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SOME ANALYTICAL PROPERTIES OF THE MATRIX RELATED TO Q-COLOURED DELANNOY NUMBERS

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Abstract The q-coloured Delannoy numbers $D_{n,k}(q)$ count the number of lattice paths from (0, 0) to (n, k) using steps (0, 1), (1, 0) and (1, 1), among which the (1, 1) steps are coloured with q colours. The focus of this paper is to study some analytical properties of the polynomial matrix $D(q) = [d_{n,k}(q)]_{n,k\geq 0} = [D_{n-k,k}(q)]_{n,k\geq 0}$, such as the strong q-log-concavity of polynomial sequences located in a ray or a transversal line of $D(q)$ and the q-total positivity of $D(q)$. We show that the zeros of all row sums $R_n(q) = \sum_{k=0}^{n} d_{n,k}(q)$ are in $(-\infty, -1)$ and are dense in the corresponding semi-closed interval. We also prove that the zeros of all antidiagonal sums $A_n(q) = \sum_{k=0}^{\lfloor n/2\rfloor} d_{n-k,k}(q)$ are in the interval $(-\infty, -1]$ and are dense there.

Keywords: strong q-log-concavity; q-total positivity; polynomial matrix; Delannoy triangle; polynomial with only real zeros

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1. Introduction

Delannoy numbers correspond to the number of paths from (0, 0) to (n, k), with jumps (0, 1), (1, 1), or (1, 0), which we denote by the sequence $(D_{n,k})_{n,k\in N}$. Then a recurrence follows that

$$D_{n,k} = D_{n-1,k} + D_{n,k-1} + D_{n-1,k-1}, \tag{1.1}$$

or a further expression

$$D_{n,k} = \sum_i \binom{k}{i}\binom{n+k-i}{k} = \sum_i \binom{n}{i}\binom{k}{i} 2^i. \tag{1.2}$$

For the historic and academic backgrounds of Delannoy numbers, and the biography of Henri Delannoy, we refer the reader to [1] and the bibliographic references therein. There have been a lot of research interests in Delannoy numbers for their nice properties. For instance, a recent work is dedicated to their analytic properties [21].

In particular, when $n = k$, $D(n, n)$ denotes the *central Delannoy numbers*. We refer the reader to [1, 16] for their combinatorial properties [17, 18], for their work on certain number-theoretic properties [19], for their biological applications in the alignments between DNA sequences, etc.

If all the (1, 1) steps, i.e. the diagonal ones, of a Delannoy path are coloured with q colours ($q \geq 0$), then we call it *q-coloured Delannoy path*. Let $D_{n,k}(q)$ denote the number of q-coloured Delannoy paths from (0, 0) to (n, k) in this case. Then, analogous to (1.1) and (1.2), respectively, we have

$$D_{n,k}(q) = D_{n-1,k}(q) + D_{n,k-1}(q) + qD_{n-1,k-1}(q), \tag{1.3}$$

and

$$D_{n,k}(q) = \sum_i \binom{k}{i}\binom{n+k-i}{k} q^i = \sum_i \binom{n}{i}\binom{k}{i}(q+1)^i. \tag{1.4}$$

As a polynomial, $D_{n,k}(q)$ has some nice properties, which is partly due to the fact that it is both a Gaussian hypergeometric function ${}_2F_1(-n, -k; 1; q+1)$ and a special Jacobi polynomial $P_n^{(0,-n-k-1)}(-2q-1)$. $D_{n,k}(q)$ also appears in chemical graph theory, as the Clar covering polynomial of one kind of hexagonal systems [7]. Moreover, $D_{n,k}(q)$ can be proved to have only real zeros by the Maló Theorem [10], which states that if both $\sum_{i=0}^{n} a_i q^i$ and $\sum_{j=0}^{m} b_j q^j$ have only real zeros then $\sum_{k=0}^{min\{n,m\}} a_k b_k q^k$ has only real zeros. It is also worth noting that many well-known combinatorial counting sequences are q-coloured Delannoy numbers. For example, $D_{n,k}(0)$ are the binomial coefficients and $D_{n,k}(1)$ are the Delannoy numbers [1]. In a sense, that endowing the diagonal steps with *being q-coloured* pleasantly brings more research materials to the existing setting. Our paper is to study some analytical properties of the matrix related to q-coloured Delannoy numbers.

The q-coloured Delannoy numbers constitute the square matrix

$$[D_{n,k}(q)]_{n,k\geq 0} = \begin{bmatrix} 1 & 1 & 1 & \cdots \\ 1 & 2+q & 3+2q & \\ 1 & 3+2q & 6+6q+q^2 & \\ \vdots & & & \ddots \end{bmatrix},$$

whereas our paper focuses on the following triangular matrix

$$D(q) := [d_{n,k}(q)]_{n,k\geq 0} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2+q & 1 & & \\ 1 & 3+2q & 3+2q & 1 & \\ \vdots & & & & \ddots \end{bmatrix}$$

which is derived by arranging the q-coloured Delannoy numbers in a triangle array, i.e. $d_{n,k}(q) = D_{n-k,k}(q)$. This matrix is more convenient for the following investigation than the former one (albeit more natural), and therefore is our protagonist here. It is interesting

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to mention in passing that $D(q)$ can unify some combinatorial triangles. For example, $D(0)$ is the well-known Pascal triangle, $D(1)$ is the Delannoy triangle [21, 25], $D(2)$ and $D(3)$ also could be found in [2] and [14, A081577, A081578].

The paper is organized as follows. In the next section, we show that the polynomial sequences located in a ray or a transversal line of $D(q)$ are strongly q-log-concave. Section 3 proves that $D(q)$ is q-totally positive. In § 4, we first show, for the row sums $R_n(q)$, that all their zeros lie in the open interval $(-\infty, -1)$ and are dense in the semi-closed interval $(-\infty, -1]$. And then, for the antidiagonal sums $A_n(q)$, we show that all zeros are in the interval $(-\infty, -1]$ and are dense there. At the end of this paper, we finish with a remark that the coefficients $r_{n,i}$ are asymptotically normal by central and local limit theorems.

2. Strong q-log-concavity of $D(q)$

Let $f(q)$ and $g(q)$ be two real polynomials in q. We say that $f(q)$ is *q-non-negative* if $f(q)$ has non-negative coefficients. Denote $f(q) \geq_q g(q)$ if the difference $f(q) - g(q)$ is q-non-negative. For a polynomial sequence $(f_n(q))_{n\geq 0}$, it is called *q-log-concave* (or *q-log-convex*) if

$$f_n(q)^2 \geq_q f_{n+1}(q)f_{n-1}(q) \ \left(\text{or } f_n(q)^2 \leq_q f_{n+1}(q)f_{n-1}(q)\right)$$

for $n \geq 1$. It is called *strongly q-log-concave* (or *strongly q-log-convex*) if

$$f_n(q)f_m(q) \geq_q f_{n+1}(q)f_{m-1}(q) \ \ (\text{or } f_n(q)f_m(q) \leq_q f_{n+1}(q)f_{m-1}(q))$$

for $n \geq m \geq 1$. Clearly, the strong q-log-concavity (strong q-log-convexity) of polynomial sequences implies the q-log-concavity (q-log-convexity), which further implies the log-concavity (log-convexity) for any fixed $q \geq 0$, however, not vice versa. The (strong) q-log-concavity has been extensively studied; see [5, 8, 13].

It is known that $D(0)$ is the Pascal triangle P. Su and Wang [15] proved the log-concavity of the sequence located in a transversal line of P or a line parallel to the boundary of P. Yu [24] pointed out that such properties also hold in $D(1)$ which is the Delannoy triangle.

The central coefficients $d_{2n,n}(q)$ of $D(q)$ are q-central Delannoy numbers

$$D_n(q) = \sum_{i\geq 0} \binom{n}{i}\binom{2n-i}{n} q^i.$$

Liu and Wang [9] proved that the sequence of q-central Delannoy numbers $(D_n(q))_{n\geq 0}$ is q-log-convex. Zhu [26, 27] later proved that $(D_n(q))_{n\geq 0}$ is strongly q-log-convex. Wang and Zhu [23] gave a stronger result that $(D_n(q))_{n\geq 0}$ forms a q-Stieltjes moment sequence, i.e., all minors of the corresponding Hankel matrix $[D_{i+j}(q)]$ are q-non-negative.

In this section, we aim to study the strong q-log-concavity of a polynomial sequence located in a ray or a transversal line of $D(q)$. Let $(d_{n_i,k_i}(q))_{i\geq 0}$ be such a sequence. Then $(n_i)_{i\geq 0}$ and $(k_i)_{i\geq 0}$ form two arithmetic sequences (see Figure 1). Clearly, the common difference of $(n_i)_{i\geq 0}$ can be assumed to be non-negative. Meanwhile the common difference of $(k_i)_{i\geq 0}$ can also be assumed to be non-negative without loss of generality since the symmetry of $D(q)$ leads to the fact that the sequences $(d_{n_i,k_i}(q))_{i\geq 0}$ and $(d_{n_i,n_i-k_i}(q))_{i\geq 0}$

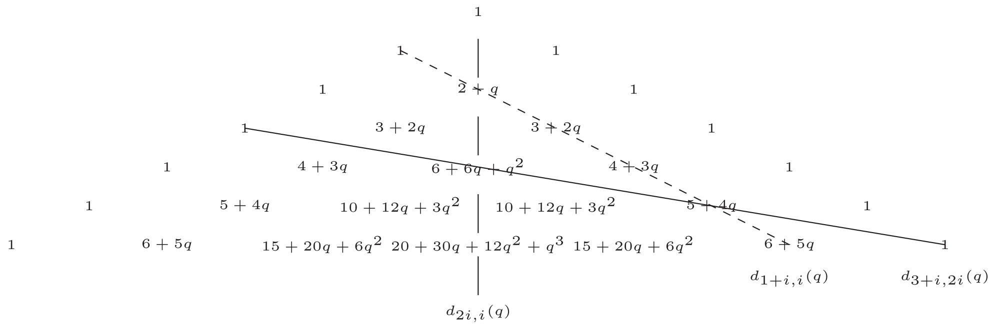

Figure 1. The symmetric isosceles triangle $D(q)$ of q-coloured Delannoy numbers.

are the same. Thus, to achieve our aim, it suffices to investigate the strong q-log-concavity of the sequence $(d_{n_0+ai,k_0+bi}(q))_{i\geq 0}$ for non-negative integers a and b, giving rise to our first main result of this paper.

Theorem 2.1. *Let n_0, k_0, a and b be four non-negative integers and $n_0 \geq k_0$, $a + b \neq 0$. Define the sequence*

$$S_i(q) = d_{n_0+ai,k_0+bi}(q), \quad i = 0, 1, 2, \ldots.$$

If $a \leq b$, then the polynomial sequence $(S_i(q))_{i\geq 0}$ is strongly q-log-concave.

Before a combinatorial proof of this theorem, we need to introduce a few notions. Let $\mathfrak{D}(n, k)$ denote the set of all q-coloured Delannoy paths from (s, t) to $(s + n, t + k)$ for fixed s and t. Note that $S_i(q)$ count the number of q-coloured Delannoy paths from $(0, 0)$ to $(n_0 - k_0 + (a - b)i,\ k_0 + bi)$. Hence, for convenience we let

$$\mathfrak{D}_i := \mathfrak{D}(n_0 - k_0 + (a - b)i, k_0 + bi).$$

Then we have

$$S_i(q) = \sum_{P\in\mathfrak{D}_i} w(P),$$

where the weight of path P, denoted by $w(P)$, is defined as the product of the weights of all its steps. Suppose that P has exactly k diagonal steps (i.e. $(1, 1)$ steps). Then

$$w(P) = q^k,$$

since the weight of each diagonal step in P is q, and the others 1. Moreover, $w(P, Q) = w(P)w(Q)$ is to denote the weight of a pair of q-coloured Delannoy paths in the following.

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Proof of Theorem 2.1. To show the strong q-log-concavity of $(S_i(q))_{i\geq 0}$, it suffices to show that

$$S_i(q)S_j(q) \geq_q S_{i+1}(q)S_{j-1}(q),$$

for $i \geq j$, i.e.,

$$\sum_{P\in\mathfrak{D}_i} w(P) \sum_{P\in\mathfrak{D}_j} w(P) \geq_q \sum_{P\in\mathfrak{D}_{i+1}} w(P) \sum_{P\in\mathfrak{D}_{j-1}} w(P). \tag{2.1}$$

It is equivalent to

$$\sum_{(Q_1,Q_2)\in(\mathfrak{D}_i,\mathfrak{D}_j)} w(Q_1,Q_2) \geq_q \sum_{(P_1,P_2)\in(\mathfrak{D}_{i+1},\mathfrak{D}_{j-1})} w(P_1,P_2). \tag{2.2}$$

Let $N_k(\mathfrak{D}_i, \mathfrak{D}_j)$ denote the number of pairs of paths with exactly k diagonal steps in the set $(\mathfrak{D}_i, \mathfrak{D}_j)$. So it needs to prove

$$\sum_k N_k(\mathfrak{D}_i,\mathfrak{D}_j)q^k \geq_q \sum_k N_k(\mathfrak{D}_{i+1},\mathfrak{D}_{j-1})q^k. \tag{2.3}$$

To this end, we construct an injection from $(\mathfrak{D}_{i+1}, \mathfrak{D}_{j-1})$ to $(\mathfrak{D}_i, \mathfrak{D}_j)$, i.e.,

$$\phi: \quad (P_1,P_2) \to (Q_1,Q_2),$$

such that

$$N_k(\mathfrak{D}_{i+1},\mathfrak{D}_{j-1}) \leq N_k(\mathfrak{D}_i,\mathfrak{D}_j). \tag{2.4}$$

Each pair of (P_1, P_2) in $(\mathfrak{D}_{i+1}, \mathfrak{D}_{j-1})$, as shown in Figure 2, follows such rules:

P$_1$: $(0, 0) \to (n_0 - k_0 + (a-b)(i+1),\ k_0 + b(i+1))$;
P$_2$: $((a-b)(i-j+1),\ b(i-j+1)) \to (n_0 - k_0 + (a-b)i,\ k_0 + bi)$.

Clearly, P_1 and P_2 must intersect at least one lattice point in the shadow area. Let A denote the first intersection point. Then we define the operation ϕ on (P_1, P_2) at the point A:

"Switch the initial segments of the two paths",

as shown in Figure 3. With this operation ϕ, we could obtain a corresponding pair $(Q_1, Q_2) \in (\mathfrak{D}_i, \mathfrak{D}_j)$, and

Q$_1$: $((a-b)(i-j+1),\ b(i-j+1)) \to (n_0 - k_0 + (a-b)(i+1),\ k_0 + b(i+1))$;
Q$_2$: $(0, 0) \to (n_0 - k_0 + (a-b)i,\ k_0 + bi)$.

For instance, let $n_0 = 10$, $k_0 = 3$, $a = 0$ and $b = 1$, and take $i = 2$, $j = 1$. Then $(P_1, P_2) \in (\mathfrak{D}_3, \mathfrak{D}_0)$, where P_1 goes from $(0, 0)$ to $(4, 6)$ and P_2 from $(-2, 2)$ to $(5, 5)$, as shown in Figure 4. The operation ϕ on $(P_1, P_2) \in (\mathfrak{D}_3, \mathfrak{D}_0)$ at the point A will lead to a pair $(Q_1, Q_2) \in (\mathfrak{D}_2, \mathfrak{D}_1)$ as shown in Figure 4.

Note that the location of the first intersection point remains invariant under the operation ϕ, which means ϕ is invertible and so that it is an injection. Meanwhile, it is easy

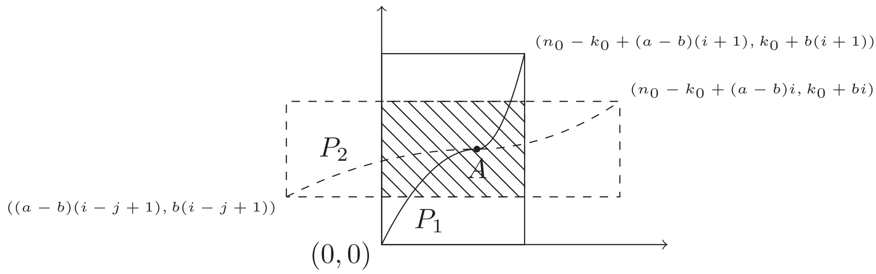

Figure 2. $(P_1, P_2) \in (\mathfrak{D}_{i+1}, \mathfrak{D}_{j-1})$.

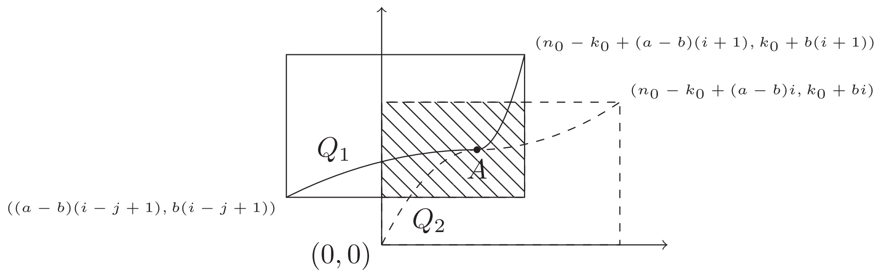

Figure 3. Operation ϕ on (P_1, P_2) in Figure 2.

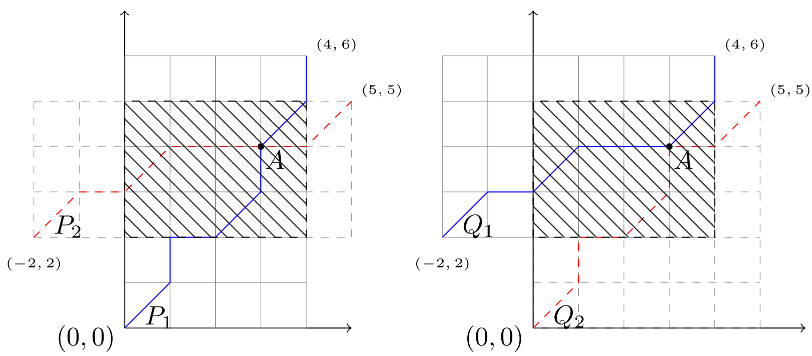

Figure 4. $(P_1, P_2) \in (\mathfrak{D}_3, \mathfrak{D}_0) \to (Q_1, Q_2) \in (\mathfrak{D}_2, \mathfrak{D}_1)$.

to check that the number of diagonal steps also remains invariant under the injection ϕ, i.e., the number of diagonal steps in (P_1, P_2) is the same as that in (Q_1, Q_2). Therefore, (2.4) follows, by which (2.1) can be obtained as desired. □

From Theorem 2.1, we have the following corollary immediately.

Corollary 2.2. *All the polynomial sequences located in a transversal of $D(q)$ or in a line parallel to the boundary of $D(q)$ are strongly q-log-concave.*

Note that $D(0)$ and $D(1)$ are Pascal triangle and Delannoy triangle, respectively. The log-concavity of the sequences in these two triangles was mentioned at the beginning

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of this section. $D(2)$ and $D(3)$ are also Pascal-like triangles and could be found in [14, A081577, A081578]. By Theorem 2.1, we can get the log-concavity of sequences in these two triangles.

Corollary 2.3. *All the sequences located in a transversal of $D(2)$ (or $D(3)$) or in a line parallel to the boundary of $D(2)$ (or $D(3)$) are log-concave.*

Remark 2.4. A polynomial sequence $(a_i(q))_{i\geq 0}$ is called a *q-Pólya frequency* (*q-PF* for short) sequence if all minors of the corresponding Toeplitz matrix $[a_{i-j}(q)]_{i,j\geq 0}$ are q-non-negative. In fact, the polynomial sequence $(S_i(q))_{i\geq 0}$ forms a q-PF sequence, which could be proved by the same technique used in the proof of Theorem 2 in [24].

3. q-total positivity of $D(q)$

Let $f(q)$ and $g(q)$ be two real polynomials in q. Let $M(q) = [m_{n,k}]_{n,k\geq 0}$ be the matrix whose entries are all real polynomials in q. We say that $M(q)$ is *q-totally positive* (*q-TP* for short) if all minors are q-non-negative.

Note that, since (1.4), the square matrix $[D_{n,k}(q)]_{n,k\geq 0} = PDP^T$, where P is the Pascal triangle and $D = \text{diag}(1, 1+q, (1+q)^2, (1+q)^3, \ldots)$. Hence the q-total positivity of $[D_{n,k}(q)]_{n,k\geq 0}$ follows immediately from the Cauchy–Binet formula and the total positivity of the Pascal triangle (i.e., all its minors are non-negative).

It is known that the triangle $D(q)$ is a Riordan array $(\frac{1}{1-x}, \frac{x+qx^2}{1-x})$ (see [12] for details). A *(proper) Riordan array*, denoted by $(d(x), h(x))$, is an infinite lower triangular matrix whose generating function of the kth column is $d(x)h^k(x)$ for $k = 0, 1, 2, \ldots$, where $d(0) = 1$, $h(0) = 0$ and $h'(0) \neq 0$. In this section, we consider the q-total positivity of $D(q)$. We first prove a lemma which is a q-analogy of Theorem 3 in [11].

Lemma 3.1. *Let $M(q) = (d(x), h(x))$ be a Riordan array, where $d(x) = \sum_{n\geq 0} d_n(q)x^n$ and $h(x) = \sum_{n\geq 0} h_n(q)x^n$. If the matrix*

$$\begin{bmatrix} d_0(q) & h_0(q) & & & \\ d_1(q) & h_1(q) & h_0(q) & & \\ d_2(q) & h_2(q) & h_1(q) & h_0(q) & \\ \vdots & & & & \ddots \end{bmatrix}$$

is q-TP, then so is the Riordan array $M(q)$.

Proof. Let $T(q) = (h(x), x) = [h_{i-j}(q)]_{i,j\geq 0}$ and $v(q) = (d_0(q), d_1(q), \ldots)^T$. Then

$$M(q) = (d(x), d(x)h(x), d(x)h^2(x), \ldots) = (v(q), T(q)v(q), T(q)^2v(q), \ldots).$$

Let $M_k(q)$ denote the submatrix $(v(q), T(q)v(q), \ldots, T(q)^{k-1}v(q))$ consisting of the first k columns of $M(q)$. Then

$$M_{k+1}(q) = (v(q), T(q)v(q), \ldots, T(q)^k v(q)) = (v(q), T(q)M_k(q))$$
$$= (v(q), T(q)) \begin{bmatrix} 1 & 0 \\ 0 & M_k(q) \end{bmatrix}.$$

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If $M_k(q)$ is q-TP, then so is $\left[\begin{smallmatrix}1 & 0\\ 0 & M_k(q)\end{smallmatrix}\right]$. The condition states that $(v(q), T(q))$ is q-TP. It follows that the product $M_{k+1}(q)$ is also q-TP from the classic Cauchy–Binet formula. Thus, the statement follows. □

Theorem 3.2. *The triangle $D(q)$ is q-totally positive.*

Proof. Note that $D(q) = (d(x), h(x)) = (\frac{1}{1-x}, \frac{x+qx^2}{1-x})$. Let $T(q) = (h(x), x)$ and $v(q) = (d_0(q), d_1(q), d_2(q), \ldots)^T$. By Lemma 3.1, it suffices to show that $(v(q), T(q))$ is q-TP. We have

$$(v(q), T(q)) = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1+q & 1 & & \\ 1 & 1+q & 1+q & 1 & \\ \vdots & & & & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & \\ \vdots & & & & \ddots \end{bmatrix} \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & q & 1 & & \\ & & q & 1 & \\ & & & \ddots & \ddots \end{bmatrix}.$$

One can check that both matrices on the right-hand side are q-TP. Therefore, $(v(q), T(q))$ is q-TP by the classic Cauchy–Binet formula, as required. □

4. Zeros of row sums

Let $R_n(q) = \sum_i r_{n,i} q^i$ be the sum of the nth row of $D(q)$, i.e.,

$$R_n(q) = \sum_{k=0}^{n} d_{n,k}(q).$$

The first few entries of $(R_n(q))_{n\geq 0}$ are $(1, 2, 4+q, 8+4q, \ldots)$. The coefficient matrix of $R_n(q)$ is defined by the matrix

$$[r_{n,i}]_{n,i\geq 0} = \begin{bmatrix} 1 & & \\ 2 & & \\ 4 & 1 & \\ 8 & 4 & \\ \vdots & & \ddots \end{bmatrix}.$$

Note that the polynomial $D_{n,k}(q)$ satisfies the recurrence (1.3), hence

$$d_{n,k}(q) = d_{n-1,k-1}(q) + d_{n-1,k}(q) + q d_{n-2,k-1}(q). \tag{4.1}$$

Thus, the row sum $R_n(q)$ satisfies the simple recurrence

$$R_n(q) = 2R_{n-1}(q) + qR_{n-2}(q)$$

with $R_1(q) = 1$, $R_2(q) = 2$.

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Let $(f_n(z))_{n\geq 0}$ be a sequence of complex polynomials. We say that the complex number z is a *limit of zeros* of the sequence $(f_n(z))_{n\geq 0}$ if there is such a sequence $(z_n)_{n\geq 0}$ that $f_n(z_n) = 0$ and $z_n \to z$ as $n \to +\infty$. Suppose now that $(f_n(z))_{n\geq 0}$ is a sequence of polynomials satisfying the recursion

$$f_{n+k}(z) = -\sum_{j=1}^{k} c_j(z) f_{n+k-j}(z)$$

where $c_j(z)$ are polynomials in z. Let $\lambda_j(z)$ be all roots of the associated characteristic equation $\lambda^k + \sum_{j=1}^{k} c_j(z)\lambda^{k-j} = 0$. It is well known that if $\lambda_j(z)$ are distinct, then

$$f_n(z) = \sum_{j=1}^{k} \alpha_j(z)\lambda_j^n(z), \tag{4.2}$$

where $\alpha_j(z)$ is determined from the initial conditions.

Lemma 4.1 (Beraha *et al.* [4, Theorem]**).** *Under the non-degeneracy requirements that in (4.2) no $\alpha_j(z)$ is identically zero and that no pair $i \neq j$ is $\lambda_i(z) \equiv \omega\lambda_j(z)$ for some $\omega \in \mathbb{C}$ of unit modulus, then z is a limit of zeros of $(f_n(z))_{n\geq 0}$ if and only if either*

(i) *two or more of the $\lambda_i(z)$ are of equal modulus, and strictly greater (in modulus) than the others; or*

(ii) *for some j, $\lambda_j(z)$ has modulus strictly greater than all the other $\lambda_i(z)$ have, and $\alpha_j(z) = 0$.*

Theorem 4.2. *Zeros of row sum $R_n(q)$ are real, distinct in $(-\infty, -1)$ and are dense in the corresponding semi-closed interval $(-\infty, -1]$.*

Proof. We first need to prove that

$$R_n(q) = 4\prod_{k=1}^{\lfloor n/2\rfloor}\left(1 + q\cos^2\frac{k\pi}{n+1}\right), \tag{4.3}$$

for which we only demonstrate the case that n is even in the following since it is quite similar for odd n.

Note that

$$R_n(q) = 2R_{n-1}(q) + qR_{n-2}(q)$$

with $R_1 = 1$, $R_2 = 2$. Hence the Binet form of the row sums is

$$R_n(q) = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}, \tag{4.4}$$

where

$$\lambda_{1,2} = 1 \pm \sqrt{1+q} \tag{4.5}$$

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are the roots of the characteristic equation $\lambda^2 - 2\lambda - q = 0$. Let $\omega_k = e^{\frac{2k\pi i}{n+1}}$. Then $\lambda^{n+1} - 1 = \prod_{k=1}^{n+1}(\lambda - \omega_k)$. Note that

$$(\lambda - \omega_k)(\lambda - \omega_{n+1-k}) = \lambda^2 - 2\lambda \cos\frac{k\pi}{n+1} + 1 = (\lambda+1)^2 - 4\lambda\cos^2\frac{k\pi}{n+1}.$$

Since n is even, we have

$$\lambda^{n+1} - 1 = (\lambda - 1)\prod_{k=1}^{n/2}\left((\lambda+1)^2 - 4\lambda\cos^2\frac{k\pi}{n+1}\right),$$

and hence

$$\lambda_1^{n+1} - \lambda_2^{n+1} = (\lambda_1 - \lambda_2)\prod_{k=1}^{n/2}\left((\lambda_1+\lambda_2)^2 - 4\lambda_1\lambda_2\cos^2\frac{k\pi}{n+1}\right).$$

Since $\lambda_1 + \lambda_2 = 2$ and $\lambda_1\lambda_2 = -q$, we have

$$R_n(q) = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2} = \prod_{k=1}^{n/2}\left(4 + 4q\cos^2\frac{k\pi}{n+1}\right).$$

Denote $z_{n,k} = -1/\cos^2\frac{k\pi}{n+1}$, $k = 1, 2, \cdots, n/2$. Then the polynomial $R_n(q)$ has distinct real zeros $z_{n,1} > z_{n,2} > \cdots > z_{n,n/2}$. Since

$$\lim_{n\to\infty} z_{n,1} = -\infty \text{ and } \lim_{n\to\infty} z_{n,n/2} = -1,$$

all zeros of $R_n(q)$ are in $(-\infty, -1)$.

We proceed to prove that each $q \in (-\infty, -1]$ is a limit of zeros of the sequence $(R_n(q))_{n\geq 0}$. The non-degeneracy conditions of Lemma 4.1 are clearly satisfied by (4.4). So the limits of zeros of $(R_n(q))_{n\geq 0}$ are those q for which $|\lambda_1(q)| = |\lambda_2(q)|$, i.e.,

$$|1 + \sqrt{q+1}| = |1 - \sqrt{q+1}|$$

by (4.5). In other words, $\sqrt{q+1}$ must be a pure imaginary. It follows that $q + 1 \leq 0$, i.e., $q \leq -1$. Then the proof is completed. □

Let $A_n(q)$ be the sum of the nth antidiagonal row of $D(q)$, i.e.,

$$A_n(q) = \sum_{k=0}^{\lfloor n/2 \rfloor} d_{n-k,k}(q).$$

The first few entries of $(A_n(q))_n$ are $(1, 1, 2, 3+q, 5+2q, \ldots)$. By (4.1), it is easy to check that $A_n(q)$ satisfies

$$A_n(q) = A_{n-1}(q) + A_{n-2}(q) + qA_{n-3}(q), \tag{4.6}$$

with $A_1 = 1$, $A_2 = 1$ and $A_3 = 2$.

Theorem 4.3. *Zeros of antidiagonal row sum $A_n(q)$ are in $(-\infty, -1]$ and are dense there.*

To prove this, we need the following lemma which can be found in [20, Theorem 3].

Lemma 4.4. *Consider the sequence of polynomials $\{P_n(q)\}_{n=0}^{\infty}$ generated by*

$$\sum_{n=0}^{\infty} P_n(q)x^n = \frac{1}{1+x+ax^2+qx^3}, \tag{4.7}$$

where $a \in \mathbb{R}$. If $-1 \le a \le 1/3$, then all the zeros of $P_n(q)$ are in the real interval

$$I_a = \left(-\infty, \frac{-2+9a-2\sqrt{(1-3a)^3}}{27}\right].$$

and are also dense in I_a.

Proof of Theorem 4.3. By (4.6), the generating function of $A_n(q)$ follows that

$$\sum_{n=0}^{\infty} A_n(q)x^n = \frac{1}{1-x-x^2-qx^3},$$

which can also be derived from (4.7) with substitutions $x \to -x$ and $a \to -1$. So $A_n(q)$ meets the condition of Lemma 4.4, and therefore, the zeros of $A_n(q)$ are in $(-\infty, -1]$ and are dense there. □

5. Remarks

In this section, we give some remarks on the asymptotic normality of coefficients of row sums. Let $a_{n,k}$ be a double-indexed sequence of non-negative numbers and let

$$p_{n,k} = \frac{a_{n,k}}{\sum_{j=0}^{n} a_{n,j}}$$

denote the normalized probabilities. Following Bender [3], we say that the sequence $a_{n,k}$ is *asymptotically normal by a central limit theorem* if

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \sum_{k\le \mu_n + x\sigma_n} p_{n,k} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \right| = 0, \tag{5.1}$$

where μ_n and σ_n^2 are the mean and variance of $a_{n,k}$, respectively. We say that $a_{n,k}$ is *asymptotically normal by a local limit theorem* on $\mathbb{R}$ if

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \sigma_n p_{n,\lfloor \mu_n + x\sigma_n \rfloor} - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| = 0. \tag{5.2}$$

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In this case,

$$a_{n,k} \sim \frac{e^{-x^2/2}\sum_{j=0}^{n} a_{n,j}}{\sigma_n\sqrt{2\pi}} \text{ as } n \to \infty,$$

where $k = \mu_n + x\sigma_n$ and $x = O(1)$. Clearly, the validity of (5.2) implies that of (5.1).

Many well-known combinatorial sequences enjoy central and local limit theorems, such as the binomial coefficients $\binom{n}{k}$, the signless Stirling numbers $c(n, k)$ of the first kind, the Stirling numbers $S(n, k)$ of the second kind, the Eulerian numbers $A(n, k)$ [6], and the Delannoy numbers $d(n, k)$ [21]. Besides, the asymptotic normality of Laplacian coefficients of graphs was discovered in [22]. A standard approach to demonstrating asymptotic normality is the following criterion (see [3, Theorem 2] for instance and [6, Example 3.4.2] for historical remarks).

Lemma 5.1. *Suppose that $S_n(q) = \sum_{k=0}^{n} a_{n,k}q^k$ have only real zeros and $S_n(q) = \prod_{i=1}^{n}(q + r_i)$, where all $a_{n,k}$ and r_i are non-negative. Let*

$$\mu_n = \sum_{i=1}^{n} \frac{1}{1 + r_i}$$

and

$$\sigma_n^2 = \sum_{i=1}^{n} \frac{r_i}{(1 + r_i)^2}.$$

Then if $\sigma_n^2 \to +\infty$, the numbers $a_{n,k}$ are asymptotically normal (by central and local limit theorems) with the mean μ_n and variance σ_n^2.

For the asymptotic normality of $r_{n,i}$ (the coefficients of row sums $R_n(q)$), we have the following result.

Theorem 5.2. *The coefficients $r_{n,i}$ are asymptotically normal (by central and local limit theorems) with the mean $\mu_n \sim \frac{(2-\sqrt{2})n}{4}$ and variance $\sigma_n^2 \sim \frac{n}{8\sqrt{2}}$.*

Its proof can be similarly produced by referring to [21, Theorem 3.2].

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