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Complete representation of some functionals showing the Lavrentieff phenomenon

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The functional $F(u) = \int_B f(x, Du)$ is considered, where B is the unit ball in $\mathbb{R}^2$, u varies in the set of the locally Lipschitz functions on $\mathbb{R}^2$ and f belongs to a family containing, as model case, the following integrand:

$$f(x,z) = \frac{|x_2|}{|x|^3}|\langle z, x\rangle| + |z|^p, \quad x = (x_1, x_2),\ z = (z_1, z_2) \in \mathbb{R}^2, \quad 1 < p < 2.$$

The computation of the relaxed functional $\bar{F}$ is provided yielding an explicit representation formula.

This formula nevertheless is not integral, because $\bar{F}$ is not a measure and does not coincide with the obvious extension of F over all $W^{1,p}(B)$.

This phenomenon is essentially due to the non-standard growth behaviour of $f(x, z)$ in the variable z.

1. Introduction

In 1926 (cf. [28]) an unexpected phenomenon concerning an integral functional of the calculus of variations was pointed out.

The considered functional was naturally defined and lower semicontinuous (with respect to the L^1 topology) on the set of absolutely continuous functions defined on the interval $[0, 1]$. Moreover, on the set of the functions u of this kind and such that $u(0) = 0$ and $u(1) = 1$, a minimum value was attained. This value, surprisingly enough, was strictly lower than the infimum value of the same functional computed on the set of Lipschitz functions with the same boundary conditions (Lavrentieff phenomenon); this fact implies that, for example, this minimum value cannot be approximated by a finite-elements method. Other examples of the same phenomenon concerning much simpler functionals were shown in [30].

Starting from these papers, an extensive literature was developed both to discover more instances of this phenomenon and to determine conditions to avoid its presence (cf. [5–8, 12, 20, 25–27, 29]).

On the other hand, given a topological space (U, τ) satisfying the first countability axiom, a τ-dense subset X of U and a functional F defined on X, a standard procedure of the calculus of variations is to define the relaxed functional $\bar{F}$ of F, i.e.

$$\bar{F}(u) = \inf\{\liminf_h F(u_h) : \{u_h\}_h \subseteq X,\ u_h \xrightarrow{\tau} u\}, \quad u \in U. \tag{1.1}$$

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This functional is τ-lower semicontinuous and, provided it has minimum value, we have

$$\inf_{u\in X} F(u) = \min_{u\in U} \bar{F}(u). \tag{1.2}$$

The Lavrentieff phenomenon can then be seen in the following way. There is a functional G defined on a topological space (U,τ) as before, the restriction F of G to a subset X τ-dense in U is considered and the minimum values of both G and $\bar{F}$ on U are compared. Then, even if we consider integral functionals of the calculus of variations, lower semicontinuous with respect to a topology of the kind L^1, and we take the restriction to a class of regular (say Lipschitzian) functions, it may happen that

$$\min_{u\in U} G(u) < \min_{u\in U} \bar{F}(u). \tag{1.3}$$

Such an approach has led to a search for functionals of this kind G, naturally defined on a large class of functions, that are different from the corresponding relaxed functionals $\bar{F}$ (cf. [2,11,14,16,17,21]).

The usual procedure is to then find some explicit function u such that $G(u) < \bar{F}(u)$ and some representation formula for $\bar{F}$, known in the one-dimensional case.

Starting from a important example of this kind, given in a two-dimensional case (cf. [21]), in the present paper we intend to give a complete representation (i.e. for every $u\in L^1$) of $\bar{F}$ for a class of functionals G showing the Lavrentieff phenomenon.

The phenomenon is well enlightened by the appearance of an odd-looking non-integral representation formula of $\bar{F}$ (while on the Lipschitz functions, $\bar{F}$ naturally coincides with G).

More precisely, let us denote by B the unit ball in $\mathbb{R}^2$ and by $\mathrm{Lip}_{\mathrm{loc}}$ the set of locally Lipschitz functions on $\mathbb{R}^2$.

In some papers (see, for example, [14,21]), the following setting has been considered: $(U,\tau) = L^1(B)$ endowed with the strong topology, the integrand function

$$f(x,z) = \frac{|x_2|}{|x|^3}|\langle z,x\rangle| + |z|^p, \quad x=(x_1,x_2),\ z=(z_1,z_2)\in\mathbb{R}^2, \quad 1<p<2, \tag{1.4}$$

and the functional

$$G(u) = \begin{cases} \displaystyle\int_B f(x,Du)\,\mathrm{d}x, & u\in W^{1,p}(B), \\ +\infty, & u\in L^1(B)\setminus W^{1,p}(B) \end{cases} \tag{1.5}$$

(observe that $G(u)$ is naturally $+\infty$ on $W^{1,1}(B)\setminus W^{1,p}(B)$).

Then, if $X=\mathrm{Lip}_{\mathrm{loc}}$, $F=G|_X$ and $u^*(x_1,x_2)=|x_2|/|x|$, both $G(u^*)$ and $\bar{F}(u^*)$ have been computed, resulting in

$$G(u^*) = \int_B |Du^*|^p < \pi + \int_B |Du^*|^p = \bar{F}(u^*). \tag{1.6}$$

We can observe that such an integrand $f(x,z)$ only verifies a non-standard growth condition (cf. [21]),

$$|z|^p \leqslant f(x,z) \leqslant a(x) + |z|^q \quad \text{where } q>2,\ a\in L^1_{\mathrm{loc}}(\mathbb{R}^2). \tag{1.7}$$

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In the present paper we consider a slightly more general family of integrand functions,

$$f(x,z)=g\biggl(\frac{x}{|x|}\biggr)\frac{|\langle z,x\rangle|}{|x|^2}+|z|^p,\quad x=(x_1,x_2),\ z=(z_1,z_2)\in\mathbb{R}^2,\quad 1<p<2, \tag{1.8}$$

where $g:S^1\to\mathbb{R}$ is a Hölder function, with Hölder exponent $s>((2-p)(p-1))/p$, non-negative and positive almost everywhere; such integrand functions show the same growth behaviour of the example, and we perform, for this family, the explicit computation of $\bar F$ over all $L^1(B)$.

For every u in $W^{1,p}(B)$, we will define

$$w(\rho,\theta)=u(\rho\cos\theta,\rho\sin\theta),\quad \rho\in(0,1),\quad \theta\in\mathbb{R}, \tag{1.9}$$

and ξ as the unique 2π-periodic function such that $g(\cos\theta,\sin\theta)=\xi(\theta)$.

We will prove that whenever $G(u)<+\infty$, w has the trace for $\rho=0$ (a.e. $\theta\in\mathbb{R}$), namely w^+. Moreover, $\xi w^+\in L^1([0,2\pi])$ and

$$\bar F(u)=\begin{cases} G(u)+\min\limits_{c\in\mathbb{R}}\displaystyle\int_0^{2\pi}\xi(\theta)|w^+(\theta)-c|\,\mathrm{d}\theta & \text{if } G(u)<+\infty,\\ +\infty & \text{otherwise.}\end{cases} \tag{1.10}$$

Let us note that if $g(\cos\theta,\sin\theta)=|\sin\theta|$ and $u^*(x_1,x_2)=|x_2|/|x|$, equation (1.10) gives the last equality in (1.6).

We observe that $\bar F(u)$ shows an extra piece, essentially due to the diversity between the coercivity exponents of the lower and upper controls of $f(x,z)$ in (1.7) (cf. also [15]). Moreover, the use of polar coordinates enables us to clarify the nature of this piece.

Finally, we prove that, in general, $\bar F$ is not a measure.

2. Notation and preliminary results

In this section we describe the notation we use throughout the paper.

In particular, we need to consider a space of functions slightly more general than the space of BV functions, and we just summarize some standard results for BV functions that are still valid for this space.

We will need two different copies of $\mathbb{R}^2$: $\mathbb{R}^2(x_1,x_2)$ and $\mathbb{R}^2(\rho,\theta)$. We will consider a function denoted by u as $u=u(x_1,x_2)$ and a function w as $w=w(\rho,\theta)$.

We will denote by

$$B=B(0,1)\subset\mathbb{R}^2(x_1,x_2)\qquad\text{the unit ball,} \tag{2.1}$$

$$Q=(a,b)\times(c,d)\subset\mathbb{R}^2(\rho,\theta)\qquad\text{any open interval,} \tag{2.2}$$

$$\tilde Q=(2a-b,b)\times(c,d)\qquad\text{for every } Q \text{ as above,} \tag{2.3}$$

$$Q_\lambda=(a+\lambda,b)\times(c,d)\qquad\text{for every } 0<\lambda<b-a, \tag{2.4}$$

$$R=(0,1)\times(0,2\pi)\subset\mathbb{R}^2(\rho,\theta), \tag{2.5}$$

$$R_\lambda=(\lambda,1)\times(0,2\pi)\qquad\text{for every } 0<\lambda<1, \tag{2.6}$$

$$\tilde R=(-1,1)\times(0,2\pi). \tag{2.7}$$

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Moreover, we will denote by $\tau = \tau(\rho,\theta)$ a positive symmetric mollifier, i.e. a function that enjoys the following properties:

(i) $\tau(\rho,\theta) \in C_c^\infty(S)$, $S = \{(\rho,\theta) : |(\rho,\theta)| \leqslant 1\}$;

(ii) $\displaystyle\int_{\mathbb{R}^2} \tau(\rho,\theta)\,\mathrm{d}\rho\,\mathrm{d}\theta = 1$;

(iii) $\tau(\rho,\theta) \geqslant 0$;

(iv) $\tau(\rho,\theta) = \sigma(|(\rho,\theta)|)$ for some function $\sigma : \mathbb{R} \to \mathbb{R}$.

We will then define

$$\tau_\varepsilon(\rho,\theta) = \frac{1}{\varepsilon^2}\tau\biggl(\frac{1}{\varepsilon}(\rho,\theta)\biggr) \quad \text{for every } \varepsilon > 0, \tag{2.8}$$

and the convolution product of a function w in $L^1_{\mathrm{loc}}(\mathbb{R}^2(\rho,\theta))$ with τ_ε as

$$(\tau_\varepsilon * w)(\rho,\theta) = \int_{\mathbb{R}^2} \tau_\varepsilon(\rho-\rho',\theta-\theta')w(\rho',\theta')\,\mathrm{d}\rho'\mathrm{d}\theta'. \tag{2.9}$$

We refer to [24, pp. 10, 11] for the standard properties of convolutions with mollifiers.

Definition 2.1. Let Ω be an open set of $\mathbb{R}^2(\rho,\theta)$ and let w be in $L^1(\Omega)$. We say that w is in $\mathcal{C}(\Omega)$ if

$$\sup_{\psi\in I}\int_\Omega w\frac{\partial\psi}{\partial\rho}\,\mathrm{d}\rho\,\mathrm{d}\theta < +\infty, \qquad I = \{\psi \in C_c^1(\Omega) : |\psi| \leqslant 1\}. \tag{2.10}$$

Furthermore, in this case, we define

$$\int_\Omega\biggl|\frac{\partial w}{\partial\rho}\biggr| = \sup_{\psi\in I}\int_\Omega w\frac{\partial\psi}{\partial\rho}\,\mathrm{d}\rho\,\mathrm{d}\theta. \tag{2.11}$$

Remark 2.2. If $w \in \mathcal{C}(\Omega)$, then $\partial w/\partial\rho$ (in the sense of distributions) is a bounded Radon measure with total variation expressed by (2.11).

Example 2.3. If $\partial w/\partial\rho \in L^1(\Omega)$, then

$$\int_\Omega\biggl|\frac{\partial w}{\partial\rho}\biggr| = \biggl\|\frac{\partial w}{\partial\rho}\biggr\|_{L^1(\Omega)}. \tag{2.12}$$

Proof. We refer the interested reader to example 1.2 in [24, p. 3]. □

Theorem 2.4. *Let $\{w_h\}_h$ be a sequence of functions in $\mathcal{C}(\Omega)$ such that $w_h \to w$ in $L^1_{\mathrm{loc}}(\Omega)$. Then we have*

$$\int_\Omega\biggl|\frac{\partial w}{\partial\rho}\biggr| \leqslant \liminf_{h\to\infty}\int_\Omega\biggl|\frac{\partial w_h}{\partial\rho}\biggr|. \tag{2.13}$$

In particular, if the lim inf *in the right-hand side is finite, then $w \in \mathcal{C}(\Omega)$.*

Proof. Adapt the proof of theorem 1.9 in [24, p. 7]. □

DEFINITION 2.5. Let w be in $\mathcal{C}(\Omega)$. We define

$$\|w\|_{\mathcal{C}(\Omega)} = \|w\|_{L^1(\Omega)} + \int_\Omega \left|\frac{\partial w}{\partial \rho}\right|. \tag{2.14}$$

We note that $\|\cdot\|_{\mathcal{C}(\Omega)}$ is clearly a norm on $\mathcal{C}(\Omega)$.

THEOREM 2.6. *$(\mathcal{C}(\Omega), \|\cdot\|_{\mathcal{C}(\Omega)})$ is a Banach space.*

Proof. We refer the interested reader to remark 1.12 in [24, p. 9]. □

LEMMA 2.7. *Let $Q = (a, b) \times (c, d)$ and $w \in \mathcal{C}(Q)$. Then there exists a (unique) function $w^+ \in L^1((c, d))$ such that*

$$\lim_{\eta \to 0^+} \frac{1}{\eta^2} \int_{\sigma-\eta}^{\sigma+\eta} \int_a^{a+\eta} |w(\rho, \theta) - w^+(\theta)| \, d\rho \, d\theta = 0 \quad a.e.\ \sigma \in (c, d). \tag{2.15}$$

Moreover, if $\psi \in C^1_c(\tilde{Q})$, we have (in distributional sense)

$$\int_Q w \frac{\partial \psi}{\partial \rho} \, d\rho \, d\theta = -\int_Q \psi \frac{\partial w}{\partial \rho} - \int_c^d w^+(\theta) \psi(a, \theta) \, d\theta. \tag{2.16}$$

Proof. Adapt the proof of lemma 2.4 in [24, p. 32]. □

The function w^+ defined on (c, d) is the trace of w on the left-hand side of the rectangle Q; in the same way, we define the trace w^- on the right-hand side.

PROPOSITION 2.8. *Let $Q_1 = (r, s) \times (c, d)$, $Q_2 = (s, t) \times (c, d)$ and let $w_1 \in \mathcal{C}(Q_1)$, $w_2 \in \mathcal{C}(Q_2)$. Let $Q = (r, t) \times (c, d)$ and let $w : Q \to \mathbb{R}$, defined by*

$$w = \begin{cases} w_1 & \text{in } Q_1, \\ w_2 & \text{in } Q_2. \end{cases} \tag{2.17}$$

Then $w \in \mathcal{C}(Q)$ and

$$\int_Q \left|\frac{\partial w}{\partial \rho}\right| = \int_{Q_1} \left|\frac{\partial w_1}{\partial \rho}\right| + \int_{Q2} \left|\frac{\partial w_2}{\partial \rho}\right| + \int_c^d |w_1^-(\theta) - w_2^+(\theta)| \, d\theta. \tag{2.18}$$

Proof. We refer the interested reader to proposition 2.8 in [24, p. 36]. □

LEMMA 2.9. *If $w \in \mathcal{C}(R)$ and $\xi(\theta)$ is a continuous function on $[0, 2\pi]$, then $w\xi \in \mathcal{C}(R)$.*

Moreover,

$$\frac{\partial (w\xi)}{\partial \rho} = \xi \frac{\partial w}{\partial \rho}, \tag{2.19}$$

as measures on R, and if given a measure ν we denote by $|\nu|$ its total variation measure, we have

$$\left|\frac{\partial (w\xi)}{\partial \rho}\right| = |\xi| \left|\frac{\partial w}{\partial \rho}\right|. \tag{2.20}$$

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Proof. $w\xi$ is obviously in $L^1(R)$. Moreover, if $\psi \in C_c^1(R)$, $|\psi| \leqslant 1$ and we set $h = \psi\xi$, then we have

$$\int_R w\xi \frac{\partial \psi}{\partial \rho}\,\mathrm{d}\rho\,\mathrm{d}\theta = \int_R w \frac{\partial h}{\partial \rho}\,\mathrm{d}\rho\,\mathrm{d}\theta \leqslant \max_{\theta\in[0,2\pi]} |\xi(\theta)| \int_R \left|\frac{\partial w}{\partial \rho}\right| < +\infty, \tag{2.21}$$

whence $w\xi \in \mathcal{C}(R)$.

Now we prove (2.19). Observe that for every $\varphi \in C_c^1(R)$ we have

$$\left\langle \frac{\partial(w\xi)}{\partial\rho}, \varphi \right\rangle = -\left\langle w\xi, \frac{\partial\varphi}{\partial\rho} \right\rangle = -\int_R w\xi \frac{\partial\varphi}{\partial\rho}\,\mathrm{d}\rho\,\mathrm{d}\theta, \tag{2.22}$$

$$\left\langle \frac{\partial w}{\partial\rho}\xi, \varphi \right\rangle = \left\langle \frac{\partial w}{\partial\rho}, \varphi\xi \right\rangle = -\left\langle w, \frac{\partial(\varphi\xi)}{\partial\rho} \right\rangle = -\int_R w\xi \frac{\partial\varphi}{\partial\rho}\,\mathrm{d}\rho\,\mathrm{d}\theta. \tag{2.23}$$

Equation (2.19) follows from (2.22) and (2.23), recalling that for every measure ν we have

$$\nu(A) = \sup_{\varphi\in C_c^1(A),\,|\varphi|\leqslant 1} \langle \nu, \varphi \rangle \quad \text{for every open set } A. \tag{2.24}$$

Finally, equation (2.20) comes from (2.19) and the fact that $|\nu\xi| = |\xi||\nu|$ for every measure ν on R. □

LEMMA 2.10. *Let w be in $L^1(R)$ and let $\xi(\theta)$ a continuous function on $[0, 2\pi]$ such that $w\xi \in \mathcal{C}(R)$ (therefore, $w\xi$ owns the trace $(w\xi)^+$ a.e. on $(0, 2\pi)$). Let $\mu > 0$ and let $C^\mu = \{\theta \in [0, 2\pi] : |\xi(\theta)| \leqslant \mu\}$. Then $w \in \mathcal{C}(R \setminus ((0, 1) \times C^\mu))$, the trace $w^{+,\mu}$ is defined on $(0, 2\pi) \setminus C^\mu$ and we have the equality*

$$w^{+,\mu} = \frac{(w\xi)^+}{\xi} \quad a.e\ \theta \in (0, 2\pi) \setminus C^\mu. \tag{2.25}$$

Proof. The only non-trivial thing is equality (2.25), which comes from (2.15) applied to $w\xi$ and $w^{+,\mu}$. □

DEFINITION 2.11. Let w be in $L^1(R)$ and let $\xi(\theta)$ be a continuous function on $[0, 2\pi]$ such that $w\xi \in \mathcal{C}(R)$. Let $Z = \{\theta \in [0, 2\pi] : \xi(\theta) = 0\}$ and assume that Z has zero Lebesgue measure. We define the trace w^+ of w on the left-hand side of R as

$$w^+ = \frac{(w\xi)^+}{\xi} \quad \text{a.e. in } (0, 2\pi). \tag{2.26}$$

REMARK 2.12. By lemma 2.10, for every $\mu > 0$, we have that $w^+ = w^{+,\mu}$ in $(0, 2\pi) \setminus C^\mu$. Moreover, $w^+ \in L^1_{\mathrm{loc}}((0, 2\pi) \setminus Z)$.

3. Some further consideration on $\mathcal{C}$

PROPOSITION 3.1. *Let $w \in L^\infty((0, 1) \times \mathbb{R})$ be 2π periodic in θ and assume that*

$$\left|\frac{\partial w}{\partial \rho}\right| \in L^1_{\mathrm{loc}}((0, 1) \times \mathbb{R}). \tag{3.1}$$

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Let $0 \leqslant \varepsilon < 1$ *and set*

$$w^{\varepsilon,c}(\rho,\theta) = \begin{cases} c, & \rho \leqslant \varepsilon,\ \theta \in \mathbb{R}, \\ w(\rho,\theta), & \varepsilon < \rho \leqslant 1,\ \theta \in \mathbb{R}, \\ w(2-\rho,\theta), & 1 < \rho \leqslant 2,\ \theta \in \mathbb{R} \end{cases} \tag{3.2}$$

and

$$w^c_{\varepsilon,\alpha} = \tau_{\varepsilon^\alpha} * w^{\varepsilon,c}, \quad \alpha \geqslant 1. \tag{3.3}$$

Then $w \in \mathcal{C}(R)$ *if and only if*

$$\liminf_{\varepsilon\to 0^+} \int_R \left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right| \mathrm{d}\rho\,\mathrm{d}\theta < +\infty.$$

In this case, we also have

$$\lim_{\varepsilon\to 0^+} \int_R \left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right| \mathrm{d}\rho\,\mathrm{d}\theta = \int_R \left|\frac{\partial w}{\partial\rho}\right| \mathrm{d}\rho\,\mathrm{d}\theta + \int_0^{2\pi} |w^+(\theta) - c|\,\mathrm{d}\theta. \tag{3.4}$$

Proof. First of all, let us observe that, by (3.2) and (3.3), $w^c_{\varepsilon,\alpha}(\rho,\theta)$ is defined for every $\rho \leqslant 1$, $\theta \in \mathbb{R}$ and it is clear that

$$w^c_{\varepsilon,\alpha} \xrightarrow{\varepsilon\to 0^+} w^{0,c} \quad \text{in } L^1(\tilde{R}).$$

By theorem 2.4 applied to $\{w^c_{\varepsilon,\alpha}\} \subset \mathcal{C}(\tilde{R})$, we obtain

$$\liminf_{\varepsilon\to 0^+} \int_R \left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right| \mathrm{d}\rho\,\mathrm{d}\theta = \liminf_{\varepsilon\to 0^+} \int_{\tilde{R}} \left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right| \mathrm{d}\rho\,\mathrm{d}\theta \geqslant \int_{\tilde{R}} \left|\frac{\partial w^{0,c}}{\partial\rho}\right|. \tag{3.5}$$

If

$$\liminf_{\varepsilon\to 0^+} \int_R \left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right| \mathrm{d}\rho\,\mathrm{d}\theta$$

is finite, then $w^{0,c} \in \mathcal{C}(\tilde{R})$, $w \in \mathcal{C}(R)$, the trace w^+ of w for $\rho = 0$ is defined and, by proposition 2.8,

$$\int_{\tilde{R}} \left|\frac{\partial w^{0,c}}{\partial\rho}\right| = \int_R \left|\frac{\partial w}{\partial\rho}\right| \mathrm{d}\rho\,\mathrm{d}\theta + \int_0^{2\pi} |w^+(\theta) - c|\,\mathrm{d}\theta. \tag{3.6}$$

On the other hand, if $w \in \mathcal{C}(R)$, let us define

$$\left.\begin{aligned} R^\varepsilon &= (-\varepsilon^\alpha, 1+\varepsilon^\alpha) \times (-\varepsilon^\alpha, 2\pi+\varepsilon^\alpha), \\ R'^\varepsilon &= (\varepsilon, 1+\varepsilon^\alpha) \times (-\varepsilon^\alpha, 2\pi+\varepsilon^\alpha), \end{aligned}\right\} \tag{3.7}$$

then, by the definition of $w^{\varepsilon,c}$, the θ-periodicity of w and proposition 2.8, we have that $w^{\varepsilon,c} \in \mathcal{C}(R^\varepsilon)$ and, by hypothesis (3.1), $\partial w^{\varepsilon,c}/\partial\rho \in L^1(R'^\varepsilon)$. Because of standard properties of mollification (cf. [24, p. 12]), we have

$$\int_R \left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right| \mathrm{d}\rho\,\mathrm{d}\theta \leqslant \int_{R^\varepsilon} \left|\frac{\partial w^{\varepsilon,c}}{\partial\rho}\right|. \tag{3.8}$$

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Moreover, again by proposition 2.8 and the boundedness of w,

$$\int_{R^\varepsilon}\left|\frac{\partial w^{\varepsilon,c}}{\partial\rho}\right| \leqslant \int_R\left|\frac{\partial w^{\varepsilon,c}}{\partial\rho}\right| + 2\varepsilon(\|w\|_{L^\infty(B)}+c) + \int_{R'^{\varepsilon}\setminus R}\left|\frac{\partial w}{\partial\rho}\right|\mathrm{d}\rho\,\mathrm{d}\theta. \tag{3.9}$$

Since $|R'^{\varepsilon}\setminus R| \to 0$ as $\varepsilon\to 0^+$, we get

$$\limsup_{\varepsilon\to 0^+}\int_{R^\varepsilon}\left|\frac{\partial w^{\varepsilon,c}}{\partial\rho}\right| \leqslant \liminf_{\varepsilon\to 0^+}\int_R\left|\frac{\partial w^{\varepsilon,c}}{\partial\rho}\right|. \tag{3.10}$$

Moreover,

$$\int_R\left|\frac{\partial w^{\varepsilon,c}}{\partial\rho}\right| = \int_0^{2\pi}|w_\varepsilon^+(\theta)-c|\,\mathrm{d}\theta + \int_{R_\varepsilon}\left|\frac{\partial w}{\partial\rho}\right|\mathrm{d}\rho\,\mathrm{d}\theta, \tag{3.11}$$

where w_ε^+ is the trace of w on the left-hand side of R_ε and it can be easily proved (cf. [24, p. 33]) that

$$\lim_{\varepsilon\to 0^+}\int_0^{2\pi}|w_\varepsilon^+(\theta)-c|\,\mathrm{d}\theta = \int_0^{2\pi}|w^+(\theta)-c|\,\mathrm{d}\theta. \tag{3.12}$$

Therefore,

$$\lim_{\varepsilon\to 0^+}\int_R\left|\frac{\partial w^{\varepsilon,c}}{\partial\rho}\right| = \int_R\left|\frac{\partial w}{\partial\rho}\right|\mathrm{d}\rho\,\mathrm{d}\vartheta + \int_0^{2\pi}|w^+(\theta)-c|\,\mathrm{d}\theta. \tag{3.13}$$

By (3.8), (3.10) and (3.13) it follows that

$$\limsup_{\varepsilon\to 0^+}\int_R\left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right|\mathrm{d}\rho\,\mathrm{d}\vartheta \leqslant \int_R\left|\frac{\partial w}{\partial\rho}\right|\mathrm{d}\rho\,\mathrm{d}\vartheta + \int_0^{2\pi}|w^+(\theta)-c|\,\mathrm{d}\theta. \tag{3.14}$$

Then the left-hand side is finite. Finally, equality (3.4) follows from (3.5), (3.6) and (3.14). □

Obtaining the same proposition for the product ξw is straightforward if ξ never takes the value zero. Otherwise, the proof is more delicate and some lemmas are needed.

LEMMA 3.2. *Let $w\in L^\infty((0,1)\times\mathbb{R})$ be 2π periodic in θ and assume that there exists $p>1$ such that*

$$\rho\left|\frac{\partial w}{\partial\rho}\right|^p \in L^1((0,1)\times(0,3\pi)). \tag{3.15}$$

Let $0<\varepsilon<1$ and $w^{\varepsilon,c}$ be defined by (3.2).

Now let $Q=(0,1)\times(a,b)\subseteq R$, $0<\sigma<1$, $Q^\sigma=(-\sigma,1+\sigma)\times(a-\sigma,b+\sigma)$, $Q^{\sigma,\varepsilon}=(\varepsilon,1+\sigma)\times(a-\sigma,b+\sigma)$.

Then there exist some constants $c_1,\dots,c_3$, depending only on c, p and w, such that

$$\int_Q\left|\frac{\partial w^{\varepsilon,c}}{\partial\rho}\right| \leqslant c_1\varepsilon^{2/p'-1}, \tag{3.16}$$

$$\int_{Q^\sigma}\left|\frac{\partial w^{\varepsilon,c}}{\partial\rho}\right| \leqslant \int_Q\left|\frac{\partial w^{\varepsilon,c}}{\partial\rho}\right| + c_2\sigma + c_3\sigma^{1/p'}\varepsilon^{2/p'-1}\left\|\rho^{1/p}\left|\frac{\partial w}{\partial\rho}\right|\right\|_{L^p(Q^{\sigma,\varepsilon}\setminus Q)}. \tag{3.17}$$

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Proof. We have, by proposition 2.8 and by the boundedness of w,

$$\begin{aligned}\int_Q\left|\frac{\partial w^{\varepsilon,c}}{\partial\rho}\right| &= \int_a^b |w_\varepsilon^+(\theta)-c|\,\mathrm{d}\theta + \int_{Q_\varepsilon}\left|\frac{\partial w}{\partial\rho}\right|\mathrm{d}\rho\,\mathrm{d}\vartheta \\ &\leqslant 2\pi(\|w\|_{L^\infty(B)}+c) + \int_{Q_\varepsilon}\left|\frac{\partial w}{\partial\rho}\right|\mathrm{d}\rho\,\mathrm{d}\vartheta, \qquad (3.18)\end{aligned}$$

where w_ε^+ is the trace of w on the left-hand side of Q_ε. Moreover, by Hölder's inequality, we get

$$\begin{aligned}\int_{Q_\varepsilon}\left|\frac{\partial w}{\partial\rho}\right|\mathrm{d}\rho\,\mathrm{d}\vartheta &\leqslant \|\rho^{1/p}\left|\frac{\partial w}{\partial\rho}\right|\|_{L^p(Q_\varepsilon)}\|\rho^{-1/p}\|_{L^{p'}(Q_\varepsilon)} \\ &\leqslant \left(\left\|\rho\left|\frac{\partial w}{\partial\rho}\right|^p\right\|_{L^1(R)}\right)^{1/p}\|\rho^{-1/p}\|_{L^{p'}(Q_\varepsilon)}. \qquad (3.19)\end{aligned}$$

Finally,

$$\|\rho^{-1/p}\|_{L^{p'}(Q_\varepsilon)} = \left(\int_{Q_\varepsilon}\rho^{1-p'}\,\mathrm{d}\rho\,\mathrm{d}\theta\right)^{1/p'} \leqslant \left(\frac{2\pi}{p'-2}(\varepsilon^{2-p'}-1)\right)^{1/p'}. \qquad (3.20)$$

By (3.18), (3.19) and (3.20), we easily obtain (3.16). Similar computations lead to (3.17). □

LEMMA 3.3. *Let $w\in L^\infty((0,1)\times\mathbb{R})$ be 2π periodic in θ and assume (3.15) holds. Let $\varepsilon>0$ and let $w^{\varepsilon,c}$ and $w^c_{\varepsilon,\alpha}$ be, respectively, defined by (3.2) and (3.3). Let Q be as in the previous lemma and assume $\alpha\geqslant p'-2=(2-p)/(p-1)$.*

Then $w\in\mathcal{C}(Q)$ if and only if

$$\liminf_{\varepsilon\to0^+}\int_Q\left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right|\mathrm{d}\rho\,\mathrm{d}\vartheta<+\infty.$$

In this case, we also have

$$\lim_{\varepsilon\to0^+}\int_Q\left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right|\mathrm{d}\rho\,\mathrm{d}\vartheta = \int_Q\left|\frac{\partial w}{\partial\rho}\right|\mathrm{d}\rho\,\mathrm{d}\vartheta + \int_a^b|w^+(\theta)-c|\,\mathrm{d}\theta. \qquad (3.21)$$

Proof. As in proposition 3.1, it is clear that $w^c_{\varepsilon,\alpha}\xrightarrow{\varepsilon\to0^+}w^{0,c}$ in $L^1(\tilde{Q})$. By theorem 2.4 applied to $\{w^c_{\varepsilon,\alpha}\}\subset\mathcal{C}(\tilde{Q})$, we obtain

$$\liminf_{\varepsilon\to0^+}\int_Q\left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right|\mathrm{d}\rho\,\mathrm{d}\vartheta = \liminf_{\varepsilon\to0^+}\int_{\tilde{Q}}\left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right|\mathrm{d}\rho\,\mathrm{d}\vartheta \geqslant \int_{\tilde{Q}}\left|\frac{\partial w^{0,c}}{\partial\rho}\right|, \qquad (3.22)$$

so that if

$$\liminf_{\varepsilon\to0^+}\int_Q\left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right|\mathrm{d}\rho\,\mathrm{d}\vartheta$$

is finite, then $w\in\mathcal{C}(Q)$. In this case, the trace w^+ on the left-hand side of Q is defined and, by proposition 2.8,

$$\int_{\tilde{Q}}\left|\frac{\partial w^{0,c}}{\partial\rho}\right| = \int_Q\left|\frac{\partial w}{\partial\rho}\right|\mathrm{d}\rho\,\mathrm{d}\vartheta + \int_a^b|w^+(\theta)-c|\,\mathrm{d}\theta. \qquad (3.23)$$

On the other hand, if $w \in \mathcal{C}(Q)$, let us define Q^σ and $Q^{\sigma,\varepsilon}$ as in the previous lemma. As in proposition 3.1, we get $w \in \mathcal{C}(Q^{\varepsilon^\alpha})$ and $\partial w^{\varepsilon,c}/\partial\rho \in L^1(Q^{\varepsilon^\alpha,\varepsilon})$.

Now observe that, because of standard properties of mollification (cf. [24, p. 12]), we have

$$\int_Q \left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right| \mathrm{d}\rho\,\mathrm{d}\vartheta \leqslant \int_{Q^{\varepsilon^\alpha}} \left|\frac{\partial w^{\varepsilon,c}}{\partial\rho}\right|. \tag{3.24}$$

By lemma 3.2 applied for $\sigma = \varepsilon^\alpha$, we obtain

$$\int_{Q^{\varepsilon^\alpha}} \left|\frac{\partial w^{\varepsilon,c}}{\partial\rho}\right| \leqslant \int_Q \left|\frac{\partial w^{\varepsilon,c}}{\partial\rho}\right| + c_2\varepsilon^\alpha + c_3\varepsilon^{(\alpha+2)/p'-1}\left\|\rho^{1/p}\left|\frac{\partial w}{\partial\rho}\right|\right\|_{L^p(Q^{\varepsilon^\alpha,\varepsilon}\setminus Q)}. \tag{3.25}$$

Since $\alpha > 0$, $(\alpha+2)/p' - 1 \geqslant 0$ and $|Q^{\varepsilon^\alpha,\varepsilon} \setminus Q| \to 0$ as $\varepsilon \to 0^+$, we have

$$\limsup_{\varepsilon\to 0^+} \int_{Q^{\varepsilon^\alpha}} \left|\frac{\partial w^{\varepsilon,c}}{\partial\rho}\right| \leqslant \limsup_{\varepsilon\to 0^+} \int_Q \left|\frac{\partial w^{\varepsilon,c}}{\partial\rho}\right|. \tag{3.26}$$

Moreover,

$$\int_Q \left|\frac{\partial w^{\varepsilon,c}}{\partial\rho}\right| = \int_a^b |w^+_\varepsilon(\theta) - c|\,\mathrm{d}\theta + \int_{Q_\varepsilon} \left|\frac{\partial w}{\partial\rho}\right| \mathrm{d}\rho\,\mathrm{d}\vartheta, \tag{3.27}$$

and it can be easily proved (cf. [24, p. 33]) that

$$\lim_{\varepsilon\to 0^+} \int_a^b |w^+_\varepsilon(\theta) - c|\,\mathrm{d}\theta = \int_a^b |w^+(\theta) - c|\,\mathrm{d}\theta. \tag{3.28}$$

Therefore,

$$\lim_{\varepsilon\to 0^+} \int_Q \left|\frac{\partial w^{\varepsilon,c}}{\partial\rho}\right| = \int_Q \left|\frac{\partial w}{\partial\rho}\right| + \int_a^b |w^+(\theta) - c|\,\mathrm{d}\theta. \tag{3.29}$$

By (3.24), (3.26) and (3.29), it follows that

$$\limsup_{\varepsilon\to 0^+} \int_Q \left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right| \leqslant \int_Q \left|\frac{\partial w}{\partial\rho}\right| + \int_a^b |w^+(\theta) - c|\,\mathrm{d}\theta. \tag{3.30}$$

Then the left-hand side is also finite. Finally, equality (3.21) follows from (3.22), (3.23) and (3.30). □

PROPOSITION 3.4. *Let ξ a non-negative periodic Hölder function on $[0, 2\pi]$, positive almost everywhere. Let w, $w^{\varepsilon,c}$, $w^c_{\varepsilon,\alpha}$, p, c, α be as in lemma 3.3.*

Then there exists α, depending on the Hölder exponent s of ξ, such that $\alpha > 1$, $\alpha \geqslant p' - 2$ and $w\xi \in \mathcal{C}(R)$ if and only if

$$\liminf_{\varepsilon\to 0^+} \int_R \xi \left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right| \mathrm{d}\rho\,\mathrm{d}\vartheta < +\infty.$$

In this case, there exists the trace w^+ on the left-hand side of R and

$$\int_0^{2\pi} \xi(\theta)|w^+(\theta) - c|\,\mathrm{d}\theta < +\infty.$$

Moreover,

$$\lim_{\varepsilon\to 0^+}\int_R \xi\left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right| \mathrm{d}\rho\,\mathrm{d}\vartheta = \int_R \xi\left|\frac{\partial w}{\partial\rho}\right| + \int_0^{2\pi} \xi(\theta)|w^+(\theta)-c|\,\mathrm{d}\theta. \tag{3.31}$$

Finally, if the Hölder exponent s of ξ is greater than $((2-p)(p-1))/p$, then α can be chosen smaller than $p'-1$.

Proof. By theorem 2.4 applied to $\{w^c_{\varepsilon,\alpha}\xi\}\subset\mathcal{C}(\tilde R)$, we obtain that

$$\liminf_{\varepsilon\to 0^+}\int_R \xi\left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right| \mathrm{d}\rho\,\mathrm{d}\vartheta \geqslant \int_{\tilde R}\left|\frac{\partial w^{0,c}\xi}{\partial\rho}\right|, \tag{3.32}$$

and consequently, if

$$\liminf_{\varepsilon\to 0^+}\int_R \xi\left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right| \mathrm{d}\rho\,\mathrm{d}\vartheta$$

is finite, by (3.32), we have that $w^{0,c}\xi\in\mathcal{C}(\tilde R)$ and $w\xi\in\mathcal{C}(R)$.

In this case, by proposition 2.8, lemma 2.9 and definition 2.11, we have

$$\int_{\tilde R}\left|\frac{\partial w^{0,c}\xi}{\partial\rho}\right| = \int_R \xi\left|\frac{\partial w}{\partial\rho}\right| + \int_0^{2\pi} \xi(\theta)|w^+(\theta)-c|\,\mathrm{d}\theta. \tag{3.33}$$

On the other hand, we have

$$\int_R\left|\frac{\partial w^{\varepsilon,c}\xi}{\partial\rho}\right| \leqslant \int_R \xi\left|\frac{\partial w}{\partial\rho}\right| \mathrm{d}\rho\,\mathrm{d}\vartheta + \int_0^{2\pi} \xi(\theta)|w^+(\theta)-c| d\theta + \int_0^{2\pi} \xi(\theta)|w^+(\theta)-w^+_\varepsilon(\theta)|\,\mathrm{d}\theta, \tag{3.34}$$

where $w^+_\varepsilon(\theta)$ is the trace of w on the left-hand side of R_ε.

Since the last integral converges to zero as $\varepsilon\to 0^+$, we obtain

$$\limsup_{\varepsilon\to 0^+}\int_R\left|\frac{\partial w^{\varepsilon,c}\xi}{\partial\rho}\right| \leqslant \int_R \xi\left|\frac{\partial w}{\partial\rho}\right| \mathrm{d}\rho\,\mathrm{d}\vartheta + \int_0^{2\pi} \xi(\theta)|w^+(\theta)-c|\,\mathrm{d}\theta. \tag{3.35}$$

Let us now divide the interval $[0,2\pi]$ into $n=n(\varepsilon)$ sub-intervals of length $2\pi/n$, namely (t_{i-1},t_i), $i=1,\dots,n(\varepsilon)$. Let us take a number α such that $\alpha>1$ and $\alpha\geqslant p'-2$. Moreover, if

$$\alpha+\frac{1}{s}\left(\frac{2}{p'}-1\right)>0,$$

it is certainly possible to assume both

$$\frac{\varepsilon^{2/p'-1}}{n(\varepsilon)^s}\xrightarrow{\varepsilon\to 0^+} 0 \quad\text{and}\quad n(\varepsilon)\varepsilon^\alpha \xrightarrow{\varepsilon\to 0^+} 0. \tag{3.36}$$

Let $Q_i=(0,1)\times(t_{i-1},t_i)$ and let $a_i=\inf\{\xi(\theta):\theta\in(t_{i-1},t_i)\}$. We have

$$\int_R \xi\left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right| \mathrm{d}\rho\,\mathrm{d}\vartheta \leqslant \sum_{i=1}^n\left(a_i+\lambda\left(\frac{2\pi}{n(\varepsilon)}\right)^s\right)\int_{Q_i}\left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right| \mathrm{d}\rho\,\mathrm{d}\vartheta, \tag{3.37}$$

where λ is the Hölder constant of ξ. Now, as in (3.24), we have

$$\int_{Q_i}\left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right| \mathrm{d}\rho\,\mathrm{d}\vartheta \leqslant \int_{Q_i^{\varepsilon^\alpha}}\left|\frac{\partial w^{\varepsilon,c}}{\partial\rho}\right|. \tag{3.38}$$

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By lemma 3.2 applied with $Q = Q_i$ and $\sigma = \varepsilon^\alpha$, we obtain

$$\int_{Q_i^{\varepsilon^\alpha}} \left|\frac{\partial w^{\varepsilon,c}}{\partial \rho}\right| \leqslant \int_{Q_i} \left|\frac{\partial w^{\varepsilon,c}}{\partial \rho}\right| + c_2\varepsilon^\alpha + c_3\varepsilon^{(\alpha+2)/p'-1} \left\|\rho^{1/p}\left|\frac{\partial w}{\partial \rho}\right|\right\|_{L^p((Q_i^{\varepsilon^\alpha,\varepsilon}\setminus Q_i))}, \tag{3.39}$$

and setting $\gamma = (\alpha+2)/p' - 1$, we have

$$\begin{aligned}\sum_{i=1}^n \int_{Q_i^{\varepsilon^\alpha}} \left|\frac{\partial w^{\varepsilon,c}}{\partial \rho}\right| &\leqslant \int_R \left|\frac{\partial w^{\varepsilon,c}}{\partial \rho}\right| + n(\varepsilon)c_2\varepsilon^\alpha + c_3\varepsilon^\gamma \sum_{i=1}^n \left\|\rho^{1/p}\left|\frac{\partial w}{\partial \rho}\right|\right\|_{L^p((Q_i^{\varepsilon^\alpha,\varepsilon}\setminus Q_i))} \\ &\leqslant \int_R \left|\frac{\partial w^{\varepsilon,c}}{\partial \rho}\right| + n(\varepsilon)c_2\varepsilon^\alpha + 2c_3\varepsilon^\gamma\psi(\varepsilon),\end{aligned} \tag{3.40}$$

where

$$\psi(\varepsilon) = \left\|\rho^{1/p}\left|\frac{\partial w}{\partial \rho}\right|\right\|_{L^p(\bigcup_{i=1}^n(Q_i^{\varepsilon^\alpha,\varepsilon}\setminus Q_i))},$$

and the last inequality holds (if ε is small enough) because no more than two sets overlap at the same time.

In the same way,

$$\begin{aligned}\sum_{i=1}^n a_i \int_{Q_i^{\varepsilon^\alpha}} \left|\frac{\partial w^{\varepsilon,c}}{\partial \rho}\right| &\leqslant \int_R \xi\left|\frac{\partial w^{\varepsilon,c}}{\partial \rho}\right| + n(\varepsilon)c_4\varepsilon^\alpha + c_5\varepsilon^\gamma \sum_{i=1}^n \left\|\rho^{1/p}\left|\frac{\partial w}{\partial \rho}\right|\right\|_{L^p((Q_i^{\varepsilon^\alpha,\varepsilon}\setminus Q_i))} \\ &\leqslant \int_R \xi\left|\frac{\partial w^{\varepsilon,c}}{\partial \rho}\right| + n(\varepsilon)c_4\varepsilon^\alpha + 2c_5\varepsilon^\gamma\psi(\varepsilon).\end{aligned} \tag{3.41}$$

Then, by (3.37), (3.38), (3.39), (3.40), (3.41) and again by lemma 3.2, we get, for suitable constants,

$$\int_R \xi\left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial \rho}\right| \mathrm{d}\rho\,\mathrm{d}\vartheta \leqslant \int_R \xi\left|\frac{\partial w^{\varepsilon,c}}{\partial \rho}\right| + n(\varepsilon)c_4\varepsilon^\alpha + 2c_5\varepsilon^\gamma\psi(\varepsilon) + \frac{c_6}{n(\varepsilon)^s}\varepsilon^{2/p'-1}. \tag{3.42}$$

Now, since $\alpha \geqslant p' - 2$, we have $\gamma \geqslant 0$. Moreover, by (3.36), we have

$$\left|\bigcup_{i=1}^n J^{-1}(Q_i^{\varepsilon^\alpha} \setminus \bar{Q}_i)\right| \xrightarrow{\varepsilon\to 0^+} 0 \quad \text{and} \quad \lim_{\varepsilon\to 0^+}\psi(\varepsilon) = 0. \tag{3.43}$$

By (3.42), (3.43), (3.36) and (3.35), we deduce

$$\begin{aligned}\limsup_{\varepsilon\to 0^+}\int_R \xi\left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial \rho}\right| \mathrm{d}\rho\,\mathrm{d}\vartheta &\leqslant \limsup_{\varepsilon\to 0^+}\int_R \xi\left|\frac{\partial w^{\varepsilon,c}}{\partial \rho}\right| \\ &\leqslant \int_R \xi\left|\frac{\partial w}{\partial \rho}\right| \mathrm{d}\rho\mathrm{d}\vartheta + \int_0^{2\pi} \xi(\theta)|w^+(\theta) - c|\,\mathrm{d}\theta.\end{aligned} \tag{3.44}$$

Then the left-hand side is finite. Moreover, equation (3.31) follows by (3.32), (3.33) and (3.44).

Let us finally observe that

$$p' - 1 + \frac{p}{(2-p)(p-1)}\left(\frac{2}{p'} - 1\right) = 0.$$

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If $s > ((2-p)(p-1))/p$, then we have $p'-1+(1/s)(2/p'-1) > 0$ and the last part of the thesis easily follows. □

PROPOSITION 3.5. *Let B the unit ball in $\mathbb{R}^2$, $u \in W^{1,p}(B) \cap L^\infty(B)$, $1 < p < 2$, and let $w(\rho,\theta)$ be defined by*

$$w(\rho,\theta) = u(\rho\cos\theta, \rho\sin\theta), \quad \rho \in (0,1), \quad \theta \in \mathbb{R}. \tag{3.45}$$

Then $w \in L^\infty((0,1)\times\mathbb{R})$ is 2π periodic in θ and (3.15) is satisfied.

Proof. The only thing to verify is (3.15). It can be easily seen that

$$\|\nabla w(\rho,\theta)\| \leqslant \|\nabla u(\rho\cos\theta, \rho\sin\theta)\| \sqrt{1+\rho^2}. \tag{3.46}$$

Therefore,

$$\int_0^{3\pi}\int_0^1 \rho\left|\frac{\partial w}{\partial\rho}\right|^p \mathrm{d}\rho\,\mathrm{d}\theta \leqslant 2\int_B |\sqrt{2}\nabla u|^p\,\mathrm{d}x < +\infty, \tag{3.47}$$

so that (3.15) is satisfied. □

4. The computation of the relaxed functional

Let B be the unit ball in $\mathbb{R}^2$, $(U,\tau) = L^1(B)$ endowed with the strong topology and $X = \mathrm{Lip}_{\mathrm{loc}}$ the set of locally Lipschitz functions on $\mathbb{R}^2$. Let

$$f(x,z) = g\left(\frac{x}{|x|}\right)\frac{|\langle z,x\rangle|}{|x|^2} + |z|^p, \quad x=(x_1,x_2),\ z=(z_1,z_2)\in\mathbb{R}^2, \quad 1<p<2, \tag{4.1}$$

where $g : S^1 \to \mathbb{R}$ is a Hölder function, non-negative and positive almost everywhere, with Hölder exponent $s > ((2-p)(p-1))/p$, and ξ is the unique 2π-periodic function such that $g(\cos\theta,\sin\theta) = \xi(\theta)$ for every $\theta\in\mathbb{R}$. Let

$$G(u) = \begin{cases} \displaystyle\int_B f(x,Du)\,\mathrm{d}x, & u \in W^{1,p}(B), \\ +\infty & u \in L^1(B)\setminus W^{1,p}(B), \end{cases} \tag{4.2}$$

and $F = G|_X$ and $\bar{F}(u)$ be defined by (1.1) for every $u \in L^1(B)$.

We will follow this notation throughout this section.

LEMMA 4.1. *If $u \notin W^{1,p}(B)$, then $\bar{F}(u) = +\infty$.*

Proof. By contradiction, let $u \notin W^{1,p}(B)$ and $\bar{F}(u) < +\infty$.

Then there exists a sequence $\{u_h\}_h \subset \mathrm{Lip}_{\mathrm{loc}}$, $m > 0$, such that

(i) $u_h \to u$ in $L^1(B)$ as $h \to +\infty$,

(ii) $\displaystyle\int_B |Du_h|^p\,\mathrm{d}x \leqslant F(u_h) \leqslant m$ for every $h \in \mathbb{N}$.

Since $u_h \to u$ in $L^1(B)$, we have

$$\bar{u}_h = \frac{1}{|B|}\int_B u_h\,\mathrm{d}x \to \bar{u} = \frac{1}{|B|}\int_B u\,\mathrm{d}x. \tag{4.3}$$

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By the Poincaré–Wirtinger inequality, there exists $m_1 \in \mathbb{R}$ such that

$$\|u_h - \bar{u}_h\|_{W^{1,p}(B)} \leqslant m_1 \int_B |Du_h|^p \, \mathrm{d}x \leqslant m_1 m. \tag{4.4}$$

By (4.3) and (4.4), we can suppose that $\{u_h\}_h$ is bounded in $W^{1,p}(B)$, so that, by the reflexivity of this space, we can assume that

$$u_h \rightharpoonup u \quad \text{weakly in } W^{1,p}(B). \tag{4.5}$$

Then we have $u \in W^{1,p}(B)$; a contradiction. □

PROPOSITION 4.2. *Let $u \in W^{1,p}(B)$ and let w be defined by (3.45). If $\bar{F}(u) < +\infty$, then $\xi w \in \mathcal{C}(R)$ and*

$$\bar{F}(u) \geqslant \int_B |Du|^p \, \mathrm{d}x + \int_R \xi \left|\frac{\partial w}{\partial \rho}\right| + \min_{c \in \mathbb{R}} \int_0^{2\pi} \xi(\theta)|w^+(\theta) - c| \, \mathrm{d}\theta. \tag{4.6}$$

Proof. Let $\{u_h\}_h \subset \mathrm{Lip}_{\mathrm{loc}}$ be such that $u_h \to u$ in $L^1(B)$. Then we have

$$\int_B |Du|^p \, \mathrm{d}x \leqslant \liminf \int_B |Du_h|^p \, \mathrm{d}x. \tag{4.7}$$

Moreover, we have, for every $h \in \mathbb{N}$,

$$\int_B g\left(\frac{x}{|x|}\right) \frac{|\langle Du_h, x\rangle|}{|x|^2} \, \mathrm{d}x = \int_R \xi \left|\frac{\partial w_h}{\partial \rho}\right| \mathrm{d}\rho \, \mathrm{d}\theta, \tag{4.8}$$

where

$$w_h(\rho, \theta) = u_h(\rho\cos\theta, \rho\sin\theta), \quad \rho \in (0,1), \quad \theta \in \mathbb{R}. \tag{4.9}$$

Let $c_h = u_h(0) = w_h(0, \theta)$, $\theta \in \mathbb{R}$.

Let us prove that $\{c_h\}_h$ has a converging subsequence.

If not, we will have $\lim_h |c_h| = +\infty$. Let $A \subset R$ be such that

$$\left.\begin{aligned} &|A| > 0, \\ &\exists\, M > 0 : w(\rho, \theta) \leqslant M \quad \text{a.e. } (\rho, \theta) \in A. \end{aligned}\right\} \tag{4.10}$$

Since we can assume that $u_h \rightharpoonup u$ weakly in $W^{1,p}(B)$, we have that $w_h \to w$ in $L^1(R)$. Therefore, $w_h \to w$ almost uniformly in R and we can find A' such that

$$\left.\begin{aligned} &|A'| > 0, \\ &|w_h(\rho, \theta)| \leqslant M + 1 \quad \text{for every } h \geqslant \bar{h},\ (\rho, \theta) \in A'. \end{aligned}\right\} \tag{4.11}$$

If π_2 is defined by $\pi_2(\rho, \theta) = \theta$, we have $|\pi_2(A')| > 0$.

Let $\mu > 0$ be such that $\delta = |\{\theta \in [0, 2\pi] : \xi(\theta) < \mu\}| < |\pi_2(A')|$.

Now, if $\lim_h |c_h| = +\infty$, we get

$$\begin{aligned} \int_R \xi \left|\frac{\partial w_h}{\partial \rho}\right| \mathrm{d}\rho \, \mathrm{d}\theta &\geqslant \mu \int_0^1 \int_{\{\theta \in \pi_2(A') : \xi(\theta) \geqslant \mu\}} \left|\frac{\partial w_h}{\partial \rho}\right| \mathrm{d}\theta \, \mathrm{d}\rho \\ &= \mu \int_{\{\theta \in \pi_2(A') : \xi(\theta) \geqslant \mu\}} \int_0^1 \left|\frac{\partial w_h}{\partial \rho}\right| \mathrm{d}\rho \, \mathrm{d}\theta \\ &\geqslant \mu(|c_h| - M - 1)(|\pi_2(A')| - \delta) \to +\infty \quad \text{as } h \to \infty. \end{aligned} \tag{4.12}$$

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In fact, for every $\theta \in \pi_2(A')$, there exists $\rho(\theta)$ such that $(\rho(\theta), \theta) \in A'$ and

$$\int_0^1 \left|\frac{\partial w_h}{\partial \rho}\right| \mathrm{d}\rho \geqslant \int_0^{\rho(\theta)} \left|\frac{\partial w_h}{\partial \rho}\right| \mathrm{d}\rho \geqslant |c_h| - M - 1. \tag{4.13}$$

Therefore, if $\liminf_h F(u_h) < +\infty$, then we can assume (up to a subsequence) that $c_h \to \bar{c}$. If we set $y_h(\rho, \theta) = w_h + \bar{c} - c_h$, we have that $\xi y_h \to \xi w^{0,\bar{c}}$ in $L^1(\tilde{R})$.

Then, by theorem 2.4, proposition 2.8, lemma 2.9 and definition 2.11, we get

$$\begin{aligned}
\liminf_h \int_R \xi \left|\frac{\partial w_h}{\partial \rho}\right| \mathrm{d}\rho\,\mathrm{d}\theta &= \liminf_h \int_{\tilde{R}} \xi \left|\frac{\partial w_h}{\partial \rho}\right| \mathrm{d}\rho\,\mathrm{d}\theta \\
&= \liminf_h \int_{\tilde{R}} \xi \left|\frac{\partial y_h}{\partial \rho}\right| \mathrm{d}\rho\,\mathrm{d}\theta \\
&\geqslant \int_{\tilde{R}} \left|\frac{\partial (w\xi)}{\partial \rho}\right| \\
&= \int_R \xi \left|\frac{\partial w}{\partial \rho}\right| + \int_0^{2\pi} \xi(\theta)|w^+(\theta) - \bar{c}|\,\mathrm{d}\theta \\
&\geqslant \int_R \xi \left|\frac{\partial w}{\partial \rho}\right| + \min_{c \in \mathbb{R}} \int_0^{2\pi} \xi(\theta)|w^+(\theta) - c|\,\mathrm{d}\theta.
\end{aligned} \tag{4.14}$$

The thesis easily follows by (4.7) and (4.14). □

REMARK 4.3. If we set

$$\psi(c) = \int_0^{2\pi} \xi(\theta)|w^+(\theta) - c|\,\mathrm{d}\theta,$$

then if $\psi \neq +\infty$, it can easily be seen that ψ is continuous and coercive so that there exists $\bar{c}$ such that $\psi(\bar{c}) = \min_{c\in\mathbb{R}} \psi(c)$.

In order to prove the opposite inequality we need some lemmas.

LEMMA 4.4. *Let $\{a_{i,j}\}_{i,j\in\mathbb{N}}$ be a double-indexed sequence of non-negative real numbers satisfying the following properties*

$$a_{i,j} \geqslant a_{i+1,j} \quad \forall i, j \in \mathbb{N} \quad \textit{and} \quad \lim_i a_{i,j} = 0 \quad \forall j \in \mathbb{N}, \tag{4.15}$$

$$\limsup_j a_{i,j} = c_i \quad \forall i \in \mathbb{N} \qquad \textit{and} \quad \lim_i c_i = 0. \tag{4.16}$$

Let us set

$$b_i = \sup_{j\in\mathbb{N}} a_{i,j}. \tag{4.17}$$

Then

$$\lim_i b_i = 0. \tag{4.18}$$

Proof. Let $\sigma > 0$. By (4.16), there exists i_0 such that $0 \leqslant c_i < \frac{1}{2}\sigma$ $\forall i \geqslant i_0$ and there exists j_0 such that $a_{i_0,j} < \sigma$ $\forall j \geqslant j_0$. Therefore, by (4.15), we get

$$0 \leqslant a_{i,j} < \sigma \quad \forall i \geqslant i_0, \quad j \geqslant j_0. \tag{4.19}$$

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Moreover, again by (4.15), there exists i_1 such that

$$0 \leqslant a_{i,j} < \sigma \quad \forall i \geqslant i_1, \quad j = 1, \dots, j_0 - 1. \tag{4.20}$$

By (4.19) and (4.20), we have

$$0 \leqslant b_i \leqslant \sigma \quad \forall i \geqslant \max(i_0, i_1) \tag{4.21}$$

and, by the arbitrariness of σ, the thesis. □

LEMMA 4.5. *Let $\mathcal{L}(\mathbb{R}^n)$ be the family of Lebesgue measurable sets of $\mathbb{R}^n$ and let $\{\psi_{A,j}\}_{A\in\mathcal{L}(\mathbb{R}^n),j\in\mathbb{N}}$ be a family of functions defined on $\mathbb{R}^n$ satisfying the following properties*

$$\psi_{A,j} \text{ is a measurable function } \forall A \in \mathcal{L}(\mathbb{R}^n),\ j \in \mathbb{N}; \tag{4.22}$$

there exists a function $\phi(x) \in L^1(\mathbb{R}^n)$ such that

$$0 \leqslant \psi_{A,j}(x) \leqslant \phi(x) \quad \text{a.e. } x \in \mathbb{R}^n \quad \forall A \in \mathcal{L}(\mathbb{R}^n), \quad j \in \mathbb{N}; \tag{4.23}$$

$$A \subseteq A' \quad \Rightarrow \quad \psi_{A,j}(x) \leqslant \psi_{A',j}(x) \quad \text{a.e. } x \in \mathbb{R}^n; \tag{4.24}$$

$$|A| \to 0 \quad \Rightarrow \quad \int_{\mathbb{R}^n} \psi_{A,j}(x)\,\mathrm{d}x \to 0 \quad \forall j \in \mathbb{N}; \tag{4.25}$$

$$\psi_{A,j}(x) \underset{j}{\to} 0 \quad \text{a.e. } x \notin A. \tag{4.26}$$

Then, if we set

$$\varphi(\delta) = \sup_{|A|<\delta,\, j\in\mathbb{N}} \int_{\mathbb{R}^n} \psi_{A,j}(x)\,\mathrm{d}x, \tag{4.27}$$

we have that $\varphi(\delta)$ decreases to zero as $\delta \to 0^+$.

Proof. Since φ is an increasing function of δ, to prove the thesis is sufficient to show that there exists a sequence $\{\delta_h\}_h$ such that $\delta_h \to 0^+$ and $\lim_h \varphi(\delta_h) = 0$.

Let $\{\delta_h\}_h$ be a sequence of non-negative real numbers such that $\sum_{h=0}^{\infty} \delta_h < +\infty$. By (4.27), for every $h \in \mathbb{N}$ there exists $A_h \in \mathcal{L}(\mathbb{R}^n)$, $j_h \in \mathbb{N}$, such that

$$|A_h| < \delta_h, \qquad \int_{\mathbb{R}^n} \psi_{A_h,j_h}(x)\,\mathrm{d}x > \varphi(\delta_h) - \frac{1}{h}. \tag{4.28}$$

Let us now consider $B_i = \bigcup_{k\geqslant i} A_k$ and let us define

$$a_{i,j} = \int_{\mathbb{R}^n} \psi_{B_i,j}(x)\,\mathrm{d}x \quad \forall i, j \in \mathbb{N}. \tag{4.29}$$

By (4.23), (4.26) and Fatou's lemma, we have

$$\limsup_j a_{i,j} \leqslant \int_{B_i} \phi(x)\,\mathrm{d}x. \tag{4.30}$$

Moreover, since $\{B_i\}_i$ is decreasing and

$$|B_i| \leqslant \sum_{h\geqslant i} |A_h| \leqslant \sum_{h\geqslant i} \delta_h \underset{i}{\to} 0,$$

by (4.25) we obtain

$$\lim_i a_{i,j} = \lim_i \int_{\mathbb{R}^n} \psi_{B_i,j}(x)\,\mathrm{d}x = 0 \quad \forall j \in \mathbb{N}. \tag{4.31}$$

Let us now set $b_h = \sup_{j\in\mathbb{N}} a_{h,j}$. By (4.30) and (4.31), the hypotheses of lemma 4.4 are satisfied. We then get

$$\lim_h b_h = 0, \tag{4.32}$$

and by the definition of b_h and by (4.24) we have

$$b_h \geqslant a_{h,j_h} \geqslant \int_{\mathbb{R}^n} \psi_{A_h,j_h}(x)\,\mathrm{d}x > \varphi(\delta_h) - \frac{1}{h}. \tag{4.33}$$

By (4.33), we get $\lim_h \varphi(\delta_h) = 0$ and the thesis. □

PROPOSITION 4.6. *Let $u \in W^{1,p}(B) \cap L^\infty(B)$ and let w be defined by (3.45). If $\xi w \in \mathcal{C}(R)$, then $\bar{F}(u) < +\infty$ and*

$$\bar{F}(u) \leqslant \int_B |Du|^p\,\mathrm{d}x + \int_R \xi(\theta)\left|\frac{\partial w}{\partial \rho}\right| \mathrm{d}\rho\,\mathrm{d}\theta + \min_{c\in\mathbb{R}} \int_0^{2\pi} \xi(\theta)|w^+(\theta) - c|\,\mathrm{d}\theta. \tag{4.34}$$

Proof. Let $w^{\varepsilon,c}$, $w^c_{\varepsilon,\alpha}$ be defined by (3.2) and (3.3). We define

$$u_{\varepsilon,\alpha}(x_1,x_2) = w^c_{\varepsilon,\alpha}(J(x_1,x_2)), \tag{4.35}$$

where J is the usual change of variables in polar coordinates. By (3.2), $u_{\varepsilon,\alpha}$ is just defined in a neighbourhood of $\bar{B}$, but we can easily modify it outside of $\bar{B}$ in order to have $u_{\varepsilon,\alpha} \in \mathrm{Lip}_{\mathrm{loc}}$.

We intend to prove that, if $\alpha < p' - 1 = p'/p$ and ε takes its values in a sequence decreasing to zero, then

$$\int_B |Du_{\varepsilon,\alpha}|^p\,\mathrm{d}x \xrightarrow[\varepsilon\to 0^+]{} \int_B |Du|^p\,\mathrm{d}x. \tag{4.36}$$

For every $\lambda > 0$, by (3.46), $w \in W^{1,p}(R_\lambda)$. Therefore, we can assume, up to subsequences, that

$$\left.\begin{aligned} w^c_{\varepsilon,\alpha} &\xrightarrow[\varepsilon\to 0^+]{} w && \text{a.e. } (\rho,\theta)\in R_\lambda,\\ \frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho} &\xrightarrow[\varepsilon\to 0^+]{} \frac{\partial w}{\partial \rho} && \text{a.e. } (\rho,\theta)\in R_\lambda,\\ \frac{\partial w^c_{\varepsilon,\alpha}}{\partial\theta} &\xrightarrow[\varepsilon\to 0^+]{} \frac{\partial w}{\partial \theta} && \text{a.e. } (\rho,\theta)\in R_\lambda. \end{aligned}\right\} \tag{4.37}$$

Then, by a diagonal method, we can assume that

$$\left.\begin{aligned} w^c_{\varepsilon,\alpha} &\xrightarrow[\varepsilon\to 0^+]{} w && \text{a.e. } (\rho,\theta)\in R,\\ Du_{\varepsilon,\alpha} &\xrightarrow[\varepsilon\to 0^+]{} Du && \text{a.e. } x\in B. \end{aligned}\right\} \tag{4.38}$$

Therefore, by Vitali's convergence theorem, to get (4.36), we only need to check that $\int |Du_{\varepsilon,\alpha}|^p\,\mathrm{d}x$ are uniformly absolutely continuous. If we express the integrals

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in polar coordinates, we have to show that

$$\int\left(\left(\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right)^2+\frac{1}{\rho^2}\left(\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\theta}\right)^2\right)^{p/2}\rho\,\mathrm{d}\rho\,\mathrm{d}\theta \tag{4.39}$$

are uniformly absolutely continuous. Equivalently, we will prove that

$$\int\left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right|^p\rho\,\mathrm{d}\rho\,\mathrm{d}\theta,\qquad \int\left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\theta}\right|^p\rho^{1-p}\,\mathrm{d}\rho\,\mathrm{d}\theta \tag{4.40}$$

are also.

We begin with

$$\int\left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial\rho}\right|^p\rho\,\mathrm{d}\rho\,\mathrm{d}\theta.$$

Let $A\subseteq R$ and set $\eta=\varepsilon^\alpha$.

Then

$$I^{A,\varepsilon}=\int_A\left|\int_{\mathbb{R}^2}\frac{\partial\tau_\eta}{\partial\rho}(\rho-\rho',\theta-\theta')w^{\varepsilon,c}(\rho',\theta')\,\mathrm{d}\rho'\mathrm{d}\theta'\right|^p\rho\,\mathrm{d}\rho\,\mathrm{d}\theta\leqslant I_1(\varepsilon)+I_2(\varepsilon),$$

where

$$I_1(\varepsilon)=\int_0^{2\pi}\int_{\varepsilon-\eta}^{\varepsilon+\eta}\left|\int_{\mathbb{R}^2}\frac{\partial\tau_\eta}{\partial\rho}(\rho-\rho',\theta-\theta')w^{\varepsilon,c}(\rho',\theta')\,\mathrm{d}\rho'\mathrm{d}\theta'\right|^p\rho\,\mathrm{d}\rho\,\mathrm{d}\theta \tag{4.41}$$

and

$$I_2(\varepsilon)=\int_{A\cap R_{\varepsilon+\eta}}\left|\int_{\mathbb{R}^2}\tau_\eta(\rho-\rho',\theta-\theta')\frac{\partial w}{\partial\rho}(\rho',\theta')\,\mathrm{d}\rho'\mathrm{d}\theta'\right|^p\rho\,\mathrm{d}\rho\,\mathrm{d}\theta. \tag{4.42}$$

Since $w^{\varepsilon,c}$ is bounded,

$$\left|\left(\frac{\partial\tau_\eta}{\partial\rho}\right)*w^{\varepsilon,c}\right|\leqslant\frac{m_1}{\eta},\qquad m_1\in\mathbb{R}^+,$$

and

$$I_1(\varepsilon)\leqslant m_2\eta^{-p}\int_{\varepsilon-\eta}^{\varepsilon+\eta}\rho\,\mathrm{d}\rho\leqslant m_3\varepsilon\eta^{1-p}=m_3\varepsilon^{1+\alpha(1-p)}. \tag{4.43}$$

We note that

$$1+\alpha(1-p)>0\quad\Leftrightarrow\quad\alpha<\frac{1}{p-1}=\frac{p'}{p}.$$

Then we have

$$\begin{aligned}I_2(\varepsilon)&\leqslant\int_{A\cap R_{\varepsilon+\eta}}\rho\left(\int_{\mathbb{R}^2}\tau_\eta(\rho-\rho',\theta-\theta')\left|\frac{\partial w}{\partial\rho}(\rho',\theta')\right|^p\mathrm{d}\rho'\mathrm{d}\theta'\right)\mathrm{d}\rho\,\mathrm{d}\theta\\&=\int_{\mathbb{R}^2}\left|\frac{\partial w}{\partial\rho}(\rho',\theta')\right|^p\left(\int_{A\cap R_{\varepsilon+\eta}}\rho\tau_\eta(\rho-\rho',\theta-\theta')\,\mathrm{d}\rho\,\mathrm{d}\theta\right)\mathrm{d}\rho'\mathrm{d}\theta'.\end{aligned} \tag{4.44}$$

Let $\{\varepsilon_j\}_j$ be the sequence of values taken by ε and let us define

$$\psi_{A,j}(\rho',\theta')=\left|\frac{\partial w}{\partial\rho}(\rho',\theta')\right|^p\left(\int_{A\cap R_{\varepsilon_j+\eta}}\rho\tau_\eta(\rho-\rho',\theta-\theta')\,\mathrm{d}\rho\,\mathrm{d}\theta\right). \tag{4.45}$$

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Taking into account that, by the linearity of ρ, the last integral is less than ρ', we have

$$\psi_{A,j} \leqslant \left|\frac{\partial w}{\partial \rho}\right|^p \rho' \chi_{(0,2)\times(-\pi,3\pi)} \in L^1(\mathbb{R}^2). \tag{4.46}$$

Hypotheses (4.24) and (4.25) are quite obviously satisfied.

If $(\rho', \theta') \in A^c$ is a Lebesgue point of χ_A, we get

$$\begin{aligned}\psi_{A,j}(\rho', \theta') &= \left|\frac{\partial w}{\partial \rho}(\rho', \theta')\right|^p \left(\int_{A\cap R_{\varepsilon_j+\eta}} \rho \tau_\eta(\rho-\rho', \theta-\theta')\,\mathrm{d}\rho\,\mathrm{d}\theta\right) \\ &\leqslant m\left|\frac{\partial w}{\partial \rho}(\rho', \theta')\right|^p \frac{1}{|B((\rho', \theta'), \eta)|}\int_{B((\rho',\theta'),\eta)} \chi_A(\rho, \theta)\,\mathrm{d}\rho\,\mathrm{d}\theta \underset{j}{\to} 0.\end{aligned} \tag{4.47}$$

Therefore, the hypotheses of lemma 4.5 are satisfied.

Let now take $\sigma > 0$. By (4.43), there exists j_0 such that

$$I_1 \leqslant \tfrac{1}{2}\sigma \quad \forall j \geqslant j_0. \tag{4.48}$$

By lemma 4.5, we have that there exists δ_0 such that

$$\sup_{|A|<\delta_0, j\in\mathbb{N}} \int_{\mathbb{R}^n} \psi_{A,j}(x)\,\mathrm{d}x < \tfrac{1}{2}\sigma. \tag{4.49}$$

By (4.48) and (4.49), we obtain that

$$|A| < \delta_0, \qquad j \geqslant j_0 \quad \Rightarrow \quad \int_A \left|\frac{\partial w^c_{\varepsilon_j,\alpha}}{\partial \rho}\right|^p \rho\,\mathrm{d}\rho\,\mathrm{d}\theta < \sigma. \tag{4.50}$$

By the absolute continuity of

$$\int \left|\frac{\partial w^c_{\varepsilon_j,\alpha}}{\partial \rho}\right|^p \rho\,\mathrm{d}\rho\,\mathrm{d}\theta, \quad \text{for } j = 1, \dots, j_0 - 1,$$

we have that there exists δ_1 such that

$$|A| < \delta_1, \qquad j < j_0 \quad \Rightarrow \quad \int_A \left|\frac{\partial w^c_{\varepsilon_j,\alpha}}{\partial \rho}\right|^p \rho\,\mathrm{d}\rho\,\mathrm{d}\theta < \sigma. \tag{4.51}$$

By equations (4.50) and (4.51), we get the uniformly absolute continuity of

$$\int \left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial \rho}\right|^p \rho\,\mathrm{d}\rho\,\mathrm{d}\theta.$$

The uniform absolute continuity of

$$\int \left|\frac{\partial w^c_{\varepsilon,\alpha}}{\partial \theta}\right|^p \rho^{1-p}\,\mathrm{d}\rho\,\mathrm{d}\theta$$

can be proved in a similar and easier way. In this case, we do not have to consider the first term I_1 and we define

$$\psi_{A,j}(\rho', \theta') = \left|\frac{\partial w}{\partial \theta}(\rho', \theta')\right|^p \left(\int_{A\cap R_{\varepsilon_j-\eta}} \rho^{1-p} \tau_\eta(\rho-\rho', \theta-\theta')\,\mathrm{d}\rho\,\mathrm{d}\theta\right). \tag{4.52}$$

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If $\rho' > \varepsilon_j - \eta$, then $B(\rho', \eta) \subset \{(\rho, \theta) : \rho > \frac{1}{2}\rho'\}$ (if j is large enough). Therefore, by (4.52), we have

$$\psi_{A,j} \leqslant 2^{p-1}\left|\frac{\partial w}{\partial \theta}\right|^p (\rho')^{1-p}\chi_{(0,2)\times(-\pi,3\pi)} \in L^1(\mathbb{R}^2). \tag{4.53}$$

The rest of the proof is analogous.

Now Vitali's convergence theorem gives (4.36). By (4.36), proposition 3.4 and arbitrariness of c, we easily obtain (4.34), provided that α satisfies $p'-2 \leqslant \alpha < p'-1$ and $\alpha > 1$. □

THEOREM 4.7. *Let $u \in W^{1,p}(B)$. Then $\bar{F}(u)$ is finite if and only if $\xi w \in \mathcal{C}(R)$. Moreover,*

$$\bar{F}(u) = \int_B |Du|^p + \int_R |\xi|\left|\frac{\partial w}{\partial \rho}\right| + \min_{c\in\mathbb{R}} \int_0^{2\pi} |\xi(\theta)||w^+(\theta) - c| \,\mathrm{d}\theta. \tag{4.54}$$

Proof. Equation (4.54) follows by propositions 4.2 and 4.6 and lemma 2.2 of [15]. □

Finally, we want to show that if we consider

$$f(x, z) = \frac{|x_2|}{|x|^3}|\langle z, x\rangle| + |z|^p$$

and we define $\bar{F}(\Omega, u)$ as the relaxed functional of $F(\Omega, u) = \int_\Omega f(x, Du)$ in the same setting of the previous section, $\bar{F}(\cdot, u^*)$ is not a measure, where $u^*(x_1, x_2) = |x_2|/|x|$.

THEOREM 4.8. *The functional $\bar{F}(\cdot, u^*)$, defined as above, is not sub-additive.*

Proof. Since $f(x, y) \geqslant 0$, we have that $\bar{F}(\cdot, u^*)$ is an increasing set function. Let us set

$$u^*(x_1, x_2) = \frac{|x_2|}{|x|},$$
$$B_1 = \{(x_1, x_2) \in B : x_2 > 0\},$$
$$B_2 = \{(x_1, x_2) \in B : x_2 < 0\}.$$

By observing that every sequence $\{u_h\}_h \subset \mathrm{Lip}_{\mathrm{loc}}$ converges to u^* in $L^1(B_1 \cup B_2)$, and is obviously also converging to u^* in $L^1(B)$, we deduce that

$$\bar{F}(B, u^*) = \bar{F}(B_1 \cup B_2, u^*). \tag{4.55}$$

We now prove that

$$\bar{F}(B, u^*) > \bar{F}(B_1, u^*) + \bar{F}(B_2, u^*). \tag{4.56}$$

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In fact, we have

$$\begin{aligned}\bar{F}(B_1,u^*) &= \bar{F}(B_2,u^*)\\ &= \int_{B_1} |Du^*|^p + \min_{c\in\mathbb{R}} \int_0^\pi |\sin(\theta)||w^+(\theta)-c|\,d\theta\\ &= \int_{B_1} |Du^*|^p + \tfrac{1}{6}(3\sqrt{3}-\pi), \end{aligned} \tag{4.57}$$

the second equality being achieved in a similar way to the computation already made for B.

Therefore, (4.56) follows by (1.6), (4.57) and the inequality $\pi > \frac{1}{3}(3\sqrt{3}-\pi)$.

From (4.56) and (4.55), we obtain

$$\bar{F}(B_1 \cup B_2, u^*) > \bar{F}(B_1,u^*) + \bar{F}(B_2,u^*) \tag{4.58}$$

and the thesis. □

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