Y is the orthogonal projection onto Y.*

Proof Let

$$Q = \frac{1}{|G|} \sum_{g\in G} g^{-1} P g.$$

Clearly $\|Q\| \le \|P\|$. Moreover, Q is a projection onto Y, since $g(Y) = Y$ for all $g \in G$. It suffices to show that $Q = P_Y$. The proof of this makes use of the following observations:

https://doi.org/10.4153/S0008414X19000087 Published online by Cambridge University Press

(1) $gQ = Qg$ for all $g \in G$.
(2) $g_i h_i = -h_i$ for all $i \geq 1$.
(3) $(g_i f, h_i) = -(f, h_i)$ for all $f \in X_n$ and for all $i \geq 1$.
(4) If $0 \leq i < j$ and $\operatorname{supp} h_j \subset \operatorname{supp} h_i$, then $(g_j f, h_i) = (f, h_i)$ for all $f \in X_n$.
(5) If $i > j \geq 0$ or if h_i and h_j are disjointly supported, then $g_i h_j = h_j$.

Suppose that $h_j \notin Y$. We have to show that $Qh_j = 0$. Since Q is a projection onto Y, it suffices to show that if $h_i \in Y$ then $(Qh_j, h_i) = 0$. If $0 \leq i < j$ and $\operatorname{supp} h_j \subset \operatorname{supp} h_i$ then

$$\begin{aligned}(Qh_j, h_i) &= (g_j Q h_j, h_i) && \text{(by (4))}\\ &= (Q g_j h_j, h_i) && \text{(by (1))}\\ &= -(Qh_j, h_i) && \text{(by (2))}.\end{aligned}$$

Hence, $(Qh_j, h_i) = 0$ in this case. Now suppose that $i > j \geq 0$ or that h_i and h_j are disjointly supported. Then

$$\begin{aligned}(Qh_j, h_i) &= (Q g_i h_j, h_i) && \text{(by (5))}\\ &= (g_i Q h_j, h_i) && \text{(by (1))}\\ &= -(Qh_j, h_i) && \text{(by (3))}.\end{aligned}$$

So $(Qh_j, h_i) = 0$. ∎

Lemma 6.4 *Let P be a projection from $\ell_1(D_n)$ onto $Z(D_n)$. Then $\|P\| \geq (2n+1)/3$.*

Proof By Theorem 6.1, we have $\ell_1(D_n) = \operatorname{span}(\{h_0\} \cup (\bigcup_{k=0}^{2n-1} H_k))$ and $Z(D_n) = \operatorname{span}(\bigcup_{k=0}^{n-1} H_{2k})$. By Lemma 6.3, it suffices to show that the orthogonal projection Q satisfies $\|Q\| \geq (2n+1)/3$. This is well known, but for completeness we recall the proof. Consider

$$f = h_0 + h_1 + 2h_2 + 2^2 h_4 + \cdots + 2^{2n-2} h_{2^{2n-2}}.$$

Note that f is the sum over the first Haar functions (normalized in $L_1[0,1]$) in each level. Then

$$Qf = h_1 + 2^2 h_4 + 2^4 h_{16} + \cdots + 2^{2n-2} h_{2^{2n-2}}.$$

It is easily seen that $\|f\| = 1$ and $\|Qf\| \geq (2n+1)/3$. ∎

Theorem 6.5 *The Banach–Mazur distance d from the Lipschitz-free space $LF(D_n)$ to the ℓ_1^N space of the same dimension satisfies*

$$4n + 4 \geq d \geq (2n+1)/3.$$

Proof Let $X_n = \operatorname{span}(\{h_0\} \cup (\bigcup_{k=0}^{2n-1} H_k))$. Using the inner product in $L_2[0,1]$ we may identify $\ell_1(D_n)^*$ with $(X_n, \|\cdot\|_\infty)$. Under this identification, $Z(D_n)^\perp = \operatorname{span}(\{h_0\} \cup (\bigcup_{k=1}^{n} H_{2k-1}))$. A calculation similar to that of the previous result, but now using the L_∞ norm, shows that any projection P from $(X_n, \|\cdot\|_\infty)$ onto $Z(D_n)^\perp$ satisfies $\|P\| \geq (2n+1)/3$. Since an ℓ_∞^N space is contractively complemented in any superspace, it follows that the Banach–Mazur distance from $LF(D_n)^* = Z(D_n)^\perp$ to an ℓ_∞^N space is at least $(2n+1)/3$. Dualizing again gives $d \geq (2n+1)/3$.

To get the upper estimate, note that $\{h_0\} \cup (\bigcup_{k=1}^{n} H_{2k-1})$ is a monotone basis for $LF(D_n)$ in the quotient norm of $LF(D_n)$ and that $\{2^{2k-1}h_i : h_i \in H_{2k-1}\}$ is 2-equivalent to the unit vector basis of the ℓ_1^N space of the same dimension. Let $x \in LF(D_n)$ and write $x = \sum_{k=0}^{n} x_k$, where $x_0 \in \operatorname{span}(\{h_0\})$ and $x_k \in \operatorname{span}(H_{2k-1})$. Then

$$\sum_{k=0}^{n} \|x_k\| \geq \|x\| \geq \frac{1}{2} \max_{0\leq k\leq n} \|x_k\| \geq \frac{1}{2n+2} \sum_{k=0}^{n} \|x_k\|,$$

which gives $d \leq 4n + 4$. ∎

Problem 6.6 *Do $\{LF(D_n)\}$ admit embeddings into ℓ_1 with uniformly bounded distortions?*

Problem 6.7 *Do $\{\ell_\infty^k\}$ admit embeddings into $\{LF(D_n)\}$ with uniformly bounded distortions?*

Problem 6.8 *Are $\{Z(D_n)\}_{n=1}^{\infty}$ uniformly isomorphic to $\{\ell_1^{k(n)}\}_{n=1}^{\infty}$ of the corresponding dimensions? This is a finite version of the longstanding open question as to whether the even levels of the Haar system in $L_1[0,1]$ span a subspace isomorphic to L_1 [45].*

Remark 6.9 It is curious that the subspaces generated by all the even/odd levels of the Haar functions appear in the study of quasi-greedy basic sequences in $L_1[0,1]$. The notion of quasi-greedy bases, which generalizes unconditional bases, was introduced by S. Konyagin and V. Temlyakov [38]; see also [18]. Although the Haar basis is not quasi-greedy in $L_1[0,1]$ [19], S. Gogyan [27] showed the subsequence consisting of all Haar functions from the even/odd levels is a quasi-greedy subsequence in $L_1[0,1]$.

Finally, we generalize the argument to handle the multi-branching diamond graphs $D_{n,k}$. The proof is similar to the case $k = 2$, so we omit some of the details.

Theorem 6.10 *The Banach–Mazur distance $d_{n,k}$ from the Lipschitz-free space $LF(D_{n,k})$ to the ℓ_1^N space of the same dimension satisfies*

$$4n + 4 \geq d_{n,k} \geq \frac{k-1}{2k} n.$$

Proof It will be convenient to identify the edge space of $D_{n,k}$ with a subspace of $L_1[0,1]$ as follows. For $n = 1$ and $1 \leq j \leq k$, we identify the pair of edge vectors of the j^{th} path of length 2 from u to v with the L_1-normalized indicator functions $2k\mathbf{1}_{(j-1)/k,(2j-1)/(2k)]}$ and $2k\mathbf{1}_{((2j-1)/(2k),j/k]}$. For $n \geq 2$, the edge space of $D_{n,k}$ is obtained from that of $D_{n,k-1}$ by subdividing the intervals corresponding to edge vectors of $D_{n,k-1}$ into $2k$ subintervals each of length $(2k)^{-n}$. Each of the k consecutive disjoint pairs of L_1-normalized indicator functions of the subintervals corresponds to each pair of edge vectors of the k paths of length 2 from the u and v vertices of the copy of $D_{1,k}$ that replaces the edge vector of $D_{n-1,k}$ corresponding to the interval of length $(2k)^{n-1}$ that is subdivided. We have now identified the edge vectors of $D_{n,k}$ with the normalized indicator functions

$$e_{n,j} = (2k)^n \mathbf{1}_{((j-1)/(2k)^n, j/(2k)^n]} \quad (1 \leq j \leq (2k)^n).$$

Arguing as in the case $k = 2$, one can show that a basis for the cycle space corresponds to the L_∞-normalized system $\bigcup_{i=1}^n \{g_{i,j}: 1 \le j \le (2k)^{i-1}(k-1)\}$, where, setting $j = a(k-1) + b$ with $0 \le a \le (2k)^{i-1} - 1$ and $1 \le b \le k-1$,

$$g_{i,j} = (2k)^{-i}(e_{a2^k+2b-1} + e_{a2^k+2b} - e_{a2^k+2b+1} - e_{a2^k+2b}).$$

Note that for $k = 2$, this agrees with the previous description of the cycle space of $D_{n,2}$ in terms of alternate levels of the Haar system. But for $k \ge 3$, note that $g_{i,j}$ *overlaps* with $g_{i,j+1}$ when $b \le k-2$, and hence this is not an orthogonal basis.

Recall that the *cut space* of a finite unweighted graph G is defined as the orthogonal complement in $\ell_2(E(G))$ of the cycle space.

It is easy to see that an orthogonal basis for the cut space corresponds to the L_∞-normalized system $\{h_0\} \cup (\bigcup_{i=1}^n \{h_{i,j}: 1 \le j \le (2k)^i/2\}$, where $h_0 = \mathbf{1}_{[0,1]}$, and

$$h_{i,j} = (2k)^{-i}(e_{i,2j-1} - e_{i,2j}).$$

Let $P_{n,k}$ denote the orthogonal projection from the edge space of $D_{n,k}$ onto the cut space. Then

$$P_{n,k}(e_{n,1}) = h_0 + \frac{1}{2}\sum_{i=1}^n (2k)^i h_{i,1}.$$

Note that for $1 \le i \le n$,

$$P_{n,k}(e_{n,1})|_{(2(2k)^{-i-1},(2k)^{-i}]} = 1 + \frac{1}{2}\sum_{j=1}^i (2k)^j \ge \frac{(2k)^i}{2}.$$

Hence,

$$\|P_{n,k}\|_1 \ge \|P_{n,k}(e_{n,1})\|_1 \ge \sum_{i=1}^n \Big(1 - \frac{1}{k}\Big)(2k)^{-i}\frac{(2k)^i}{2} = \Big(1 - \frac{1}{k}\Big)\frac{n}{2}.$$

Since $P_{n,k}$ is self-adjoint, it follows that $P_{n.k}$ is a projection from the edge space $E(D_{n,k})$, equipped with the L_∞ norm, onto the cut space $Z(D_{n,k})^\perp$ satisfying $\|P_{n,k}\|_\infty \ge (1 - 1/k)n/2$.

As in the case $k = 2$, one can show that if P is any projection onto the cut space (in the L_∞ norm), then $\|P\|_\infty \ge \|P_{n,k}\|_\infty$. By duality, as in the case $k = 2$, it follows that $d_{n,k} \ge (1 - 1/k)n/2$.

To get the upper estimate, note that $\{h_0\} \cup (\bigcup_{i=1}^n \{h_{i,j}: 1 \le j \le (2k)^i/2\})$ is a monotone basis for $LF(D_{n,k})$ in the quotient norm of $LF(D_{n,k})$ and that, for each i, $(h_{i,j})_{j=1}^{(2k)^i}$ is 2-equivalent to the unit vector basis of the ℓ_1^N space of the same dimension. As in the case $k = 2$, this gives $d_{n,k} \le 4n + 4$. ∎

Acknowledgments The authors would like to thank the referee for careful reading of the paper and numerous corrections.

References

[1] A. Andoni, K. Do Ba, P. Indyk, and D. Woodruff, *Efficient sketches for earth-mover distance, with applications*. In: *2009 50th Annual IEEE Symposium on Foundations of Computer Science FOCS 2009, IEEE Computer Soc.*, Los Alamitos, CA, 2009, pp. 324–330. https://doi.org/10.1109/FOCS.2009.25

[2] A. Andoni, P. Indyk, and R. Krauthgamer, *Earth mover distance over high-dimensional spaces*. In: *Proceedings of the Nineteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, ACM, New York, 2008, pp. 343–352.

[3] A. Andoni, A. Naor, and O. Neiman, *Snowflake universality of Wasserstein spaces*. Ann. Sci. Éc. Norm. Supér. (4) 51(2018), no. 3, 657–700. https://doi.org/10.24033/asens.2363

[4] A. D. Andrew, *On subsequences of the Haar system in* $C(\Delta)$. Israel J. Math. 31(1978), 85–90. https://doi.org/10.1007/BF02761382

[5] R. F. Arens and J. Eells Jr., *On embedding uniform and topological spaces*. Pacific J. Math. 6(1956), 397–403.

[6] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis*, Vol. 1. American Mathematical Society Colloquium Publications, 48, American Mathematical Society, Providence, RI, 2000.

[7] N. L. Biggs, *Algebraic potential theory on graphs*. Bull. London Math. Soc. 29(1997), no. 6, 641–682. https://doi.org/10.1112/S0024609397003305

[8] J. A. Bondy and U. S. R. Murty, *Graph theory*, Graduate Texts in Mathematics, 244, Springer, New York, 2008. https://doi.org/10.1007/978-1-84628-970-5

[9] J. Bourgain, *On Lipschitz embedding of finite metric spaces in Hilbert space*. Israel J. Math. 52(1985), no. 1–2, 46–52. https://doi.org/10.1007/BF02776078

[10] J. Bourgain, *The metrical interpretation of superreflexivity in Banach spaces*. Israel J. Math. 56(1986), no. 2, 222–230. https://doi.org/10.1007/BF02766125

[11] J. Bourgain and S. J. Szarek, *The Banach-Mazur distance to the cube and the Dvoretzky-Rogers factorization*. Israel J. Math. 62(1988), no. 2, 169–180. https://doi.org/10.1007/BF02787120

[12] G. Carlsson and F. Mémoli, *Characterization, stability and convergence of hierarchical clustering methods*. J. Mach. Learn. Res. 11(2010), 1425–1470.

[13] M. Cúth and M. Doucha, *Lipschitz-free spaces over ultrametric spaces*. Mediterr. J. Math. 13(2016), no. 4, 1893–1906. https://doi.org/10.1007/s00009-015-0566-7

[14] M. Cúth, M. Doucha, and P. Wojtaszczyk, *On the structure of Lipschitz-free spaces*. Proc. Amer. Math. Soc. 144 no. 9, 3833–3846. https://doi.org/10.1090/proc/13019

[15] A. Dalet, *Free spaces over some proper metric spaces*. Mediterr. J. Math. 12(2015), no. 3, 973–986. https://doi.org/10.1007/s00009-014-0455-5

[16] R. Diestel, *Graph theory*, Fifth ed., Graduate Texts in Mathematics, 173, Springer, Berlin, 2017. https://doi.org/10.1007/978-3-662-53622-3

[17] S. J. Dilworth, N. J. Kalton, and D. Kutzarova, *On the existence of almost greedy bases in Banach spaces*. Studia Math. 159(2003), 67–101. https://doi.org/10.4064/sm159-1-4

[18] S. J. Dilworth, N. J. Kalton, D. Kutzarova, and V. N. Temlyakov, *The thresholding greedy algorithm, greedy basis, and duality*. Constr. Approx. 19(2003), no. 4, 575–597. https://doi.org/10.1007/s00365-002-0525-y

[19] S. J. Dilworth, D. Kutzarova, and P. Wojtaszczyk, *On approximate* ℓ_1 *systems in Banach spaces*. J. Approx. Theory 114(2002), 214–241. https://doi.org/10.1006/jath.2001.3641

[20] R. L. Dobrushin, *Definition of a system of random variables by means of conditional distributions*. Teor. Veroyatnost. i Primenen. 15(1970), 469–497; English translation: Theor. Probability Appl. **15** (1970), 458–486.

[21] I. Doust, S. Sánchez, and A. Weston, *Asymptotic negative type properties of finite ultrametric spaces*. J. Math. Anal. Appl. 446(2017), no. 2, 1776–1793. https://doi.org/10.1016/j.jmaa.2016.09.069

[22] P. Erdős and L. Pósa, *On the maximal number of disjoint circuits of a graph*. Publ. Math. Debrecen 9(1962), 3–12.

[23] A. A. Giannopoulos, *A note on the Banach–Mazur distance to the cube*. In: *Geometric aspects of functional analysis (Israel, 1992–1994)*, Oper. Theory Adv. Appl., 77, Birkhäuser, Basel, 1995, pp. 67–73.

[24] A. Godard, *Tree metrics and their Lipschitz-free spaces*. Proc. Amer. Math. Soc. 138(2010), no. 12, 4311–4320. https://doi.org/10.1090/S0002-9939-2010-10421-5

[25] G. Godefroy, *A survey on Lipschitz-free Banach spaces*. Comment. Math. 55(2015), no. 2, 89–118. https://doi.org/10.14708/cm.v55i2.1104

[26] G. Godefroy and N. J. Kalton, *Lipschitz-free Banach spaces*. Studia Math. 159(2003), no. 1, 121–141. https://doi.org/10.4064/sm159-1-6

[27] S. Gogyan, *Greedy algorithm with regard to Haar subsystems*. East J. Approx. 11(2005), 221–236.

[28] B. Grünbaum, *Projection constants*. Trans. Amer. Math. Soc. 95(1960), 451–465. https://doi.org/10.2307/1993567

[29] A. Gupta, *Steiner points in tree metrics don't (really) help*. In: *Proceedings of the Twelfth Annual ACM-SIAM Symposium on Discrete Algorithms (Washington, DC, 2001)*, SIAM, Philadelphia, PA, 2001, pp. 220–227.

[30] A. Gupta, I. Newman, Y. Rabinovich, and A. Sinclair, *Cuts, trees and ℓ_1-embeddings of graphs*. Combinatorica 24(2004), 233–269. https://doi.org/10.1007/s00493-004-0015-x

[31] P. Indyk and J. Matoušek, *Low-distortion embeddings of finite metric spaces*. In: *Handbook of discrete and computational geometry*, Chapman and Hall/CRC, Boca Raton, FL, 2004, pp. 177–196. https://doi.org/10.1201/9781420035315

[32] W. B. Johnson and G. Schechtman, *Diamond graphs and super-reflexivity*. J. Topol. Anal. 1(2009), no. 2, 177–189. https://doi.org/10.1142/S1793525309000114

[33] M. I. Kadets and M. G. Snobar, *Certain functionals on the Minkowski compactum (Russian)*. Mat. Zametki 10(1971), 453–457.

[34] N. J. Kalton, *The nonlinear geometry of Banach spaces*. Rev. Mat. Complut. 21(2008), no. 1, 7–60. https://doi.org/10.5209/rev_REMA.2008.v21.n1.16426

[35] L. V. Kantorovich, *On mass transportation (Russian)*. Doklady Acad. Naus SSSR, (N.S.) 37(1942), 199–201; English transl.: J. Math. Sci. (N. Y.) **133**(2006), no. 4, 1381–1382. https://doi.org/10.1007/s10958-006-0049-2

[36] L. V. Kantorovich and G. S. Rubinstein, *On a space of completely additive functions. (Russian)*. Vestnik Leningrad. Univ. 13(1958), no. 7, 52–59.

[37] S. Khot and A. Naor, *Nonembeddability theorems via Fourier analysis*. Math. Ann. 334(2006), 821–852. https://doi.org/10.1007/s00208-005-0745-0

[38] S. V. Konyagin and V. N. Temlyakov, *A remark on greedy approximation in Banach spaces*. East. J. Approx. 5(1999), 365–379.

[39] J. B. Kruskal Jr., *On the shortest spanning subtree of a graph and the traveling salesman problem*. Proc. Amer. Math. Soc. 7(1956), 48–50. https://doi.org/10.2307/2033241

[40] T. J. Laakso, *Ahlfors Q-regular spaces with arbitrary $Q > 1$ admitting weak Poincare inequality*. Geom. Funct. Anal. 10(2000), no. 1, 111–123. https://doi.org/10.1007/s000390050003

[41] U. Lang and C. Plaut, *Bilipschitz embeddings of metric spaces into space forms*. Geom. Dedicata 87(2001), 285–307. https://doi.org/10.1023/A:1012093209450

[42] J. R. Lee and P. Raghavendra, *Coarse differentiation and multi-flows in planar graphs*. Discrete Comput. Geom. 43(2010), no. 2, 346–362. https://doi.org/10.1007/s00454-009-9172-4

[43] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. I. Sequence spaces*. Ergebnisse der Mathematik und ihrer Grenzgebiete, 92, Springer-Verlag, Berlin-New York, 1977.

[44] N. Linial, E. London, and Y. Rabinovich, *The geometry of graphs and some of its algorithmic applications*. Combinatorica 15(1995), no. 2, 215–245. https://doi.org/10.1007/BF01200757

[45] A. Martínez-Abejón, E. Odell, and M. M. Popov, *Some open problems on the classical function space L_1*. Mat. Stud 24(2005), 173–191.

[46] A. Naor and Y. Rabani, *On Lipschitz extension from finite subsets*. Israel J. Math. 219(2017), no. 1, 115–161. https://doi.org/10.1007/s11856-017-1475-1

[47] A. Naor and G. Schechtman, *Planar Earthmover is not in L_1*. SIAM J. Comput. 37(2007), 804–826. https://doi.org/10.1137/05064206X.

[48] S. Ostrovska and M. I. Ostrovskii, *Nonexistence of embeddings with uniformly bounded distortions of Laakso graphs into diamond graphs*. Discrete Math. 340(2017), no. 2, 9–17. https://doi.org/10.1016/j.disc.2016.08.003

[49] M. I. Ostrovskii, *Metric embeddings: Bilipschitz and coarse embeddings into Banach spaces*. de Gruyter Studies in Mathematics, 49, Walter de Gruyter, Berlin, 2013. https://doi.org/10.1515/9783110264012

[50] M. I. Ostrovskii and B. Randrianantoanina, *A new approach to low-distortion embeddings of finite metric spaces into non-superreflexive Banach spaces*. J. Funct. Anal. 273(2017), no. 2, 598–651. https://doi.org/10.1016/j.jfa.2017.03.017

[51] D. Peleg, *Distributed computing. A locality-sensitive approach*. SIAM Monographs on Discrete Mathematics and Applications, 5, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000.

[52] W. Rudin, *Projections on invariant subspaces*. Proc. Amer. Math. Soc. 13(1962), 429–432. https://doi.org/10.2307/2034952

[53] S. J. Szarek, *Spaces with large distance to ℓ_∞^n and random matrices*. Amer. J. Math. 112(1990), no. 6, 899–942. https://doi.org/10.2307/2374731

[54] S. J. Szarek and M. Talagrand, *An "isomorphic" version of the Sauer-Shelah lemma and the Banach-Mazur distance to the cube*. In: *Geometric aspects of functional analysis (1987–88)*, Lecture Notes in Math., 1376, Springer, Berlin, 1989, pp. 105–112.

[55] K. Tikhomirov, *On the Banach-Mazur distance to cross-polytope*. Adv. Math. 345(2019), 598–617. https://doi.org/10.1016/j.aim.2019.01.013
[56] L. N. Vasershtein, *Markov processes over denumerable products of spaces describing large system of automata*. Problems of Information Transmission 5(1969), no. 3, 47–52; translated from: Problemy Peredachi Informatsii 5(1969), no. 3, 64–72. https://doi.org/10.1016/s0016-0032(33)90010-1
[57] C. Villani, *Topics in optimal transportation*, Graduate Studies in Mathematics, 58, American Mathematical Society, Providence, RI, 2003. https://doi.org/10.1007/b12016
[58] C. Villani, *Optimal transport: Old and new*, Grundlehren der Mathematischen Wissenschaften, 338, Springer-Verlag, Berlin, 2009. https://doi.org/10.1007/978-3-540-71050-9
[59] N. Weaver, *Lipschitz algebras*. Second ed., World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018.
[60] P. Youssef, *Restricted invertibility and the Banach-Mazur distance to the cube*. Mathematika 60(2014), no. 1, 201–218. https://doi.org/10.1112/S0025579313000144

Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA
e-mail: dilworth@math.sc.edu

Department of Mathematics University of Illinois at Urbana-Champaign Urbana, IL 61801, USA

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences
e-mail: denka@math.uiuc.edu

Department of Mathematics and Computer Science, St. John's University, 8000 Utopia Parkway, Queens, NY 11439, USA
e-mail: ostrovsm@stjohns.edu