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RELIABILITY ANALYSIS OF k-OUT-OF-n SYSTEMS BASED ON A GROUPING OF COMPONENTS

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Abstract

In this paper we treat a two-stage grouping procedure of building a k-out-of-n system from several clusters of components. We use a static framework in which the component reliabilities are fixed. Under such a framework, we address the impact of the selecting strategies, the sampling probabilities, and the component reliabilities on the constructed system's reliability. An interesting finding is that the level of component reliabilities could be identified as a decisive factor in determining how the selecting strategies and the component reliabilities affect the system reliability. The new results generalize and extend those established earlier in the literature such as Di Crescenzo and Pellerey (2011), Hazra and Nanda (2014), Navarro, Pellerey, and Di Crescenzo (2015), and Hazra, Finkelstein, and Cha (2017). Several Monte Carlo simulation experiments are provided to illustrate the theoretical results.

Keywords: k-out-of-n structure; reliability; two-stage sampling

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1. Introduction

In reliability theory and practice the k-out-of-n system, including the series and parallel systems as its special cases, is a very popular fault-tolerant structure. Such a system functions if and only if k or more among the n components function. To evaluate the performance of a system, one widely employed method is to study the system state. One classical and popular model is to characterize the state of a system/component by 'on' or 'off', in accordance with it being working or not. Usually, the state of a system or component is random in practical applications, and the randomness of this binary state could be captured by a Bernoulli random variable. Specifically, denote ϕ_k as the structure function of a k-out-of-n system with component states $Z_1, \ldots, Z_n$. Then the random variable $\phi_k(Z_1, \ldots, Z_n)$ can be regarded as the system state and $q_i = \mathbb{E}Z_i$ is called the reliability of the ith component. Let $\boldsymbol{q} = (q_1, \ldots, q_n)$ be the vector of component reliabilities. If all components of the system function independently then the function $h_{k,n}\colon [0, 1]^n \mapsto [0, 1]$,

$$h_{k,n}(\boldsymbol{q}) = \mathbb{E}\phi_k(Z_1, \ldots, Z_n) = \sum_{\varepsilon_1+\cdots+\varepsilon_n \geq k} \prod_{i=1}^{n} q_i^{\varepsilon_i}(1-q_i)^{1-\varepsilon_i} \tag{1.1}$$

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is called the reliability of a k-out-of-n system, where ε_i is either 0 or 1 for $i = 1, \ldots, n$. Note, for example, that with this notation $h_{0,n}(\boldsymbol{q}) = 1$ and $h_{n+1,n}(\boldsymbol{q}) = 0$. For a comprehensive exposition on the reliability of k-out-of-n systems, we refer the reader to Kuo and Zuo (2003).

During the past two decades, numerous researches have studied the performance of a system, aiming to achieve maximal system reliability. Most of the literature on this topic concentrates on the stochastic comparisons of system lifetimes or improvements in system reliability from the viewpoint of deterministic assemble strategies. See, for example, Shaked and Shanthikumar (1992), Singh and Singh (1997), Misra *et al.* (2009), Ding and Li (2012), Da and Ding (2016), Fang and Li (2017), and the references therein. Typically, the most reliable system is found to be the one consisting of the most reliable components. However, as pointed out in Navarro *et al.* (2015), using the most reliable components to construct a system may not be possible in most practical situations, simply because it is hard to determine which components are the most reliable, and it may be necessary to use multiple types of components. In these situations, random assembling strategies could be the optimal alternative. There are few studies on system performance taking the viewpoint of random assemble strategies. Di Crescenzo (2007) considered the comparison of a pair of two-component series systems, wherein the units of the first system are less reliable than those of the second. It was shown that, by allowing each unit in the first system to be randomly selected from a set of components identical to the previous components, under suitable conditions, the first system's reliability can be improved and unexpectedly higher than that of the second system, even if each single unit is less reliable than those of the second system. This reveals that in some situations, a random assemble strategy might be a better option. This astonishing finding was further confirmed for series systems with dependent components by Navarro and Spizzichino (2010). Motivated by their works, Di Crescenzo and Pellerey (2011), Hazra and Nanda (2014), and Navarro *et al.* (2015) further considered the random assemble of components in coherent systems, and investigated the effect of random assemble strategy on the reliability of resulting systems, where it is assumed that the components could be randomly selected from two clusters.

Recently, Hazra *et al.* (2017) dealt with the problem of random assembling through the viewpoint of 'optimal grouping of components'. In their framework, it was assumed that there are m clusters of components, and all components in the same cluster have independent and identically distributed lifetimes, while components coming from different clusters could have different lifetime distributions. Following a two-stage selecting procedure, an n-component system is assembled by randomly grouping components from these clusters. Formally, the sampling procedure is carried out as follows.

- Firstly, select d ($d \le n$) clusters from the m clusters by simple random sampling with replacement having the sampling probabilities $p = (p_1, \ldots, p_m)$, where for each $i \in \{1, \ldots, m\}$, p_i is the probability of selecting the ith cluster.
- Secondly, draw ℓ_j components from the jth selected cluster, $j = 1, \ldots, d$. And then the selected components are used to construct a k-out-of-n system.

Following the notation in Hazra *et al.* (2017), we denote the two-stage sampling model as $\mathcal{M}_n^m(d \mid \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{q})$, where $\boldsymbol{q}$ is the *component reliabilities* of the m clusters, $\boldsymbol{p}$ is the *sampling probabilities*, and $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_d)$ is the *selecting strategy* such that $\sum_{i=1}^d \ell_i = n$ for $1 \le \ell_i \le n$.

In many engineering applications it is not uncommon that a reliability system could be constructed from a two-stage sampling model. Take the following two situations for example.

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- Consider a multicomponent network system formed by several groups of components suffering from external shock processes. The system will fail when at least a certain number of components fail. For simplicity, assume that shocks constitute the only cause of component failure, and one kind of shock only affects one component group. Furthermore, assume that when a shock arrives, it will 'kill' all components from the corresponding vulnerable group in the system. In this context, the whole system's resilience to shocks could be increased by randomly selecting a certain number of groups, and then construct the system by using components from the selected groups. Likely, for series systems it is optimal to make the whole system exposed to one single shock process, i.e. selecting all the components from one component group. As for parallel systems, selecting as many component groups as possible could be the optimal decision.
- In material science, with modern technologies, textile fabrics are usually made of blended yarns, for example, jute/cotton and polyester. Consider a yarn made of several different kinds of fibre from different producers. Assume that the strengths of the fibres from the same producer follow a common distribution and that the strengths of the fibres from different producers have different distributions. According to the well-known weakest-link theory, the strength of the yarn is determined by that of the fibre with the lowest strength, which could be essentially viewed as a series structure. Since it may not be possible to know which fibre from which producer has the largest strength, to obtain a stronger blended yarn, randomly choosing several producers and using fibres produced by them may be a wise choice.

Typically, many two-stage sampling models could be employed for assembling systems. All the admissible models are contained in the following set:

$$\mathfrak{M}=\bigg\{\boldsymbol{\mathcal{M}}_n^m(d \mid \boldsymbol{\ell},\boldsymbol{p},\boldsymbol{q})\colon \boldsymbol{\ell}=(\ell_1,\ldots,\ell_d),\sum_{i=1}^{d}\ell_i=n;\boldsymbol{p}\in[0,1]^m,\sum_{i=1}^{m}p_i=1;\boldsymbol{q}\in[0,1]^m\bigg\}.$$

The model with $d=1$ and $m=n$ is called 'mixing at the system level', whereas the model with $d=m=n$ is called 'mixing at the component level'. Focusing on series and parallel systems constructed from admissible two-stage sampling models, Hazra *et al.* (2017) investigated how the variation of selecting strategy $\boldsymbol{\ell}$ and sampling probabilities $\boldsymbol{p}$ have an impact on the reliability of the resulting series and parallel systems. However, not much is known for coherent systems having a more general structure, such as the k-out-of-n system.

In this paper we aim to further probe into the effects of different two-stage sampling strategies on the reliability of k-out-of-n systems. We will tackle this problem through a static and classic viewpoint, instead of the dynamic one focusing on stochastic behavior as the time varies employed in Hazra *et al.* (2017). Overall, we generalize the work of Hazra *et al.* (2017) to a more general k-out-of-n framework. Specifically, we fix the component reliabilities and study how the corresponding system reliability changes under different two-stage sampling models. In this vein, an interesting finding is that, when the component reliabilities are in different levels (i.e. $\boldsymbol{q}\in[0,(k-1)/(n-1)]^n$ and $\boldsymbol{q}\in[(k-1)/(n-1),1]^n$), the reliability of the constructed k-out-of-n could be affected by the selecting strategies/component reliabilities in entirely opposite directions. This provides us with some insightful observations into stochastic behavior of the k-out-of-n systems constructed from two-stage sampling.

The rest of this paper is organized as follows. In Section 2 we recall two important concepts and several technical lemmas. In Section 3 we investigate the effect of different selecting strategies, sampling probabilities, and component reliabilities on the performance of the constructed k-out-of-n systems. Several simulation experiments illustrating the theoretical results are presented in Section 4. Finally, some concluding remarks are presented in Section 5.

Throughout the paper, we use 'increasing' instead of 'nondecreasing' and 'decreasing' instead of 'nonincreasing'. Define $\mathrm{Re} = (-\infty, +\infty)$, $\mathcal{N}_m = \{1, 2, \ldots, m\}$, and $\mathbf{1} = (\underbrace{1, \ldots, 1}_{n})$. For $\mathcal{A} \subseteq \mathrm{Re}$,

$$\mathcal{I}_{\mathcal{A}^n} = \{(a_1, \ldots, a_n) \in \mathcal{A}^n : a_1 \leq \cdots \leq a_n\}$$

denotes the set of all increasing vectors on $\mathcal{A}^n$, and

$$\mathcal{D}_{\mathcal{A}^n} = \{(a_1, \ldots, a_n) \in \mathcal{A}^n : a_1 \geq \cdots \geq a_n\}$$

denotes the set of all decreasing vectors on $\mathcal{A}^n$. Moreover, for a real vector z, we respectively denote by z^i and z^{ij} the vectors obtained by deleting the ith argument, and the ith and jth arguments of z.

2. Preliminaries

In this section we recall several pertinent concepts and lemmas to be used in the sequel. We first introduce several majorization-type orders.

Definition 2.1. Let $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ be the increasing arrangement of components of the vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$. For vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{Re}^n$,

(i) $\boldsymbol{x}$ is said to majorize $\boldsymbol{y}$, denoted by $\boldsymbol{x} \succeq_{\mathrm{m}} \boldsymbol{y}$, if $\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$ for $j = 1, 2, \ldots, n-1$, and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$;

(ii) $\boldsymbol{x}$ is said to weakly supermajorize $\boldsymbol{y}$, denoted by $\boldsymbol{x} \succeq^{\mathrm{w}} \boldsymbol{y}$, if $\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$ for $j = 1, 2, \ldots, n$;

(iii) $\boldsymbol{x}$ is said to weakly submajorize $\boldsymbol{y}$, denoted by $\boldsymbol{x} \succeq_{\mathrm{w}} \boldsymbol{y}$, if $\sum_{i=j}^{n} x_{(i)} \leq \sum_{i=j}^{n} y_{(i)}$ for $j = 1, 2, \ldots, n$.

For any two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, it is evident that $\boldsymbol{x} \succeq_{\mathrm{m}} \boldsymbol{y}$ implies both $\boldsymbol{x} \succeq_{\mathrm{w}} \boldsymbol{y}$ and $\boldsymbol{x} \succeq^{\mathrm{w}} \boldsymbol{y}$, while the reverse is not true in general. The next two lemmas are involved in the preservation of majorization orders. The first lemma focuses on differentiable symmetric functions, and the second lemma focuses on possibly asymmetric functions. It should be noted that, unlike those given by Marshall *et al.* (2011), in Lemma 2.1 we provide a new version for functions defined on the set of increasing vectors rather than decreasing vectors. For a comprehensive discussion on the theory and applications of the majorization order, we refer the reader to Marshall *et al.* (2011).

Lemma 2.1. (Theorem 3.A.3 of Marshall *et al.* (2011).) *Let ϕ be a real-valued function, defined on $\mathcal{I}_{\mathcal{A}^n}$ and continuously differentiable on the interior of $\mathcal{I}_{\mathcal{A}^n}$. Then*

$$\phi(\boldsymbol{x}) \leq (\geq)\, \phi(\boldsymbol{y}) \quad \textit{whenever } \boldsymbol{x} \preceq_{\mathrm{m}} \boldsymbol{y}$$

if and only if $\phi_{(i)}(z)$ is increasing (decreasing) in $i \in \{1, \ldots, n\}$, where $\phi_{(i)}(z) = \partial\phi(z)/\partial z_i$ denotes the partial derivative of ϕ with respect to its ith argument.

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Lemma 2.2. (Theorem 3.A.7 of Marshall *et al.* (2011).) *Let ϕ be a real-valued function, defined and continuous on $\mathcal{I}_{\mathcal{A}^n}$ and continuously differentiable on the interior $\mathcal{I}_{\mathcal{A}^n}$. Let $\phi_{(i)}(z) = \partial\phi(z)/\partial z_i$ denote the partial derivative of ϕ with respect to the ith argument. Then*

$$\phi(\boldsymbol{x}) \le \phi(\boldsymbol{y}) \quad \textit{whenever } \boldsymbol{x} \preceq_{\mathrm{w}} \boldsymbol{y}$$

if and only if

$$\phi_{(n)}(z) \ge \phi_{(n-1)}(z) \ge \cdots \ge \phi_{(1)}(z) \ge 0$$

for all z in the interior of $\mathcal{I}_{\mathcal{A}^n}$. Similarly,

$$\phi(\boldsymbol{x}) \le \phi(\boldsymbol{y}) \quad \textit{whenever } \boldsymbol{x} \preceq^{\mathrm{w}} \boldsymbol{y}$$

if and only if

$$0 \ge \phi_{(n)}(z) \ge \phi_{(n-1)}(z) \ge \cdots \ge \phi_{(1)}(z)$$

for all z in the interior of $\mathcal{I}_{\mathcal{A}^n}$.

The following two lemmas are useful in deriving the main results in Section 3. The first lemma is due to Boland and Proschan (1983), and the second lemma turns out to be a direct consequence of the first lemma. Recall that $h_{k,n}(\boldsymbol{q})$ is the probability that at least k of n components with reliabilities $\boldsymbol{q} = (q_1, \ldots, q_n) \in [0, 1]^n$ work.

Lemma 2.3. (Lemma 2.3 of Boland and Proschan (1983).) *For $0 \le k \le n$ and $\boldsymbol{q} \in [0, 1]^n$, let*

$$h^*_{k,n}(\boldsymbol{q}) = h_{k,n}(\boldsymbol{q}) - h_{k+1,n}(\boldsymbol{q})$$

represent the probability that exactly k components work in a k-out-of-n system. By convention, $h^*_{-1,n}(\boldsymbol{q}) = 0$. Then

$$h^*_{k-1,n}(\boldsymbol{q}) \le (\ge)\, h^*_{k,n}(\boldsymbol{q})$$

whenever $q_i \ge (\le)\, k/(n+1)$ for all $i = 1, \ldots, n$.

Lemma 2.4. *For $0 \le k \le n$ and $\boldsymbol{q} \in [0, 1]^n$, $h^*_{k,n}(\boldsymbol{q}) = h_{k,n}(\boldsymbol{q}) - h_{k+1,n}(\boldsymbol{q})$ is increasing on $\boldsymbol{q} \in [0, k/n]^n$ and decreasing on $\boldsymbol{q} \in [k/n, 1]^n$.*

Proof. Note that, for any $1 \le i \le n-1$ and $\boldsymbol{q} \in [0, 1]^n$,

$$h^*_{k,n}(\boldsymbol{q}) = q_i h^*_{k-1,n-1}(\boldsymbol{q}^i) + (1 - q_i) h^*_{k,n-1}(\boldsymbol{q}^i).$$

Taking the partial derivative with respect to q_i, we obtain

$$\frac{\partial h^*_{k,n}(\boldsymbol{q})}{\partial q_i} = h^*_{k-1,n-1}(\boldsymbol{q}^i) - h^*_{k,n-1}(\boldsymbol{q}^i).$$

Consequently, the desired result immediately follows from Lemma 2.3. □

Lemma 2.4 is of independent interest. Let $h(\boldsymbol{q})$ be the reliability function of a coherent system, Barlow and Proschan (1981) defined the reliability importance of component i as $I_h(i) = \partial h(\boldsymbol{q})/\partial q_i$. For a k-out-of-n system with component reliabilities q, the reliability importance of the ith component can be represented as

$$I_{h_{k,n}}(i) = \frac{\partial h_{k,n}(\boldsymbol{q})}{\partial q_i} = h_{k-1,n-1}(\boldsymbol{q}^i) - h_{k,n-1}(\boldsymbol{q}^i).$$

Then, through Lemma 2.4, we know that one component would be more (less) important to the system as its reliability increases if all of the components have relatively low (high) reliabilities.

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3. Main results

In this section we discuss how different selecting strategies, sampling probabilities, and component reliabilities affect the reliability of the constructed k-out-of-n systems. Before comparing results, we first present the reliability of a k-out-of-n system composed of components arising from the two-stage sampling model $\mathcal{M}_n^m(d \mid \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{q})$.

For convenience, for $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_d)$, let

$$\boldsymbol{\ell}_{-s} = (\ell_1, \ldots, \ell_{s-1}, \ell_s - 1, \ell_{s+1}, \ldots, \ell_d)$$

denote the real vector obtained by subtracting 1 from the sth argument of $\boldsymbol{\ell}$. Also, define

$$h_{k,n}(i_1, i_2, \ldots, i_d \mid \boldsymbol{\ell}) = h_{k,n}(\underbrace{q_{i_1}}_{\ell_1}, \underbrace{q_{i_2}}_{\ell_2}, \ldots, \underbrace{q_{i_d}}_{\ell_d}),$$

$$h^*_{k,n}(i_1, i_2, \ldots, i_d \mid \boldsymbol{\ell}) = h_{k,n}(i_1, i_2, \ldots, i_d \mid \boldsymbol{\ell}) - h_{k+1,n}(i_1, i_2, \ldots, i_d \mid \boldsymbol{\ell}).$$

Proposition 3.1. *The reliability of the resulting k-out-of-n system based on the model $\mathcal{M}_n^m(d \mid \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{q})$ is given by*

$$\overline{H}^{(k)}_{\mathcal{M}_n^m(d \mid \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{q})} = \sum_{(i_1, \ldots, i_d) \in \mathcal{N}_m^d} p_{i_1} \ldots p_{i_d} h_{k,n}(i_1, \ldots, i_d \mid \boldsymbol{\ell}).$$

Proof. Denote the random variable I_i as the type of the ith selected component cluster, $i = 1, \ldots, d$. By the double expectation principle, it follows from (1.1) that

$$\begin{aligned}
\overline{H}^{(k)}_{\mathcal{M}_n^m(d \mid \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{q})} &= \mathbb{E}\phi_k(\underbrace{Z_{I_1}}_{\ell_1}, \ldots, \underbrace{Z_{I_d}}_{\ell_d}) \\
&= \mathbb{E}\Big[\mathbb{E}\phi_k(\underbrace{Z_{I_1}}_{\ell_1}, \ldots, \underbrace{Z_{I_d}}_{\ell_d}) \mid I_1, \ldots, I_d\Big] \\
&= \mathbb{E}h_{k,n}(\underbrace{q_{I_1}}_{\ell_1}, \ldots, \underbrace{q_{I_d}}_{\ell_d}) \\
&= \sum_{(i_1, \ldots, i_d) \in \mathcal{N}_m^d} p_{i_1} \cdots p_{i_d} h_{k,n}(\underbrace{q_{i_1}}_{\ell_1}, \ldots, \underbrace{q_{i_d}}_{\ell_d}) \\
&= \sum_{(i_1, \ldots, i_d) \in \mathcal{N}_m^d} p_{i_1} \cdots p_{i_d} h_{k,n}(i_1, \ldots, i_d | \boldsymbol{\ell}).
\end{aligned}$$

This completes the proof. □

3.1. Effect of the selecting strategy

We now investigate how different dispersion levels of the selecting strategies affect the constructed system's performance. In this regard, the majorization ordering will be used to describe the dispersion of a selected strategy.

Theorem 3.1. *For $\mathcal{M}_n^m(d \mid \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{q})$, $\mathcal{M}_n^m(d \mid \boldsymbol{\ell}', \boldsymbol{p}, \boldsymbol{q}) \in \mathfrak{M}$,*

(i) *if $q_i \geq (k-1)/(n-1)$ for all $i \in \{1, \ldots, m\}$ then*

$$\boldsymbol{\ell} \succeq_m \boldsymbol{\ell}' \quad \Longrightarrow \quad \overline{H}^{(k)}_{\mathcal{M}_n^m(d \mid \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{q})} \leq \overline{H}^{(k)}_{\mathcal{M}_n^m(d \mid \boldsymbol{\ell}', \boldsymbol{p}, \boldsymbol{q})};$$

(ii) *if* $q_i \le (k-1)/(n-1)$ *for all* $i \in \{1, \dots, m\}$ *then*

$$\boldsymbol{\ell} \succeq_m \boldsymbol{\ell}' \quad \Longrightarrow \quad \overline{H}^{(k)}_{\boldsymbol{\mathcal{M}}^m_n(d \mid \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{q})} \ge \overline{H}^{(k)}_{\boldsymbol{\mathcal{M}}^m_n(d \mid \boldsymbol{\ell}', \boldsymbol{p}, \boldsymbol{q})}.$$

Proof. According to Lemma D.1 of Marshall *et al.* (2011, p. 195), for any two vectors $\boldsymbol{\ell}$ and $\boldsymbol{\ell}'$ with increasing arranged elements such that $\boldsymbol{\ell} \succeq_{\mathrm{m}} \boldsymbol{\ell}'$, there exist s vectors such that

$$\boldsymbol{\ell}' = \boldsymbol{\ell}^{(1)} \preceq_{\mathrm{m}} \boldsymbol{\ell}^{(2)} \preceq_{\mathrm{m}} \cdots \preceq_{\mathrm{m}} \boldsymbol{\ell}^{(s)} = \boldsymbol{\ell},$$

and, for $u = 1, \dots, s-1$, $\boldsymbol{\ell}^{(u)} = (\ell_1^{(u)}, \dots, \ell_n^{(u)})$ and $\boldsymbol{\ell}^{(u+1)} = (\ell_1^{(u+1)}, \dots, \ell_n^{(u+1)})$ satisfy, for some $1 \le i < j \le n$,

$$\ell_i^{(u)} = \ell_i^{(u+1)} + 1, \qquad \ell_j^{(u)} = \ell_j^{(u+1)} - 1, \quad \text{and} \quad \ell_k^{(u)} = \ell_k^{(u+1)} \quad \text{for } k \ne i, j.$$

Thus, without loss of generality, we assume that, for $\boldsymbol{\ell} = (\ell_1, \dots, \ell_d)$ and $\boldsymbol{\ell}' = (\ell_1', \dots, \ell_d')$,

$$\ell_1 = \ell_1' + 1, \qquad \ell_2 = \ell_2' - 1, \qquad \ell_i = \ell_i' \quad \text{for } i \ne 1, 2.$$

From Proposition 3.1, we have

$$\begin{aligned}
&\bar{H}^{(k)}_{\boldsymbol{\mathcal{M}}^m_n(d \mid \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{q})} - \bar{H}^{(k)}_{\boldsymbol{\mathcal{M}}^m_n(d \mid \boldsymbol{\ell}', \boldsymbol{p}, \boldsymbol{q})} \\
&\quad = \sum_{(i_1, \dots, i_d) \in \boldsymbol{\mathcal{N}}^d_m} p_{i_1} \cdots p_{i_d} h_{k,n}(i_1, \dots, i_d \mid \boldsymbol{\ell}) - \sum_{(i_1, \dots, i_d) \in \boldsymbol{\mathcal{N}}^d_m} p_{i_1} \cdots p_{i_d} h_{k,n}(i_1, \dots, i_d \mid \boldsymbol{\ell}') \\
&\quad = \sum_{(i_1, \dots, i_d) \in \boldsymbol{\mathcal{N}}^d_m} p_{i_1} \cdots p_{i_d} \\
&\qquad\qquad \times [q_{i_1} h_{k-1,n-1}(i_1, \dots, i_d \mid \boldsymbol{\ell}_{-1}) + (1 - q_{i_1}) h_{k,n-1}(i_1, \dots, i_d \mid \boldsymbol{\ell}_{-1})] \\
&\qquad - \sum_{(i_1, \dots, i_d) \in \boldsymbol{\mathcal{N}}^d_m} p_{i_1} \cdots p_{i_d} \\
&\qquad\qquad \times [q_{i_2} h_{k-1,n-1}(i_1, \dots, i_d \mid \boldsymbol{\ell}_{-1}) + (1 - q_{i_2}) h_{k,n-1}(i_1, \dots, i_d \mid \boldsymbol{\ell}_{-1})] \\
&\quad = \sum_{(i_1, \dots, i_d) \in \boldsymbol{\mathcal{N}}^d_m} p_{i_1} \cdots p_{i_d} (q_{i_1} - q_{i_2}) [h_{k-1,n-1}(i_1, \dots, i_d \mid \boldsymbol{\ell}_{-1}) - h_{k,n-1}(i_1, \dots, i_d \mid \boldsymbol{\ell}_{-1})] \\
&\quad = \sum_{i<j} \sum_{(i_3, \dots, i_d) \in \boldsymbol{\mathcal{N}}^{d-2}_m} p_{i_3} \cdots p_{i_d} p_i p_j (q_i - q_j) \\
&\qquad\qquad \times [h_{k-1,n-1}(i, j, i_3, \dots, i_d \mid \boldsymbol{\ell}_{-1}) - h_{k,n-1}(i, j, i_3, \dots, i_d \mid \boldsymbol{\ell}_{-1}) \\
&\qquad\qquad\quad - h_{k-1,n-1}(j, i, i_3, \dots, i_d \mid \boldsymbol{\ell}_{-1}) + h_{k,n-1}(j, i, i_3, \dots, i_d \mid \boldsymbol{\ell}_{-1})] \\
&\quad = \sum_{i<j} \sum_{(i_3, \cdots, i_d) \in \boldsymbol{\mathcal{N}}^{d-2}_m} p_{i_3} \cdots p_{i_d} p_i p_j (q_i - q_j) \\
&\qquad\qquad \times [h^*_{k-1,n-1}(i, j, i_3, \dots, i_d \mid \boldsymbol{\ell}_{-1}) - h^*_{k-1,n-1}(j, i, i_3, \dots, i_d \mid \boldsymbol{\ell}_{-1})].
\end{aligned}$$

In terms of the symmetry of $h^*_{k,n}$ and $\ell_1 \le \ell_2$, it follows from Lemma 2.4 that, for each pair of (i, j),

$$\begin{aligned}
&(q_i - q_j)[h^*_{k-1,n-1}(i, j, i_3, \dots, i_d \mid \boldsymbol{\ell}_{-1}) - h^*_{k-1,n-1}(j, i, i_3, \dots, i_d \mid \boldsymbol{\ell}_{-1})] \\
&\quad = (q_i - q_j)[h^*_{k-1,n-1}(\underbrace{q_i}_{\ell_1 - 1}, \underbrace{q_j}_{\ell_2}, \dots, \underbrace{q_{i_d}}_{\ell_d}) - h^*_{k-1,n-1}(\underbrace{q_i}_{\ell_1 - 1}, \underbrace{q_j}_{\ell_1 - 1}, \underbrace{q_i}_{\ell_2 - \ell_1 + 1}, \dots, \underbrace{q_{i_d}}_{\ell_d})] \\
&\quad \le (\ge)\, 0,
\end{aligned}$$

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whenever $\boldsymbol{q} \in [(k-1)/(n-1), 1]^n$ ($\boldsymbol{q} \in [0, (k-1)/(n-1)]^n$). The desired result follows immediately. □

The above theorem tells an interesting and reasonable result. When the available clusters of components all have relatively high reliability, then a less balanced selecting strategy may result in a less reliable k-out-of-n system. Therefore, it is optimal to pick components from each of the selected clusters as evenly as possible. On the contrary, when all the clusters of components have low reliability, a more concentrated selecting strategy may construct a more reliable system. Intuition behind this phenomenon is that, with highly reliable components at hand, a more even selecting strategy inclines to exploit more from the component reliability and, hence, benefits the constructed system more. However, when selecting from less reliable components, one should avoid obtaining too many components having relatively low reliability, and, hence, an even selecting strategy is not desirable.

It should be pointed out for the series system, namely the n-out-of-n system, Theorem 3.1(ii) shows that the most concentrated selected strategy will result in the most reliable system. Conversely, for the parallel system, Theorem 3.1(i) shows that the evenly selecting strategy is the optimal. This finding coincides with the results in Hazra *et al.* (2017), and, hence, the results in Theorem 3.1 serve to be a generalization of them.

As a direct consequence of Theorem 3.1, we have the following corollary.

Corollary 3.1. *For any two-stage sampling model* $\mathcal{M}_n^m(\cdot \mid \cdot, p, \boldsymbol{q}) \in \mathfrak{M}$, *we have*

(i) *for* $q_i \geq (k-1)/(n-1)$ *for all* i,

$$\overline{H}^{(k)}_{\mathcal{M}_n^m(1 \mid n, \boldsymbol{p}, \boldsymbol{q})} \leq \overline{H}^{(k)}_{\mathcal{M}_n^m(\cdot \mid \cdot, \boldsymbol{p}, \boldsymbol{q})} \leq \overline{H}^{(k)}_{\mathcal{M}_n^m(n \mid \mathbf{1}, \mathbf{p}, \mathbf{q})};$$

(ii) *if* $q_i \leq (k-1)/(n-1)$ *for all* i,

$$\overline{H}^{(k)}_{\mathcal{M}_n^m(n \mid \mathbf{1}, \boldsymbol{p}, \boldsymbol{q})} \leq \overline{H}^{(k)}_{\mathcal{M}_n^m(\cdot \mid \cdot, \mathbf{p}, \mathbf{q})} \leq \overline{H}^{(k)}_{\mathcal{M}_n^m(1 \mid n, \boldsymbol{p}, \boldsymbol{q})}.$$

The scenario studied in Theorem 3.1 and Corollary 3.1 concentrates on a fixed time point. As discussed previously, when the constructed system has parallel (series) structure, only the first (second) case in Theorem 3.1 and Corollary 3.1 holds, namely, a less majorized selecting strategy always leads to a more (less) reliable system. Hence, we can obtain a corresponding result in the dynamic models by replacing component reliabilities by the component lifetimes, i.e. models in which the states of components vary over time. For convenience, we denote the set of all admissible dynamic models by

$$\left\{\mathcal{M}_n^m(d \mid \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{X})\colon \boldsymbol{\ell} \in \mathcal{N}_m^d, \sum_{i=1}^{d} \ell_i = n; p \in [0, 1]^m, \sum_{i=1}^{m} p_i = 1\right\}, \tag{3.1}$$

where $\boldsymbol{X} = (X_1, \ldots, X_m)$ denotes the lifetimes of components in the ith cluster, $i = 1, \ldots, m$.

Recall that a random variable X with survival function $\bar{F}$ is said to be smaller than the random variable Y with survival function $\bar{G}$ in the usual stochastic order (denoted by $X \leq_{\text{st}} Y$) if $\bar{F}(t) \leq \bar{G}(t)$ for all t. We refer the reader to Shaked and Shanthikumar (2007) for a comprehensive discussion. Denote by $S_{\mathcal{M}_n^m(d \mid \boldsymbol{\ell}, \boldsymbol{p}, \mathbf{X})}$ and $P_{\mathcal{M}_n^m(d \mid \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{X})}$ the lifetimes of series and parallel systems constructed from the model $\mathcal{M}_n^m(d \mid \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{X})$, respectively. Then, from Corollary 3.1, the following results are obvious.

Corollary 3.2. *If* $\boldsymbol{\ell} \succeq_m \boldsymbol{\ell}'$ *then*

$$P_{\mathcal{M}_n^m(d \mid \boldsymbol{\ell},\boldsymbol{p},\boldsymbol{X})} \leq_{st} P_{\mathcal{M}_n^m(d \mid \boldsymbol{\ell}',\boldsymbol{p},\boldsymbol{X})} \quad \text{and} \quad S_{\mathcal{M}_n^m(d \mid \boldsymbol{\ell},\boldsymbol{p},\boldsymbol{X})} \geq_{st} S_{\mathcal{M}_n^m(d \mid \boldsymbol{\ell}',\boldsymbol{p},\boldsymbol{X})}.$$

In particular, we have

$$P_{\mathcal{M}_n^m(1 \mid n,\boldsymbol{p},\boldsymbol{X})} \leq_{st} P_{\mathcal{M}_n^m(\cdot \mid \cdot,\boldsymbol{p},\boldsymbol{X})} \leq_{st} P_{\mathcal{M}_n^m(n \mid \mathbf{1},\mathbf{p},\mathbf{X})},$$
$$S_{\mathcal{M}_n^m(n \mid \mathbf{1},\boldsymbol{p},\mathbf{X})} \leq_{st} S_{\mathcal{M}_n^m(\cdot \mid \cdot,\mathbf{p},\boldsymbol{X})} \leq_{st} S_{\mathcal{M}_n^m(1 \mid n,\mathbf{p},\boldsymbol{X})}.$$

Corollary 3.2 covers Theorems 1 and 7 of Hazra *et al.* (2017) by providing the best selecting strategy and the worst selecting strategy at the same time.

3.2. Effect of sampling probabilities

In what follows we investigate the effect of sampling probabilities on the reliability of the constructed k-out-of-n systems. Intuitively, to achieve maximal reliability of a system, the cluster with more reliable components should be selected with higher probability. The following theorem confirms this intuition.

Theorem 3.2. *Suppose that* $\boldsymbol{p}, \boldsymbol{p}' \in \mathcal{I}_{[0,1]^m}$ *and* $\boldsymbol{q} \in \mathcal{I}_{[0,1]^m}$ $(\boldsymbol{q} \in \mathcal{D}_{[0,1]^m})$*. Then, for* $\mathcal{M}_n^m(d \mid \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{q})$ *and* $\mathcal{M}_n^m(d \mid \boldsymbol{\ell}, \boldsymbol{p}', \boldsymbol{q})$*, we have*

$$\boldsymbol{p} \succeq_m \boldsymbol{p}' \quad \Longrightarrow \quad \overline{H}^{(k)}_{\mathcal{M}_n^m(d \mid \boldsymbol{\ell},\boldsymbol{p},\boldsymbol{q})} \geq (\leq)\, \overline{H}^{(k)}_{\mathcal{M}_n^m(d \mid \boldsymbol{\ell},\boldsymbol{p}',\boldsymbol{q})}.$$

Proof. Note that, for any p_α, the decomposition

$$\begin{aligned}
&\overline{H}^{(k)}_{\mathcal{M}_n^m(d \mid \boldsymbol{\ell},\boldsymbol{p},\boldsymbol{q})} \\
&\quad = \sum_{s=1}^{d} \sum_{(i_1,\dots,i_d)^s \in \mathcal{N}_m^{d-1}} p_{i_1} \cdots p_{i_{s-1}} p_\alpha p_{i_{s+1}} \cdots p_{i_d} h_{k,n}(i_1, \dots, i_{s-1}, \alpha, i_{s+1}, \dots, i_d \mid \boldsymbol{\ell}) \\
&\qquad + \sum_{(i_1,\dots,i_d) \in \{\mathcal{N}_m \setminus \{\alpha\}\}^d} p_{i_1} \cdots p_{i_d} h_{k,n}(i_1, \dots, i_d \mid \boldsymbol{\ell}),
\end{aligned}$$

holds, where s means the cluster α is the sth selected one. Taking the partial derivative with respect to the αth argument p_α of p, we have, for all $\alpha \in \{1, \dots, m\}$,

$$\begin{aligned}
&\frac{\partial}{\partial p_\alpha} \overline{H}^{(k)}_{\mathcal{M}_n^m(d \mid \boldsymbol{\ell},\boldsymbol{p},\boldsymbol{q})} \\
&\quad = \sum_{s=1}^{d} \sum_{(i_1,\dots,i_d)^s \in \mathcal{N}_m^{d-1}} p_{i_1} \cdots p_{i_{s-1}} p_{i_{s+1}} \cdots p_{i_d} h_{k,n}(i_1, \dots, i_{s-1}, \alpha, i_{s+1}, \dots, i_d \mid \boldsymbol{\ell}).
\end{aligned}$$

Similarly, for p_β, we have

$$\begin{aligned}
&\frac{\partial}{\partial p_\beta} \overline{H}^{(k)}_{\mathcal{M}_n^m(d \mid \boldsymbol{\ell},\boldsymbol{p},\boldsymbol{q})} \\
&\quad = \sum_{s=1}^{d} \sum_{(i_1,\dots,i_d)^s \in \mathcal{N}_m^{d-1}} p_{i_1} \cdots p_{i_{s-1}} p_{i_{s+1}} \cdots p_{i_d} h_{k,n}(i_1, \dots, i_{s-1}, \beta, i_{s+1}, \dots, i_d \mid \boldsymbol{\ell}).
\end{aligned}$$

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Then, if $\beta > \alpha$,

$$\frac{\partial}{\partial p_\alpha}\overline{H}^{(k)}_{\mathcal{M}^m_n(d\,|\,\boldsymbol{\ell},\boldsymbol{p},\boldsymbol{q})} - \frac{\partial}{\partial p_\beta}\overline{H}^{(k)}_{\mathcal{M}^m_n(d\,|\,\boldsymbol{\ell},\boldsymbol{p},\boldsymbol{q})}$$
$$= \sum_{s=1}^{d} \sum_{(i_1,\dots,i_d)^s \in \mathcal{N}^{d-1}_m} p_{i_1}\cdots p_{i_{s-1}} p_{i_{s+1}} \cdots p_{i_d} [h_{k,n}(i_1,\dots,i_{s-1},\alpha,i_{s+1},\dots,i_d \mid \boldsymbol{\ell})$$
$$- h_{k,n}(i_1,\dots,i_{s-1},\beta,i_{s+1},\dots,i_d \mid \boldsymbol{\ell})]$$

Because the reliability function $h_{k,n}(q)$ is increasing in each argument, for any $\boldsymbol{q} \in \mathcal{I}_{[0,1]}$ ($\boldsymbol{q} \in \mathcal{D}_{[0,1]}$),

$$\frac{\partial}{\partial p_\alpha}\overline{H}^{(k)}_{\mathcal{M}^m_n(d\,|\,\boldsymbol{\ell},\boldsymbol{p},\boldsymbol{q})} - \frac{\partial}{\partial p_\beta}\overline{H}^{(k)}_{\mathcal{M}^m_n(d\,|\,\boldsymbol{\ell},\boldsymbol{p},\boldsymbol{q})} \le (\ge)\, 0,$$

which implies that $\partial\overline{H}^{(k)}_{\mathcal{M}^m_n(d\,|\,\boldsymbol{\ell},\boldsymbol{p},\boldsymbol{q})}/\partial p_i$ is increasing (decreasing) in $i \in \{1,\dots,m\}$. Then the proof can be completed by using Lemma 2.1. □

Theorem 3.2 compares the effect of two different sampling probability vectors given the component reliability vector. By using the dynamic models discussed in Corollary 3.2, we could also obtain the following dynamic version of Theorem 3.2. Following the notation in (3.1), denote the lifetime of a k-out-of-n system constructed by a dynamic model $\mathcal{M}^m_n(d \mid \boldsymbol{\ell}, p', \boldsymbol{X})$ as $L_{\mathcal{M}^m_n(d\,|\,\boldsymbol{\ell},p',\boldsymbol{X})}$. We will further illustrate this through a simulation experiment in Section 4.

Corollary 3.3. *Suppose that $\boldsymbol{p}, \boldsymbol{p}' \in \mathcal{I}_{[0,1]^m}$ and $X_1 \le_{st} \cdots \le_{st} X_m$ ($X_1 \ge_{st} \cdots \ge_{st} X_m$). Then, for $\mathcal{M}^m_n(d \mid \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{X})$ and $\mathcal{M}^m_n(d \mid \boldsymbol{\ell}, p', \boldsymbol{X})$, we have*

$$\boldsymbol{p} \succeq_m \boldsymbol{p}' \quad\Longrightarrow\quad L_{\mathcal{M}^m_n(d\,|\,\boldsymbol{\ell},\boldsymbol{p},\boldsymbol{X})} \ge_{st} (\le_{st})\, L_{\mathcal{M}^m_n(d\,|\,\boldsymbol{\ell},p',\boldsymbol{X})}.$$

3.3. Effect of component reliability

So far we have discussed the impact of different selecting strategies and sampling probabilities on the reliability of the constructed k-out-of-n system. Another interesting problem is: *Ceteris paribus*, what kind of component reliability leads to a more reliable system? The next theorem provides a partial answer under the scenarios where all components have relative low or high reliability.

Theorem 3.3. *Suppose that $\boldsymbol{q}, \boldsymbol{q}' \in \mathcal{I}_{[0,1]^m}$. For $\mathcal{M}^m_n(d \mid \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{q})$ and $\mathcal{M}^m_n(d \mid \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{q}')$.*

(i) *If $\boldsymbol{p} \in \mathcal{D}_{[0,1]^m}$ and $q_i \ge (k-1)/(n-1)$ for all $i \in \{1,\dots,m\}$, then*

$$\boldsymbol{q} \succeq^{w} \boldsymbol{q}' \quad\Longrightarrow\quad \overline{H}^{(k)}_{\mathcal{M}^m_n(d\,|\,\boldsymbol{\ell},\boldsymbol{p},\boldsymbol{q})} \le \overline{H}^{(k)}_{\mathcal{M}^m_n(d\,|\,\boldsymbol{\ell},\boldsymbol{p},\boldsymbol{q}')}.$$

(ii) *If $\boldsymbol{p} \in \mathcal{I}_{[0,1]^m}$ and $q_i \le (k-1)/(n-1)$ for all $i \in \{1,\dots,m\}$, then*

$$\boldsymbol{q} \succeq_{w} \boldsymbol{q}' \quad\Longrightarrow\quad \overline{H}^{(k)}_{\mathcal{M}^m_n(d\,|\,\boldsymbol{\ell},\boldsymbol{p},\boldsymbol{q})} \ge \overline{H}^{(k)}_{\mathcal{M}^m_n(d\,|\,\boldsymbol{\ell},\boldsymbol{p},\boldsymbol{q}')}.$$

Proof. We only prove case (i), as the other case can be verified in a similar manner.

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Taking the partial derivative of $\overline{H}^{(k)}_{\mathcal{M}^m_n(d\,|\,\boldsymbol{\ell},\boldsymbol{p},\boldsymbol{q})}$ with respect to the αth argument q_α of $\boldsymbol{q}$ yields

$$\begin{aligned}
&\frac{\partial}{\partial q_\alpha}\overline{H}^{(k)}_{\mathcal{M}^m_n(d\,|\,\boldsymbol{\ell},\boldsymbol{p},\boldsymbol{q})}\\
&\quad=\sum_{s=1}^{d}\sum_{(i_1,\ldots,i_d)^s\in\mathcal{N}^{d-1}_m} p_{i_1}\cdots p_{i_{s-1}}p_\alpha p_{i_{s+1}}\cdots p_{i_d}\frac{\partial}{\partial q_\alpha}h_{k,n}(i_1,\ldots,i_{s-1},\alpha,i_{s+1},\ldots,i_d\mid\boldsymbol{\ell})\\
&\quad=\sum_{s=1}^{d}\sum_{(i_1,\ldots,i_d)^s\in\mathcal{N}^{d-1}_m} p_{i_1}\cdots p_{i_{s-1}}p_\alpha p_{i_{s+1}}\cdots p_{i_d}\\
&\qquad\qquad\times\ell_s[h_{k-1,n-1}(i_1,\ldots,i_{s-1},\alpha,i_{s+1},\ldots,i_d\mid\boldsymbol{\ell}_{-s})\\
&\qquad\qquad\qquad-h_{k,n-1}(i_1,\ldots,i_{s-1},\alpha,i_{s+1},\ldots,i_d\mid\boldsymbol{\ell}_{-s})]\\
&\quad=\sum_{s=1}^{d}\sum_{(i_1,\ldots,i_d)^s\in\mathcal{N}^{d-1}_m} p_{i_1}\cdots p_{i_{s-1}}p_\alpha p_{i_{s+1}}\cdots p_{i_d}\\
&\qquad\qquad\times\ell_s h^*_{k-1,n-1}(i_1,\ldots,i_{s-1},\alpha,i_{s+1},\ldots,i_d\mid\boldsymbol{\ell}_{-s}).
\end{aligned}$$

Thus, for any pair of (α,β) such that $\alpha\le\beta$, we have

$$\begin{aligned}
&\frac{\partial}{\partial q_\alpha}\overline{H}^{(k)}_{\mathcal{M}^m_n(d\,|\,\boldsymbol{\ell},\boldsymbol{p},\boldsymbol{q})}-\frac{\partial}{\partial q_\beta}\overline{H}^{(k)}_{\mathcal{M}^m_n(d\,|\,\boldsymbol{\ell},\boldsymbol{p},\boldsymbol{q})}\\
&\quad=\sum_{s=1}^{d}\sum_{(i_1,\ldots,i_d)^s}\ell_s p_{i_1}\cdots p_{i_{s-1}}p_{i_{s+1}}\cdots p_{i_d}\\
&\qquad\qquad\times[p_\alpha h^*_{k-1,n-1}(i_1,\ldots,i_{s-1},\alpha,i_{s+1},\ldots,i_d\mid\boldsymbol{\ell}_{-s})\\
&\qquad\qquad\qquad-p_\beta h^*_{k-1,n-1}(i_1,\ldots,i_{s-1},\beta,i_{s+1},\ldots,i_d\mid\boldsymbol{\ell}_{-s})].
\end{aligned}$$

Since $q_\alpha\le q_\beta$, it follows from Lemma 2.4 that when $q_i\ge(k-1)/(n-1)$ for $i=1,\ldots,n$,

$$\begin{aligned}
&h^*_{k-1,n-1}(i_1,\ldots,i_{s-1},\alpha,i_{s+1},\ldots,i_d\mid\boldsymbol{\ell}_{-s})-h^*_{k-1,n-1}(i_1,\ldots,i_{s-1},\beta,i_{s+1},\ldots,i_d\mid\boldsymbol{\ell}_{-s})\\
&\quad\ge 0.
\end{aligned}$$

In combination with $p_\alpha\ge p_\beta$, we obtain

$$\frac{\partial}{\partial q_\alpha}\overline{H}^{(k)}_{\mathcal{M}^m_n(d\,|\,\boldsymbol{\ell},\boldsymbol{p},\boldsymbol{q})}-\frac{\partial}{\partial q_\beta}\overline{H}^{(k)}_{\mathcal{M}^m_n(d\,|\,\boldsymbol{\ell},\boldsymbol{p},\boldsymbol{q})}\ge 0,$$

which means $\partial(-\overline{H}^{(k)}_{\mathcal{M}^m_n(d\,|\,\boldsymbol{\ell},\boldsymbol{p},\boldsymbol{q})})/\partial q_i$ is increasing in $i\in\{1,\ldots,m\}$. Moreover,

$$\frac{\partial}{\partial q_i}(-\overline{H}^{(k)}_{\mathcal{M}^m_n(d\,|\,\boldsymbol{\ell},\boldsymbol{p},\boldsymbol{q})})\le 0\quad\text{for any }\alpha\in\{1,\ldots,m\}.$$

Then, by employing Lemma 2.2, the desired result follows immediately. □

Observing the relation among the majorization orders, the weakly submajorization order and the weakly supermajorization order, leads directly to the following corollary of Theorem 3.3.

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Corollary 3.4. *Suppose that* q, $q' \in \mathcal{I}_{[0,1]^m}$. *For* $\mathcal{M}_n^m(d \mid \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{q})$ *and* $\mathcal{M}_n^m(d \mid \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{q}')$,

(i) *if* $\boldsymbol{p} \in \mathcal{D}_{[0,1]^m}$ *and* $q_i \geq (k-1)/(n-1)$ *for all* $i \in \{1, \ldots, m\}$, *then*

$$\boldsymbol{q} \succeq_m \boldsymbol{q}' \quad \Longrightarrow \quad \overline{H}^{(k)}_{\mathcal{M}_n^m(d \mid \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{q})} \leq \overline{H}^{(k)}_{\mathcal{M}_n^m(d \mid \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{q}')};$$

(ii) *if* $\boldsymbol{p} \in \mathcal{I}_{[0,1]^m}$ *and* $q_i \leq (k-1)/(n-1)$ *for all* $i \in \{1, \ldots, m\}$, *then*

$$\boldsymbol{q} \succeq_m \boldsymbol{q}' \quad \Longrightarrow \quad \overline{H}^{(k)}_{\mathcal{M}_n^m(d \mid \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{q})} \geq \overline{H}^{(k)}_{\mathcal{M}_n^m(d \mid \boldsymbol{\ell}, \boldsymbol{p}, \boldsymbol{q}')}.$$

4. Simulations

As shown in Proposition 3.1, the reliability of the system constructed from a two-stage sampling model is in fact the expectation of the reliability of the randomly constructed k-out-of-n system. This motivates us to perform Monte Carlo simulation experiments to empirically illustrate the main findings in the previous section.

4.1. Experiment 1

The first simulation experiment intends to illustrate Theorem 3.1. In this experiment, we consider a 3-out-of-7 system. Suppose that there are eight different clusters of components and that we employ the two-stage sampling model discussed in the previous section to select seven components and construct the 3-out-of-7 system. Given component reliabilities in these eight clusters, we can obtain the constructed system's reliability. Specifically, the experiment goes through the following steps.

Step (i). Due to the symmetry of the concerned system structure, we only consider selecting strategies having elements arranged increasingly. Since there are seven components to be selected, by standard calculation we have 15 selecting strategies in total. Denote them as $\boldsymbol{\ell}_i$ for $i = 1, \ldots, 15$. All the 15 selecting strategies are

$$\begin{aligned}
\boldsymbol{\ell}_1 &= (0, 0, 0, 0, 0, 0, 7), & \boldsymbol{\ell}_2 &= (0, 0, 0, 0, 0, 1, 6), & \boldsymbol{\ell}_3 &= (0, 0, 0, 0, 0, 2, 5),\\
\boldsymbol{\ell}_4 &= (0, 0, 0, 0, 0, 3, 4), & \boldsymbol{\ell}_5 &= (0, 0, 0, 0, 1, 1, 5), & \boldsymbol{\ell}_6 &= (0, 0, 0, 0, 1, 2, 4),\\
\boldsymbol{\ell}_7 &= (0, 0, 0, 0, 1, 3, 3), & \boldsymbol{\ell}_8 &= (0, 0, 0, 0, 2, 2, 3), & \boldsymbol{\ell}_9 &= (0, 0, 0, 1, 1, 1, 4),\\
\boldsymbol{\ell}_{10} &= (0, 0, 0, 1, 1, 2, 3), & \boldsymbol{\ell}_{11} &= (0, 0, 0, 1, 2, 2, 2), & \boldsymbol{\ell}_{12} &= (0, 0, 1, 1, 1, 1, 3),\\
\boldsymbol{\ell}_{13} &= (0, 0, 1, 1, 1, 2, 2), & \boldsymbol{\ell}_{14} &= (0, 1, 1, 1, 1, 1, 2), & \boldsymbol{\ell}_{15} &= (1, 1, 1, 1, 1, 1, 1).
\end{aligned}$$

We can easily check that these selecting strategies can be ordered in the sense of majorization order, which is presented in Figure 1. The selecting strategies connected with a line are majorization ordered, and the strategy at the right endpoint or the upper endpoint is larger than the other one in the same line. For example, $\boldsymbol{\ell}_8 \succeq_{\mathrm{m}} \boldsymbol{\ell}_{10}$ but $\boldsymbol{\ell}_8$ and $\boldsymbol{\ell}_9$ cannot be majorization ordered.

Step (ii). For the eight clusters of components, we consider three different component reliability settings:

$$\begin{aligned}
\boldsymbol{q}_1 &= (0.4, 0.5, 0.6, 0.7, 0.7, 0.8, 0.9, 0.5),\\
\boldsymbol{q}_2 &= (0.1, 0.2, 0.3, 0.05, 0.15, 0.25, 0.2, 0.1),\\
\boldsymbol{q}_3 &= (0.1, 0.2, 0.3, 0.5, 0.6, 0.25, 0.2, 0.1).
\end{aligned}$$

It is routine to verify that for a 3-out-of-7 system, the component reliability vectors $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ fulfill the conditions in Theorem 3.1(i) and (ii), respectively, and $\boldsymbol{q}_3$ violates both conditions in Theorem 3.1.

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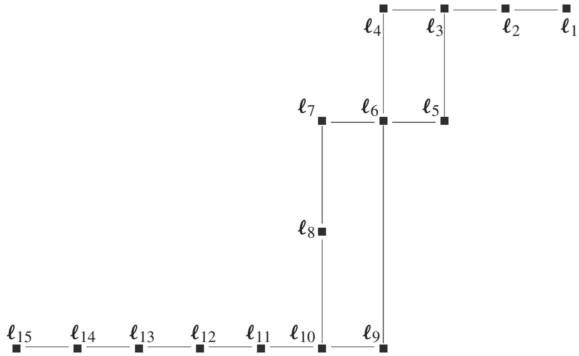

FIGURE 1: Ordering properties for the majorization order from the selected strategies

Step (iii). For each of the selecting strategies $\boldsymbol{\ell}_i$, denote the corresponding number of positive elements by d_i and the positive elements as $\ell_{i,1}, \ldots, \ell_{i,d_i}$ for $i = 1, \ldots, 15$. We randomly sample d_i substocks of components from the eight component clusters by performing random sampling with replacement using the sampling probability vector

$$\boldsymbol{p} = \left(\frac{1}{36}, \frac{1}{18}, \frac{1}{12}, \frac{1}{9}, \frac{5}{36}, \frac{1}{6}, \frac{7}{36}, \frac{2}{9}\right).$$

For the *j*th selected cluster, we take $\ell_{i,j}$ number of components in this selected cluster, and then calculate the corresponding system reliability based on the component reliability vector $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, and $\boldsymbol{q}_3$, respectively.

Step (iv). For each $\boldsymbol{q}_j$ and $\boldsymbol{\ell}_i$, we repeat the above step 1000 times to obtain a sample of observations of the random system reliability, denoted by $x_k^{(i,j)}$, $k = 1, \ldots, 1000$. Then we use the sample mean

$$\widehat{\overline{H}}^{(3)}_{\mathcal{M}_7^8(\boldsymbol{\ell}_i\,|\,p,\boldsymbol{q}_j)} = \frac{1}{1000}\sum_{k=1}^{1000} x_k^{(i,j)}$$

as an empirical approximation of the true system reliability $\overline{H}^{(3)}_{\mathcal{M}_7^8(\boldsymbol{\ell}_i\,|\,p,\boldsymbol{q}_j)}$ for $i = 1, \ldots, 15$ and $j = 1, 2, 3$.

Step (v). In the last step we compare the obtained empirical system reliabilities to illustrate the results in Theorem 3.1.

In Table 1 we list the empirical system reliabilities for different combinations of selecting strategies and component reliability vectors.

In Figure 2 we graphically display these empirical system reliabilities, and as can be seen, for a component reliability vector satisfying the conditions in Theorem 3.1, the empirical system reliabilities under selecting strategies which could be ordered in majorization order indeed have the desired ordering relation given in Theorem 3.1. When a component reliability vector violates the conditions in Theorem 3.1, a more balanced selecting strategy may or may not lead to a larger system reliability. This indicates that, in general, the effect of selecting a strategy's dispersion level on the system reliability is rather complicated.

4.2. Experiment 2

The second simulation experiment focuses on the findings of Theorem 3.2. In this experiment we intend to empirically show the dynamic version of the conclusion. Note that, when a component has lifetime X, its reliability at any time point t could be evaluated as

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TABLE 1: Empirical system reliabilities.

$\boldsymbol{\ell}_i$	$\widehat{\overline{H}}^{(3)}_{\mathcal{M}^8_7(\boldsymbol{\ell}_i \mid \boldsymbol{p},\boldsymbol{q}_j)}$		
	$\boldsymbol{q}_1$	$\boldsymbol{q}_2$	$\boldsymbol{q}_3$
(0,0,0,0,0,0,7)	0.9076	0.121	0.3345
(0,0,0,0,0,1,6)	0.9196	0.113	0.3318
(0,0,0,0,0,2,5)	0.9308	0.1083	0.3327
(0,0,0,0,0,3,4)	0.9369	0.1061	0.336
(0,0,0,0,1,1,5)	0.9353	0.1087	0.3228
(0,0,0,0,1,2,4)	0.9434	0.1045	0.3258
(0,0,0,0,1,3,3)	0.9462	0.1029	0.3303
(0,0,0,0,2,2,3)	0.9487	0.102	0.3235
(0,0,0,1,1,1,4)	0.9449	0.1039	0.3139
(0,0,0,1,1,2,3)	0.9512	0.1017	0.3153
(0,0,0,1,2,2,2)	0.9545	0.1006	0.3178
(0,0,1,1,1,1,3)	0.955	0.0991	0.3155
(0,0,1,1,1,2,2)	0.9564	0.0986	0.3113
(0,1,1,1,1,1,2)	0.9578	0.0966	0.3081
(1,1,1,1,1,1,1)	0.961	0.0954	0.3066

$\mathrm{P}(X > t)$. Therefore, one natural dynamic version of the reliability vector $\boldsymbol{q} = (q_1, \ldots, q_m)$ may be $(\overline{F}_1(t), \ldots, \overline{F}_m(t))$, where $\overline{F}_i$ is the survival function of some random lifetime, $i = 1, \ldots, m$. As for the system level, the system with a consistent larger system reliability at any time point t will have a larger lifetime in the sense of the usual stochastic order.

As in the first simulation experiment, we also consider a 3-out-of-7 system and eight component clusters with lifetimes having survival functions $\overline{F}_1, \ldots, \overline{F}_8$. Moreover, we focus on the selecting strategy $\boldsymbol{\ell} = (\ell_1, \ell_2, \ell_3) = (1, 2, 4)$. Specifically, the detailed setting is as follows.

Step (i). For the survival functions $\overline{F}_1, \ldots, \overline{F}_8$, we consider the scenarios where $\overline{F}_i$ is the survival function of the component lifetime having gamma distribution G*amma*(ν_i, λ_i) with shape parameter ν_i and scale parameters λ_i, $i = 1, \ldots, 8$. Specifically, for gamma component lifetimes, we consider three parameter combinations:

(a) $(\nu_i, \lambda_i) = (0.25 + 0.25i, 8.5 - i)$;

(b) $(\nu_i, \lambda_i) = (2.5 - 0.25i, -0.5 + i)$;

(c) $(\nu_i, \lambda_i) = (0.25 + 0.25i, 8.5 - i)$ for $i = 1, \ldots, 4$, and $(\nu_i, \lambda_i) = (3.3 - 0.4i, -7 + 2i)$ for $i = 5, \ldots, 8$.

It is routine to verify that, under case (a), we have $\overline{F}_1 \le_{st} \cdots \le_{st} \overline{F}_8$, which means that the component reliability vector at any time point belongs to $\mathcal{I}_{[0,1]^8}$. Similarly, under case (b), we have $\overline{F}_1 \ge_{st} \cdots \ge_{st} \overline{F}_8$, which means that the component reliability vector at any time point belongs to $\mathcal{D}_{[0,1]^8}$. Under case (c), we have $\overline{F}_1 \ge_{st} \cdots \ge_{st} \overline{F}_5 \le_{st} \cdots \le_{st} \overline{F}_8$, which means that the component reliability vector at any time point does not belong to either $\mathcal{D}_{[0,1]^8}$ or $\mathcal{I}_{[0,1]^8}$.

Step (ii). We consider the following two ways of sampling probabilities for the eight clusters of components:

$$p_1 = \left(\tfrac{1}{36}, \tfrac{1}{18}, \tfrac{1}{18}, \tfrac{5}{36}, \tfrac{5}{36}, \tfrac{1}{6}, \tfrac{7}{36}, \tfrac{2}{9}\right); \qquad p_2 = \left(\tfrac{1}{12}, \tfrac{1}{12}, \tfrac{5}{36}, \tfrac{5}{36}, \tfrac{5}{36}, \tfrac{5}{36}, \tfrac{5}{36}, \tfrac{5}{36}\right).$$

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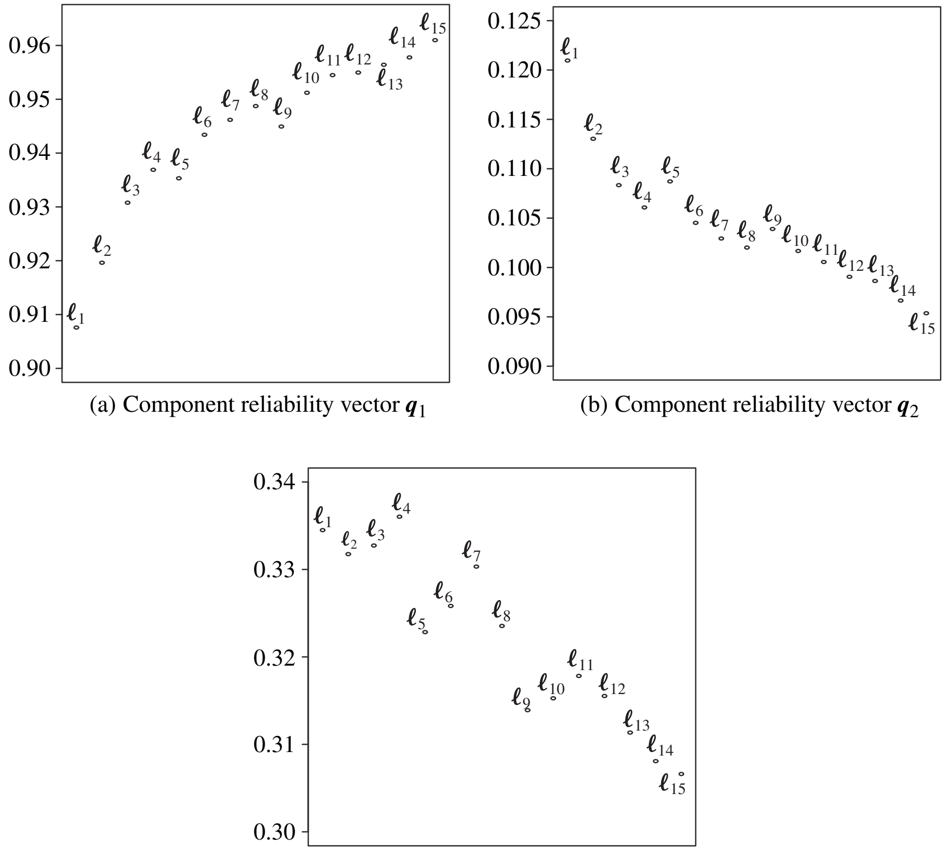

(a) Component reliability vector $\boldsymbol{q}_1$

(b) Component reliability vector $\boldsymbol{q}_2$

(c) Component reliability vector $\boldsymbol{q}_3$

FIGURE 2: Reliability of a 3-out-of-7 system for different selecting strategies.

Clearly, we have $\boldsymbol{p}_1 \succeq_{\mathrm{m}} \boldsymbol{p}_2$.

Step (iii). Given the element ℓ_i in the selecting strategy $\boldsymbol{\ell}$ and the sampling probability $\boldsymbol{p}_j$, we randomly sample one observation from the eight clusters of components by using the sampling probability $\boldsymbol{p}_j$, and then generate a sample of ℓ_i observations having the same distribution of the selected component lifetime. We then pick the fifth smallest observations from all the seven sampled observations as one realization of the desired 3-out-of-7 system lifetime.

Step (iv). For each $\boldsymbol{p}_i$ and ℓ_j, the above step is iterated 300 times to generate a 300-size sample of the concerned 3-out-of-7 system lifetime. Denote the obtained observations as $x_k^{(i,j)}$, $k = 1, \ldots, 300$. Then, for any $t \geq 0$, we use the empirical cumulative distribution

$$\widehat{\overline{F}}_{i,j}(t) = \frac{1}{300} \sum_{k=1}^{300} \mathbf{1}\,(x_k^{(i,j)} > t)$$

as an estimation of the real reliability at time t for $i = 1, 2$ and $j = 1, 2, 3$.

Step (v). Finally, the obtained empirical cumulative distributions are compared with each other to justify the theoretical findings in Theorem 3.2.

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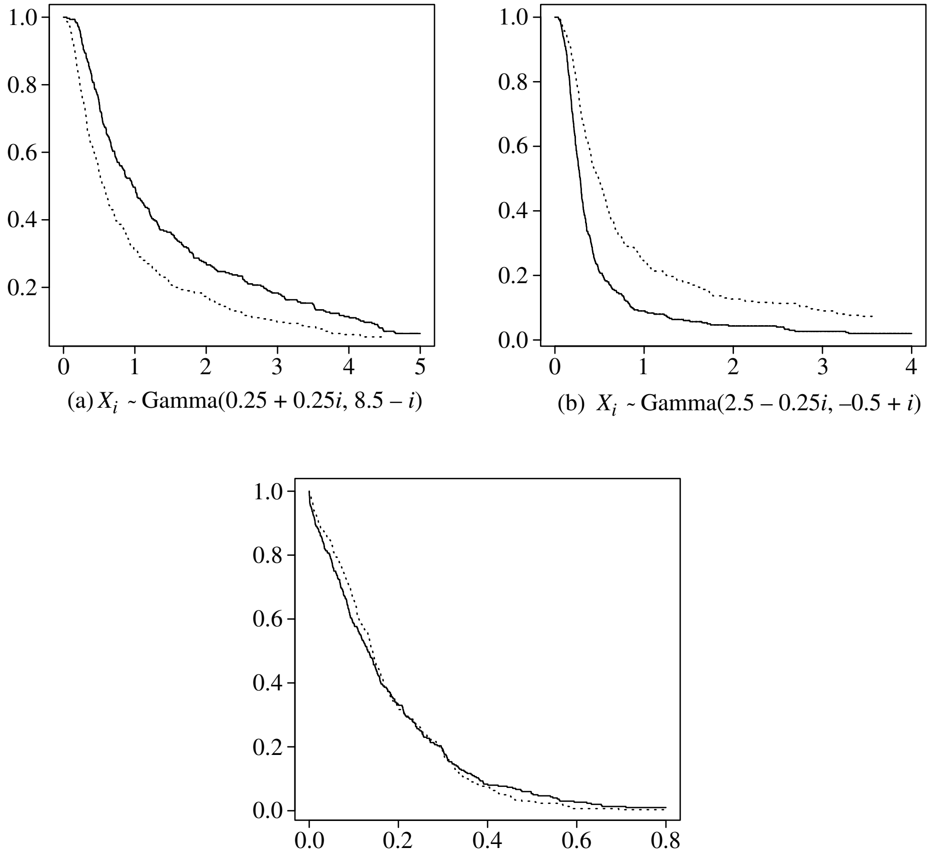

(a) $X_i \sim \text{Gamma}(0.25 + 0.25i, 8.5 - i)$

(b) $X_i \sim \text{Gamma}(2.5 - 0.25i, -0.5 + i)$

(c) $X_i \sim \text{Gamma}(0.25 + 0.25i, -8.5 - i)$, i = 1,…,4, and
$X_i \sim \text{Gamma}(3.3 - 0.4i, -7 + 2i)$, i = 5,…,8

FIGURE 3: Reliability functions of a 3-out-of-7 system for different sampling probabilities: $\boldsymbol{p}_1$ (*solid line*) and $\boldsymbol{p}_2$ (*dotted line*).

In Figure 3 we plot the empirical cumulative distribution curves for the 3-out-of-7 system lifetime constructed from eight clusters of components having gamma lifetimes. As can be seen, the sampling probability vector that is larger in the majorization order produces a stochastically larger (smaller) system lifetime when the component lifetimes are ordered increasingly (decreasingly) in the sense of the usual stochastic order. However, when the condition on the component reliability vector $\boldsymbol{q}$ is violated, the lifetimes of the constructed systems may not be ordered in the usual stochastic order any more.

4.3. Experiment 3

In this experiment we empirically illustrate the findings of Theorem 3.3. For this purpose, we consider a 3-out-of-7 system with three clusters of components. The detailed setting is as follows.

Step (i). We focus on the selecting strategy $\boldsymbol{\ell} = (1, 2, 4)$.

Step (ii). Two scenarios are considered. One assumes 100 component reliability vectors

$$\boldsymbol{q}_{1,i} = \left(\frac{1}{3} + \frac{i}{300}, \frac{2}{3}, 1 - \frac{i}{300}\right), \qquad i = 1, \ldots, 100,$$

and the other assumes another 100 component reliability vectors

$$\boldsymbol{q}_{2,i} = \left(\frac{i}{600}, \frac{1}{6}, \frac{1}{3} - \frac{i}{600}\right), \qquad i = 1, \ldots, 100.$$

We can verify that $\boldsymbol{q}_{1,i}, \boldsymbol{q}_{2,i} \in \mathcal{I}_{[0,1]^2}$ for $i = 1, \ldots, 100$, and both $\boldsymbol{q}_{1,i}$ and $\boldsymbol{q}_{2,i}$ are decreasing in i with respect to the majorization order. Therefore, for $j = 1, \ldots, 99$,

$$\boldsymbol{q}_{1,j} \succeq^{\mathrm{w}} \boldsymbol{q}_{1,j+1} \quad \text{and} \quad \boldsymbol{q}_{2,j} \succeq_{\mathrm{w}} \boldsymbol{q}_{2,j+1}.$$

Moreover, all the elements of $\boldsymbol{q}_{1,i}$ larger than $(3-1)/(7-1) = \frac{1}{3}$, while all the elements of $\boldsymbol{q}_{2,i}$ are less than $\frac{1}{3}$ for $i = 1, \ldots, 100$.

Step (iii). Given one component reliability vector, we randomly sample three substocks of components from the three component clusters by performing random sampling with replacement using the probability vector $\boldsymbol{p}$. For the first selected cluster, we take one component; for the second selected cluster, we take two components; and for the third selected cluster, we take four components. Then the corresponding system reliability based on the component reliability vector is calculated.

Step (iv). For each of the 200 component reliability vectors $\boldsymbol{q}_{i,j}$, the above step is iterated 1000 times to generate a sample of system reliability $x_k^{(i,j)}$, and then the sample mean

$$\widehat{H}^{(3)}_{\mathcal{M}_7^2(\boldsymbol{\ell} \mid p, \boldsymbol{q}_{i,j})} = \frac{1}{1000} \sum_{k=1}^{1000} x_k^{(i,j)}$$

is employed as an estimation of the real system reliability, $i = 1, 2$, $j = 1, \ldots, 100$, and $k = 1, \ldots, 1000$.

Step (v). We consider the four configurations

(a) $\boldsymbol{q}_{1,i} = (\frac{1}{3} + i/300, \frac{1}{3}, 1 - i/300)$, $p_1 = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$;

(b) $\boldsymbol{q}_{2,i} = (i/600, \frac{1}{6}, \frac{1}{3} - i/600)$, $p_2 = (\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$;

(c) $\boldsymbol{q}_{1,i} = (\frac{1}{3} + i/300, \frac{1}{3}, 1 - i/300)$, $p_3 = (\frac{1}{9}, \frac{4}{9}, \frac{4}{9})$;

(d) $\boldsymbol{q}_{2,i} = (i/600, \frac{1}{6}, \frac{1}{3} - i/600)$, $p_4 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$

In Figure 4 we plot the empirical system reliabilities under different component reliability vectors. As can be seen in Figure 4(a), for $\boldsymbol{q}_{1,i}$, the height of depicted points are increasing in i, confirming the result in Theorem 3.3(i); and for $\boldsymbol{q}_{2,i}$, the decreasing trend in Figure 4(b) illustrates the finding in Theorem 3.3(ii). Configurations (c) and (d) concern the scenarios in which some of the conditions of Theorem 3.3 are violated. For the case where the components all have high reliability, with increasing sampling probabilities, in Figure 4(c) we plot the system reliability under different $\boldsymbol{q}_{1,i}$. As can be seen, the height of depicted points is not monotonic in i, indicating that the conclusion in Theorem 3.3(i) no longer holds. As for nonmonotonic sampling probabilities, in Figure 4(d) we plot the system reliability under $\boldsymbol{q}_{2,i}$. As can be seen, the conclusion in Theorem 3.3(ii) no longer holds either. This implies that the assumptions on monotonic sampling probabilities in Theorem 3.3 may not be dropped.

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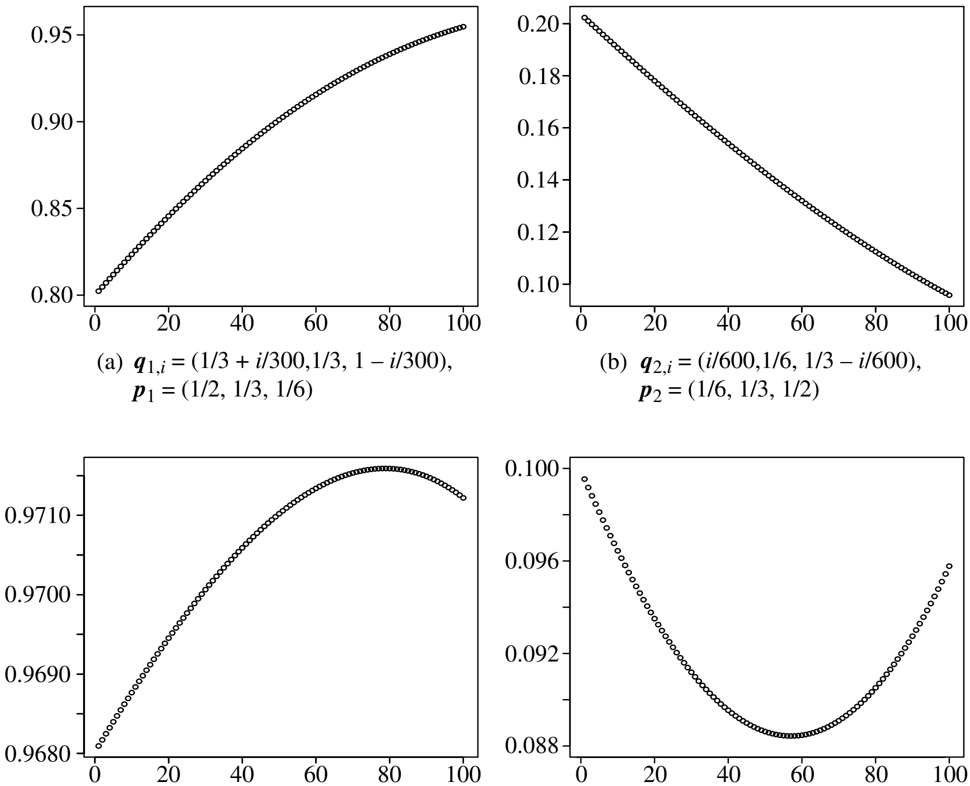

(a) $\boldsymbol{q}_{1,i} = (1/3 + i/300, 1/3, 1 - i/300)$, $\boldsymbol{p}_1 = (1/2, 1/3, 1/6)$

(b) $\boldsymbol{q}_{2,i} = (i/600, 1/6, 1/3 - i/600)$, $\boldsymbol{p}_2 = (1/6, 1/3, 1/2)$

(c) $\boldsymbol{q}_{1,i} = (1/3 + i/300, 1/3, 1 - i/300)$, $\boldsymbol{p}_3 = (1/9, 4/9, 4/9)$

(d) $\boldsymbol{q}_{2,i} = (i/600, 1/6, 1/3 - i/600)$, $\boldsymbol{p}_4 = (1/2, 1/6, 1/3)$

FIGURE 4: Reliability of a 3-out-of-7 system.

5. Concluding remarks

In this paper we consider k-out-of-n systems built from two-stage grouping models. How the constructed system reliability reacts to the change of selecting strategy, sampling probabilities, and component reliabilities is investigated. It is found that the level of component reliability plays a vital role in determining the effect of selecting strategies and component reliabilities on the system reliability. When all the components are of relatively high (low) reliability, a more concentrated selecting strategy or component reliability would lead to a lower (higher) system reliability. As for the sampling probabilities, a larger one in the sense of the majorization order may result in a more (less) reliable system when the component reliabilities are arranged in the same (opposite) direction to the sampling probabilities. Our findings may provide some guidance for practices. For example, when building a yarn from several fibre types, usually it is unknown which fibres have the highest strength. According to Theorem 3.1, by the weakest-link principle, one should choose all the fibres from one selected type in order to obtain a yarn with the highest reliability.

Note that, in this study, only the k-out-of-n system structure is treated, and it is of both theoretical and practical interest to pursue extensions to general coherent structures in future work.

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