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HOMOLOGY THEORIES FOR COMPLEXES BASED ON FLATS

LI LIANG
School of Mathematics and Physics, Lanzhou Jiaotong University, Lanzhou 730070, China
e-mail: lliangnju@gmail.com

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Abstract. In this paper, we introduce and study the Gorenstein relative homology theory for unbounded complexes of modules over arbitrary associative rings, which is defined using special Gorenstein flat precovers. We compare the Gorenstein relative homology to the Tate/unbounded homology and get some results that improve the known ones.

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1. Introduction. Gorenstein relative cohomology for finitely generated modules M of finite Gorenstein dimension over Noetherian rings has been studied explicitly by Avramov and Martsinkovsky in [2]. Holm extended the definition of Gorenstein relative cohomology groups to the case where M admits a proper resolution $G \xrightarrow{\simeq} M$ by Gorenstein projective modules over an arbitrary associative ring in [10], where the Gorenstein relative cohomology of M is defined as $\mathrm{Ext}^n_{\mathcal{GP}}(M, -) = \mathrm{H}_{-n}(\mathrm{Hom}_R(G, -))$. This theory was further treated by Veliche in [20], where she pointed out some obstacles to define Gorenstein relative cohomology groups for complexes of modules: For a complex M that has a special Gorenstein projective resolution $G \simeq M$, Veliche [20, Remark 6.7] noticed that it is tempting to use $\mathrm{H}_{-n}(\mathrm{Hom}_R(G, -))$ to define Gorenstein relative cohomology groups, but it is not known whether this construction has the necessary uniqueness and functoriality. Iacob [12] proposed an approach to define Gorenstein relative cohomology groups for complexes of modules over Gorenstein rings: Let M be a complex with $G \to M$ a special Gorenstein projective precover. The Gorenstein relative cohomology of M is defined as $\mathrm{Ext}^n_{\mathcal{GP}}(M, -) = \mathrm{H}_{-n}(\mathrm{Hom}_R(G, -))$. Recently, Liu [17] further extended this definition to complexes of modules that have special Gorenstein projective precovers over arbitrary associative rings.

In this paper, we investigate the homological side. As Holm [10] defined for modules, we give definitions of the Gorenstein relative homology $\mathrm{Tor}^{\mathcal{GF}_R}$, $\mathrm{Tor}^{_R\mathcal{GF}}$, $\mathrm{Tor}^{\mathcal{GP}_R}$, and $\mathrm{Tor}^{_R\mathcal{GP}}$ for unbounded complexes of modules over an arbitrary associate ring; see Definition 3.4. In Section 3, we study the relationships between these relative homology theories; see Theorem 3.8.

THEOREM A. *Let R be a ring, and let M be an R°-complex and N an R-complex.*

(1) *If $\mathrm{Gpd}_{\mathcal{C}}\, M < \infty$ and $\mathrm{Gpd}_{\mathcal{C}}\, N < \infty$, then for each $i \in \mathbb{Z}$ there is an isomorphism $\mathrm{Tor}_i^{\mathcal{GP}_R}(M, N) \cong \mathrm{Tor}_i^{_R\mathcal{GP}}(M, N)$.*

(2) *If $\mathrm{Gpd}_{\mathcal{C}}\, M < \infty$ and $\mathrm{Gfd}_{\mathcal{C}}\, N < \infty$, then for each $i \in \mathbb{Z}$ there is an isomorphism $\mathrm{Tor}_i^{\mathcal{GP}_R}(M, N) \cong \mathrm{Tor}_i^{_R\mathcal{GF}}(M, N)$.*

(3) *If* $\mathrm{Gfd}_{\mathcal{C}}\, M < \infty$ *and* $\mathrm{Gpd}_{\mathcal{C}}\, N < \infty$, *then for each* $i \in \mathbb{Z}$ *there is an isomorphism* $\mathrm{Tor}_i^{\mathcal{GF}_R}(M, N) \cong \mathrm{Tor}_i^{R\mathcal{GP}}(M, N)$.

(4) *If* $\mathrm{Gfd}_{\mathcal{C}}\, M < \infty$ *and* $\mathrm{Gfd}_{\mathcal{C}}\, N < \infty$, *then for each* $i \in \mathbb{Z}$ *there is an isomorphism* $\mathrm{Tor}_i^{\mathcal{GF}_R}(M, N) \cong \mathrm{Tor}_i^{R\mathcal{GF}}(M, N)$.

We recall the definition of the Gorenstein projective/flat dimension in 2.8. This result improves [10, Theorem 4.8] by removing the coherent assumption on the ring.

In Section 4, we study the generalized Tate homology $\widehat{\mathrm{Tor}}^{\mathcal{GF},\mathcal{F}}$ for complexes of modules using the ideals proposed by Iacob [11] in the case of modules. In particular, we build an isomorphism between the generalized Tate homology and the classical Tate homology; see Theorem 4.9. As an application, in the case of module arguments, we give an exact sequence connecting the absolute homology Tor, the Gorenstein relative homology $\mathrm{Tor}^{\mathcal{GF}_R}$, and the Tate homology $\widehat{\mathrm{Tor}}^{\mathcal{F}}$; see Corollary 4.10.

THEOREM B. *Let M be an R°-module of finite Gorenstein flat dimension d. For each R-module N, there exists an exact sequence*

$$0 \to \widehat{\mathrm{Tor}}_d^{\mathcal{F}}(M, N) \to \mathrm{Tor}_d^R(M, N) \to \mathrm{Tor}_d^{\mathcal{GF}_R}(M, N) \to \cdots$$

$$\to \widehat{\mathrm{Tor}}_1^{\mathcal{F}}(M, N) \to \mathrm{Tor}_1^R(M, N) \to \mathrm{Tor}_1^{\mathcal{GF}_R}(M, N) \to 0.$$

The definition of the Tate homology group $\widehat{\mathrm{Tor}}_i^{\mathcal{F}}(M, N)$ can be found in 2.5. We notice that this result improves Liang [14, Theorem 4.4] by removing the assumptions that the ring R should be coherent and the module M should be cotorsion.

Finally, in Section 5 we compare the Gorenstein relative homology $\mathrm{Tor}^{\mathcal{GF}_R}$ to the unbounded homology $\overline{\mathrm{Tor}}$ developed by Celikbas, Christensen, Liang, and Piepmeyer in [4]. The following result is contained in Theorem 5.2.

THEOREM C. *Let R be a left coherent ring, and let M be an R°-module of finite Gorenstein flat dimension and N an R-module. Then for each $n > 1$, there is an isomorphism* $\mathrm{Tor}_n^{\mathcal{GF}_R}(M, N) \cong \overline{\mathrm{Tor}}_n^R(M, N)$.

This result improves Liang [15, Proposition 4.10] by removing the cotorsion assumption on the module M.

2. Preliminaries.

We begin with some notation and terminology for use throughout this paper.

2.1. Throughout this work, R is assumed to be an associative ring with identity, and we employ the convention that R acts on the left. That is, an R-module is a left R-module, and right R-modules are treated as modules over the opposite ring, denoted R°. By an R-complex M, we mean a complex of R-modules as follows:

$$\cdots \longrightarrow M_{i+1} \xrightarrow{\partial_{i+1}^M} M_i \xrightarrow{\partial_i^M} M_{i-1} \longrightarrow \cdots .$$

We frequently (and without warning) identify R-modules with R-complexes concentrated in degree 0. For an R-complex M, we set $\sup M = \sup\{i \in \mathbb{Z} \mid M_i \neq 0\}$ and $\inf M = \inf\{i \in \mathbb{Z} \mid M_i \neq 0\}$. An R-complex M is called *bounded* if $\sup M < \infty$ and $\inf M > -\infty$. For $n \in \mathbb{Z}$, the symbol $\Sigma^n M$ denotes the complex with $(\Sigma^n M)_i = M_{i-n}$ and $\partial_i^{\Sigma^n M} = (-1)^n \partial_{i-n}^M$ for all $i \in \mathbb{Z}$. We set $\Sigma M = \Sigma^1 M$. The symbol $\mathrm{Z}_n(M)$ (resp., $\mathrm{B}_n(M)$, $\mathrm{C}_n(M)$) denotes the

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kernel of ∂_n^M (resp., the image of ∂_{n+1}^M, the cokernel of ∂_{n+1}^M), and $\mathrm{H}_n(M)$ denotes the nth *homology* of M, i.e., $\mathrm{Z}_n(M)/\mathrm{B}_n(M)$. An R-complex M with $\mathrm{H}(M)=0$ is called *acyclic*. For an R-complex M, the symbol $M_{\leq n}$ denotes the subcomplex of M with $(M_{\leq n})_i = M_i$ for $i \leq n$ and $(M_{\leq n})_i = 0$ for $i > n$, and the symbol $M_{\geqslant n}$ denotes the quotient complex of M with $(M_{\geqslant n})_i = M_i$ for $i \geqslant n$ and $(M_{\geqslant n})_i = 0$ for $i < n$.

If M and N are both R-complexes, then by a *morphism* $\alpha : M \to N$ we mean a sequence $\{\alpha_n\}_{n\in\mathbb{Z}}$ of homomorphisms of R-modules $\alpha_n : M_n \to N_n$ such that $\alpha_{n-1}\partial_n^M = \partial_n^N \alpha_n$ for each $n \in \mathbb{Z}$. We let $\mathrm{Cone}(\alpha)$ denote the *mapping cone* of α. $\mathrm{Hom}_{\mathcal{C}}(M, N)$ denotes the set of morphisms of R-complexes from M to N, and $\mathrm{Ext}^i_{\mathcal{C}}(-,-)$ is the right derived functor of $\mathrm{Hom}_{\mathcal{C}}(-,-)$. A morphism $M \to N$ of R-complexes that induces an isomorphism $\mathrm{H}(M) \to \mathrm{H}(N)$ is called a *quasi-isomorphism*, and the symbol $\simeq$ is used to decorate quasi-isomorphisms. The symbol $\mathrm{Ch}(R)$ denotes the abelian category of R-complexes.

2.2. For R-complexes M and N, the *Hom complex* $\mathrm{Hom}_R(M, N)$ is the $\mathbb{Z}$-complex with the degree-n term

$$\mathrm{Hom}_R(M, N)_n = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(M_i, N_{i+n})$$

and the differential given by $\partial(\alpha) = \partial^N\alpha - (-1)^{|\alpha|}\alpha\partial^M$ for a homogeneous element α. We notice that $\mathrm{Hom}_{\mathcal{C}}(M, N)$ is actually the group $\mathrm{Z}_0(\mathrm{Hom}_R(M, N))$.

Let X be an R°-complex and Y an R-complex. The *tensor product complex* $X \otimes_R Y$ is the $\mathbb{Z}$-complex with the degree-n term

$$(X \otimes_R Y)_n = \coprod_{i\in\mathbb{Z}} (X_i \otimes_R Y_{n-i})$$

and the differential given by $\partial(x \otimes y) = \partial^X(x) \otimes n + (-1)^{|x|}(x \otimes \partial^Y(y))$ for homogeneous elements x and y.

2.3. A complex P of projective R-modules is called *semi-projective* if the functor $\mathrm{Hom}_R(P, -)$ preserves quasi-isomorphisms. A complex F of flat R-modules is called *semi-flat* if the functor $- \otimes_R F$ preserves quasi-isomorphisms. Semi-projective (resp., semi-flat) complexes are also called DG-projective (resp., DG-flat); see for example Gillespie [8].

Let M be an R-complex. A *semi-projective resolution* of M is a quasi-isomorphism $P \to M$, where P is a semi-projective R-complex. A *semi-flat replacement* of M is an isomorphism $F \simeq M$ in the derived category, where F is a semi-flat R-complex. Every R-complex has a semi-projective resolution and hence a semi-flat replacement.

Let M be an R°-complex with $P \xrightarrow{\simeq} M$ a semi-projective resolution, and let N be an R-complex. For all $i \in \mathbb{Z}$ the $\mathbb{Z}$-modules

$$\mathrm{Tor}_i^R(M, N) = \mathrm{H}_i(P \otimes_R N)$$

make up the *absolute homology* of M and N over R. It is clear that the definition of $\mathrm{Tor}_i^R(M, N)$ is functorial and homological in either argument. We notice that if $F \simeq M$ is a semi-flat replacement of M, then one has $\mathrm{Tor}_i^R(M, N) \cong \mathrm{H}_i(F \otimes_R N)$ for each $i \in \mathbb{Z}$.

2.4. An acyclic complex T of projective R°-modules is called *totally acyclic* if $\mathrm{Hom}_{R^\circ}(T, P)$ is acyclic for every projective R°-module P. An R°-module G is called *Gorenstein projective* if there exists a totally acyclic complex T of projective R°-modules such that $G \cong \mathrm{Coker}(T_1 \to T_0)$. Following [20], a *complete projective resolution* of an R°-complex M is a diagram $T \xrightarrow{\tau} P \xrightarrow{\simeq} M$, where T is a totally acyclic complex of projective R°-modules, $P \xrightarrow{\simeq} M$ is a semi-projective resolution, and τ_i is an isomorphism for all $i \gg 0$.

Let M be an R°-complex that has a complete projective resolution $T \to P \xrightarrow{\simeq} M$, and let N be an R-complex. For all $i \in \mathbb{Z}$, the $\mathbb{Z}$-modules

$$\widehat{\mathrm{Tor}}_i^R(M, N) = \mathrm{H}_i(T \otimes_R N)$$

are the *Tate homology* modules of M and N over R; see Christensen and Jorgensen [5, 2.4].

2.5. An acyclic complex T of flat R°-modules is called *F-totally acyclic* if $T \otimes_R E$ is acyclic for every injective R-module E. An R°-module G is called *Gorenstein flat* if there exists a F-totally acyclic complex T of flat R°-modules such that $G \cong \mathrm{Coker}(T_1 \to T_0)$. A *Tate flat resolution* of an R°-complex M is a pair (T, F) where T is an F-totally acyclic complex of flat R°-modules and $F \simeq M$ is a semi-flat replacement with $T_{\geqslant g} \cong F_{\geqslant g}$ for some $g \in \mathbb{Z}$. Furthermore, if there exists a morphism $\tau : T \to F$ such that τ_i is an isomorphism for each $i \geq g$, then the Tate flat resolution (T, F) is said to be a *complete flat resolution* of M; see [15].

Let M be an R°-complex that has a Tate flat resolution (T, F), and let N be an R-module. For all $i \in \mathbb{Z}$, the $\mathbb{Z}$-modules

$$\widehat{\mathrm{Tor}}_i^{\mathcal{F}}(M, N) = \mathrm{H}_i(T \otimes_R N)$$

are called the *Tate homology based on flats* of M and N over R.

Using the same argument as in the proof of [14, Proposition 3.7], one gets that the above definition is independent (up to isomorphism) of the choice of Tate flat resolutions. The relationship between the Tate homology $\widehat{\mathrm{Tor}}^{\mathcal{F}}$ and $\widehat{\mathrm{Tor}}^{R^\circ}$ can be found in [15, Theorem A].

2.6. An R-complex P is called *projective* (resp., *injective*, *flat*, and *cotorsion*) if P is acyclic and each $\mathrm{Z}_i(P)$ is projective (resp., injective, flat, and cotorsion) for $i \in \mathbb{Z}$.

An R-complex C is called *DG-cotorsion* if C_i is cotorsion for each $i \in \mathbb{Z}$ and the Hom complex $\mathrm{Hom}_R(F, C)$ is acyclic whenever F is a flat R-complex. From [8, Corollary 3.13], acyclic semi-flat R-complexes are actually flat R-complexes, and acyclic DG-cotorsion R-complexes are actually cotorsion R-complexes.

Following García Rozas [7], an R-complex M is called *Gorenstein projective* if there exists an exact sequence $\cdots \to P^{-1} \to P^0 \to P^1 \to \cdots$ of R-complexes with each P^i projective and $M \cong \mathrm{Ker}(P^0 \to P^1)$, such that it remains exact after applying $\mathrm{Hom}_{\mathcal{C}}(-, P)$ to it for each projective R-complex P. An R-complex G is called *Gorenstein flat* if there is an exact sequence $\cdots \to F^{-1} \to F^0 \to F^1 \to \cdots$ of R-complexes with each F^i flat and $G \cong \mathrm{Ker}(F^0 \to F^1)$, such that it remains exact after applying $F \otimes^\bullet -$ to it for each flat R°-complex F. Here $F \otimes^\bullet -$ is the functor with $F \otimes^\bullet N = (F \otimes_R N)/\mathrm{B}(F \otimes_R N)$ for each R-complex N; see [7] for more details.

LEMMA 2.7. *Let M be an R-complex. Then the following assertions hold:*

(1) *M is Gorenstein projective if and only if for each $i \in \mathbb{Z}$ the R-module M_i is Gorenstein projective.*

(2) *M is Gorenstein flat if and only if for each $i \in \mathbb{Z}$ the R-module M_i is Gorenstein flat.*

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Proof. Part (1) was proved by Yang and Liu; see [24, Theorems 2.2]. From a very recent result by Šaroch and Šťovíček [19, Corollary 3.12], all rings are GF-closed in the sense of Bennis [3], so part (2) holds by [23, Theorem 3.11]. □

2.8. The *Gorenstein projective dimension* of R-complex M, denoted $\mathrm{Gpd}_{\mathcal{C}}\, M$, is defined by declaring that $\mathrm{Gpd}_{\mathcal{C}}\, M \le n$ ($n \in \mathbb{N}$) if and only if there is an exact sequence $0 \to P^n \to \cdots \to P^0 \to M \to 0$ of R-complexes with each P^i Gorenstein projective. The *Gorenstein flat dimension* (resp., *projective dimension*, *flat dimension*, and *injective dimension*) of R-complex M, denoted $\mathrm{Gfd}_{\mathcal{C}}\, M$ (resp., $\mathrm{pd}_{\mathcal{C}}\, M$, $\mathrm{fd}_{\mathcal{C}}\, M$, $\mathrm{id}_{\mathcal{C}}\, M$) can be defined similarly. It is easy to check that if M has finite projective (flat, injective) dimension, then M is acyclic. Moreover, one has $\mathrm{pd}_{\mathcal{C}}\, M = \max\{\mathrm{pd}_R\, \mathrm{Z}_i(M) \mid i \in \mathbb{Z}\}$, $\mathrm{fd}_{\mathcal{C}}\, M = \max\{\mathrm{fd}_R\, \mathrm{Z}_i(M) \mid i \in \mathbb{Z}\}$, and $\mathrm{id}_{\mathcal{C}}\, M = \max\{\mathrm{id}_R\, \mathrm{Z}_i(M) \mid i \in \mathbb{Z}\}$. From Lemma 2.7, one has $\mathrm{Gpd}_{\mathcal{C}}\, M = \max\{\mathrm{Gpd}_R\, M_i \mid i \in \mathbb{Z}\}$, and $\mathrm{Gfd}_{\mathcal{C}}\, M = \max\{\mathrm{Gfd}_R\, M_i \mid i \in \mathbb{Z}\}$.

2.9. Given a class $\mathcal{A}$ of R-complexes, we write

$$\mathcal{A}^{\perp} = \{B \in \mathrm{Ch}(R) \mid \mathrm{Ext}^1_{\mathcal{C}}(A, B) = 0 \text{ for all } A \in \mathcal{A}\}$$

and

$$^{\perp}\mathcal{A} = \{B \in \mathrm{Ch}(R) \mid \mathrm{Ext}^1_{\mathcal{C}}(B, A) = 0 \text{ for all } A \in \mathcal{A}\}.$$

A pair $(\mathcal{A}, \mathcal{B})$ of classes of R-complexes is called a *cotorsion pair* if $\mathcal{A}^{\perp} = \mathcal{B}$ and $^{\perp}\mathcal{B} = \mathcal{A}$. Recall that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *hereditary* if $\mathrm{Ext}^i_{\mathcal{C}}(A, B) = 0$ for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and for each $i \ge 1$. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is called *complete* if for each R-complex X there are exact sequences $0 \to X \to B \to A \to 0$ and $0 \to B' \to A' \to X \to 0$ with $B, B' \in \mathcal{B}$ and $A, A' \in \mathcal{A}$.

We let $\widetilde{\mathcal{F}}$ (resp., $dg\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{C}}$, and $dg\widetilde{\mathcal{C}}$) denote the class of flat (resp., semi-flat, cotorsion, and DG-cotorsion) R-complexes. The following result holds by [8, Corollary 3.13] and [22, Theorem 3.5].

LEMMA 2.10. *The pairs* $(dg\widetilde{\mathcal{F}}, \widetilde{\mathcal{C}})$, *and* $(\widetilde{\mathcal{F}}, dg\widetilde{\mathcal{C}})$ *are complete hereditary cotorsion pairs.*

2.11. Let $\mathcal{A}$ be a class of R-complexes. Following Enochs [6], a morphism $\phi : A \to X$ of R-complexes is called an $\mathcal{A}$-*precover* of X if $A \in \mathcal{A}$ and

$$\mathrm{Hom}_{\mathcal{C}}(A', A) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A', X) \longrightarrow 0$$

is exact for each $A' \in \mathcal{A}$. We notice that if $\mathcal{A}$ contains all projective R-complexes, then $\mathcal{A}$-precovers are always surjective.

The next result is for use in Proposition 3.3 and Lemma 4.1.

LEMMA 2.12. *Let M and N be R-complexes. Let $\varphi : A \to M$ be a morphism with $A \in \mathcal{A}$, and let $\psi : A' \to N$ be a surjective $\mathcal{A}$-precover of N. If* $\mathrm{Ext}^1_{\mathcal{C}}(\Sigma A, \mathrm{Ker}\,\psi) = 0$, *then for each morphism $f : M \to N$, there exists a unique, up to homotopy, morphism $g : A \to A'$ such that $f\varphi = \psi g$.*

Proof. From the definition of $\mathcal{A}$-precovers, there exists a morphism $g : A \to A'$ such that $f\varphi = \psi g$. Assume that there is another morphism $h : A \to A'$ such that $f\varphi = \psi h$. Then $g - h$ is a morphism from A to $K = \mathrm{Ker}\,\psi$. Since $\mathrm{Ext}^1_{\mathcal{C}}(\Sigma A, K) = 0$, the exact sequence $0 \to K \to \mathrm{Cone}(g - h) \to \Sigma A \to 0$ is split, and so $g - h$ is null-homotopic by [7, Lemma 2.3.2]. Hence, g is homotopic to h. □

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2.13. Let $\mathcal{F} \subseteq \mathcal{G}$ be two classes of objects of an abelian category. Recall from Auslander and Buchweitz [1] that $\mathcal{F}$ is a cogenerator of $\mathcal{G}$ if for each $X \in \mathcal{G}$ there exists an exact sequence $0 \to X \to F \to X' \to 0$ with $F \in \mathcal{F}$ and $X' \in \mathcal{G}$.

Let $\mathcal{GF}(\mathcal{M})$ be the class of Gorenstein flat R-modules and $\mathcal{FC}(\mathcal{M})$ the class of flat and cotorsion R-modules. Then $\mathcal{FC}(\mathcal{M})$ is a cogenerator of $\mathcal{GF}(\mathcal{M})$. Actually, for $G \in \mathcal{GF}(\mathcal{M})$ there is an exact sequence $0 \to G \to X \to G' \to 0$ with $X \in \mathcal{GF}(\mathcal{M})^{\perp}$ and $G' \in \mathcal{GF}(\mathcal{M})$ by [19, Corollary 3.12], so one has $X \in \mathcal{GF}(\mathcal{M}) \cap \mathcal{GF}(\mathcal{M})^{\perp} = \mathcal{FC}(\mathcal{M})$ again by [19, Corollary 3.12].

Part (1) of the next result improves [24, Lemma 3.12] by removing the assumption that R should be right coherent. Let $\mathcal{GF}(\mathcal{C})$ be the class of Gorenstein flat R-complexes and $\mathcal{FC}(\mathcal{C})$ the class of flat and DG-cotorsion R-complexes. Then part (2) of the next result shows that $\mathcal{FC}(\mathcal{C})$ is a cogenerator of $\mathcal{GF}(\mathcal{C})$.

LEMMA 2.14. *Let M be a Gorenstein flat R-complex. Then the following hold:*

(1) $\mathrm{Ext}^i_{\mathcal{C}}(M, N) = 0$ *for each $i \geq 1$ and each DG-cotorsion R-complex N of finite flat dimension.*

(2) *There is an exact sequence $0 \to M \to C \to M' \to 0$ of R-complexes such that C is flat and DG-cotorsion, and M' is Gorenstein flat.*

Proof. (1) From [19, Corollary 3.12], one has $\mathrm{Ext}^i_R(A, B) = 0$ for each Gorenstein flat R-module A and each flat and cotorsion R-module B, and all $i \geq 1$. Thus, $\mathrm{Ext}^i_{\mathcal{C}}(M, K) = 0$ for each flat and cotorsion R-complex K by Lemma 2.7(2) and [16, Lemma 4.4(1)]. From Lemma 2.10, there is an exact sequence $0 \to K_s \to \cdots \to K_0 \to N \to 0$ of R-complexes for some $s \in \mathbb{Z}$ such that each K_i is flat and DG-cotorsion. Since flat and DG-cotorsion R-complexes are flat and cotorsion (see 2.6), one has $\mathrm{Ext}^i_{\mathcal{C}}(M, N) = 0$ for each $i \geq 1$ by dimension shifting.

(2) Since M is a Gorenstein flat R-complex, there is an exact sequence $0 \to M \to F \to G \to 0$ with F flat and G Gorenstein flat. We notice that $(\widetilde{\mathcal{F}}, \mathrm{dg}\widetilde{\mathcal{C}})$ is a complete cotorsion pair; see Lemma 2.10. So there is an exact sequence $0 \to F \to C \to F' \to 0$ with C DG-cotorsion and F' flat. Consider the following push-out diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & F & \longrightarrow & G & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & C & \longrightarrow & M' & \longrightarrow & 0. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & F' & = & F' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since the class of Gorenstein flat R-modules is closed under extensions by [19, Corollary 3.12], so is the class of Gorenstein flat R-complexes by Lemma 2.7(2). Thus, M' is Gorenstein flat. It is clear that C is flat, so the second nonzero row in the above diagram is as desired. □

2.15. It is well known that every R-complex M admits a surjective semi-projective precover $\varphi : P \to M$ with $\mathrm{Ker}\,\varphi$ acyclic. From Lemma 2.10, every R-complex M admits a

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surjective semi-flat precover $\psi : F \to M$ with Ker ψ cotorsion. These semi-projective and semi-flat precovers are always called *special*.

Recall from [12] that a Gorenstein projective precover $f : G \to M$ is special if f is an epimorphism and Ker f is an R-complex of finite projective dimension. A Gorenstein flat precover $g : G' \to M$ is said to be special if g is an epimorphism and Ker g is a DG-cotorsion R-complex of finite flat dimension.

EXAMPLE 2.16. (1) If M is an R-module of finite Gorenstein projective dimension n, then there is an exact sequence

$$0 \to P_n \to \cdots \to P_1 \to G_0 \to M \to 0, \qquad (2.16.1)$$

such that G_0 is Gorenstein projective and each P_i is projective; see Holm [9, Theorem 2.10]. Let $G = \cdots \to 0 \to P_n \to \cdots \to P_1 \to G_0 \to 0$. Then $G \to M$ is a special Gorenstein projective precover of M, and it is clear that $G \to M$ is a proper Gorenstein projective resolution in the sense of Holm [10], that is, the sequence (2.16.1) remains exact after applying $\mathrm{Hom}_R(L, -)$ to it for each Gorenstein projective R-module L.

(2) If M is an R-module of finite Gorenstein flat dimension n, then it follows from [19, Corollary 3.12] that $\mathcal{GF}(\mathcal{M})$ is closed under extensions and direct summands. $\mathcal{FC}(\mathcal{M})$ is closed under direct summands, and $\mathcal{FC}(\mathcal{M})$ is a cogenerator of $\mathcal{GF}(\mathcal{M})$; see 2.13. Thus, by [1, Theorem 1.1] there is an exact sequence $0 \longrightarrow K_1 \longrightarrow G_0 \xrightarrow{\delta_0} M \longrightarrow 0$ such that G_0 is a Gorenstein flat and K_1 is cotorsion with $\mathrm{fd}_R K_1 = n-1$. For K_1, there is an exact sequence $0 \longrightarrow K_2 \longrightarrow F_1 \xrightarrow{\delta_1} K_1 \longrightarrow 0$ such that F_1 is flat and K_2 is cotorsion with $\mathrm{fd}_R K_2 = n-2$. Hence, F_1 is cotorsion. Continuing this process, one gets an exact sequence

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \xrightarrow{\delta_{n-1}} \cdots \longrightarrow F_1 \xrightarrow{\delta_1} G_0 \xrightarrow{\delta_0} M \longrightarrow 0, \qquad (2.16.2)$$

such that G_0 is Gorenstein flat, each F_i is flat and cotorsion, and each Ker δ_i is cotorsion with $\mathrm{fd}_R \operatorname{Ker} \delta_i < \infty$. Let

$$G = \cdots \to 0 \to F_n \to \cdots \to F_1 \to G_0 \to 0 \to \cdots .$$

Then by Lemma 2.7(2), G is a Gorenstein flat R-complex, and there is a surjective morphism $\alpha : G \to M$. It is clear that Ker α is a cotorsion R-complex of finite flat dimension, so $G \to M$ is a special Gorenstein flat precover of M by Lemma 2.14(1). It follows from [19, Corollary 3.12] that $G \to M$ is a proper Gorenstein flat resolution in the sense of Holm [10], that is, the sequence (2.16.2) remains exact after applying $\mathrm{Hom}_R(L, -)$ to it for each Gorenstein flat R-module L.

3. Gorenstein relative homology of complexes.

In this section, we focus on the Gorenstein relative homology of unbounded complexes. The following result holds by [24, Proposition 2.6].

PROPOSITION 3.1. *Let M be an R-complex. The following are equivalent.*

(*i*) $\mathrm{Gpd}_{\mathcal{C}}\, M < \infty$.

(*ii*) *M admits a special Gorenstein projective precover in* $\mathrm{Ch}(R)$.

The next result improves [24, Theorem 3.13] by removing the coherent assumption on the ring R.

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PROPOSITION 3.2. *Let M be an R-complex. The following are equivalent.*

(*i*) $\mathrm{Gfd}_{\mathcal{C}}\, M < \infty$.

(*ii*) *M admits a special Gorenstein flat precover in* $\mathrm{Ch}(R)$.

Proof. $(ii) \Longrightarrow (i)$ is clear. Next we prove $(i) \Longrightarrow (ii)$.

The class $\mathcal{GF}(\mathcal{C})$ of Gorenstein flat R-complexes is closed under extensions and direct summands by [19, Corollary 3.12] and Lemma 2.7(2), and the class $\mathcal{FC}(\mathcal{C})$ of flat and DG-cotorsion R-complexes is closed under direct summands. By Lemma 2.14(2), $\mathcal{FC}(\mathcal{C})$ is a cogenerator of $\mathcal{GF}(\mathcal{C})$. Thus, from [1, Theorem 1.1] there is an exact sequence $0 \to K \to G \xrightarrow{g} M \to 0$ of R-complexes such that G is Gorenstein flat and K is of finite $\mathcal{FC}(\mathcal{C})$-dimension, that is, there exists an exact sequence $0 \to K_n \to \cdots \to K_0 \to K \to 0$ for some $n \in \mathbb{Z}$ with each K_i in $\mathcal{FC}(\mathcal{C})$. So K is a DG-cotorsion R-complex of finite flat dimension. Thus, g is a special Gorenstein flat precover by Lemma 2.14(1). □

By [12, Remark 4], a special Gorenstein projective precover is unique up to homotopy equivalence. The next result shows that a special Gorenstein flat precover is also unique up to homotopy equivalence.

PROPOSITION 3.3. *Let M and N be R-complexes. Let $\varphi : G \to M$ and $\psi : G' \to N$ be special Gorenstein flat precovers of M and N, respectively. For each morphism $f : M \to N$, there exists a unique, up to homotopy, morphism $g : G \to G'$ such that $f\varphi = \psi g$. In particular, a special Gorenstein flat precover is unique up to homotopy equivalence.*

Proof. Since $K = \mathrm{Ker}\,\psi$ is a DG-cotorsion R-complex of finite flat dimension, one has $\mathrm{Ext}^1_{\mathcal{C}}(G, \Sigma^{-1} K) = 0$ by Lemma 2.14(1). Thus, the conclusion holds by Lemma 2.12. □

DEFINITION 3.4. Let R be a ring.

(1) Let M be an R°-complex that has a special Gorenstein projective precover $G \to M$. For each R-complex N and each $i \in \mathbb{Z}$, we define

$$\mathrm{Tor}_i^{\mathcal{GP}_R}(M, N) := \mathrm{H}_i(G \otimes_R N).$$

(2) Let N be an R-complex that has a special Gorenstein projective precover $G' \to N$. For each R°-complex M and each $i \in \mathbb{Z}$, we define

$$\mathrm{Tor}_i^{{}_R\mathcal{GP}}(M, N) := \mathrm{H}_i(M \otimes_R G').$$

(3) Let M be an R°-complex that has a special Gorenstein flat precover $G \to M$. For each R-complex N and each $i \in \mathbb{Z}$, we define

$$\mathrm{Tor}_i^{\mathcal{GF}_R}(M, N) = \mathrm{H}_i(G \otimes_R N).$$

(4) Let N' be an R-complex that has a special Gorenstein flat precover $G' \to N$. For each R°-complex M and each $i \in \mathbb{Z}$, we define

$$\mathrm{Tor}_i^{{}_R\mathcal{GF}}(M, N) = \mathrm{H}_i(M \otimes_R G').$$

REMARK 3.5. From [12, Remark 4] and Proposition 3.3, the Gorenstein relative homology groups defined above are well defined. If M (resp., N) is a R°-module (resp., R-module) as complex at 0, then it is easy to see that $\mathrm{Tor}_i^{\mathcal{GP}_R}(M, N)$, $\mathrm{Tor}_i^{{}_R\mathcal{GP}}(M, N)$, $\mathrm{Tor}_i^{\mathcal{GF}_R}(M, N)$, and $\mathrm{Tor}_i^{{}_R\mathcal{GF}}(M, N)$ are the Gorenstein relative homology groups given by Holm in [10]. Actually, if M is an R°-module of finite Gorenstein projective dimension,

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then there is a special Gorenstein projective precover $G \to M$ as in Example 2.16(1), and it is a proper Gorenstein projective resolution. So $\mathrm{Tor}_i^{\mathcal{GP}_R}(M, N)$ is the group defined in [**10**]. If M is an R°-module of finite Gorenstein flat dimension, then there is a special Gorenstein flat precover $G \to M$ as in Example 2.16(2), and it is a proper Gorenstein flat resolution. So $\mathrm{Tor}_i^{\mathcal{GF}_R}(M, N)$ is the group defined in [**10**]. The cases for $\mathrm{Tor}_i^{_R\mathcal{GP}}(M, N)$ and $\mathrm{Tor}_i^{_R\mathcal{GF}}(M, N)$ are similar.

LEMMA 3.6. *Let* $0 \longrightarrow A \longrightarrow B \longrightarrow E \longrightarrow 0$ *be an exact sequence of R-modules with* $\mathrm{id}_R E < \infty$, *and let G be an R-module. Then the sequence*

$$0 \longrightarrow \mathrm{Hom}_R(G, A) \longrightarrow \mathrm{Hom}_R(G, B) \longrightarrow \mathrm{Hom}_R(G, E) \longrightarrow 0$$

is exact, provided that

(1) *G is Gorenstein flat, and A and E are cotorsion; or*

(2) *G is Gorenstein projective.*

Proof. Assume that G is Gorenstein flat, and A and E are cotorsion. Let f be a morphism from G to E. There is an exact sequence $0 \longrightarrow G \xrightarrow{\alpha} F \longrightarrow G' \longrightarrow 0$ of R-modules with F flat and G' Gorenstein flat. Note that E is a cotorsion R-module with $\mathrm{id}_R E < \infty$. By dimension shifting one has $\mathrm{Ext}_R^1(G', E) = 0$, and so there is a morphism $g : F \to E$ such that $g\alpha = f$. Since A is cotorsion, one has $\mathrm{Ext}_R^1(F, A) = 0$, and so there is a morphism $h : F \to B$ such that $\beta h = g$, where β is the morphism from B to E. Thus, $h\alpha$ is a morphism from G to B satisfying $\beta h\alpha = f$. This yields the exact sequence in the statement.

If G is Gorenstein projective, then by dimension shifting one has $\mathrm{Ext}_R^1(G, E) = 0$ for each R-module E with $\mathrm{id}_R E < \infty$. Thus, as proved in the above paragraph, one gets the desired exact sequence in the statement. □

LEMMA 3.7. *Let* $0 \longrightarrow K \longrightarrow G \xrightarrow{\tau} M \longrightarrow 0$ *be an exact sequence of* R°*-complexes with* $\mathrm{fd}_{\mathcal{C}} K < \infty$, *and let* G' *be a Gorenstein flat R-complex* (*or a Gorenstein projective R-complex*). *Then the morphism*

$$\tau \otimes_R G' : G \otimes_R G' \to M \otimes_R G'$$

is a quasi-isomorphism.

Proof. Let G' be a Gorenstein flat R-complex. Set $(-)^+ = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$. For each integer p, consider the exact sequence $0 \to M_p^+ \to G_p^+ \to K_p^+ \to 0$. Since $\mathrm{Ext}_R^1(F, M_p^+) = 0 = \mathrm{Ext}_R^1(F, K_p^+)$ for each flat R-module F, one gets that M_p^+ and K_p^+ are cotorsion. The module K_p^+ has finite injective dimension as $\mathrm{fd}_{\mathcal{C}} K < \infty$; see 2.8. For each integer n, G'_{n-p} is Gorenstein flat by Lemma 2.7(2), so the sequence

$$0 \to \mathrm{Hom}_R(G'_{n-p}, M_p^+) \to \mathrm{Hom}_R(G'_{n-p}, G_p^+) \to \mathrm{Hom}_R(G'_{n-p}, K_p^+) \to 0$$

is exact by Lemma 3.6. Thus, the sequence

$$0 \to K_p \otimes_R G'_{n-p} \to G_p \otimes_R G'_{n-p} \to M_p \otimes_R G'_{n-p} \to 0$$

is exact. This yields that the sequence

$$0 \longrightarrow K \otimes_R G' \longrightarrow G \otimes_R G' \xrightarrow{\tau \otimes_R G'} M \otimes_R G' \longrightarrow 0$$

of $\mathbb{Z}$-complexes is exact. Since $\mathrm{fd}_C K < \infty$, one gets that K is acyclic; see 2.8. Hence, K^+ is a cotorsion R-complex. Moreover, one has $\mathrm{id}_C K^+ < \infty$, so $\mathrm{Ext}^1_C(G', \Sigma^i K^+) = 0$ for each $i \in \mathbb{Z}$ by dimension shifting. Thus, $\mathrm{Hom}_R(G', K^+)$ is an acyclic $\mathbb{Z}$-complex by [8, Lemma 2.1], and so $K \otimes_R G'$ is acyclic. Hence, $\tau \otimes_R G' : G \otimes_R G' \to M \otimes_R G'$ is a quasi-isomorphism.

If G' is a Gorenstein projective R-complex, then one has $\mathrm{Ext}^1_C(G', E) = 0$ for each R-complex E with $\mathrm{id}_C E < \infty$. Thus, as proved in the above paragraph, one gets that $\tau \otimes_R G'$ is a quasi-isomorphism using Lemmas 2.7(1) and 3.6. □

The next result advertised as Theorem A in the introduction improves [10, Theorem 4.8] by removing the assumption that the ring R should be coherent.

THEOREM 3.8. *Let R be a ring, and let M be an R°-complex and N an R-complex.*

(1) *If $\mathrm{Gpd}_C M < \infty$ and $\mathrm{Gpd}_C N < \infty$, then for each $i \in \mathbb{Z}$ there is an isomorphism $\mathrm{Tor}_i^{\mathcal{GP}_R}(M, N) \cong \mathrm{Tor}_i^{{}_R\mathcal{GP}}(M, N)$.*
(2) *If $\mathrm{Gpd}_C M < \infty$ and $\mathrm{Gfd}_C N < \infty$, then for each $i \in \mathbb{Z}$ there is an isomorphism $\mathrm{Tor}_i^{\mathcal{GP}_R}(M, N) \cong \mathrm{Tor}_i^{{}_R\mathcal{GF}}(M, N)$.*
(3) *If $\mathrm{Gfd}_C M < \infty$ and $\mathrm{Gfd}_C N < \infty$, then for each $i \in \mathbb{Z}$ there is an isomorphism $\mathrm{Tor}_i^{\mathcal{GF}_R}(M, N) \cong \mathrm{Tor}_i^{{}_R\mathcal{GF}}(M, N)$.*
(4) *If $\mathrm{Gfd}_C M < \infty$ and $\mathrm{Gpd}_C N < \infty$, then for each $i \in \mathbb{Z}$ there is an isomorphism $\mathrm{Tor}_i^{\mathcal{GF}_R}(M, N) \cong \mathrm{Tor}_i^{{}_R\mathcal{GP}}(M, N)$.*

Proof. (1) By Proposition 3.1, there are exact sequences $0 \to K \to G \to M \to 0$ and $0 \to K' \to G' \to N \to 0$, where G (resp., G') is a Gorenstein projective R°-complex (resp., R-complex), and K (resp., K') is an R°-complex (resp., R-complex) of finite projective dimension. Then for each $i \in \mathbb{Z}$ one has $\mathrm{H}_i(G \otimes_R N) \cong \mathrm{H}_i(M \otimes_R G')$ by Lemma 3.7, and so $\mathrm{Tor}_i^{\mathcal{GP}_R}(M, N) \cong \mathrm{Tor}_i^{{}_R\mathcal{GP}}(M, N)$.

(2) By Proposition 3.1, there is an exact sequence $0 \to K \to G \to M \to 0$ where G is a Gorenstein projective R°-complex, and K is an R°-complex of finite projective dimension. From Proposition 3.2, one gets an exact sequence $0 \to K' \to G' \to N \to 0$ where G' is a Gorenstein flat R-complex, and K' is an R-complex of finite flat dimension. Thus, as proved in (1) one gets the isomorphism in the statement.

(3) and (4) can be proved similarly. □

We recall the invariant $\mathrm{splf}\, R = \sup\{\mathrm{pd}_R F \mid F \text{ is a flat } R\text{-module}\}$. Since an arbitrary direct sum of flat R-modules is flat, the invariant $\mathrm{splf}\, R$ is finite if and only if every flat R-module has finite projective dimension. If R is commutative Noetherian of finite Krull dimension d, then one has $\mathrm{splf}\, R \leqslant d$ by Jensen [13, Proposition 6]. Osofsky [18, 3.1] gave examples of rings for which the splf invariant is infinite. By the proof of [9, Proposition 3.4], one gets that if R is a right coherent ring with $\mathrm{splf} R < \infty$, then all Gorenstein projective R-modules are Gorenstein flat, and so for each R-complex M one has $\mathrm{Gfd}_C M \leq \mathrm{Gpd}_C M$ by Lemma 2.7. Thus, the following result holds by Theorem 3.8.

COROLLARY 3.9. *Let R be a coherent ring with $\mathrm{splf} R$ and $\mathrm{splf} R^\circ$ finite, and let M be an R°-complex with $\mathrm{Gpd}_C M < \infty$ and N an R-complex with $\mathrm{Gpd}_C N < \infty$. Then for all $i \in \mathbb{Z}$ there are isomorphisms*

$$\mathrm{Tor}_i^{\mathcal{GP}_R}(M, N) \cong \mathrm{Tor}_i^{{}_R\mathcal{GF}}(M, N) \cong \mathrm{Tor}_i^{\mathcal{GF}_R}(M, N) \cong \mathrm{Tor}_i^{{}_R\mathcal{GP}}(M, N).$$

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4. Generalized Tate homology of complexes. We start with the following lemmas, which are to show that the generalized Tate homology considered in this section is well defined.

LEMMA 4.1. *Let M and N be R°-complexes. Let $\varphi : F \to M$ and $\psi : P \to N$ be special semi-flat precovers of M and N, respectively. For each morphism $f : M \to N$, there exists a unique, up to homotopy, morphism $g : F \to P$ such that $f\varphi = \psi g$. In particular, a special semi-flat precover is unique up to homotopy equivalence.*

Proof. Since $K = \operatorname{Ker}\psi$ is a cotorsion R°-complex, one has $\operatorname{Ext}^1_{\mathcal{C}}(F, \Sigma^{-1}K) = 0$; see Lemma 2.10. Thus, the conclusion holds by Lemma 2.12. □

LEMMA 4.2. *Let M be an R°-complex. Let $\varphi : G \to M$ be a special Gorenstein flat precover and $\theta : F \to G$ a special semi-flat precover. Then $\varphi\theta : F \to M$ is a special semi-flat precover.*

Proof. Let $K = \operatorname{Ker}\varphi$ and $L = \operatorname{Ker}\theta$. Then K is a DG-cotorsion R°-complex of finite flat dimension, and L is a cotorsion R°-complex. Let $T = \operatorname{Ker}\varphi\theta$. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & F & \xrightarrow{\varphi\theta} & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\theta} & & \| & & \\ 0 & \longrightarrow & K & \longrightarrow & G & \xrightarrow{\varphi} & M & \longrightarrow & 0. \end{array}$$

By the classical "Snake Lemma," there is an exact sequence $0 \to L \to T \to K \to 0$. Since $\operatorname{fd}_{\mathcal{C}} K < \infty$, K is an acyclic R°-complex; see 2.8. Hence, K is cotorsion; see 2.6. Thus, T is cotorsion, and so $\varphi\theta : F \to M$ is a special semi-flat precover. □

LEMMA 4.3. *Let M and N be R°-complexes, and let $\varphi : F \to M$ be a special semi-flat precover and $\psi : G \to N$ a special Gorenstein flat precover. For each morphism $f : M \to N$, there exists a unique, up to homotopy, morphism $g : F \to G$ such that $f\varphi = \psi g$.*

Proof. By Lemma 2.7(2), F is Gorenstein flat, so there is a morphism $g : F \to G$ such that $f\varphi = \psi g$. Assume that there is another morphism $h : F \to G$ such that $f\varphi = \psi h$. Consider the exact sequence $0 \longrightarrow K \longrightarrow P \xrightarrow{\tau} G \longrightarrow 0$ where $\tau : P \to G$ is a special semi-flat precover of G; see 2.15. Since F is semi-flat and K is cotorsion, one has $\operatorname{Ext}^1_{\mathcal{C}}(F, K) = 0$ by Lemma 2.10. Thus, there are two morphisms α and β from F to P such that $\tau\alpha = h$ and $\tau\beta = g$, and so one has $\psi\tau\alpha = f\varphi$ and $\psi\tau\beta = f\varphi$. By Lemma 4.2, $\psi\tau : P \to N$ is a special semi-flat precover, so α is homotopic to β by Lemma 4.1. Thus, h is homotopic to g. □

DEFINITION 4.4. Let M be an R°-complex that has a special Gorenstein flat precover $G \to M$, and let $F \to M$ be a special semi-flat precover. By Lemma 4.3, there is a morphism $\alpha : F \to G$ induced by 1_M. For each R-complex N and each $i \in \mathbb{Z}$, the ith *generalized Tate homology based on flats* is defined by

$$\widehat{\operatorname{Tor}}_i^{\mathcal{GF},\mathcal{F}}(M, N) = \operatorname{H}_{i+1}(\operatorname{Cone}\alpha \otimes_R N).$$

REMARK 4.5. Using the method analogous to that used for the generalized Tate cohomology (see [12, Section 4]), and using Lemma 4.3, one can prove that the generalized Tate homology group $\widehat{\operatorname{Tor}}_i^{\mathcal{GF},\mathcal{F}}(M, N)$ is well defined. For an R°-module M of finite Gorenstein flat dimension, there is a special Gorenstein flat precover $G \to M$ as in

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Example 2.16(1), and it is a proper Gorenstein flat resolution. Fix a proper flat resolution $\cdots \to F_1 \to F_0 \to M \to 0$ such that $K_i = \mathrm{Ker}(F_i \to F_{i-1})$ is cotorsion. Let $F = \cdots \to F_1 \to F_0 \to 0 \to \cdots$. Then $F \to M$ is a special semi-flat precover of M. So for each R-module N, $\widehat{\mathrm{Tor}}_i^{\mathcal{GF},\mathcal{F}}(M, N)$ is the group defined by Iacob in [11].

For a special semi-flat precover $F \xrightarrow{\alpha} M$, $\mathrm{Ker}\,\alpha$ is a cotorsion R°-complex, and hence acyclic. So $F \to M$ is a semi-flat replacement of M. Thus, for each $i \in \mathbb{Z}$ one has $\mathrm{Tor}_i^R(M, N) \cong \mathrm{H}_i(F \otimes_R N)$; see 2.3. Hence, we have the following result that was proved by Iacob in [11, Proposition 6] for modules over a commutative Noetherian ring.

PROPOSITION 4.6. *Let M be an R°-complex with $\mathrm{Gfd}_{\mathcal{C}}\, M < \infty$. For each R-complex N, there exists an exact sequence*

$$\cdots \to \widehat{\mathrm{Tor}}_i^{\mathcal{GF},\mathcal{F}}(M, N) \to \mathrm{Tor}_i^R(M, N) \to \mathrm{Tor}_i^{\mathcal{GF}_R}(M, N) \to \widehat{\mathrm{Tor}}_{i-1}^{\mathcal{GF},\mathcal{F}}(M, N) \to \cdots.$$

Proof. Fix a special Gorenstein flat precover $G \to M$ and a special semi-flat precover $F \to M$. By Lemma 4.3, one has a morphism $\alpha : F \to G$ induced by 1_M. The sequence $0 \to G \to \mathrm{Cone}(\alpha) \to \Sigma F \to 0$ is degree-wise split, so one gets an exact sequence $0 \to G \otimes_R N \to \mathrm{Cone}(\alpha) \otimes_R N \to \Sigma F \otimes_R N \to 0$ of $\mathbb{Z}$-complexes, which yields the desired exact sequence of homology modules in the statement. □

LEMMA 4.7. *Let M be a bounded R°-complex with $\mathrm{Gfd}_{\mathcal{C}}\, M < \infty$. Then M has a special Gorenstein flat precover $G \to M$ such that G is bounded with $\inf G = \inf M$ and $\sup G \leq \mathrm{Gfd}_{\mathcal{C}}\, M + \sup M$, and G_i is flat for each $i > \sup M$.*

Proof. Set $g = \mathrm{Gfd}_{\mathcal{C}}\, M < \infty$. Without loss of generality, we may assume that $\inf M = 0$ and $\sup M = s$. We argue by induction on s. If $s = 0$, then one has $M = M_0$ with $\mathrm{Gfd}_{R^\circ}\, M_0 = g$. Thus, there exists a special Gorenstein flat precover $G \to M$ of M such that $\inf G = 0$ and $\sup G \leq g$, and G_i is flat for each $i > 0$; see Example 2.16(2).

We let $s > 0$. Then there exists a morphism $f : \Sigma^{s-1} M_s \to M_{\leqslant s-1}$. By induction hypothesis, there are special Gorenstein flat precovers $G' \xrightarrow{\alpha'} \Sigma^{s-1} M_s$ and $G'' \xrightarrow{\alpha''} M_{\leqslant s-1}$ of $\Sigma^{s-1} M_s$ and $M_{\leqslant s-1}$ respectively, such that G' and G'' are bounded with $\inf G' = s-1$, $\inf G'' = 0$, $\sup G' \leq g+s-1$ and $\sup G'' \leq g+s-1$, and G'_i and G''_i are flat for $i > s-1$.

Let $K' = \mathrm{Ker}\,\alpha'$ and $K'' = \mathrm{Ker}\,\alpha''$. Then K' and K'' are DG-cotorsion R°-complexes of finite flat dimension. By Lemma 2.14(1), one has $\mathrm{Ext}_{\mathcal{C}}^1(G', K'') = 0$, so there is a commutative diagram of R°-complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K' & \longrightarrow & G' & \longrightarrow & \Sigma^{s-1} M_s & \longrightarrow & 0 \\ & & \downarrow h & & \downarrow g & & \downarrow f & & \\ 0 & \longrightarrow & K'' & \longrightarrow & G'' & \longrightarrow & M_{\leqslant s-1} & \longrightarrow & 0. \end{array}$$

This yields an exact sequence

$$0 \longrightarrow \mathrm{Cone}\, h \longrightarrow \mathrm{Cone}\, g \longrightarrow \mathrm{Cone}\, f \longrightarrow 0$$

of R°-complexes. We notice that $\mathrm{Cone}\, f = M$. Let $G = \mathrm{Cone}\, g$ and $K = \mathrm{Cone}\, h$. Then G is a bounded Gorenstein flat R°-complex with $\inf G = 0$ and $\sup G \leq g + s$, and G_i is flat for each $i > s$. It is clear that K is a DG-cotorsion R°-complex of finite flat dimension, so $G \to M$ is a special Gorenstein flat precover of M by Lemma 2.14(1). □

The following result is proved similarly as in Lemma 4.7. Here we give its proof for the convenience of the readers.

LEMMA 4.8. *Let M be a bounded R°-complex. Then M has a special semi-flat precover $F \to M$ such that* $\inf F = \inf M$.

Proof. Without loss of generality, we may assume that $\inf M = 0$ and $\sup M = s$. We argue by induction on s. If $s = 0$, then one has $M = M_0$. There is an exact sequence $\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M_0 \longrightarrow 0$ with each F_i flat and $K_i = \operatorname{Coker}(F_{i+2} \to F_{i+1})$ cotorsion for $i \geq 0$. Let $F = \cdots \to F_1 \to F_0 \to 0 \to \cdots$. Then F is semi-flat, and there is a surjective morphism $\alpha : F \to M$ with $\operatorname{Ker} \alpha$ cotorsion. Thus, α is a special semi-flat precover of M with $\inf F = 0$.

We let $s > 0$. Then there exists a morphism $f : \Sigma^{s-1} M_s \to M_{\leqslant s-1}$. By induction hypothesis, there are exact sequences $0 \to K' \to F' \to \Sigma^{s-1} M_s \to 0$ and $0 \to K'' \to F'' \to M_{\leqslant s-1} \to 0$ such that F' and F'' are semi-flat R°-complexes with $\inf F' = s-1$ and $\inf F'' = 0$, and K' and K'' are cotorsion. By Lemma 2.10 one has $\operatorname{Ext}^1_{\mathcal{C}}(F', K'') = 0$, so there is a commutative diagram of R°-complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K' & \longrightarrow & F' & \longrightarrow & \Sigma^{s-1} M_s & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & K'' & \longrightarrow & F'' & \longrightarrow & M_{\leqslant s-1} & \longrightarrow & 0.
\end{array}$$

This yields an exact sequence

$$0 \longrightarrow \operatorname{Cone} h \longrightarrow \operatorname{Cone} g \longrightarrow \operatorname{Cone} f \longrightarrow 0$$

of R°-complexes. We notice that $\operatorname{Cone} f = M$. Let $F = \operatorname{Cone} g$ and $K = \operatorname{Cone} h$. Then F is semi-flat with $\inf F = 0$ and K is cotorsion, and so $F \to M$ is a special semi-flat precover of M. □

The next result was proved in [**14**, Proposition 4.5] for a cotorsion R°-module M of finite Gorenstein flat dimension. Here we don't need the cotorsion assumption.

THEOREM 4.9. *Let M be a bounded R°-complex with* $\operatorname{Gfd}_{\mathcal{C}} M < \infty$. *Then M has a Tate flat resolution, and for each $n >$* $\sup M$ *and each R-module N there is an isomorphism*

$$\widehat{\operatorname{Tor}}_n^{\mathcal{GF},\mathcal{F}}(M, N) \cong \widehat{\operatorname{Tor}}_n^{\mathcal{F}}(M, N).$$

Proof. Let $\operatorname{Gfd}_{\mathcal{C}} M = g$ and $\sup M = s$. By Lemma 4.7, there exists a special Gorenstein flat precover $\pi : G \to M$ such that G is bounded with $\inf G = \inf M$ and $\sup G \leq g + s$, and G_i is flat for $i > s$. On the other hand, M admits a special semi-flat precover $\pi' : F \to M$ such that $\inf F = \inf M$ by Lemma 4.8. Since F is a Gorenstein flat R°-complex by Lemma 2.7(2), and π and π' are quasi-isomorphisms as $\operatorname{Ker} \pi$ and $\operatorname{Ker} \pi'$ are acyclic, there is a quasi-isomorphism $\alpha : F \to G$ such that $\pi\alpha = \pi'$. Let $X = \Sigma^{-1} \operatorname{Cone} \alpha$. Then X is an acyclic complex of Gorenstein flat R-modules with X_i flat for $i \geq s$ and $X_i = F_i$ for $i \geq g + s$, and there is a degree-wise split surjective morphism $f : X \to F$ such that $f_i = \mathrm{id}_{F_i}$ for $i \geq g + s$, which yields a morphism $\tau : \mathrm{C}_s(X) \to \mathrm{C}_s(F)$. Since X is bounded below, one gets that $\mathrm{C}_s(X)$ is Gorenstein flat by [**19**, Corollary 3.12], and so there is an exact sequence

$$0 \to \mathrm{C}_s(X) \to P_{s-1} \to P_{s-2} \to \cdots \qquad (4.1)$$

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with each P_i flat, such that it remains exact when applying the functor $- \otimes_R I$ to it for each injective R-module I. Assembling the sequence (1) and the sequence $\cdots \to X_{s+1} \to X_s \to \mathrm{C}_s(X) \to 0$, one gets an F-totally acyclic complex T of flat R°-modules with $T_{\geqslant s+g} = F_{\geqslant s+g}$ and $T_{\geqslant s} = X_{\geqslant s}$. Since $\pi' : F \to M$ is a semi-flat replacement of M, the pair (T, F) is a Tate flat resolution of M. Thus, for each R-module N and each $n > s$, one has

$$\begin{aligned} \widehat{\mathrm{Tor}}_n^{\mathcal{GF},\mathcal{F}}(M, N) &\cong \mathrm{H}_{n+1}(\mathrm{Cone}\,\alpha \otimes_R N) \\ &= \mathrm{H}_n(\Sigma^{-1}\,\mathrm{Cone}\,\alpha \otimes_R N) \\ &= \mathrm{H}_n(X \otimes_R N) \\ &= \mathrm{H}_n(T \otimes_R N) \\ &\cong \widehat{\mathrm{Tor}}_n^{\mathcal{F}}(M, N)\,. \end{aligned}$$

□

The next result improves [**14**, Theorem 4.4] by removing the assumptions that R should be right coherent and M should be cotorsion.

COROLLARY 4.10. *Let M be an R°-module of finite Gorenstein flat dimension d. For each R-module N, there exists an exact sequence*

$$0 \to \widehat{\mathrm{Tor}}_d^{\mathcal{F}}(M, N) \to \mathrm{Tor}_d^R(M, N) \to \mathrm{Tor}_d^{\mathcal{GF}_R}(M, N) \to \cdots$$

$$\to \widehat{\mathrm{Tor}}_1^{\mathcal{F}}(M, N) \to \mathrm{Tor}_1^R(M, N) \to \mathrm{Tor}_1^{\mathcal{GF}_R}(M, N) \to 0.$$

Proof. By Proposition 4.6, there exists an exact sequence

$$\cdots \to \widehat{\mathrm{Tor}}_i^{\mathcal{GF},\mathcal{F}}(M, N) \to \mathrm{Tor}_i^R(M, N) \to \mathrm{Tor}_i^{\mathcal{GF}_R}(M, N) \to \widehat{\mathrm{Tor}}_{i-1}^{\mathcal{GF},\mathcal{F}}(M, N) \to \cdots.$$

From Theorem 4.9, one has $\widehat{\mathrm{Tor}}_n^{\mathcal{GF},\mathcal{F}}(M, N) \cong \widehat{\mathrm{Tor}}_n^{\mathcal{F}}(M, N)$ for each $n > 0$, and it is clear that $\widehat{\mathrm{Tor}}_0^{\mathcal{GF},\mathcal{F}}(M, N) = 0$. Since $\mathrm{Gfd}_R M = d$, one has $\mathrm{Tor}_{d+1}^{\mathcal{GF}_R}(M, N) = 0$. Thus, the desired exact sequence in the statement follows. □

DEFINITION 4.11. Let M be an R°-complex that has a special Gorenstein projective precover $G \to M$, and let $P \to M$ be a special semi-projective precover. By Lemma 2.7(1), P is Gorenstein projective, and so there is a morphism $\alpha : P \to G$ induced by 1_M. For each R-complex N and $i \in \mathbb{Z}$, the ith *generalized Tate homology* is defined by

$$\widehat{\mathrm{Tor}}_i^{\mathcal{GP},\mathcal{P}}(M, N) = \mathrm{H}_{i+1}(\mathrm{Cone}\,\alpha \otimes_R N).$$

REMARK 4.12. As proved in [**12**, Section 4], one gets that the generalized Tate homology group $\widehat{\mathrm{Tor}}_i^{\mathcal{GP},\mathcal{P}}(M, N)$ is well defined. Using the same argument as in Remark 4.5, one gets that if M is an R°-module of finite Gorenstein projective dimension, then for each R-module N, $\widehat{\mathrm{Tor}}_i^{\mathcal{GP},\mathcal{P}}(M, N)$ is the group defined in [**11**].

We notice that $P \to M$ is a semi-projective resolution of M. Then for each $i \in \mathbb{Z}$ one has $\mathrm{Tor}_i^R(M, N) \cong \mathrm{H}_i(P \otimes_R N)$. Now we have the following similar results, where Theorem 4.14 was proved for modules by Iacob in [**11**, Proposition 1].

PROPOSITION 4.13. *Let M be an R°-complex with $\mathrm{Gpd}_{\mathcal{C}} M < \infty$. For each R-complex N there exists an exact sequence*

$$\cdots \to \widehat{\mathrm{Tor}}_i^{\mathcal{GP},\mathcal{P}}(M, N) \to \mathrm{Tor}_i^R(M, N) \to \mathrm{Tor}_i^{\mathcal{GP}_R}(M, N) \to \widehat{\mathrm{Tor}}_{i-1}^{\mathcal{GP},\mathcal{P}}(M, N) \to \cdots.$$

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THEOREM 4.14. *Let M be a bounded R°-complex with $\operatorname{Gpd}_{\mathcal{C}} M < \infty$. Then M has a complete projective resolution, and for each $n > \sup M$ and each R-module N there is an isomorphism*

$$\widehat{\operatorname{Tor}}_n^{\mathcal{GP},\mathcal{P}}(M,N) \cong \widehat{\operatorname{Tor}}_n^R(M,N).$$

As an immediate consequence of the above results, we have the following corollary that was proved by Iacob in [**11**, Theorem 1].

COROLLARY 4.15. *Let M be an R°-module of finite Gorenstein projective dimension d. For each R-module N, there exists an exact sequence*

$$0 \to \widehat{\operatorname{Tor}}_d^R(M,N) \to \operatorname{Tor}_d^R(M,N) \to \operatorname{Tor}_d^{\mathcal{GP}_R}(M,N) \to \cdots$$

$$\to \widehat{\operatorname{Tor}}_1^R(M,N) \to \operatorname{Tor}_1^R(M,N) \to \operatorname{Tor}_1^{\mathcal{GP}_R}(M,N) \to 0.$$

5. Unbounded homology of complexes. Let M be an R°-complex and N an R-complex. Let $P \xrightarrow{\simeq} M$ be a semi-projective resolution and let $N \xrightarrow{\simeq} I$ be a semi-injective resolution. From [**4**], for each $i \in \mathbb{Z}$, the $\mathbb{Z}$-modules

$$\overline{\operatorname{Tor}}_i^R(M,N) = \operatorname{H}_i(P \overline{\otimes}_R I)$$

are called the *unbounded homology* of M and N over R. Here $P \overline{\otimes}_R I$ is the $\mathbb{Z}$-complex with the degree-n term $(P \overline{\otimes}_R I)_n = \prod_{i\in\mathbb{Z}}(P_i \otimes_R I_{n-i})$ and the differential defined as in 2.2. In this section, we compare the unbounded homology $\overline{\operatorname{Tor}}$ to the Gorenstein relative homology $\operatorname{Tor}^{\mathcal{GF}_R}$.

LEMMA 5.1. *Let R be a left coherent ring and M an R°-module. If M is Gorenstein flat, then there exists an exact sequence*

$$0 \to M \to F_0 \to F_{-1} \to \cdots$$

with each F_i flat, such that it remains exact when applying functors $\operatorname{Hom}_{R^\circ}(-,Q)$ and $-\otimes_R I$ to it for each flat R°-module Q and each injective R-module I.

Proof. Since R is left coherent, there is a flat preenvelope $f: M \to F_0$ by Xu [**21**, Theorem 2.5.1]. On the other hand, M is Gorenstein flat, so there exists a monomorphism from M to a flat R°-module. Thus, the above flat preenvelope f is a monomorphism. Consider the exact sequence

$$0 \longrightarrow M \xrightarrow{f} F_0 \longrightarrow C_0 \longrightarrow 0 \tag{5.1}$$

of R°-modules with $C_0 = \operatorname{Coker}(f)$. Then it remains exact after applying the functor $\operatorname{Hom}_{R^\circ}(-,Q)$ to it for each flat R°-module Q. Let I be an injective R-module. Then one has

$$\operatorname{Hom}_{\mathbb{Z}}(\operatorname{Tor}_1^R(C_0,I),\mathbb{Q}/\mathbb{Z}) \cong \operatorname{Ext}^1_{R^\circ}(C_0,\operatorname{Hom}_{\mathbb{Z}}(I,\mathbb{Q}/\mathbb{Z})) = 0,$$

where the equality holds since $\operatorname{Hom}_{\mathbb{Z}}(I,\mathbb{Q}/\mathbb{Z})$ is a flat and cotorsion R°-module. Hence, $\operatorname{Tor}_1^R(C_0,I) = 0$, and so C_0 is Gorenstein flat by [**9**, Proposition 3.8], and the sequence (1) remains exact after applying the functor $-\otimes_R I$ to it. Continuing this process one gets the desired exact sequence in the statement. □

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The next result contains Theorem C from the introduction, which is proved in [15, Proposition 4.10] for a cotorsion R°-module M of finite Gorenstein flat dimension. Here we prove it for a bounded complex without the cotorsion assumption.

THEOREM 5.2. *Let R be a left coherent ring, and let M be a bounded R°-complex with $\mathrm{Gfd}_{\mathcal{C}} M < \infty$ and N an R-module. Then for each $n > \sup M + 1$, there is an isomorphism $\overline{\mathrm{Tor}}_n^R(M, N) \cong \mathrm{Tor}_n^{\mathcal{GF}_R}(M, N)$.*

Proof. Let $\mathrm{Gfd}_{\mathcal{C}} M = g$ and $\sup M = s$. By Lemma 4.7, there exists a special Gorenstein flat precover $\pi : G \to M$ such that G is bounded with $\inf G = \inf M$ and $\sup G \leq g + s$, and G_i is flat for $i > s$. On the other hand, M admits a semi-flat precover $\pi' : F \to M$ such that $\inf F = \inf M$ by Lemma 4.8. Since F is a Gorenstein flat R°-complex by Lemma 2.7(2), and π and π' are quasi-isomorphisms as $\mathrm{Ker}\,\pi$ and $\mathrm{Ker}\,\pi'$ are acyclic, there is a quasi-isomorphism $\alpha : F \to G$ such that $\pi\alpha = \pi'$. Let $X = \Sigma^{-1}\,\mathrm{Cone}\,\alpha$. Then X is an acyclic complex of Gorenstein flat R-modules with X_i flat for $i \geq s$ and $X_i = F_i$ for $i \geq g + s$, and there is a degree-wise split surjective morphism $f : X \to F$ such that $f_i = \mathrm{id}_{F_i}$ for $i \geq g + s$, which yields a morphism $\tau : \mathrm{C}_s(X) \to \mathrm{C}_s(F)$. Since X is bounded below, one gets that $\mathrm{C}_s(X)$ is Gorenstein flat, and so by Lemma 5.1 there is an exact sequence

$$0 \to \mathrm{C}_s(X) \to P_{s-1} \to P_{s-2} \to \cdots \qquad (1)$$

with each P_i flat, such that it remains exact when applying functors $\mathrm{Hom}_{R^{\circ}}(-, Q)$ and $- \otimes_R I$ to it for each flat R°-module Q and each injective R-module I. Thus, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{C}_s(X) & \longrightarrow & P_{s-1} & \longrightarrow & P_{s-2} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle\tau} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{C}_s(F) & \longrightarrow & F_{s-1} & \longrightarrow & F_{s-2} & \longrightarrow & \cdots . \end{array}$$

Consider the exact sequence

$$\cdots \to X_{s+1} \to X_s \to \mathrm{C}_s(X) \to 0 \qquad (2)$$

and the commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X_{s+1} & \longrightarrow & X_s & \longrightarrow & \mathrm{C}_s(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_{s+1}} & & \downarrow{\scriptstyle f_s} & & \downarrow{\scriptstyle\tau} & & \\ \cdots & \longrightarrow & F_{s+1} & \longrightarrow & F_s & \longrightarrow & \mathrm{C}_s(F) & \longrightarrow & 0. \end{array}$$

Assembling the sequences (1) and (2), one gets an F-totally acyclic complex T' of flat R°-modules, and a morphism $\tau' : T' \to F$ with $\tau'_i = f_i$ for $i \geq s$. Set $T'' = \Sigma^{-1}\,\mathrm{Cone}(\,\mathrm{id}_{F_{\leqslant s-1}})$. Then T'' is a contractible complex and there is a degree-wise split surjective morphism $\kappa : T'' \to F_{\leqslant s-1}$. Let $\tau'' = \varepsilon\kappa : T'' \to F$, where $\varepsilon : F_{\leqslant s-1} \to F$ is the natural morphism. Then τ''_i is split surjective for each $i \leq s-1$ and $\tau''_i = 0$ for all $i \geq s$. Let $T = T' \oplus T''$ and $\tau = (\tau', \tau'') : T \to F$. Then T is an F-totally acyclic complex of flat R°-modules with $T_{\geqslant s} = X_{\geqslant s}$, and τ_i is split surjective for each $i \leq s-1$ and $\tau_i = \tau'_i = f_i$ for all $i \geq s$. Thus, $T \xrightarrow{\tau} F$ is a complete flat resolution of M such that τ_i is split surjective for each $i \in \mathbb{Z}$; see 2.5. Let $K = \mathrm{Ker}\,\tau$. One see that $(\Sigma K)_{\geqslant s+1} = G_{\geqslant s+1}$. Thus, for each R-module N and each

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$n > s+1$ the second equation in the next computation holds:

$$\begin{aligned}\overline{\mathrm{Tor}}_n^R(M,N) &\cong \mathrm{H}_{n-1}(K\otimes_R N)\\ &= \mathrm{H}_n(\Sigma\, K\otimes_R N)\\ &= \mathrm{H}_n(G\otimes_R N)\\ &\cong \mathrm{Tor}_n^{\mathcal{GF}_R}(M,N)\,,\end{aligned}$$

where the first isomorphism holds by [15, Theorem 3.9 and Proposition 4.3]. □

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