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Tying hairs for structurally stable exponentials

RANJIT BHATTACHARJEE and ROBERT L. DEVANEY
Department of Mathematics, Boston University, Boston MA 02215, USA
(e-mail: bob@bu.edu)

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Abstract. Our goal in this paper is to describe the structure of the Julia set of complex exponential functions that possess an attracting cycle. When the cycle is a fixed point, it is known that the Julia set is a 'Cantor bouquet', a union of uncountably many distinct curves or 'hairs'. When the period of the cycle is greater than one, infinitely many of the hairs in the bouquet become pinched or attached together. In this paper, we develop an algorithm to determine which of these hairs are attached. Of crucial importance in this construction is the kneading invariant, a sequence that is derived from the topology of the basins of attraction of the attracting cycle.

1. *Introduction*

In this paper we will discuss the topology of the Julia set for certain complex exponential maps of the form $E_\lambda(z) = \lambda \exp(z)$ where $\lambda \in \mathbb{C}$. We will restrict attention to those exponentials for which E_λ has an attracting periodic orbit. It is known that, since E_λ has a unique asymptotic value (0) and no critical values, E_λ has at most one attracting cycle.

The Julia set of E_λ is the set of points at which the family of iterates of E_λ fails to be a normal family in the sense of Montel. It is known that each point in the Julia set whose orbit has bounded imaginary part lies on a curve in the Julia set [**5**]. This curve is the image of a homeomorphism $h : [0, \infty) \to \mathbb{C}$ with the following properties: (i) the orbit of $h(0)$ is bounded; (ii) if $t > 0$, the orbit of $E_\lambda^n(h(t))$ tends to ∞ as $n \to \infty$; (iii) $\lim_{t\to\infty} \operatorname{Re} h(t) = \infty$.

These curves are called *hairs*. Points whose orbits escape lie on the *tail* of the hair. The point $h(0)$ whose orbit is bounded is called the *endpoint* of the hair.

It is known [**2**] that the Julia set of E_λ is also the closure of the repelling periodic points of E_λ. These points therefore lie at the endpoints of the hairs, since all other points on the hairs escape to infinity. Hence these endpoints must accumulate on all other points on the Julia set. Another established fact is that the tails of different hairs cannot meet in $\mathbb{C}$. However, the endpoints of certain hairs may coincide [**7**]. In fact, it often happens that more than one tail meets at a given endpoint. When this happens, we say that the hairs are *attached* or *tied together*.

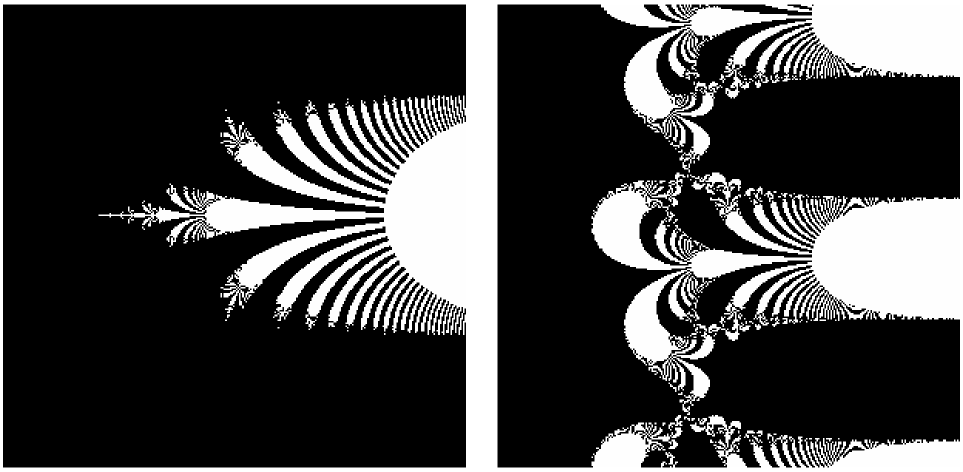

FIGURE 1. The Julia sets for $\lambda = 1/e$ and $\lambda = 5 + \pi i$.

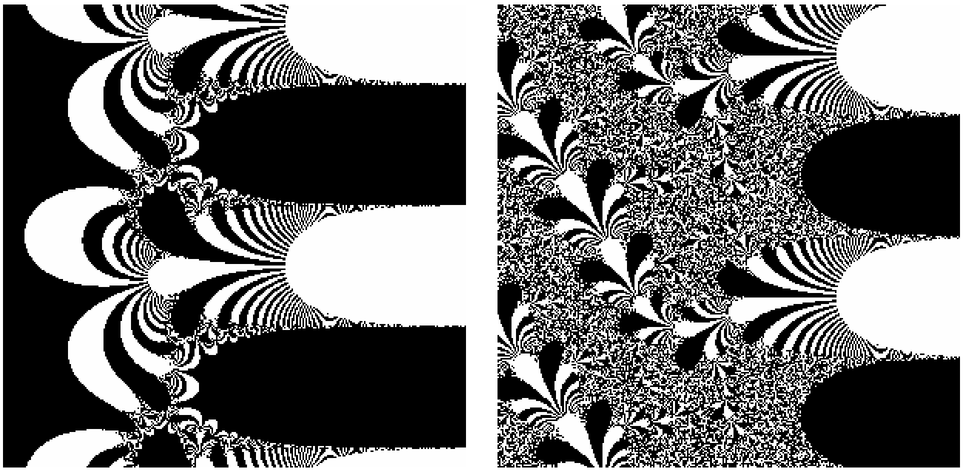

FIGURE 2. The Julia set for $\lambda = 10 + 3\pi i$ and $\lambda = 3.14i$.

For example, in Figure 1, we display the Julia set when $\lambda = 1/e$. For this λ-value, E_λ has an attracting fixed point. The basin of attraction of this fixed point (the complement of the Julia set) is displayed in black. In this figure, it appears that the Julia set contains open sets. However, this is not the case. In fact, $J(E_\lambda)$ is a 'Cantor bouquet' which consists of an uncountable collection of hairs, none of which are tied together. It is known [**9**] that the Hausdorff dimension of this Julia set is two.

In Figure 1, we also display the Julia set when $\lambda = 5 + i\pi$, which is one of the fundamental examples we deal with below. As we will see, this exponential has an attracting cycle of period three. In this case it appears that there are trios of hairs that are attached at a number of distinct points in the plane.

In Figure 2, we display the Julia set when $\lambda = 10 + 3\pi i$. This map also has an attracting cycle of period three. Note that different hairs now seem to be attached. In contrast, the Julia set for $\lambda = 3.14i$ (Figure 2) shows that the structure of the attached hairs can be extremely complicated.

Our goal in this paper is to develop an algorithm that specifies exactly which hairs are attached at which endpoints in the Julia set. Our algorithm will depend on the *kneading sequence* associated to E_λ. The kneading sequence is a sequence of $n - 2$ integers that specifies the topology of the basin of attraction of the attracting n-cycle (we assume that $n > 2$ since the period one and two cases are trivial). Given the kneading sequence, we will use symbolic dynamics to understand how the hairs are tangled together. In particular, we will prove that, if the last integer in the kneading sequence is nonzero, then the corresponding exponential must have infinitely many distinct periodic points that have multiple hairs attached.

2. *Fingers*

In this section we will describe some general properties of the complement of the Julia set. We assume that E_λ has an attracting periodic cycle $z_0, \ldots, z_n = z_0$ of period n. Throughout we assume that $n \geq 3$.

It is well known that the asymptotic value zero lies in the immediate basin of attraction of some point on the cycle. Without loss of generality, we will assume that $0 \in A^*(z_1)$ where $A^*(z)$ is the immediate basin of attraction of z. The reason for assuming $0 \in A^*(z_1)$ rather than $0 \in A^*(z_0)$ will become apparent in the following. We will define a collection of open sets B_i about each of the z_i. Starting with the point z_1, we first define a set B_{n+1} with the following properties: (i) B_{n+1} is an open and simply connected subset of $A^*(z_1)$; (ii) $0, z_1 \in B_{n+1}$; (iii) B_{n+1} has compact closure and is a fundamental domain, i.e. $E_\lambda^n(B_{n+1}) \subset B_{n+1}$.

Next we will obtain a neighborhood of z_0 by considering the preimage of B_{n+1}. Define

$$B_n = E_\lambda^{-1}(B_{n+1}).$$

The proof of the following is straightforward.

PROPOSITION 2.1. *B_n is a simply connected neighborhood of z_0 and B_n contains a half-plane* $\operatorname{Re} z \leq \xi_1$ *and is contained in a half-plane* $\operatorname{Re} z \leq \xi_2$ *for some* $\xi_1, \xi_2 \in \mathbb{R}$.

Now we can extend this construction to all the points on the cycle. For $j = 1, \ldots, n$ let B_{n-j} be the connected component of $E_\lambda^{-1}(B_{n-j+1})$ that contains z_{n-j}. Note that B_1 is contained in the immediate basin of z_1 and $B_1 \supset B_{n+1}$. Indeed, $E_\lambda^n(B_1) = B_{n+1} - \{0\}$. We also have $B_0 \supset B_n$ and $E_\lambda^n(B_0) = B_n$. The next proposition follows directly from applying appropriate branches of the logarithm.

PROPOSITION 2.2. *For $j = 1, \ldots, n-1$, B_j is a simply connected set which is mapped univalently onto B_{j+1} by E_λ.*

Note that $E_\lambda : B_0 \to B_1 - \{0\}$ is a universal covering and hence this map is not univalent.

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Definition 2.3. An unbounded, simply connected $F \subset \mathbb{C}$ is called a finger of width c if:

(i) F is bounded by a simple curve $\gamma \subset \mathbb{C}$;

(ii) there exists a $\nu > 0$ such that $F \cap \{z \mid \operatorname{Re} z > \nu\}$ is simply connected, extends to infinity, and satisfies

$$\{F \cap \{z \mid \operatorname{Re} z > \nu\}\} \subset \left\{ z \;\middle|\; \operatorname{Im} z \in \left[\xi - \frac{c}{2}, \xi + \frac{c}{2}\right]\right\}$$

for some $\xi \in \mathbb{R}$.

It is worth noting that, since γ is a simple curve, there exists a μ such that

$$F \cap \{z \mid \operatorname{Re} z \leq \mu\} = \emptyset.$$

PROPOSITION 2.4. *Suppose F is a finger of width c with $0 \notin F$. Then $E_\lambda^{-1}(F)$ consists of infinitely many disjoint fingers, each of width $d \leq 2\pi$.*

The proof is straightforward. As a consequence, we have the following.

PROPOSITION 2.5. *Let $n \geq 2$. For $j = 1, \dots, n-1$, B_j is a finger of width $b_j \leq 2\pi$.*

This construction stops at B_0, since B_0 is not a finger due to the fact that $0 \in B_1$.

PROPOSITION 2.6. *The complement of B_0 consists of infinitely many fingers of width w_0, where $w_0 \leq 2\pi$.*

Proof. Since B_1 is a finger of width w_1, $w_1 < 2\pi$, the set $B_1 \cap \operatorname{Re} z > \nu > 0$ for sufficiently large ν is also a finger of width 2π. Call this finger $\tilde{B}_1$. If $\nu > 0$, then $0 \notin \tilde{B}_1$, so Proposition 2.4 applies and $E_\lambda^{-1}(\tilde{B}_1)$ consists of infinitely many fingers of width 2π. But each of these fingers is contained in B_0 which is connected. Hence the complement of B_0 consists of infinitely many fingers which, in $\operatorname{Re} z > \nu$, are separated by the unbounded components of $E_\lambda^{-1}(\tilde{B}_1)$. Since these components are $2\pi i$ periodic, it follows that the fingers have width at most 2π. □

In this sense B_0 resembles a 'glove', since it contains a left half-plane and has infinitely many fingers extending to the right. We summarize as follows.

THEOREM 2.7. *Suppose $z_0, \dots, z_{n-1}$ is an attracting periodic orbit for E_λ with $n \geq 3$. Suppose $0 \in A^*(z_1)$. Then there exist disjoint, open, simply connected sets $B_0, \dots, B_{n-1}$ such that:*

(i) $z_j \in B_j$, $B_j \subset A^*(z_j)$;

(ii) $E_\lambda(B_j) = B_{j+1}$, $j = 0, \dots, n-2$, *and* $E_\lambda(B_{n-1}) \subset B_0$;

(iii) $B_1, \dots, B_{n-1}$ *are fingers of width* $b_j \leq 2\pi$;

(iv) *the complement of B_0 consists of infinitely many disjoint fingers.*

Since this collection of sets will become important, we will formulate the following.

Definition 2.8. A collection of open subsets $B_0, \dots, B_{n-1}$ satisfying the conditions in Theorem 2.7 is called a fundamental set of attracting domains for the cycle $z_0, \dots, z_{n-1}$. The fingers $B_1, \dots, B_{n-1}$ are called stable fingers.

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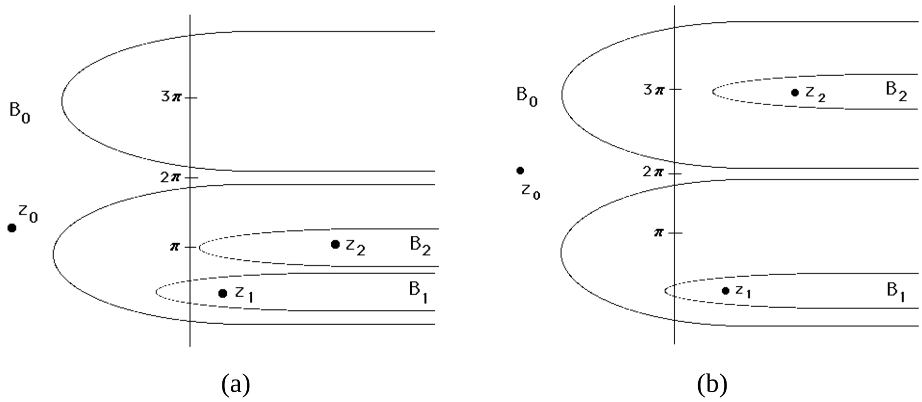

FIGURE 3. (a) Fingers for E_μ. (b) Fingers for E_ν.

Example A. Let $\mu = 5+i\pi$. Actually, the construction below works for any μ of the form $a+i\pi$ with a sufficiently large; we choose 5 merely for convenience.

The map E_μ has an attracting cycle of period three. To see this, we first note that the real part of $E_\mu^i(0)$ satisfies Re $E_\mu(0) = 5$ and Re $E_\mu^2(0) \approx -5e^5$.

Thus

$$|E_\mu^3(0)| \approx 5e^{-5e^5}$$

which is very close to zero.

Let B_δ denote the ball of radius δ centered at the origin. Then $E_\mu^j(B_\delta)$ contains a ball whose radius is: of the order of 5δ centered at $\mu = E_\mu(0)$ if $j=1$; of the order of $5e^5 \cdot 5\delta$ centered at $E_\mu^2(0)$ if $j=2$; and of the order of $5e^{-5e^5} \cdot 5e^5 \cdot 5\delta$ centered at $E_\mu^3(0)$ if $j=3$. One checks easily that this latter radius is much smaller than δ for δ of the order of $1/5$. Moreover, the distance from $E_\mu^3(0)$ to zero is much smaller than δ. Consequently, E_μ^3 maps B_δ inside itself, and so E_μ has an attracting cycle of period three.

According to the above construction, we set $B_4 = B_\delta$. Then the B_j for $j = 0, 1, 2$ form a fundamental set of attracting regions and are as displayed in Figure 3. Note that this picture is a caricature of the B_j, as the sizes of the fingers in practice are quite different.

Example B. Now let $\nu = a + 3\pi i$ where a is sufficiently large. A similar proof as in Example A shows that E_ν has an attracting cycle of period three. In Figure 3 we sketch the location of the various B_j for E_ν. Note that the only difference is the placement of B_2 relative to the fingers in the complement of B_0.

In fact there are many ways to construct a fundamental set of attracting domains. In order to simplify later computations we wish to make the boundaries of the fingers smooth and nearly horizontal in the far right half-plane.

We will describe one important property in the following.

Definition 2.9. A smooth curve $\gamma(t)$ is called horizontally asymptotic to c if:

(i) $\lim_{t\to\infty} \mathrm{Re}(\gamma(t)) = +\infty$;

(ii) $\lim_{t\to\infty} \mathrm{Im}(\gamma(t)) = c$;

(iii) $\lim_{t\to\infty} \arg(\gamma'(t)) = 0$.

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It is then straightforward to check that if $\gamma(t)$ is horizontally asymptotic to c, then $E_\lambda^{-1}(\gamma(t))$ is horizontally asymptotic to $2\pi k - \arg\lambda$ for some $k \in \mathbb{Z}$.

For any fundamental set of attracting domains the property that the fingers have boundaries that are smooth and nearly asymptotic is solely dependent on the boundary of the component B_n which includes the left half-plane. We will choose the boundary of that component to be vertical for large enough imaginary part. This yields the following proposition which we will state without a proof.

PROPOSITION 2.10. *For a cycle $z_0, \ldots, z_{n-1}$ there exists a fundamental set of attracting domains, denoted C_j for $j = 0, \ldots, n-1$, with the following properties. There are integers k_j and a parameterization $\gamma_j(t)$ of the boundary of C_j which is horizontally asymptotic to:*

(i) $2\pi k_{n-1} - \arg(\lambda) \pm \pi/2$ *if* $j = n-1$;

(ii) $2\pi k_j - \arg(\lambda)$ *if* $j = 1, \ldots, n-2$, *where* $k_j \in Z$.

For the remainder of this paper, we always assume that the fundamental set of attracting domains is chosen to satisfy the above constraints.

3. *Dynamics on the Julia set*

Our goal in this section is to describe the dynamics of E_λ on its Julia set via symbolic dynamics.

We begin by describing the itineraries of points in the Julia set as well as a collection of subsets of the Julia set, each of which is homeomorphic to a Cantor set.

Recall that the complement of C_0 consists of infinitely many closed fingers, unbounded in the right half-plane. We denote these fingers by $\mathcal{H}_k$ where $k \in \mathbb{Z}$. We index the $\mathcal{H}_k$ so that $0 \in \mathcal{H}_0$ and so that k increases with increasing imaginary parts. Note that $J(E_\lambda)$ is contained in the union of the $\mathcal{H}_k$.

We have $E_\lambda(C_0) = C_1 - \{0\}$, so it follows that $E_\lambda(\mathcal{H}_k) = \mathbb{C} - C_1$ for each k. We define $L_{\lambda,k}$ to be the inverse of E_λ on $\mathbb{C} - C_1$ which takes values in $\mathcal{H}_k$.

Let $\Sigma = \{(s) = (s_0 s_1 s_2 \ldots) \mid s_j \in \mathbb{Z} \text{ for each } j\}$. Σ is called the *sequence space*. The *shift map* σ on Σ is given by

$$\sigma(s_0 s_1 s_2 \ldots) = (s_1 s_2 s_3 \ldots).$$

We define the *itinerary* $S(z)$ of $z \in J(E_\lambda)$ by

$$S(z) = (s_0 s_1 s_2 \ldots) \quad \text{where } s_j = k \text{ iff } E_\lambda^j(z) \in \mathcal{H}_k.$$

Note that $S(E_\lambda(z)) = \sigma(S(z))$.

We will be primarily concerned with itineraries whose entries are bounded. Therefore, we set

$$\Sigma_N = \{s \in \Sigma \mid |s_j| \le N \text{ for each } j\}.$$

Each E_λ possesses a natural invariant set Γ_N that is homeomorphic to Σ_N for each N. The details of this construction may be found in [**5**], but for completeness we sketch it here also.

For $\tau \gg 0$ and $|k| \leq N$, we define

$$V_k = (\mathcal{H}_k \cap \{\mathrm{Re}\, z \leq \tau\}) - \bigcup_{j=1}^{n-1} C_j.$$

For τ large enough, each of the V_k are simply connected and have the property that

$$E_\lambda(V_k) \supset V_j$$

for each j.

Hence $L_{\lambda,k}$ is well defined and maps $\cup V_j$ into V_k. Given $s = (s_0 s_1 s_2 \dots) \in \Sigma_N$, we define

$$L_\lambda^n = L_{\lambda,s_0} \circ \cdots \circ L_{\lambda,s_n}.$$

Each L_λ^n maps any V_j into V_{s_0}. In particular, L_λ^n maps V_{s_0} into V_{s_0}. Moreover, for large enough n, the closure of $L_\lambda^n(V_{s_0})$ is contained in the interior of V_{s_0}. Indeed, any point on the boundary of V_{s_0} is mapped by E_λ^k outside of the V_j for some $k \leq n$. The Poincaré (hyperbolic) metric defined on the unit disc induces a metric on V_{s_0}, since it is simply connected. The earlier argument then implies that L_λ^n is a contraction in the Poincaré metric on V_{s_0}. It follows that

$$\gamma_s = \lim_{n\to\infty} L_\lambda^n(z)$$

exists and is independent of $z \in V_{s_0}$.

Let Γ_N denote the union of the γ_s for $s \in \Sigma_N$. Then it is straightforward to check that Γ_N is a Cantor set that is homeomorphic to Σ_N (with homeomorphism given by $s \to \gamma_s$). Moreover, Γ_N is contained in the Julia set of E_λ and is invariant under E_λ. Furthermore, the action of E_λ on Γ_N is conjugate to the shift map σ on Σ_N.

We summarize this as follows.

THEOREM 3.1. *For each $N > 0$ there is an invariant subset Γ_N of $J(E_\lambda)$ that is homeomorphic to Σ_N and on which E_λ is conjugate to the shift map.*

Remark. There are many points in $J(E_\lambda)$ besides γ_s that share the same itinerary. Indeed, as we will describe below, each point in Γ_N has at least one 'hair' attached that shares the same itinerary. This hair is a continuous curve that connects a point in Γ_N to ∞ and lies in the Julia set.

For each C_j with $1 \leq j \leq n-1$, there exists $\mathcal{H}_k$ such that $C_j \subset \mathcal{H}_k$. We define the kneading sequence for λ as follows.

Definition 3.2. Let E_λ have a attracting cycle of period $n \geq 3$. The kneading sequence as the string of $n-2$ integers is

$$K(\lambda) = 0k_1k_2 \dots k_{n-2}*$$

where $k_i = j$ iff $E_\lambda^i(0) \in \mathcal{H}_j$.

Note that the kneading sequence gives the location of $E_\lambda(0), \dots, E_\lambda^{n-2}(0)$ relative to the $\mathcal{H}_k$. For completeness we include the location of zero in $\mathcal{H}_0$. Similarly, $E_\lambda^{n-1}(0)$ lies in C_0, which is the complement of the $\mathcal{H}_k$, and so this will be denoted by $*$. Equivalently,

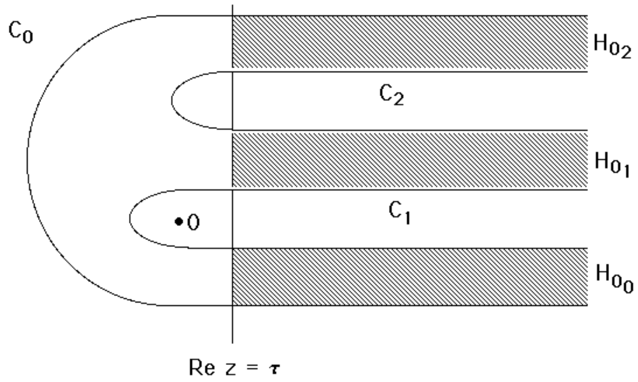

FIGURE 4. The fingers H_{0_0}, H_{0_1}, and H_{0_2} for E_μ.

the kneading sequence indicates which $\mathcal{H}_k$ contain the points $z_2, \ldots, z_{n-1}$ on the orbit of the cycle.

For $\tau \gg 0$ as defined above, the set

$$\Lambda_\tau = \{z \in \mathbb{C} \mid \operatorname{Re} z \geq \tau\} - \bigcup_{j=0}^{n-1} C_j$$

consists of infinitely many closed fingers. Each finger in Λ_τ is included in precisely one $\mathcal{H}_j$, since all of the fingers in the glove C_0 which bounds the $\mathcal{H}_k$ are removed with the other C_j. If j is not one of the entries in the kneading sequence, then there is only one finger in Λ_τ that lies in $\mathcal{H}_j$ (namely the far right portion of $\mathcal{H}_j$ itself). We denote this finger in Λ_τ by H_j. However, for j in the kneading sequence, there is more than one finger in Λ_τ that meets $\mathcal{H}_j$ since the C_i separate $\Lambda_\tau \cap \mathcal{H}_j$ into at least two fingers. The fingers that lie in such an $\mathcal{H}_j \cap \Lambda_\tau$ will be denoted H_{j_k} where j_k orders them with ascending imaginary part beginning with j_0. Note that all of these fingers lie in the half-plane $\operatorname{Re} z \geq \tau$.

Example A. Recall the example E_μ where $\mu = 5 + i\pi$ as described in the previous section. In this case both C_1 and C_2 lie in $\mathcal{H}_0$. Since the kneading sequence only involves the location of C_2 in this case, we have $K(\mu) = 00*$. Furthermore, the fingers C_1 and C_2 subdivide $\{\operatorname{Re} z \geq \tau\} \cap \mathcal{H}_0$ into three fingers which we denote by H_{0_0}, H_{0_1}, and H_{0_2}. See Figure 4.

Example B. In Example B of the previous section, the kneading sequence is now $K(\nu) = 01*$, since C_2 lies in $\mathcal{H}_1$. Thus C_1 and C_2 subdivide both $\{\operatorname{Re} z \geq \tau\} \cap \mathcal{H}_0$ and $\{\operatorname{Re} z \geq \tau\} \cap \mathcal{H}_1$ into two subfingers, denoted by H_{0_0}, H_{0_1}, H_{1_0}, and H_{1_1}. See Figure 5.

We can describe the itinerary of certain points in the Julia set even more precisely by defining an augmented itinerary for $z \in J(E_\lambda) \cap \{z \in \mathbb{C} \mid \operatorname{Re} z \geq \tau\}$. In an augmented itinerary, we specify which of the H_{j_k} the orbit of z visits. More precisely, let $\mathbb{Z}'$ denote the set whose elements are either integers not contained in the kneading sequence, or

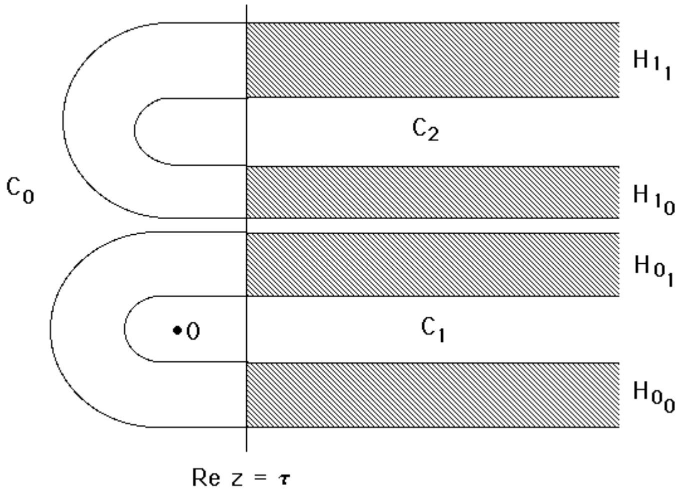

FIGURE 5. The fingers H_{0_0}, H_{0_1}, H_{1_0}, and H_{1_1} for E_ν.

subscripted integers j_k corresponding to an H_{j_k} if j is an entry in the kneading sequence. The *augmented itinerary* of z is

$$S'(z) = (s_0 s_1 s_2 \dots)$$

where each $s_j \in \mathbb{Z}'$ and s_j specifies the finger in Λ_τ containing $E_\lambda(z)$. Let Σ' denote the set of augmented itineraries. Of course, the augmented itinerary is defined only for points whose orbits remain for all time in Λ_τ.

Definition 3.3. The deaugmentation map is a map $D : \Sigma' \to \Sigma$ such that if $s_n = j_k$ then $(D(s))_n = j$. If $s_n = j$, then $(D(s))_n = j$.

That is, D simply removes the subscript from each subscripted entry in a sequence in Σ', and leaves other entries alone.

It turns out that not all augmented itineraries actually correspond to orbits in the far right half-plane. In order to describe which augmented itineraries do correspond to points in $J(E_\lambda)$, we introduce the concept of allowable transitions.

Definition 3.4. Let $s = (s_0 s_1 s_2 \dots) \in \Sigma'$. A transition is defined as any two adjacent entries (s_i, s_{i+1}) in s. The transition is called allowable if

$$E_\lambda(H_{s_i}) \cap H_{s_{i+1}} \neq \emptyset.$$

In this case we say $E_\lambda(H_{s_i})$ meets $H_{s_{i+1}}$. An allowable transition will be denoted as $s_i \to s_{i+1}$. An itinerary $s' \in \Sigma'$ will be called allowable if for all s_j it follows that $s_j \to s_{j+1}$. The set of allowable itineraries will be denoted Σ^*.

For the remainder of this paper we assume that N satisfies $|k_j| \leq N$ for all entries k_j in the kneading sequence. Let Σ^*_N denote the set of sequences in Σ^* whose deaugmentation is a sequence in Σ_N.

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We now turn to the question of which points in $J(E_\lambda)$ share the same itinerary (augmented or otherwise). Without proof we will first state the following.

PROPOSITION 3.5. *We may choose τ large enough so that if $j_\ell \to i_k$ for a sequence in Σ_N^*, then $\{\Lambda_\tau \cap L_{\lambda,s_j}(H_{i_k})\} \subset H_{j_\ell}$ is a closed finger that is bounded on the left by* $\operatorname{Re} z = \tau$ *and completely contained inside some H_{j_ℓ}.*

Using arguments developed in [**5**], we can show that given $s' \in \Sigma_N^*$,

$$\lim_{n\to\infty}\{\Lambda_\tau \cap L_{\lambda,s_0} \circ \cdots \circ \{\Lambda_\tau \cap \mathrm{L}_{\lambda,s_n}(H_{s'_{n+1}})\}\}$$

is a closed and connected set that meets ∞, $\operatorname{Re} z = \tau$ and it is a continuous curve which we may parameterize by $h_{\lambda,s'} : [t_0, \infty) \to H_{s'_0}$ with $\operatorname{Re} h_{\lambda,s'_0}(t_0) = \tau$. This curve is called the tail of a *hair* in the Julia set. Therefore, we have the following.

PROPOSITION 3.6. *Let $s \in \Sigma_N^*$. There is a unique tail of a hair in $\Lambda_\tau \cap J(E_\lambda)$ that has augmented itinerary s.*

Thus, for each allowable sequence s' in Σ_N^*, we have a well defined hair in the portion of the Julia set to the right of $\operatorname{Re} z = \tau$ that has itinerary s'. Our goal now is to see how these hairs connect to the Cantor set Γ_N constructed earlier.

Given the hair $h_{\lambda,\sigma(s)}(t)$, we may pull this curve back into the region $\operatorname{Re} z < \tau$ by applying L_{λ,s_0}. The result is a curve that extends the hair $h_{\lambda,s}(t)$ into the region $\operatorname{Re} z < \tau$. This follows since $E_\lambda \circ h_{\lambda,s}(t)$ is properly contained in the hair $h_{\lambda,\sigma(s)}(t)$ in the far right half-plane. We continue this process by applying

$$L_{\lambda,s_0} \circ \cdots \circ L_{\lambda,s_n}$$

to the hair $h_{\lambda,\sigma^{n+1}(s)}(t)$. Each time we extend the original hair. Moreover, as in the proof that Γ_N is a Cantor set, these extended hairs all tend to a unique point in Γ_N. Now there is only one point in Γ_N that has the same non-augmented itinerary as the hair, namely the point whose deaugmented itinerary is given by $D(s)$. Therefore, the hair must terminate at this point.

If we let $h^\tau_{\lambda,s}$ be the set of points on the tail of the hair $h_{\lambda,s}(t)$ where $t \in [\tau, \infty)$, then the full hair is characterized by the following definition.

Definition 3.7. The full hair corresponding to the sequence $s \in \Sigma_N^*$ is given by

$$\lim_{n\to\infty} L_{\lambda,s_0} \circ \cdots \circ L_{\lambda,s_n} h^\tau_{\lambda,\sigma^{n+1}(s)}.$$

We have shown the following.

THEOREM 3.8. *Let $s \in \Sigma_N^*$. The full hair corresponding to s is a curve in the Julia set that tends to ∞ in the right half-plane and limits on $\gamma_{D(s)} \in \Gamma_N$.*

It follows from Theorem 3.8 that hairs that correspond to different sequences in Σ_N^* that have the same deaugmentation must limit on the same point in Γ_N. In this case, we say that the hairs are attached to the same point.

Hairs can in fact be tied together, as the following examples show.

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Example A. Recall that for E_μ the kneading sequence is $K(\mu) = 00*$ and that the region H_0 contained only the two fingers C_1 and C_2. These fingers subdivide Λ_τ into the three fingers which we denoted by H_{0_0}, H_{0_1}, and H_{0_2}.

Hence there are three full hairs in H_0, one tending to infinity in each of these three fingers. As we will see in the next section, all of these hairs have deaugmented sequence $(000\ldots)$. Hence, by Theorem 3.1, each of these hairs must be attached to γ_s with $s = (000\ldots)$, which is a fixed point for E_λ. Furthermore, any preimage of γ_s must have three hairs attached, by invariance of the Julia set. These triple attachments are clear in Figure 2, which shows $J(E_\mu)$.

Example B. For the map E_ν, the kneading sequence is $K(\nu) = 01*$ and we have two fingers, $C_1 \subset H_0$ and $C_2 \subset H_1$. In H_0 we have two fingers H_{0_0} and H_{0_1}, and there are two in H_1 with indices 1_0 and 1_1. Each of these fingers contains a hair, and we will see that the pair in H_0 is attached to a point of period two with itinerary $(010101\ldots)$, while the pair in H_1 is attached to the point with itinerary $(101010\ldots)$. These, as well as many other attachments, are visible in Figure 2. Note the visible difference between $J(E_\nu)$ shown in this figure compared to $J(E_\mu)$.

4. *Untangling the hairs*

In this section, we show how to determine when two hairs are attached at the same point in the Julia set. By Proposition 3.6, if we have an allowable itinerary in $s' \in \Sigma_N^*$, then there is a unique tail of a hair in $J(E_\lambda)$ with that itinerary. If an augmented sequence is not allowable, then there is no such tail of a hair. Then, using Theorem 3.1, we can pull each of these hairs back until it lands at a point in Γ_N. The landing point is then given by the point whose deaugmented itinerary is $D(s')$. Therefore, to determine whether we have more than one hair attached to a given point, all we need to do is to determine when we have multiple allowable augmented sequences, each of which has the same deaugmentation. This reduces the geometry of the hairs to a combinatorial problem, as we show below.

Our main tool is the following lemma.

LEMMA 4.1. *Let $s_0, s_1, \ldots, s_j \in \mathbb{Z}$. Let $s'_j \in \mathbb{Z}'$ with $D(s'_j) = s_j$. Then there is a unique sequence $s'_0, s'_1, \ldots, s'_{j-1}$ such that:*

(1) *$D(s'_i) = s_i$ for $i = 0, 1, \ldots, j-1$;*

(2) *the transitions $s'_0 \to s'_1 \to \cdots \to s'_j$ are all allowable.*

Proof. Suppose that $i_j \to k_\ell$. Recall that this means that $E_\lambda(H_{i_j})$ meets H_{k_ℓ} in the far right half-plane. Equivalently, we must have

$$L_{\lambda,i}(H_{k_\ell}) \cap \Lambda_\tau \subset H_{i_j}$$

as shown in the proof of Proposition 3.6. Now if $i_m \to k_\ell$ also, we must have that $E_\lambda(H_{i_m})$ meets H_{k_ℓ} in the far right half-plane as well. But both H_{i_j} and H_{i_m} are contained in H_i and E_λ is injective on H_i. Hence there can be at most one allowable transition of the form $i_* \to k_\ell$. This shows that the sequence above is unique, if it exists.

To see that there is a transition $i_j \to k_\ell$, recall that $E_\lambda(H_i)$ covers $\mathbb{C} - C_1$. Hence $E_\lambda(H_i)$ meets all of the fingers in Λ_τ. In particular, there is a subfinger in $\Lambda_\tau \cap H_i$ that maps over H_{k_ℓ} in the far right half-plane. This proves existence. □

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Thus, according to this lemma, given any $s_j \in \mathbb{Z}'$, we can find one and only one initial portion of an allowable sequence whose jth entry is s_j. Thus we have the following corollaries.

COROLLARY 4.2. *Suppose $s \in \Sigma'_N$ contains infinitely many entries that are non-subscripted. Then there is at most one hair corresponding to this sequence.*

COROLLARY 4.3. *The only points in Γ_N that can have multiple hairs attached are those:*
(1) *whose itineraries consist only of subscripted entries in $\mathbb{Z}'$; or*
(2) *which are preimages of such points.*

Therefore, to determine which hairs are attached to which points in Γ_N, we need only consider allowable sequences that consist entirely of subscripted entries. These allowable sequences together with their preimages are the only sequences that may have multiple hairs attached. So we have reduced the question to: which sequences $s' \in \Sigma^*_N$ with only subscripted entries have the property that there is a second sequence t' with $D(s') = D(t')$. We will describe the algorithm for determining this after going over several examples.

4.1. *Example A.* Consider the function $E_\lambda(z) = \mu e^z$ where $\mu = 5 + i\pi$ as described earlier. We have $K(\mu) = 00*$ and the structure of the relevant H_{0_j} is a shown in Figure 4.

By the previous remarks, the only points in Γ_N that may have multiple hairs attached are those whose itineraries end $(s_0 \ldots s_n \overline{0} \ldots)$. That is, only the single (repelling) fixed point in H_0 (and its preimages) can have multiple hairs attached. We will show that there are exactly three hairs attached to each such point.

To determine this, we need to ask which sequences in Σ^*_N have deaugmentation $(000\ldots)$. This in turn is determined by the allowable transitions among the 0_j.

PROPOSITION 4.4. *For E_μ, the allowable entries in a sequence in Σ^*_N are $0_0, 0_1, 0_2$, and all nonzero integers. The transition rules among these entries are: (i) $0_0 \to 0_1$; (ii) $0_1 \to 0_2, k \geq 1$; (iii) $0_2 \to 0_0, k \leq -1$; and (iv) $j \to k, 0_0, 0_1, 0_2$, for any two nonzero integers j and k.*

The proof of this proposition follows immediately from the construction of the fundamental set of attracting domains shown in Figure 4.

As a consequence, the only three allowable sequences consisting of only the 0_j are $(\overline{0_0 0_1 0_2} \ldots)$, $(\overline{0_1 0_2 0_0} \ldots)$, and $(\overline{0_2 0_0 0_1} \ldots)$. Hence we have the following theorem.

THEOREM 4.5. *For $\lambda = \mu$, the only points in Γ_N with multiple hairs attached are the fixed point with itinerary $(000\ldots)$ and all of its preimages. Each of these points has exactly three hairs attached. All other points have a single hair attached.*

Notice that we can capture the information about these hairs in matrix form using a *transition matrix*. In this matrix, the (i, j) entry is either 0 or 1 depending on whether $i \to j$ is either not allowed or allowed. Here the rows and columns of the matrix are specified by the subscripted entries in $\mathbb{Z}'$. In this case, the transition matrix involves the

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entries $0_0, 0_1$, and 0_2 and is given by

$$T_\mu = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

4.2. *Example B.* Now recall the function $E_\lambda(z) = \nu e^z$, where $\nu = a + 3\pi i$ and a is sufficiently large. In this case C_1 lies in H_0 but C_2 now lies in H_1. So $K(\nu) = 01*$. Therefore, the relevant entries in Σ_N^* are $0_0, 0_1, 1_0$, and 1_1 and we need only consider sequences involving just 0's and 1's.

PROPOSITION 4.6. *For E_ν, the allowable entries in a sequence in Σ_N^* are $0_0, 0_1, 1_0, 1_1$, and all nonzero integers. The transition rules among these entries are:*
(i) $0_0 \to 0_1, 1_0$;
(ii) $0_1 \to$ *all others, i.e. the complement of* $0_1, 1_0$;
(iii) $1_0 \to 0_1, 1_0, 1_1, k > 0$; *and*
(iv) $1_1 \to$ *all others, i.e. the complement.*

Again the proof follows from the construction shown in Figure 5.

Thus the transition matrix involves the four subscripted entries in Σ_N^* and is given by

$$T_\nu = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The hair structure for E_ν is much different from that of E_μ. For example, the period two transitions $0_0 \to 0_1 \to 0_0$ and $0_1 \to 0_0 \to 0_1$ are both allowable. Also, the transitions $0_0 \to 1_0 \to 0_1$ and $0_1 \to 1_1 \to 0_0$ are also allowable. Let α denote the pair $0_0 0_1$ and β the opposite pair $0_1 0_0$. Then we can string together any number of α's, say k, follow it with a 1_1 and then repeat periodically and we obtain an allowable sequence in Σ_N^*. Similarly, the same number of β's followed by a 1_0 and then repeated periodically is also allowable. But both of these sequences have the same deaugmentation, namely

$$(\overline{0\ldots 01}\ldots)$$

with $2k$ 0's in each repeating block. Hence the hairs corresponding to each of these sequences are attached to a periodic point of period $2k + 1$.

Now none of these periodic points are preimages of each other. So, unlike the case of E_μ, we have infinitely many distinct periodic points with multiple hairs attached. Of course, each of their infinitely many preimages also has a pair of hairs attached.

Remark. Multiple hairs can be attached to nonperiodic points as well. For example, let $\alpha = 0_0 0_1$ and $\beta = 0_1 0_0$. The we have the following allowable sequences:

$$\alpha 1_1 \alpha\alpha 1_1 \alpha\alpha\alpha 1_1 \ldots, \quad \beta 1_0 \beta\beta 1_0 \beta\beta\beta 1_0 \ldots.$$

Note that each of these sequences has the same nonperiodic deaugmentation.

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5. *The general case*

In this section we prove the main result of this paper.

THEOREM 5.1. *Suppose that* $K(\lambda) = k_1k_2\dots k_{n-2}$ *where* $k_{n-2} \neq 0$. *Then the corresponding exponential has the property that there are infinitely many distinct periodic points that have multiple hairs attached.*

Before proving this result, we introduce some notation. In $\mathcal{H}_0$ there is at least one C_j, namely C_1, and perhaps stable fingers as well. We will denote the two fingers in Λ_τ directly below and above C_1 by H_μ and $H_{\mu+1}$. This means that the subscripted entries in $\mathbb{Z}'$ with deaugmentation 0 may be ordered

$$0_0 \leq \cdots \leq 0_\mu < 0_{\mu+1} \leq \cdots \leq 0_k.$$

LEMMA 5.2. *Suppose the last digit in the kneading sequence is nonzero. The following transitions are allowable:*

$$0_0 \to 0_{\mu+1} \quad \textit{and} \quad 0_k \to 0_\mu.$$

In particular, if $0_0 = 0_\mu$, *then* $0_\mu \to 0_{\mu+1}$ *is allowable. If* $0_k = 0_{\mu+1}$, *then* $0_{\mu+1} \to 0_\mu$ *is allowable.*

Proof. $E_\lambda(H_{0_0})$ is mapped over at least $H_{0_{\mu+1}}$ and perhaps other fingers in Λ_τ, as the boundary of H_{0_0} is mapped to the upper boundary of C_1. Similarly, the upper boundary of H_{0_k} is mapped over the lower boundary of C_1, so H_{0_k} is mapped over H_{0_μ}. □

LEMMA 5.3. *Suppose* $K(\lambda) = k_1 \dots k_{n-2}$ *with* $k_{n-2} \neq 0$. *Then there exists two strings* $\hat{k}_1, \dots, \hat{k}_{n-2}$ *and* $k_1^*, \dots, k_{n-2}^*$, *where* $D(\hat{k}_j) = D(k_j^*) = k_j$, *having the property that the following transitions are allowable:*

$$0_{\mu+1} \to k_1^* \to k_2^* \to \cdots \to k_{n-2}^* \to 0_i \quad \textit{with } i \leq \mu,$$
$$0_\mu \to \hat{k}_1 \to \hat{k}_2 \to \cdots \to \hat{k}_{n-2} \to 0_j \quad \textit{with } j \geq \mu + 1.$$

Proof. Let A denote the finger in $H_{k_{n-2}} \cap \operatorname{Re} z \geq \tau$ that is bounded above by the upper boundary of C_0 and below by the upper boundary of C_{n-1}. Let B denote the finger in $H_{k_{n-2}} \cap \operatorname{Re} z \geq \tau$ that is bounded below by the lower boundary of C_0 and above by the lower boundary of C_{n-1}. Since $k_{n-2} \neq 0$, both A and B do not meet the finger C_1. It follows that

$$F_A = L_{\lambda,0} \circ L_{\lambda,k_1} \circ \cdots \circ L_{\lambda,k_{n-2}}(A) \cap \{\operatorname{Re} z \geq \tau\}$$

and

$$F_B = L_{\lambda,0} \circ L_{\lambda,k_1} \circ \cdots \circ L_{\lambda,k_{n-2}}(B) \cap \{\operatorname{Re} z \geq \tau\}$$

are fingers in H_0.

Note that

$$L_{\lambda,k_{n-j}} \circ \cdots \circ L_{\lambda,k_{n-2}}(A) \quad \text{and} \quad L_{\lambda,k_{n-j}} \circ \cdots \circ L_{\lambda,k_{n-2}}(B)$$

abut the finger C_{n-j-1} for each j, so F_A (respectively F_B) is a finger bounded below (respectively above) by the boundary of C_1.

We claim that $F_A \subset H_{0_{\mu+1}}$. Certainly, F_A meets $H_{0_{\mu+1}}$ by the above observation. So suppose that F_A also meets a different C_j in H_0. Then there is an integer $j < n-1$ for which $E_\lambda^j(F_A)$ meets C_{n-1}, and thus E_λ^{j+1} maps points in F_A to points in H_υ for arbitrarily large υ. This contradicts the fact that each point in F_A has itinerary that begins $0k_1 \ldots k_{n-2}$. The same argument shows that $F_B \subset H_{0_\mu}$. In particular, we have the following allowable transitions:

$$0_{\mu+1} \to k_1^* \to k_2^* \to \cdots \to k_{n-2}^*$$

and

$$0_\mu \to \hat{k}_1 \to \hat{k}_2 \to \cdots \to \hat{k}_{n-2}.$$

Now note that $E_\lambda(F_A)$ meets H_{0_i} for each $i \le \mu$ and $E_\lambda(F_B)$ meets H_{0_j} for each $j \ge \mu+1$. This provides the desired itineraries. □

We now complete the proof of Theorem 5.1. Let β_1 denote the subscripted index 0_0 and β_2 denote the subscripted index 0_k. Also let

$$\alpha_1 = 0_{\mu+1}k_1^* \ldots k_{n-2}^* \quad \text{and} \quad \alpha_2 = 0_\mu \hat{k}_1 \ldots \hat{k}_{n-2}.$$

Then combining Lemmas 5.2 and 5.3 we can conclude that the following sequences are allowable: $\alpha_1\alpha_2$, $\alpha_2\alpha_1$, $\alpha_1\beta_1$, $\alpha_2\beta_2$, $\beta_1\alpha_1$, and $\beta_2\alpha_2$.

This allows us to construct infinitely many pairs of sequences

$$(\alpha_1\beta_1\alpha_1 \underbrace{\ldots}_{*} \alpha_2\beta_2\alpha_2 \underbrace{\ldots}_{*} \alpha_1 \ldots) \quad \text{and} \quad (\alpha_2\beta_2\alpha_2 \underbrace{\cdots}_{*} \alpha_1\beta_1\alpha_1 \underbrace{\ldots}_{*} \alpha_2 \ldots)$$

where the space $(*)$ can be filled with an arbitrary even number of α. Note that both α_1, α_2 and β_1, β_2 have the same deaugmentation. Hence we have constructed infinitely many pairs of augmented itineraries that have the same deaugmentation. These pairs correspond to the hairs that meet at the same point in Γ_N. This completes the proof. □

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