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Existence of positive solutions for a class of semipositone problem in whole $\mathbb{R}^N$

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In this paper we show the existence of solution for the following class of semipositone problem

$$\begin{cases} -\Delta u = h(x)(f(u)-a) & \text{in } \mathbb{R}^N, \\ \quad u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{P}$$

where $N \geqslant 3$, $a > 0$, $h : \mathbb{R}^N \to (0, +\infty)$ and $f : [0, +\infty) \to [0, +\infty)$ are continuous functions with f having a subcritical growth. The main tool used is the variational method together with estimates that involve the Riesz potential.

Keywords: Variational methods; semipositone problem; positive solutions; Riesz potential

2010 *Mathematics subject classification:* Primary: 35J20; 45M20; 35A08

1. Introduction

In this paper we study the existence of positive weak solutions for the semipositone problem

$$\begin{cases} -\Delta u = h(x)(f(u)-a) & \text{in } \mathbb{R}^N, \\ \quad u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{P}$$

where $N \geqslant 3$, $f : [0, +\infty) \to [0, +\infty)$ is a local Lipschitz function with subcritical growth and $a > 0$. In what follows, $h : \mathbb{R}^N \to (0, +\infty)$ is a continuous function that satisfies the following condition:
(h) There exists $P \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$0 < h(x) \leqslant P(|x|), \quad \forall x \in \mathbb{R}^N \setminus \{0\},$$

and P verifies the following assumptions:
(P_1) $\displaystyle\int_{\mathbb{R}^N} |x|^{2-N} P(|x|) \, \mathrm{d}x < +\infty$,
(P_2) $P(|.|) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$,

and

(P_3) $\displaystyle\int_{\mathbb{R}^N} P(|y|)/|x-y|^{N-2}\,\mathrm{d}y \leqslant C/|x|^{N-2}$, for all $x \in \mathbb{R}^N \setminus \{0\}$ and some $C > 0$.

An example of a function P that satisfies the hypotheses $(P_1) - (P_3)$ is as follows: Let P be a function of the form

$$P(t) = Q(t)R(t), \quad \forall t \geqslant 0$$

where, Q, R are decreasing and positive continuous functions satisfying

$$R(|\,.\,|) \in L^1(\mathbb{R}^N) \quad \text{and} \quad \sup_{x\in\mathbb{R}^N} (|x|^{N-2}Q(|x|)) < +\infty.$$

In what follows we will prove only (P_3), because $(P_1) - (P_2)$ are immediate. Note that

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{P(|y|)}{|x-y|^{N-2}}\,\mathrm{dy} &= \int_{|x-y|\leqslant|x|/2} \frac{P(|y|)}{|x-y|^{N-2}}\,\mathrm{d}y + \int_{|x-y|\geqslant|x|/2} \frac{P(|y|)}{|x-y|^{N-2}}\,\mathrm{d}y \\ &\leqslant \int_{|x-y|\leqslant|x|/2} \frac{P(|y|)}{|x-y|^{N-2}}\,\mathrm{d}y + \frac{2^{N-2}}{|x|^{N-2}} \int_{\mathbb{R}^N} P(|y|)\,\mathrm{d}y.\end{aligned}$$

For $|x-y| \leqslant |x|/2$, fixing $z = x - y$ we get

$$|x-z| \geqslant |x| - |z| \geqslant |x| - \frac{1}{2}|x| = \frac{1}{2}|x|$$

and

$$|x-z| \geqslant |x| - |z| \geqslant 2|z| - |z| = |z|.$$

As R and Q are decreasing, it follows that

$$Q(|x-z|) \leqslant Q(|x|/2) \quad \text{and} \quad R(|x-z|) \leqslant R(|z|).$$

Therefore

$$\begin{aligned}\int_{|x-y|\leqslant|x|/2} \frac{P(|y|)}{|x-y|^{N-2}}\,\mathrm{dy} &= \int_{|z|\leqslant|x|/2} \frac{P(|x-z|)}{|z|^{N-2}}\,\mathrm{d}z \\ &\leqslant \int_{|z|\leqslant|x|/2} \frac{Q(1/2|x|)R(|z|)}{|z|^{N-2}}\,\mathrm{d}z \\ &\leqslant Q\left(\frac{1}{2}|x|\right) \int_{\mathbb{R}^N} \frac{R(|z|)}{|z|^{N-2}}\,\mathrm{d}z.\end{aligned}$$

As $R(|\,.\,|) \in C(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, we know that $R(|.|)/|\,.\,|^{N-2} \in L^1(\mathbb{R}^N)$, this proves (P_3).

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Related to the function f, we assume the following conditions:

$$0 = f(0) = \min_{t \in [0,+\infty)} f(t). \tag{f_1}$$

$$\lim_{t \to 0^+} \frac{f(t)}{t} = 0. \tag{f_2}$$

There is $q \in (2, 2^*)$, where $2^* = 2N/(N-2)$, such that

$$\limsup_{t \to +\infty} \frac{f(t)}{t^{q-1}} < +\infty. \tag{f_3}$$

There are $\theta > 2$ and $t_0 > 0$ such that

$$\theta F(t) \leqslant f(t)t, \quad \forall t \geqslant t_0, \tag{f_4}$$

where $F(t) = \int_0^t f(\tau)\, \mathrm{d}\tau$.

In the sequel, we say that a function $u \in D^{1,2}(\mathbb{R}^N)$ is a *weak solution* for (P) if u is a continuous positive function that verifies

$$\int_{\mathbb{R}^N} \nabla u \nabla \varphi \, \mathrm{d}x = \int_{\mathbb{R}^N} h(x)(f(u) - a)\varphi \, \mathrm{d}x, \quad \forall \varphi \in D^{1,2}(\mathbb{R}^N).$$

The problem (P) for $a = 0$ is very simple, and it can be solved by using the mountain pass theorem due to Ambrosetti & Rabinowitz [**4**], because by supposing that $f(t) = 0$ for $t \leqslant 0$, it is possible to show that the functional

$$J(u) = \int_{\mathbb{R}^N} h(x) f(u) u \, \mathrm{d}x, \quad \forall u \in D^{1,2}(\mathbb{R}^N)$$

is weakly continuous, that is,

$$u_n \rightharpoonup u \quad \text{in } D^{1,2}(\mathbb{R}^N) \Rightarrow J(u_n) \to J(u) \quad \text{as } n \to +\infty.$$

This fact permits to prove that the energy functional verifies the well-known Palais-Smale condition.

However, for the case where (P) is semipositone, that is, when $a > 0$, the existence of positive solution is not so simple, because the standard arguments via mountain pass theorem combined with maximum principle do not ensure the existence of a positive solution for the problem, because $f(t) - a$ is negative near of $t = 0$. Here, the size of the constant a and the conditions on function h apply an important role in our arguments, in the sense that we were able to prove the existence of positive solution for (P) when a is small enough.

Many authors have studied semipositone problems in bounded domain over the years since the appearance of the paper by Castro and Shivaji [**9**] that were the first to consider this class of problem. In the literature we find different methods to prove the existence and non existence of solutions, such as sub-supersolutions, degree theory arguments, fixed point theory and bifurcation, see for example the [**1**, **2**, **5**, **6**] and their references. Besides these methods, the variational method was also used in some few papers as can be seen in [**3**, **7**, **8**, **10**–**14**].

The present work has been mainly motivated by papers [7, 10], and by the fact that the authors did not find in the literature any paper involving semipositone problem in whole $\mathbb{R}^N$ by using variational methods. In [7], Caldwell, Castro, Shivaji and Unsurangsie have studied the existence positive solutions for the following class of semipositone problem

$$\begin{cases} -\Delta u &= \mu g(u) + \lambda f(u) & \text{in} \quad \Omega, \\ u &> 0 & \text{in} \quad \Omega, \\ u &= 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, is a smooth bounded domain, μ, λ are positive parameters, $g, f : [0, +\infty) \to \mathbb{R}^+$ are differentiable and non decreasing functions verifying the following conditions:

Conditions on g: There exist $A, B > 0$ and $q \in (1, N+2/N-2)$ such that

$$At^q \leqslant g(t) \leqslant Bt^q, \quad \forall t \geqslant 0$$

There exists $\theta > 2$ such that for t large

$$0 < \theta G(t) \leqslant g(t)t,$$

where $G(t) = \displaystyle\int_0^t g(s)\,\mathrm{d}s$.

Conditions on f: There is $\alpha \in (0, 1)$ such that

$$\lim_{t \to +\infty} \frac{f(t)}{t^\alpha} = 0,$$

and $f(0) < 0$. The existence of solution has been obtained by applying the mountain pass theorem and sub-supersolutions for convenient values of λ and μ.

In [10], Castro, de Figueiredo and Lopera have established the existence of positive solution for the following class of semipositone problem involving the p-Laplacian operator

$$\begin{cases} -\Delta_p u &= \lambda f(u) & \text{in} \quad \Omega, \\ u &> 0 & \text{in} \quad \Omega, \\ u &= 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N > p > 2$, is a smooth bounded domain, $\lambda > 0$ and $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f(0) < 0$. In that paper, the authors have assumed that there exist $q \in (p-1, Np/(N-p)-1)$, $A, B > 0$ such that

$$\left. \begin{array}{l} A(t^q - 1) \leqslant f(t) \leqslant B(t^q - 1), \quad \text{for } t > 0 \\ f(t) = 0, \quad \text{for } t \leqslant -1. \end{array} \right\}$$

The existence of solution was proved by combining the mountain pass theorem with the regularity theory.

Our main result is the following

Theorem 1.1. *Assume* (h) *and* $(f_1)-(f_4)$. *Then, there exists* $a^* > 0$ *such that if* $a \in (0, a^*)$, *problem* (P) *has a positive weak solution* $u_a \in C(\mathbb{R}^N) \cap D^{1,2}(\mathbb{R}^N)$.

In the proof of theorem 1.1 we have used variational methods and estimates involving the Riesz potential. By using mountain pass theorem we found a solution u_a for all $a > 0$. By taking the limit when a goes to 0, we were able to show, via elliptic regularity theory and estimates involving the Riesz potential, that u_a is positive for a small enough. We believe that this is the first paper involving semipositone problem in whole $\mathbb{R}^N$.

Notations

- C is a positive constant which may vary line by line.
- $B_r(x)$ denotes the open ball centred at the x with radius $r > 0$ in $\mathbb{R}^N$.
- $L^s(\mathbb{R}^N)$, for $1 \leqslant s \leqslant \infty$, denotes the Lebesgue space with usual norm denoted by $\|u\|_s$.
- If H is a measurable function, $L^2_H(\mathbb{R}^N)$ denotes the class of real-valued Lebesgue measurable functions u such that

$$\int_{\mathbb{R}^N} H(x)|u(x)|^2 \,\mathrm{d}x < \infty.$$

  $L^2_H(\mathbb{R}^N)$ is a Hilbert space endowed with the inner product

$$(u, v)_{2,H} = \int_{\mathbb{R}^N} H(x)u(x)v(x)\,\mathrm{d}x, \quad \forall u, v \in L^2_H(\mathbb{R}^N).$$

  The norm associated with this inner product will denote by $|\cdot|_{2,H}$.

2. Preliminary results

In this section, we denote by $f_a : \mathbb{R} \longrightarrow \mathbb{R}$ the continuous function given by

$$f_a(t) = \begin{cases} f(t) - a & \text{if} \quad t \geqslant 0, \\ -a(t+1) & \text{if} \quad t \in [-1, 0], \\ 0 & \text{if} \quad t \leqslant -1, \end{cases}$$

$0 < a < 1$, and $-a = \min_{t\in\mathbb{R}} f_a(t)$. Our intention is to prove the existence of positive solution for the following auxiliary problem

$$\begin{cases} -\Delta u = h(x) f_a(u) & \text{in} \quad \mathbb{R}^N, \\ \quad u > 0 & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{AP}$$

because such a solution is also a solution of (P). Associated with (AP), we have the energy functional $I_a : D^{1,2}(\mathbb{R}^N) \longrightarrow \mathbb{R}$ defined by

$$I_a(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} h(x)F_a(u)\,\mathrm{d}x,$$

where

$$F_a(t) = \int_0^t f_a(\tau)\,\mathrm{d}\tau, \quad t \in \mathbb{R}.$$

Using standard arguments (see [16]), it is possible to prove that $I_a \in C^1(D^{1,2}(\mathbb{R}^N), \mathbb{R})$ with

$$I_a'(u)v = \int_{\mathbb{R}^N} \nabla u \nabla v\,\mathrm{d}x - \int_{\mathbb{R}^N} h(x) f_a(u) v\,\mathrm{d}x, \quad \forall u, v \in D^{1,2}(\mathbb{R}^N),$$

then critical points of I_a are weak solutions of (AP).

Hereafter, we will endow $D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N);\ \nabla u \in L^2(\mathbb{R}^N, \mathbb{R}^N)\}$ with its standard scalar product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} \nabla u \nabla v\,\mathrm{d}x$$

and the usual norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x\right)^{1/2}.$$

Since the Gagliardo-Nirenberg-Sobolev inequality

$$|u|_{2^*} \leqslant S_N \|u\|,$$

holds for all $u \in D^{1,2}(\mathbb{R}^N)$ for some constant $S_N > 0$, we have that the embedding

$$D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N) \tag{2.1}$$

is continuous.

By using the assumptions on h and (2.1), we have that the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^2_h(\mathbb{R}^N)$ is continuous, that is, there exists $\Lambda > 0$ such that

$$\left(\int_{\mathbb{R}^N} h|u|^2\,\mathrm{d}x\right)^{1/2} \leqslant \Lambda \|u\|, \quad \forall u \in D^{1,2}(\mathbb{R}^N). \tag{2.2}$$

The above embedding is a consequence of the following lemma

LEMMA 2.1. *Assume* $(P_1) - (P_2)$*. Then, the embedding* $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^2_h(\mathbb{R}^N)$ *is compact.*

Proof. Let $\{u_n\}$ be a sequence in $D^{1,2}(\mathbb{R}^N)$ with $u_n \rightharpoonup 0$ in $D^{1,2}(\mathbb{R}^N)$. For each $R > 0$, we have the continuous embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow H^1(B_R(0))$. Since the embedding $H^1(B_R(0)) \hookrightarrow L^2(B_R(0))$ is compact, it follows that $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^2(B_R(0))$ is a compact embedding as well. Hence,

$$u_n(x) \to 0, \ a.e. \quad \text{in } \mathbb{R}^N,$$

for some subsequence. As $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is a continuous embedding, we have $\{|u_n|^2\}$ is a bounded sequence in $L^{2^*/2}(\mathbb{R}^N)$. By a Brézis-Lieb lemma (see [16]),

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up to a subsequence if necessary,

$$|u_n|^2 \rightharpoonup 0 \text{ in } L^{2^*/2}(\mathbb{R}^N),$$

or equivalently,

$$\int_{\mathbb{R}^N} |u_n|^2 \varphi \,\mathrm{d}x \to 0, \quad \forall \varphi \in L^p(\mathbb{R}^N),$$

where $2/2^* + 1/p = 1$. As (P_2) guarantees that $h \in L^r(\mathbb{R}^N)$ for all $r \geqslant 1$, it follows that

$$\int_{\mathbb{R}^N} h(x)|u_n|^2 \,\mathrm{d}x \to 0.$$

This shows that $u_n \to 0$ in $L^2_h(\mathbb{R}^N)$, finishing the proof. □

In the next two lemmas, we will establish the mountain pass geometry for functional I_a.

LEMMA 2.2. *There exist $r > 0$ such that if $\rho \in (0, r)$ and $||u|| = \rho$, then there are $\alpha = \alpha(\rho) > 0$ and $a_1 = a_1(\rho) > 0$ such that $I_a(u) \geqslant \alpha$ for all $a \in (0, a_1)$. Moreover, the constants r, ρ are independent of $a \in (0, a_1)$.*

Proof. Given $\epsilon \in (0, 1/4\Lambda^2)$, there is a constant $C_\epsilon > 0$, which is independent of a, such that $F_a(t) \leqslant \epsilon|t|^2 + C_\epsilon|t|^{2^*} + a$ for all $t \in \mathbb{R}$. Therefore,

$$\begin{aligned}
I_a(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} h(x)F_a(u)\,\mathrm{d}x \\
&\geqslant \frac{1}{2}\|u\|^2 - \epsilon \int_{\mathbb{R}^N} h(x)|u|^2\,\mathrm{d}x - C_\epsilon \int_{\mathbb{R}^N} h(x)|u|^{2^*}\,\mathrm{d}x - a\|h\|_1 \\
&\overset{(2.2)}{\geqslant} \frac{1}{2}\|u\|^2 - \epsilon\Lambda^2\|u\|^2 - C_\epsilon\|h\|_\infty \int_{\mathbb{R}^N} |u|^{2^*}\,\mathrm{d}x - a\|h\|_1 \\
&\geqslant \frac{1}{4}\|u\|^2 - C_\epsilon\|h\|_\infty \int_{\Omega} |u|^{2^*}\,\mathrm{d}x - a\|h\|_1 \\
&\geqslant \frac{1}{4}\|u\|^2 - C_\epsilon||u||_{2^*}^{2^*} - a\|h\|_1.
\end{aligned}$$

It is well known that there exists $S_N > 0$ such that

$$||u||_{2^*} \leqslant S_N||u||, \quad \forall\, u \in D^{1,2}(\mathbb{R}^N).$$

Thus, there is $C_1 > 0$ verifying

$$I_a(u) \geqslant \frac{1}{4}||u||^2 - C_1||u||^{2^*} - a\|h\|_1.$$

Taking $r = (1/4C_1)^1/2^* - 2$ and $||u|| = \rho$ with $\rho \in (0, r)$, we get

$$I_a(u) \geqslant \rho^2(1/4 - C_1\rho^{2^*-2}) - a\|h\|_1.$$

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Now, we fix $a_1 = a_1(\rho) > 0$ and $r > 0$ such that

$$\rho^2(1/4 - C_1\rho^{2^*-2}) - a\|h\|_1 \geqslant \frac{\rho^2(1/4 - C_1\rho^{2^*-2})}{2} > 0,$$
$$\forall a \in (0, a_1) \quad \text{and } \forall \rho \in (0, r).$$

From this, $I_a(u) \geqslant \alpha > 0$ if $||u|| = \rho$ where $\alpha = \alpha_\rho := \rho^2(1 - C_1\rho^{2^*-2})/2$, proving the lemma. □

Lemma 2.3. *There exists $v \in D^{1,2}(\mathbb{R}^N)$ such that $||v|| > \rho$ and $I_a(v) < 0$, for all $a \in (0, a_1)$, where ρ was fixed in lemma 2.2.*

Proof. Fix a function

$$\varphi \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}, \quad \text{with } \varphi \geqslant 0 \quad \text{and } ||\varphi|| = 1.$$

Notice that for all $t > 0$,

$$\begin{aligned} I_a(t\varphi) &= \frac{1}{2}\int_\Omega |\nabla t\varphi|^2 \, \mathrm{d}x - \int_\Omega h(x)F_a(t\varphi)\, \mathrm{d}x \\ &= \frac{1}{2}\int_\Omega |\nabla t\varphi|^2 \, \mathrm{d}x - \int_\Omega h(x)F(t\varphi)\, \mathrm{d}x + a\int_\Omega h(x)t\varphi\, \mathrm{d}x, \end{aligned}$$

where $\Omega = supp\,\varphi$. By (f_4), there are $A_1, B_1 > 0$ verifying

$$F(t) \geqslant A_1|t|^\theta - B_1, \quad \forall t \in \mathbb{R}. \tag{2.3}$$

From this,

$$I_a(t\varphi) \leqslant \frac{t^2}{2} - t^\theta A_1 \int_\Omega h(x)|\varphi|^\theta \, \mathrm{d}x + ta\|h\|_\infty\|\varphi\|_1 + B_1\|h\|_1.$$

Since $\theta > 2$ and $a \in (0, a_1)$, we can fix $t_0 > 1$ large enough so that $I_a(v) < 0$, where $v = t_0\varphi \in D^{1,2}(\mathbb{R}^N)$. □

In the sequel, we say that I_a satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ ($(PS)_c-$condition for short), if every sequence $\{u_n\} \subset D^{1,2}(\mathbb{R}^N)$ such that

$$I_a(u_n) \to c \quad \text{and } I'_a(u_n) \to 0 \quad \text{in } \; D^{1,2}(\mathbb{R}^N)^*, \tag{2.4}$$

has a strongly convergent subsequence. Moreover, if $\{u_n\}$ only satisfies (2.4) we say that the this sequence is a Palais-Smale sequence at level c of I_a.

Now, we are going to study the boundedness of Palais-Smale sequences of I_a. To do this, we recall that f_a satisfies the following inequality:

$$\theta F_a(t) \leqslant tf_a(t) + M, \quad \forall t \in \mathbb{R}, \tag{2.5}$$

for some $M \in \mathbb{R}$. It is very important to point out that M is independent of $a \in (0, a_1)$.

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LEMMA 2.4. *The functional I_a satisfies the Palais-Smale condition for all $a > 0$.*

Proof. Let $\{u_n\}$ be a sequence in $D^{1,2}(\mathbb{R}^N)$ such that $\{I_a(u_n)\}$ is bounded and $I'_a(u_n) \to 0$. Hence, there exists $n_0 \in \mathbb{N}$ such that $|\langle I'_a(u_n), u_n\rangle| \leqslant ||u_n||$ for $n > n_0$. Thus,

$$- \|u_n\| - \|u_n\|^2 \leqslant - \int_{\mathbb{R}^N} h(x) f_a(u_n) u_n \,\mathrm{d}x. \tag{2.6}$$

On the other hand, as there exists $K > 0$ such that $|I_a(u_n)| \leqslant K$ for all $n = 1, 2, \ldots$, it follows that

$$\frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^N} h(x) F_a(u_n) \,\mathrm{d}x \leqslant K, \quad \forall n \in \mathbb{N}. \tag{2.7}$$

From (2.5) and (2.7),

$$\frac{1}{2}\|u_n\|^2 - \frac{1}{\theta}\int_{\mathbb{R}^N} h(x) f_a(u_n) u_n \,\mathrm{d}x - \frac{1}{\theta} M\|h\|_1 \leqslant K, \quad \forall n \in \mathbb{N}. \tag{2.8}$$

Thereby, by (2.6) and (2.8),

$$\frac{1}{2}\|u_n\|^2 - \frac{1}{\theta}||u_n|| - \frac{1}{\theta}\|u_n\|^2 \leqslant K + \frac{1}{\theta} M\|h\|_1,$$

or equivalently,

$$\left(\frac{1}{2} - \frac{1}{\theta}\right) ||u_n||^2 - \frac{1}{\theta}||u_n|| \leqslant K + \frac{1}{\theta} M\|h\|_1,$$

for n large enough. This shows that $\{u_n\}$ is bounded in $D^{1,2}(\mathbb{R}^N)$. Thus, without loss of generality, we may assume that

$$u_n \rightharpoonup u \ \text{ in } \ D^{1,2}(\mathbb{R}^N)$$

and

$$u_n(x) \to u(x) \quad \text{a.e. in} \quad \mathbb{R}^N.$$

By conditions $(f_1) - (f_3)$, there exists $C > 0$ that does not dependent on a such that

$$|f_a(t)| \leqslant C(|t|^{q-1} + |t|) + a, \quad \forall t \in \mathbb{R},$$

and so,

$$|h(x) f_a(u_n)(u_n - u)| \leqslant C_1 h(x)|u_n - u|(|u_n|^{q-1} + |u_n| + a).$$

for some $C_1 > 0$ independent of a.

CLAIM 2.5. The limit below holds

$$\int_{\mathbb{R}^N} h(x)|u_n - u|(|u_n|^{q-1} + |u_n| + a) \,\mathrm{d}x \to 0 \quad \text{as } n \to +\infty.$$

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In fact, we will only show the limit

$$\int_{\mathbb{R}^N} h(x)|u_n - u||u_n|^{q-1}\,\mathrm{d}x \to 0 \quad \text{as} \quad n \to +\infty, \tag{2.9}$$

because the limits involving the others terms follow with the same idea. To begin with, note that $|u_n|^{q-1} \in L^{2^*/(q-1)}(\mathbb{R}^N)$ and $h|u_n - u| \in L^{2^*/2^*-(q-1)}$. Then by Hölder inequality

$$\int_{\mathbb{R}^N} h(x)|u_n - u||u_n|^{q-1}\,\mathrm{d}x \leqslant \|h(u_n - u)\|_{2^*/2^*-(q-1)}\|u_n\|_{2^*}^{q-1}.$$

Now, using the fact that $|h|^{2^*/2^*-(q-1)} \in L^{2^*-(q-1)/2^*-q}(\mathbb{R}^N)$ and $|u_n - u|^{2^*/2^*-(q-1)} \rightharpoonup 0$ in $L^{2^*-(q-1)}(\mathbb{R}^N)$, we have that $\|h(u_n - u)\|_{2^*/2^*-(q-1)} \to 0$. As $\{u_n\}$ is bounded in $L^{2^*}(\mathbb{R}^N)$, we get (2.9).

An immediate consequence of claim 2.5 is the limit

$$\int_{\mathbb{R}^N} h(x)f_a(u_n)(u_n - u)\,\mathrm{d}x \to 0$$

that combines with the equality below

$$o_n(1) = I_a'(u_n)(u_n - u) = \int_{\mathbb{R}^N} \nabla u_n \nabla(u_n - u)\,\mathrm{d}x - \int_{\mathbb{R}^N} h(x)f_a(u_n)(u_n - u)\,\mathrm{d}x$$

to give

$$\int_{\mathbb{R}^N} \nabla u_n \nabla(u_n - u)\,\mathrm{d}x \to 0. \tag{2.10}$$

The weak convergence $u_n \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^N)$ yields

$$\int_{\mathbb{R}^N} \nabla u \nabla(u_n - u)\,\mathrm{d}x \to 0. \tag{2.11}$$

From (2.10) and (2.11),

$$\int_{\mathbb{R}^N} |\nabla u_n - \nabla u|^2\,\mathrm{d}x \to 0.$$

Therefore, $u_n \to u$ in $D^{1,2}(\mathbb{R}^N)$, finishing the proof. □

LEMMA 2.6. *If $a \in (0, a_1)$, then (P) has a solution $u_a \in D^{1,2}(\mathbb{R}^N)$ satisfying $I_a(u_a) \leqslant C$ where $C = C(a_1, \theta, \|h\|_1, \|h\|_\infty) > 0$.*

Proof. The lemmas 2.2, 2.3 and 2.4 guarantee that we can apply the mountain pass theorem due to Ambrosetti-Rabinowitz [4] to show the existence of a critical point $u_a \in D^{1,2}(\mathbb{R}^N)$ for all $a \in (0, a_1)$ with $I_a(u_a) = d_a > 0$, where d_a is the mountain

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pass level of I_a. Now, letting $\varphi \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$, $\varphi(x) \geqslant 0$ and $t > 0$, it follows from (2.3) that

$$\begin{aligned} I_a(t\varphi) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla t\varphi|^2 \, \mathrm{d}x - \int_{\mathbb{R}^N} h(x)F_a(t\varphi)\, \mathrm{d}x \\ &\leqslant \frac{t^2}{2}\int_\Omega |\nabla\varphi|^2 \, \mathrm{d}x - A_1\int_\Omega h(x)(t\varphi)^\theta \, \mathrm{d}x + B_1\int_\Omega h \, \mathrm{d}x + \int_\Omega h(x)at\varphi \, \mathrm{d}x. \end{aligned}$$

where $\Omega = supp\,\varphi$. Then,

$$I_a(t\varphi) \leqslant C_1t^2 - C_2t^\theta + C_3t + C_4,$$

where $C_1 = 1/2\|\varphi\|^2$, $C_2 = A_1\int_\Omega h\varphi^\theta \, \mathrm{d}x$, $C_3 = a_1\|h\|_\infty\|\varphi\|_1$, and $C_4 = B_1\|h\|_1$. Setting $g(t) = C_1t^2 - C_2t^\theta + C_3t + C_4$, and using the fact that $\theta > 2$, we find

$$d_a \leqslant \max\{I_a(t\varphi); t \geqslant 0\} \leqslant \max_{t\geqslant 0} g(t) = C(a_1, \theta, \|h\|_1, \|h\|_\infty) < +\infty.$$

Thus, $I_a(u_a) \leqslant C(a_1, \theta, \|h\|_1, \|h\|_\infty)$, for all $a \in (0, a_1)$. □

The next lemma shows a very important estimate involving the solution u_a for $a \in (0, a_1)$.

LEMMA 2.7. *There exists* $K = K(a_1, \theta, \|h\|_1, \|h\|_\infty, M) > 0$, *such that* $||u_a|| \leqslant K$ *for* $a \in (0, a_1)$.

Proof. To begin with, recall that

$$\begin{aligned} C(a_1, \theta, \|h\|_1, \|h\|_\infty) &\geqslant I_a(u_a) - \frac{1}{\theta}I_a'(u_a)u_a \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_a\|^2 + \int_{\mathbb{R}^N} h(x)\left(\frac{1}{\theta}f_a(u_a)u_a - F_a(u_a)\right) \mathrm{d}x. \end{aligned}$$

From (2.5),

$$C(a_1, \theta, \|h\|_1, \|h\|_\infty) \geqslant \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_a\|^2 - \int_{\mathbb{R}^N} \frac{M}{\theta}h(x)\, \mathrm{d}x,$$

that is

$$C(a_1, \theta, \|h\|_1, \|h\|_\infty) \geqslant \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_a\|^2 - \frac{M}{\theta}\|h\|_1,$$

leading to

$$||u_a|| \leqslant K, \quad \forall a \in (0, a_1),$$

where $K = K(a_1, \theta, \|h\|_1, \|h\|_\infty, M) > 0$. □

Our next result establishes that u_a belongs to $L^\infty(\Omega)$ and that $\{u_a : a \in (0, a_1)\}$ is a bounded set in $L^\infty(\Omega)$ for a_1 small enough. This fact is crucial in our approach.

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LEMMA 2.8. *There is $a_2 \in (0, a_1)$ such that $u_a \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ for all $a \in (0, a_2)$. Moreover, there is $C > 0$ such that*

$$\|u_a\|_\infty \leqslant C, \quad \forall a \in (0, a_2).$$

Proof. In order to prove the lemma, it is enough to show that for any sequence $a_j \to 0$, the sequence of solutions $u_j = u_{a_j}$ possesses a subsequence, still denoted by itself, which is a bounded sequence in $L^\infty(\mathbb{R}^N)$. By lemma 2.7, the sequence $\{u_j\}$ is bounded in $D^{1,2}(\mathbb{R}^N)$, then for some subsequence, there is $u \in D^{1,2}(\mathbb{R}^N)$ such that

$$u_j \rightharpoonup u \quad \text{in } D^{1,2}(\mathbb{R}^N)$$

and

$$u_j(x) \to u(x) \quad \text{a.e. in } \mathbb{R}^N.$$

By using the same approach explored in the proof of lemma 2.4, we have that

$$u_j \to u \quad \text{in } D^{1,2}(\mathbb{R}^N).$$

Consequently,

$$u_j \to u \quad \text{in } L^{2^*}(\mathbb{R}^N) \tag{2.12}$$

and for some subsequence, there is $g \in L^{2^*}(\mathbb{R}^N)$ such that

$$|u_j(x)| \leqslant g(x), \quad \text{a.e. in} \quad \mathbb{R}^N \quad \text{and} \quad \forall j \in \mathbb{N}.$$

Setting the function

$$A_j(x) = h(x)\frac{(1 + |u_j|^{N+2/N-2})}{1 + |u_j|},$$

it follows that

$$|A_j| \leqslant h(1 + |u_j|^{4/(N-2)}) \leqslant h(1 + |g|^{4/(N-2)}), \quad \forall j \in \mathbb{N}. \tag{2.13}$$

Thereby, the Lebesgue theorem together with (2.12) and (2.13) implies that $A_j \to A$ in $L^{N/2}(\mathbb{R}^N)$, for some $A \in L^{N/2}(\mathbb{R}^N)$. As there are $C > 0$ and $j_0 \in \mathbb{N}$ such that

$$|h(x) f_j(u_j)| \leqslant C A_j(x)(1 + |u_j|), \quad \forall j \geqslant j_0$$

and u_j is a solution of (AP), we can use [**15**, lemma B3] to deduce that $u_j \in C(\mathbb{R}^N)$ and for fixed $p \in [2^*, +\infty)$, there is $K_p > 0$ such that

$$\|u_j\|_p \leqslant K_p, \quad \forall j \in \mathbb{N}. \tag{2.14}$$

Since $u_j \to u$ in $L^{2^*}(\mathbb{R}^N)$, the above boundedness ensures that

$$u_j \to u \quad \text{in } L^p(\mathbb{R}^N), \quad \forall p \in [2^*, +\infty).$$

Now, by Riesz potential theory, we know that

$$u_j(x) = C_N \int_{\mathbb{R}^N} \frac{h(y) f_j(u_j(y))}{|x - y|^{N-2}} \, \mathrm{d}y, \quad \forall x \in \mathbb{R}^N,$$

for some $C_N > 0$. Hence,

$$|u_j(x)| \leqslant C_N \int_{B_1(x)} \frac{h(y)|f_j(u_j(y))|}{|x-y|^{N-2}}\,\mathrm{d}y + C_N \int_{B_1^c(x)} \frac{h(y)|f_j(u_j(y))|}{|x-y|^{N-2}}\,\mathrm{d}y.$$

Note that

$$\begin{aligned}\int_{B_1(x)} \frac{h(y)|f_j(u_j(y))|}{|x-y|^{N-2}}\,\mathrm{d}y &\leqslant \int_{B_1(0)} \frac{1}{|z|^{N-2}}|h(x-z)f_j(u_j(x-z))|\,\mathrm{d}z \\ &\leqslant \left|\frac{1}{|z|^{N-2}}\right|_{L^t(B_1(0))} |hf_j(u_j)|_{L^{t'}(B_1(x))},\end{aligned}$$

where $1 < t$, $t \approx 1$ and $1/t + 1/t' = 1$. Recalling that $\{u_j\}$ is bounded in $L^{t'}(\mathbb{R}^N)$ and $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, we derive that

$$C\left|\frac{1}{|z|^{N-2}}\right|_{L^t(B_1(0))} |hf_j(u_j)|_{L^{t'}(B_1(x))} \leqslant M_1, \quad \forall j \in \mathbb{N}, \tag{2.15}$$

for some $M_1 > 0$. On the other hand,

$$\begin{aligned}\int_{B_1^c(x)} \frac{h(y)|f_j(u_j(y))|}{|x-y|^{N-2}}\,\mathrm{d}y &\leqslant \int_{B_1^c(x)} h(y)|f_j(u_j(y))|\,\mathrm{d}y \\ &\leqslant C\int_{B_1^c(x)} h(y)(|u_j| + |u_j|^{q-1} + 1)\,\mathrm{d}y.\end{aligned}$$

Since $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, the Hölder inequality combines with (2.14) to give

$$\int_{B_1^c(x)} h(y)(|u_j| + |u_j|^{q-1} + 1)\,\mathrm{d}y \leqslant M_2, \quad \forall x \in \mathbb{R}^N \quad \text{and} \quad j \in \mathbb{N}, \tag{2.16}$$

for some $M_2 > 0$. From (2.15) to (2.16), there is $M_3 > 0$ such that

$$|u_j(x)| \leqslant M_3, \quad \forall x \in \mathbb{R}^N \quad \text{and } j \in \mathbb{N},$$

showing the lemma. □

In what follows, we show an estimate from below to the norm $L^\infty(\mathbb{R}^N)$ of u_a for a small enough.

LEMMA 2.9. *There exists $a_3 \in (0, a_2)$ and $\delta > 0$ that does not depend on $a \in (0, a_3)$, such that $||u_a||_\infty \geqslant \delta$ for all $a \in (0, a_3)$.*

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Proof. Since u_a is a solution de (AP), then

$$\int_{\mathbb{R}^N} \nabla u_a \nabla \varphi \, \mathrm{d}x = \int_{\mathbb{R}^N} h(x) f_a(u_a) \varphi \, \mathrm{d}x, \quad \forall \varphi \in D^{1,2}(\mathbb{R}^N).$$

For $\varphi = u_a$, we have

$$\int_{\mathbb{R}^N} |\nabla u_a|^2 \, \mathrm{d}x = \int_{\mathbb{R}^N} h(x) f_a(u_a) u_a \, \mathrm{d}x.$$

Since $I_a(u_a) \geqslant \alpha > 0$ for all $a \in (0, a_2)$, there exists $a_3 \in (0, a_2)$ and $\alpha_0 > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla u_a|^2 \, \mathrm{d}x \geqslant \alpha_0, \quad \forall a \in (0, a_3). \tag{2.17}$$

Thus,

$$\int_{\mathbb{R}^N} h(x) f_a(u_a) u_a \, \mathrm{d}x \geqslant \alpha_0 > 0, \quad \forall a \in (0, a_3).$$

Using again the inequality below

$$|f_a(t)| \leqslant C(|t|^{q-1} + |t|) + a, \quad \forall t \in \mathbb{R} \quad \text{and} \quad \forall a \in (0, a_3),$$

we get

$$\alpha_0 \leqslant \int_{\mathbb{R}^N} h(x)(C(|u_a|^{q-1} + |u_a|) + a) \, \mathrm{d}x \leqslant (C(||u_a||_\infty^{q-1} + \|u_a\|_\infty) + a)\|h\|_1.$$

This implies that $||u_a||_\infty \geqslant \delta$ for some $\delta > 0$ for all $a \in (0, a_3)$, decreasing a_3 if necessary. □

3. Proof of theorem 1.1

In order to conclude the proof of theorem 1.1, we need to show that u_a is a positive solution for $a \in (0, a_3)$, decreasing a_3 if necessary. Indeed, let $\{a_j\} \subset (0, a_3)$ be a sequence with $a_j \to 0$ as $j \to +\infty$, and let u_j be a solution of (P) with $a = a_j$. Setting $f_j(u_j) = f_{a_j}(u_j)$, we have

$$\begin{cases} -\Delta u_j = h(x) f_j(u_j) & \text{in } \mathbb{R}^N, \\ u_j \in D^{1,2}(\mathbb{R}^N). \end{cases}$$

By lemma 2.8, $u_j \in C(\mathbb{R}^N)$ and there is $C > 0$ such that $||u_j||_\infty \leqslant C$ for all $j \in \mathbb{N}$, and so, $||f_j(u_j)||_\infty \leqslant C_1$ for all $j \in \mathbb{N}$ and some $C_1 > 0$. In what follows, by

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lemma 2.7, we can assume that

$$u_j \rightharpoonup u \quad \text{in} \quad D^{1,2}(\mathbb{R}^N)$$

and

$$u_j(x) \to u(x) \quad \text{a.e. in} \quad \mathbb{R}^N.$$

From this, $u \in L^\infty(\mathbb{R}^N)$, $\|u\|_\infty \leqslant C$ and

$$|h(x) f_j(u_j)(u_j - u)| \leqslant C_1 h(x)|u_j - u|.$$

Since $|u_j - u| \rightharpoonup 0$ in $L^{2^*}(\mathbb{R}^N)$ and $h \in L^{2N/(N+2)}(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} h(x)|u_j - u| \, \mathrm{d}x \to 0$$

which yields

$$\int_{\mathbb{R}^N} h(x) f_j(u_j)(u_j - u) \, \mathrm{d}x \to 0.$$

The above limit ensures that

$$\int_{\mathbb{R}^N} \nabla u_j \nabla (u_j - u) \, \mathrm{d}x \to 0.$$

Since

$$\int_{\mathbb{R}^N} \nabla u \nabla (u_j - u) \, \mathrm{d}x \to 0$$

we deduce that

$$\int_{\mathbb{R}^N} |\nabla u_j - \nabla u|^2 \, \mathrm{d}x \to 0,$$

that is,

$$u_j \to u \quad \text{in} \quad D^{1,2}(\mathbb{R}^N).$$

Hence, as in lemma 2.8,

$$u_j \to u \in L^p(\mathbb{R}^N), \quad \forall p \in [2^*, +\infty). \tag{3.1}$$

Let v_j be the solution of problem

$$\begin{cases} -\Delta v_j = h(x) k_j & \text{in} \quad \mathbb{R}^N, \\ v_j \in D^{1,2}(\mathbb{R}^N), \end{cases}$$

where $k_j = \min\{f_j(t); t \in \mathbb{R}\} = -a_j \to 0^-$ as $j \to \infty$. Then, $u_j, v_j \in D^{1,2}(\mathbb{R}^N)$ and

$$-\Delta v_j \leqslant -\Delta u_j \quad \text{in } \mathbb{R}^N,$$

or equivalently

$$-\Delta (v_j - u_j) \leqslant 0 \quad \text{in } \mathbb{R}^N.$$

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This inequality implies that

$$\int_{\mathbb{R}^N} (\nabla v_j - u_j)\nabla\phi \, dx \leqslant 0,$$

for all nonnegative function $\phi \in D^{1,2}(\mathbb{R}^N)$. Setting $\phi = (v_j - u_j)^+$, we get

$$\int_{\mathbb{R}^N} |\nabla(v_j - u_j)^+|^2 \, dx = 0,$$

implying that $(v_j - u_j)^+ = 0$, and so,

$$v_j \leqslant u_j \ \text{ a.e. in } \ \mathbb{R}^N, \quad \forall j \in \mathbb{N}. \tag{3.2}$$

On the other hand, arguing as in lemma 2.8, we see that function

$$\Gamma(x) = \int_{\mathbb{R}^N} \frac{h(y)}{|x-y|^{N-2}} \, dy, \quad \forall x \in \mathbb{R}^N$$

belongs to $L^\infty(\mathbb{R}^N)$. Thus, since

$$v_j = C_N k_j \int_{\mathbb{R}^N} \frac{h(y)}{|x-y|^{N-2}} \, dy = C_N k_j \Gamma(x), \quad \forall x \in \mathbb{R}^N$$

and $k_j \to 0$, we have that

$$||v_j||_\infty \to 0. \tag{3.3}$$

Then, (3.2) combined with (3.3) implies that $u \geqslant 0$ a.e in $\mathbb{R}^N$. Notice that

- $\{h(x)f_j(u_j)\}$ is bounded in $L^s(\mathbb{R}^N)$, for some $s > 1$,
- $h(x)f_j(u_j) \rightharpoonup z$ in $L^s(\mathbb{R}^N)$,
- $h(x)f_j(u_j(x)) \to h(x)f_0(u(x))$ a.e. $x \in \mathbb{R}^N$

where $f_0(t) = f(t)$ if $t \geqslant 0$, and $f_0(t) = 0$ if $t < 0$. Having this in mind, we deduce that $z = h(x)f_0(u)$, and for any $\varphi \in C_0^\infty(\mathbb{R}^N)$

$$\begin{aligned}\int_{\mathbb{R}^N} \nabla u \nabla\varphi &= \lim_{j\to+\infty} \int_{\mathbb{R}^N} \nabla u_j \nabla\varphi = \lim_{j\to+\infty} \int_{\mathbb{R}^N} h(x)f_j(u_j)\varphi \, dx \\ &= \int_{\mathbb{R}^N} z\varphi \, dx = \int_{\mathbb{R}^N} hf_0(u)\varphi \, dx,\end{aligned}$$

consequently

$$\begin{cases} -\Delta u &= h(x)f_0(u) \quad \text{in} \quad \mathbb{R}^N, \\ \quad u &\geqslant 0 \quad\quad\quad\;\; \text{in} \quad \mathbb{R}^N. \end{cases}$$

From this, the Riesz potential theory ensures that

$$u_j(x) = C_N \int_{\mathbb{R}^N} \frac{h(y)f_j(u_j(y))}{|x-y|^{N-2}} \, dy, \quad \forall x \in \mathbb{R}^N \quad \text{and } \forall j \in \mathbb{N}$$

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and

$$u(x) = C_N \int_{\mathbb{R}^N} \frac{h(y) f_0(u(y))}{|x-y|^{N-2}} \, \mathrm{d}y, \quad \forall x \in \mathbb{R}^N,$$

for some positive constant C_N. Hence, arguing as in the proof of lemma 2.8, there are $t, t' \in (1, +\infty)$, $t \approx 1$ and $1/t + 1/t' = 1$, such that

$$|u_j(x) - u(x)| \leqslant C \left| \frac{1}{|z|^{N-2}} \right|_{L^t(B_1(0))} |h(f_j(u_j) - f_0(u))|_{L^{t'}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} h(y)|f_j(u_j) - f_0(u)| \, \mathrm{d}y, \quad \forall x \in \mathbb{R}^N,$$

that is,

$$\|u_j - u\|_\infty \leqslant C \left| \frac{1}{|z|^{N-2}} \right|_{L^t(B_1(0))} |h(f_j(u_j) - f_0(u))|_{L^{t'}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} h(y)|f_j(u_j) - f_0(u)| \, \mathrm{d}y, \quad \forall j \in \mathbb{N}.$$

Now, combining the fact that $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with (3.1), we get by Lebesgue theorem,

$$|h(f_j(u_j) - f_0(u))|_{L^{t'}(\mathbb{R}^N)} \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} h(y)|f_j(u_j) - f_0(u)| \, \mathrm{d}y \to 0,$$

from where it follows that

$$u_j \to u \quad \text{in} \quad L^\infty(\mathbb{R}^N). \tag{3.4}$$

As $||u_j||_\infty \geqslant C_0$ for all $j \in \mathbb{N}$, we derive that $||u||_\infty \geqslant C_0$, and so $u \neq 0$. By regularity theory and maximum principle, it follows that

$$u \in C(\mathbb{R}^N) \quad \text{and} \quad u(x) > 0 \;\text{ in }\; \mathbb{R}^N$$

and so, $f_0(u) = f(u)$. If $C = \sup_{j \in \mathbb{N}} \|u_j\|_\infty$, since $f : [0, +\infty) \to \mathbb{R}$ is a Lipschitz function in the interval $[-C, C]$, we then have

$$|f(t) - f(s)| \leqslant M|t - s|, \quad \forall s, t \in [-C, C],$$

for some constant $M > 0$. From this, for each $x \in \mathbb{R}^N$,

$$|u_j(x) - u(x)| = C_N \left| \int_{\mathbb{R}^N} \frac{h(y)(f_j(u_j(y)) - f(u(y))}{|x-y|^{N-2}} \, \mathrm{d}y \right|$$

or equivalently

$$|u_j(x) - u(x)| = C_N \left| \int_{\mathbb{R}^N} \frac{h(y)(f_j(u_j(y)) - f_j(u(y))) + h(y)a_j}{|x-y|^{N-2}} \, \mathrm{d}y \right|.$$

Thus,

$$|u_j(x) - u(x)| \leqslant C_N M \left| \int_{\mathbb{R}^N} \frac{h(y)|u_j(y) - u(y)|}{|x-y|^{N-2}} \, \mathrm{d}y \right| + C_N \int_{\mathbb{R}^N} \frac{h(y)a_j}{|x-y|^{N-2}} \, \mathrm{d}y$$

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from where if follows that

$$|u_j(x) - u(x)| \leqslant C_N(M\|u_j - u\|_\infty + a_j) \int_{\mathbb{R}^N} \frac{P(y)}{|x-y|^{N-2}}\,\mathrm{d}y.$$

Then by (P_3),

$$|u_j(x) - u(x)| \leqslant \frac{\tilde{C}(M\|u_j - u\|_\infty + a_j)}{|x|^{N-2}}, \quad \forall x \in \mathbb{R}^N.$$

Hence,

$$\sup_{x\in\mathbb{R}^N} \{|x|^{N-2}|u_j(x) - u(x)|\} \leqslant \tilde{C}(M\|u_j - u\|_\infty + a_j), \quad \forall j,k \in \mathbb{N},$$

where $\tilde{C}$ is independent of j. As $a_j \to 0$ and $u_j \to u$ in $L^\infty(\mathbb{R}^N)$, we deduce that

$$\sup_{x\in\mathbb{R}^N} \{|x|^{N-2}|u_j(x) - u(x)|\} \to 0 \quad \text{as } j \to +\infty. \tag{3.5}$$

On the other hand, by a straightforward computation,

$$\begin{aligned}\lim_{|x|\to+\infty} |x|^{N-2}u(x) &= C_N \lim_{|x|\to+\infty} \int_{\mathbb{R}^N} \frac{|x|^{N-2}h(y)f(u)}{|x-y|^{N-2}}\,\mathrm{d}y \\ &= C_N \int_{\mathbb{R}^N} h(y)f(u)\,\mathrm{d}y = C_* > 0.\end{aligned}$$

Therefore, this limit combined with (3.5) guarantees the existence of $j_0 \in \mathbb{N}$ and $R > 0$ such that

$$u_j(x) > 0 \quad \text{for } |x| \geqslant R \quad \text{and } j \geqslant j_0. \tag{3.6}$$

Now, using the limit (3.4), we also have that

$$u_j \to u \quad \text{in } C(\overline{B}_R(0)).$$

As $u > 0$ in $\overline{B}_R(0)$, increasing j_0 if necessary, we find

$$u_j(x) > 0 \quad \text{in } \overline{B}_R(0), \quad \forall j \geqslant j_0 \tag{3.7}$$

From (3.6)-(3.7),

$$u_j(x) > 0, \quad \forall x \in \mathbb{R}^N \quad \text{and } j \geqslant j_0.$$

The above analysis implies that u_a is a positive solution for $a \in (0, a_3)$. This completes the proof of theorem 1.1.

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