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On a nonlinear eigenvalue problem in Orlicz–Sobolev spaces

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We consider the eigenvalue problem

$$\begin{aligned} -\Delta_m(u) &= \lambda m(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

in an arbitrary Orlicz–Sobolev space. We show that the existence of an eigenvalue can be derived from a generalized version of Lagrange multiplier rule. Our approach also applies to more general problems. We emphasize that no Δ_2 condition is imposed.

1. Introduction

Let $m : [0, +\infty[\to [0, +\infty[$ be a non-decreasing continuous function with

$$m(0) = 0, \quad m(t) > 0 \quad \text{for } t > 0 \quad \text{and} \quad \lim_{t\to+\infty} m(t) = +\infty.$$

Associated with m we consider the operator

$$\Delta_m(u) := \operatorname{div}\left(m(|\nabla u|) \frac{\nabla u}{|\nabla u|} \right) \tag{1.1}$$

on a bounded open subset Ω of $\mathbb{R}^N$. We will refer to Δ_m as the m-Laplacian operator.

In (1.1), $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^N$. We notice that since $m(0) = 0$, the function $m(|\xi|)\xi/|\xi|$ is continuous on $\mathbb{R}^N$ and, in fact, is the gradient of $M(|\xi|)$, where

$$M(t) = \int_0^t m(s)\, \mathrm{d}s.$$

For later purposes, it will be convenient to extend m into an odd function on $\mathbb{R}$ by putting $m(-t) = -m(t)$.

When $m(t) = t^{p-1}$, $1 < p < \infty$, Δ_m reduces to the usual p-Laplacian

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

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Let us consider the eigenvalue problem

$$\left.\begin{aligned} -\Delta_m(u) = \lambda m(u) \quad & \text{in } \Omega, \\ u = 0 \qquad\quad & \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.2}$$

By a non-trivial solution of (1.2) we mean a pair (λ, u) with $\lambda \in \mathbb{R}$ and $u \not\equiv 0$ that verifies (1.2) in a suitable weak sense.

This problem, with m possibly non-homogeneous, was recently studied in [4, 5, 10, 11]. In particular, it is shown in [10] that under the sole assumptions on m described above, problem (1.2) has at least one non-trivial solution.

As observed in [4,5,10,11], problem (1.2), when properly formulated in the setting of Orlicz–Sobolev spaces, leads to several difficulties connected with the lack of homogeneity of m and the structure of the corresponding spaces (in general, they may not be reflexive). In particular, the functional naturally associated to $-\Delta_m(u)$ is, in general, neither everywhere defined nor *a fortiori* C^1. This excludes the use of the standard Lagrange multiplier rule.

One of the purposes in this paper is to show that a certain theorem on generalized multipliers, which is well known in the theory of mathematical programming in Banach spaces (cf. [12]), can be applied to deal with problem (1.2) in its full generality (i.e. without any additional assumption on the function m). This theorem involves the so-called Robinson constraint qualification condition.

One advantage of our approach is that it applies as well to more general problems. We will consider problems with an indefinite bounded weight of the form

$$\left.\begin{aligned} -\Delta_m(u) = \lambda \rho(x) m(u) \quad & \text{in } \Omega, \\ u = 0 \qquad\qquad & \text{on } \partial\Omega \end{aligned}\right\} \tag{1.3}$$

and, more generally, problems like

$$\left.\begin{aligned} -\Delta_m(u) = \lambda b(x,u) \quad & \text{in } \Omega, \\ u = 0 \qquad\quad & \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.4}$$

The lack of homogeneity of m and the fact that ρ changes sign in (1.3) lead to new difficulties, in particular, with respect to the sign of the principal eigenvalue that we construct, as we will see later (cf. remark 4.4).

The same approach can also be used to deal with a problem of the form

$$\left.\begin{aligned} &\mathcal{A}(u) = \lambda \mathcal{B}(u) && \text{in } \Omega, \\ &D^\alpha u = 0 && \text{on } \partial\Omega \quad \text{for } |\alpha| \leqslant n, \end{aligned}\right\} \tag{1.5}$$

where $\mathcal{A}$ and $\mathcal{B}$ are quasilinear operators in divergence form of order $2n$ and $2(n-1)$, respectively, given by

$$\mathcal{A}(u) := \sum_{|\alpha| \leqslant n} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \nabla u, \dots, \nabla^n u),$$

$$\mathcal{B}(u) := \sum_{|\alpha| \leqslant n-1} (-1)^{|\alpha|} D^\alpha B_\alpha(x, u, \nabla u, \dots, \nabla^{n-1} u).$$

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The main assumptions in this case are that $\mathcal{A}$ and $\mathcal{B}$ are potential operators, with $\mathcal{A}$ elliptic and monotone.

The paper is organized as follows. Preliminary results from mathematical programming are described in § 2. In § 3 we recall some tools from Orlicz and Orlicz–Sobolev spaces we will need. We then use the results of § 2 to study problems (1.3) and (1.4) in §§ 4 and 5, respectively. Section 6 is devoted to the technically more complicated problem (1.5).

2. The Lagrange multiplier rule

The Lagrange multiplier rule, in its most elementary form, asserts the following. Let f and g be real C^1 functionals on a real Banach space U. If $u_0 \in U$ minimizes $f(u)$ under the constraint $g(u) = 0$ and if $g'(u_0) \neq 0$, then there exists $\lambda \in \mathbb{R}$ such that

$$\lambda g'(u_0) = f'(u_0). \tag{2.1}$$

We will need a result of this type for the case when $f : U \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous proper convex function. In this situation, one is tempted to replace (2.1) by the inclusion

$$\lambda g'(u_0) \in \partial f(u_0),$$

where $\partial f(u_0)$ denotes the subgradient of f at u_0. Some care must, however, be taken, as the following simple example indicates: $U = \mathbb{R}$, $f(u) = -\sqrt{u}$ for $u \geqslant 0$, $f(u) = +\infty$ for $u < 0$, $g(u) = u$ (here, $u_0 = 0$ is a minimizer, with $g'(u_0) \neq 0$, but $\partial f(u_0)$ is empty).

As we will see later, the above problem of

$$\text{minimizing } f(u) \text{ under the constraint } g(u) = 0, \tag{2.2}$$

with f lower semicontinuous proper convex on the Banach space U and g of class C^1 on U, is closely related to the following mathematical programming problem in Banach spaces,

$$\text{minimize } F(x) \text{ under the constraint } G(x) \in K, \tag{2.3}$$

where F is a real C^1 functional on a real Banach space X, G is a C^1 mapping from X into another real Banach space Y and K is a non-empty closed convex subset of Y. For this latter problem, there is a classical condition, known as the Robinson constraint qualification condition (cf. (2.4) below), which guarantees the existence of generalized multipliers.

Proposition 2.1 (cf. theorem 3.1 of [12] or theorem 4.2 of [1]). *Let $F : X \to \mathbb{R}$, $G : X \to Y$ and $K \subset Y$ be as in (2.3). Suppose that $x_0 \in X$ solves (2.3) and assume in addition that*

$$G'(x_0)X - \mathbb{R}^+(K - G(x_0)) = Y. \tag{2.4}$$

Then there exists y^* in the dual space Y^* such that

$$y^* \in N_K(G(x_0)), \tag{2.5}$$

$$F'(x_0) + y^* \circ G'(x_0) = 0, \tag{2.6}$$

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where

$$N_K(G(x_0)) := \{z^* \in Y^* : \langle z^*, y - G(x_0)\rangle \leqslant 0 \ \forall y \in K\}$$

denotes the normal cone to K *at* $G(x_0)$ *and* $\circ$ *in (2.6) denotes composition.*

Recall that problem (2.2) is called feasible if there exists

$$u \in \operatorname{dom} f := \{u \in U : f(u) < +\infty\},$$

with $g(u) = 0$.

COROLLARY 2.2. *Let* $f : U \to \mathbb{R} \cup \{+\infty\}$ *and* $g : U \to \mathbb{R}$ *be as in (2.2). Suppose that problem (2.2) is feasible and that* $u_0 \in U$ *solves (2.2). Furthermore, suppose that there exist* u_1 *and* u_2 *in* $\operatorname{dom} f$ *such that*

$$g'(u_0)(u_1 - u_0) > 0 \quad \text{and} \quad g'(u_0)(u_2 - u_0) < 0. \tag{2.7}$$

Then there exists $\lambda \in \mathbb{R}$ *such that*

$$\lambda g'(u_0) \in \partial f(u_0). \tag{2.8}$$

Proof. We will apply proposition 2.1 with $X = U \times \mathbb{R}$, $F(u, r) = r$, $Y = U \times \mathbb{R} \times \mathbb{R}$, $G(u, r) = (u, r, g(u))$ and $K = \operatorname{epi}(f) \times \{0\}$, where

$$\operatorname{epi}(f) := \{(u, r) \in U \times \mathbb{R} : r \geqslant f(u)\}.$$

Clearly, $u_0 \in U$ solves (2.2) if and only if $x_0 = (u_0, f(u_0)) \in X$ solves (2.3). The verification of (2.4) amounts to proving that, for any given $(v, \alpha, \beta) \in U \times \mathbb{R} \times \mathbb{R}$, there exist $u \in U$, $r \in \mathbb{R}$, $(u', r') \in \operatorname{epi}(f)$ and $t \geqslant 0$ such that

$$u - t(u' - u_0) = v, \tag{2.9}$$

$$r - t(r' - f(u_0)) = \alpha, \tag{2.10}$$

$$g'(u_0)u = \beta. \tag{2.11}$$

Using (2.7), one first finds $t \geqslant 0$ and $u' \in \operatorname{dom} f$ such that

$$tg'(u_0)(u' - u_0) = \beta - g'(u_0)v.$$

One can then choose, for instance, $r' = f(u')$ and find u and r so as to satisfy (2.9), (2.10). Proposition 2.1 thus implies the existence of $y^* = (u^*, r, s) \in U^* \times \mathbb{R} \times \mathbb{R}$ such that (2.5) and (2.6) hold. Expressing these two latter relations in terms of f, g, u_0, one gets

$$u^* = -sg'(u_0), \quad r = -1, \quad u^* \in \partial f(u_0),$$

and the conclusion (2.8) follows. □

REMARK 2.3. In [12] it is assumed that K is a cone, but this is not really needed when the result is stated as in proposition 2.1 above. Related works involving condition (2.4), as well as proposition 2.1, can be found in [1]. The more general problem of minimizing $F(x)$ under the constraints $G(x) \in K$ and $x \in C$, C a non-empty closed convex subset of X, is also considered in [1, 12].

REMARK 2.4. A proof of corollary 2.2, with no reference to mathematical programming, can also be obtained from the results in [9].

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REMARK 2.5. The possibility of applying corollary 2.2 to eigenvalue problems in Orlicz–Sobolev spaces was already observed in [5].

3. Preliminaries on Orlicz–Sobolev spaces

Let Ω be a bounded open subset of $\mathbb{R}^N$ and let

$$M(t) = \int_0^t m(s)\,\mathrm{d}s,$$

where $m : [0, +\infty[\ \to [0, +\infty[$ satisfies the same conditions as in § 1 and is considered as extended as an odd function on $\mathbb{R}$.

The Orlicz space associated to the N-function M is denoted by $L_M(\Omega)$, the norm closure of $L^\infty(\Omega)$ in $L_M(\Omega)$ by $E_M(\Omega)$ and the Orlicz class by $\mathcal{L}_M(\Omega)$ (cf., for example, [8]).

The corresponding Sobolev spaces of functions in $L_M(\Omega)$ (respectively, $E_M(\Omega)$) with first distributional derivatives in $L_M(\Omega)$ (respectively, $E_M(\Omega)$) are denoted, respectively, by $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$. They are identified to subspaces of the product $(L_M(\Omega))^{N+1} \equiv \Pi L_M$.

We define the Orlicz–Sobolev spaces $W_0^1L_M(\Omega)$ (respectively, $W_0^1E_M(\Omega)$) as the $\sigma(\Pi L_M, \Pi E_{\bar{M}})$ closure (respectively, norm closure) of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$. Here, $\bar{M}$ denotes the N-function conjugate to M. We will also need the spaces of distributions $W^{-1}L_{\bar{M}}(\Omega)$ and $W^{-1}E_{\bar{M}}(\Omega)$, which are defined in the usual way,

$$W^{-1}L_{\bar{M}}(\Omega) := \left\{ f \in \mathcal{D}'(\Omega) : f = f_0 - \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}, \text{ with } f_0, f_i \in L_{\bar{M}}(\Omega) \right\},$$

and similarly for $W^{-1}E_{\bar{M}}(\Omega)$, where one requires f_0, $f_i \in E_{\bar{M}}(\Omega)$. These spaces are endowed with the quotient norm.

It is known that if Ω has the segment property, then the four spaces

$$(W_0^1L_M(\Omega), W_0^1E_M(\Omega); W^{-1}L_{\bar{M}}(\Omega), W^{-1}E_{\bar{M}}(\Omega))$$

form a complementary system (cf. [6, § 1]). This means that through the natural pairing

$$\langle u, f \rangle = \int_\Omega u f_0 + \sum_{i=1}^{N} \int_\Omega \frac{\partial u}{\partial x_i} f_i,$$

the dual of $W_0^1E_M(\Omega)$ can be identified (algebraically and topologically) as $W^{-1}L_{\bar{M}}(\Omega)$ and the dual of $W^{-1}E_{\bar{M}}(\Omega)$ as $W_0^1L_M(\Omega)$. In particular, $W_0^1L_M(\Omega)$ is a dual space.

Next we list some well-known properties of these spaces to which we will refer repeatedly.

(a) $m(u) \in L_{\bar{M}}(\Omega)$ if $u \in E_M(\Omega)$ (this is a consequence of the inequalities $\bar{M}(m(u)) \leqslant um(u) \leqslant M(2u)$). Moreover, for any N-function Q with $Q \ll \bar{M}$ (i.e. that grows at infinity essentially less rapidly than $\bar{M}$), the mapping $u \to m(u)$ is continuous from $E_M(\Omega)$ into $L_Q(\Omega)$ (a consequence of theorem 17.3 in [8]).

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(b) The mapping $u \in E_M(\Omega) \to M(u) \in L^1(\Omega)$ is continuous (this a consequence of the inequality

$$M(u_k) \leqslant \tfrac{1}{2}M(2u) + \tfrac{1}{2}M(2(u_k - u)),$$

where $u_k \to u$ in $E_M(\Omega)$).

(c) If $\int_\Omega M(u)$ remains bounded, then u remains bounded in $L_M(\Omega)$ (this is a consequence of the definition of the Luxemburg norm and of the inequality $M(u/t) \leqslant M(u)/t$ for $t \geqslant 1$).

(d) If $u \in E_M(\Omega)$ and has compact support in Ω, then the regularized function $u_\varepsilon \to u$ in $E_M(\Omega)$ (cf. [3]).

(e) Poincaré's inequality: $\|u\|_{L_M(\Omega)} + \||\nabla u|\|_{L_M(\Omega)}$ and $\||\nabla u|\|_{L_M(\Omega)}$ are equivalent norms in $W_0^1 L_M(\Omega)$ (cf. [6, corollary 5.8]).

Let us also recall the imbedding theorem of [3]. If

$$\int_1^\infty \frac{M^{-1}(t)}{t^{1+1/N}} = +\infty,$$

then $W_0^1 L_M(\Omega) \subset L_{M^*}(\Omega)$ continuously. Moreover, $W_0^1 L_M(\Omega) \subset E_P(\Omega)$, with compact imbedding for any N-function P with $P \ll M^*$. Here, $M^{-1} : \mathbb{R}^+ \to \mathbb{R}^+$ denotes the inverse of M and the N-function M^* is the so-called Sobolev conjugate of M. This function is defined by

$$(M^*)^{-1}(t) = \int_0^t \frac{M^{-1}(t)}{t^{1+1/N}},$$

where it is assumed, without loss of generality, that

$$\int_0^1 \frac{M^{-1}(t)}{t^{1+1/N}} < +\infty.$$

If

$$\int_1^\infty \frac{M^{-1}(t)}{t^{1+1/N}} < +\infty,$$

then $W_0^1 L_M(\Omega) \subset C(\bar{\Omega})$ with compact imbedding.

Note that, by lemma 4.14 in [6] (see also proposition 2.1 in [5]), $M \ll M^*$ and consequently $W_0^1 L_M(\Omega) \subset E_M(\Omega)$ with compact imbedding, a property which will be used repeatedly below.

4. Application to problem (1.3)

In this section we study the eigenvalue problem (1.3). It is assumed throughout that m is as in § 1, that the bounded open set Ω has the segment property and that $\rho \in L^\infty(\Omega)$, with $\rho^+ \not\equiv 0$.

By a solution u to (1.3) we mean a pair (λ, u), with $\lambda \in \mathbb{R}$ and $u \in W_0^1 L_M(\Omega)$, such that $m(|\nabla u|) \in L_{\bar{M}}(\Omega)$, which satisfies

$$\int_\Omega m(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla v = \lambda \int_\Omega \rho m(u) v \tag{4.1}$$

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for all $v \in W_0^1 L_M(\Omega)$. Note that the integral on the left-hand side makes sense since $m(|\nabla u|)$ is required to lie in $L_{\bar{M}}(\Omega)$. The integral on the right-hand side also makes sense by property (a) above.

For simplicity of notation, let us denote the complementary system

$$(W_0^1 L_M(\Omega), W_0^1 E_M(\Omega); W^{-1} L_{\bar{M}}(\Omega), W^{-1} E_{\bar{M}}(\Omega))$$

by $(Y, Y_0; Z, Z_0)$.

Let us define

$$f(u) := \int_\Omega M(|\nabla u|) \tag{4.2}$$

and

$$g(u) := \int_\Omega \rho M(u). \tag{4.3}$$

As in [5,10], our approach to problem (4.1) consists of minimizing $f(u)$ on Y under the constraint $g(u) = \mu$, $\mu > 0$, given.

We have that the function f takes values in $\mathbb{R} \cup \{+\infty\}$ and is clearly convex on Y, with $\operatorname{dom} f = \{u \in Y : |\nabla u| \in \mathcal{L}_M(\Omega)\}$. Since $Y \subset E_M(\Omega) \subset \mathcal{L}_M(\Omega)$, the function g is real valued on Y.

THEOREM 4.1. *For any $\mu > 0$, there exists $u_0 \in \operatorname{dom} f$, which minimizes f on Y under the constraint $g(u) = \mu$.*

Proof. We first show that our minimizing problem is feasible, i.e. that

$$\exists u \in \operatorname{dom} f \text{ with } g(u) = \mu. \tag{4.4}$$

This will clearly follow from property (b) if we show that there exists $v \in \mathcal{D}(\Omega)$ with $g(v) < \mu$ and $w \in \mathcal{D}(\Omega)$ with $g(w) \geqslant \mu$.

Clearly, $v \equiv 0$ satisfies $g(v) < \mu$. Let us show how to construct w. Take $K \subset \Omega$ with $\operatorname{cl} K \subset \Omega$, K of positive measure and $\rho \geqslant 0$ on K but not identically zero on K. Next let $r > 0$ be such that

$$\int_\Omega \rho M(r 1_K) > \mu,$$

where 1_K denotes the characteristic function of K. Since the regularized function $(r 1_K)_\varepsilon$ converges to $r 1_K$ in $E_M(\Omega)$ (cf. property (d)), by using property (b), the existence of w follows.

The rest of the proof can now be adapted from [5, 10]. We will, however, sketch a slightly different argument, which will turn out to be useful when considering the more general problem (1.5).

Since the minimization problem is feasible and since Y is the dual of Z_0, the conclusion of theorem 4.1 will clearly follow from the following three facts: (i) f is $\sigma(Y, Z_0)$ sequentially lower semicontinuous on Y; (ii) g is sequentially continuous on Y with respect to the $\sigma(Y, Z_0)$ topology; and (iii) any minimizing sequence is bounded in Y.

Fact (i) can be seen as a consequence of a well-known result from the calculus of variations, which says that if a Carathéodory function h on $\Omega \times \mathbb{R}^k \times \mathbb{R}^\ell$ is non-negative and convex with respect to its last variable, then $\int_\Omega h(x, u(x), v(x))$ is

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sequentially lower semicontinuous with respect to the L^1 convergence of u and the $\sigma(L^1, L^\infty)$ convergence of v (cf. theorem 3.4 in [2]). Fact (ii) follows from the compact imbedding $Y \subset E_M(\Omega)$ and property (b). Fact (iii) follows from properties (c) and (e). This ends the proof of the theorem. □

Theorem 4.2. *Let $\mu > 0$ and let $u_0 \in \operatorname{dom} f$ be a minimizer of f on Y under the constraint $g(u) = \mu$. Then $m(|\nabla u_0|) \in L_{\bar{M}}(\Omega)$ and u_0 solves (4.1) for some $\lambda \in \mathbb{R}$.*

As indicated in § 1, theorem 4.2, with $\rho \equiv 1$, was derived in [10] by some ingenious calculations based, among other things, on the implicit function theorem. We show next that theorem 4.2 can be obtained by applying the Lagrange multiplier rule stated in corollary 2.2. Theorem 4.2 (with $\rho \equiv 1$) was also obtained in [4] and [5] (using different approaches) under some additional assumptions on m (M and $\bar{M}$ satisfy a Δ_2 condition at infinity).

Proof of theorem 4.2. First we show that

$$g : Y \to \mathbb{R} \text{ is } C^1, \tag{4.5}$$

with

$$\langle g'(u), v\rangle = \int_\Omega \rho m(u) v \quad \text{for } u, v \in Y.$$

Indeed, by the mean-value theorem,

$$\frac{g(u + tv) - g(u)}{t} = \int_\Omega \rho m(u + \theta v) v$$

for some $\theta = \theta(u, v, t, x)$, with $0 < |\theta| < |t|$, where x is the integration variable. Letting $t \to 0$, the conclusion (4.5) then follows from the imbedding theorem and property (a) (where one can take, for instance, $Q = \bar{M}^*$ when M^* is defined, or Q equal to any N-function with $Q \ll \bar{M}$ when M^* is not defined).

Let us now verify that condition (2.7) from corollary 2.2 holds in our situation. We note that proving the first part of (2.7) (the second part is proved similarly) amounts to showing the existence of $u_1 \in \operatorname{dom} f$ such that

$$\int_\Omega \rho m(u_0) u_1 > \int_\Omega \rho m(u_0) u_0. \tag{4.6}$$

We have that $\rho m(u_0) \not\equiv 0$ (otherwise, $\rho M(u_0) \equiv 0$, which is impossible since $\mu > 0$). So we can take $K \subset \Omega$, with $\operatorname{cl} K \subset \Omega$ and $\operatorname{meas}(K) > 0$, so that $\rho m(u_0)$ is $\not\equiv 0$ and of one sign on K. For a suitable $r \in \mathbb{R}$, we then have

$$\int_\Omega \rho m(u_0) r 1_K > \int_\Omega \rho m(u_0) u_0.$$

Since, by property (d), the regularized function $(r1_K)_\varepsilon$ converges to $r1_K$ in $E_M(\Omega)$, and since, by property (a), $\rho m(u_0) \in L_{\bar{M}}(\Omega)$, the existence of $u_1 \in \mathcal{D}(\Omega)$ satisfying (4.6) follows from the preceding inequality.

The feasibility of our minimizing problem and the lower semicontinuity of f were already verified during the proof of theorem 4.1. We are thus in a position to apply

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corollary 2.2. This yields the existence of $\lambda \in \mathbb{R}$ such that $\lambda g'(u_0) \in \partial f(u_0)$, i.e.

$$\int_\Omega M(|\nabla v|) \geqslant \int_\Omega M(|\nabla u_0|) + \lambda \int_\Omega \rho m(u_0)(v - u_0) \tag{4.7}$$

for all $v \in Y$. Now we claim that (4.7) implies

$$m(|\nabla u_0|) \in L_{\bar{M}}(\Omega). \tag{4.8}$$

Indeed, replacing v by $u_0 + \varepsilon v$ in (4.7), $0 < \varepsilon < 1$, we obtain

$$\frac{1}{\varepsilon} \int_\Omega (M(|\nabla(u_0 + \varepsilon v)|) - M(|\nabla u_0|)) \geqslant \lambda \int_\Omega \rho m(u_0) v$$

and, by the mean-value theorem,

$$\int_\Omega m(|\nabla(u_0 + \theta v)|) \frac{\nabla(u_0 + \theta v)}{|\nabla(u_0 + \theta v)|} \nabla v \geqslant \lambda \int_\Omega \rho m(u_0) v \tag{4.9}$$

for some $\theta = \theta(u_0, v, \varepsilon, x)$ with $0 < \theta < \varepsilon$. Taking $v = -u_0$ in (4.9) gives

$$\int_\Omega m((1 - \theta)|\nabla u_0|)|\nabla u_0| \leqslant \lambda \int_\Omega \rho m(u_0) u_0.$$

Next, let $\varepsilon \to 0$ and apply the Fatou lemma to find

$$\int_\Omega m(|\nabla u_0|)|\nabla u_0| \leqslant \lambda \int_\Omega \rho m(u_0) u_0 < +\infty,$$

which implies our claim (4.8) (since $\bar{M}(m(t)) = tm(t) - M(t) \leqslant tm(t)$).

We now return to (4.9) and take $v = w - u_0$ with $w \in Y_0$. Observe that, since $0 < \theta < 1$,

$$m(|\nabla(u_0 + \theta(w - u_0))|) \leqslant m((1 - \theta)|\nabla u_0| + \theta|\nabla w|) \leqslant m(|\nabla u_0|) + m(|\nabla w|),$$

where, by (4.8), the choice of w and property (a), we have that the right-hand side belongs to $L_{\bar{M}}(\Omega)$. It follows that we can apply Lebesgue's theorem in the $v = w - u_0$ version of (4.9) when $\varepsilon \to 0$. In this way, We obtain

$$\int_\Omega m(|\nabla u_0|) \frac{\nabla u_0}{|\nabla u_0|} \nabla(w - u_0) \geqslant \lambda \int_\Omega \rho m(u_0)(w - u_0), \tag{4.10}$$

which holds for all $w \in Y_0$. Since $(Y, Y_0; Z, Z_0)$ is a complementary system, any $w \in Y$ can be approximated in the $\sigma(Y, Z)$ sense by elements in Y_0 (cf. [6, § 1]). Consequently, using (4.8) again, equation (4.10) holds for all $w \in Y$. The conclusion (4.1) now follows by replacing w in (4.10) by $u_0 \pm v$ with $v \in Y$. □

Remark 4.3. Since f and g above are invariant by replacing u by $|u|$, the minimizer in theorem 4.1 can be taken $\geqslant 0$. The eigenvalue λ provided by theorem 4.2 is thus a principal eigenvalue (i.e. corresponds to a non-negative eigenvector).

Remark 4.4. Taking u_0 as a testing function in (4.1), one deduces that

$$\lambda \int_\Omega \rho m(u_0) u_0 > 0,$$

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so that λ is $\neq 0$ and has the same sign as $\int_\Omega \rho m(u_0)u_0$. The lack of homogeneity and the presence of the possibly indefinite weight ρ prevent the comparison of $\int_\Omega \rho m(u_0)u_0$ with the constraint $\int_\Omega \rho M(u_0)$. In particular, it is not clear whether the eigenvalue λ constructed in the proof of theorem 4.2 is always greater than 0. (This is clearly the case when $m(t) = |t|^{p-1}$ or when $\rho \geqslant 0$.)

REMARK 4.5. It was proved recently in [11] that problem (1.2) admits an infinite sequence of eigenvalues going to $+\infty$. The argument in [11] involves Galerkin approximations and Ljusternik–Schnirelman theory. It would be interesting to see whether problem (1.3) with an indefinite weight admits a double sequence of eigenvalues going to $+\infty$ and to $-\infty$. A difficulty of the same type as that pointed out in remark 4.4 may appear in this respect.

5. Application to problem (1.4)

The results of theorems 4.1 and 4.2 can be extended to problem (1.4), with the same kind of proofs, as we will see in this section.

We will assume that m and Ω are as before and that b is a Carathéodory function that satisfies some suitable growth condition related to the N-function M. When the Sobolev conjugate M^* is defined, this growth condition reads

$$|b(x,t)| \leqslant a(x) + c\bar{P}^{-1}P(ct) \tag{5.1}$$

for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$, where P is an N-function with $P \ll M^*$, $0 \leqslant a(x) \in L_{\bar{P}}(\Omega)$, and $c \geqslant 0$ is a constant. When the Sobolev conjugate M^* is not defined (i.e. when $Y := W_0^1 L_M(\Omega)$ is imbedded into $C(\bar{\Omega})$), the growth condition on $b(x,t)$ reads

$$|b(x,t)| \leqslant d(x)e(t) \tag{5.2}$$

for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$, where $0 \leqslant d(x) \in L^1(\Omega)$ and $e : \mathbb{R} \to \mathbb{R}^+$ is continuous.

Note that, by the imbedding theorem and property (b), equation (5.1) implies that $b(\cdot, u(\cdot)) \in L_{\bar{P}}(\Omega) \subset L_{\bar{M}^*}(\Omega)$ for any $u \in Y$. Similarly, equation (5.2) implies that $b(\cdot, u(\cdot)) \in L^1(\Omega)$ for any $u \in Y$. Note also that (5.1) implies that the primitive

$$B(x,t) := \int_0^t b(x,s)\,\mathrm{d}s$$

satisfies

$$|B(x,t)| \leqslant a(x)|t| + c\bar{P}^{-1}(P(ct))|t| \tag{5.3}$$

for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$ (because $\bar{P}^{-1} \circ P$ is increasing). On the other hand, equation (5.2) implies, for $B(x,t)$, an estimate similar to (5.2) (with another continuous function e). So, in any case, M^* being defined or not, we deduce from the imbedding theorem and property (b) that $B(\cdot, u(\cdot)) \in L^1(\Omega)$ for any $u \in Y$.

By a solution to (1.4), we mean a pair (λ, u), with $\lambda \in \mathbb{R}$ and $u \in W_0^1 L_M(\Omega)$, such that $m(|\nabla u|) \in L_{\bar{M}}(\Omega)$ and

$$\int_\Omega m(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla v = \lambda \int_\Omega b(x,u)v \tag{5.4}$$

for all $v \in W_0^1 L_M(\Omega)$. Note that, by the preceding observations, the integral on the right-hand side makes sense.

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As in § 4, our approach to problem (5.4) consists of minimizing

$$f(u) := \int_{\Omega} M(|\nabla u|)$$

on Y under the constraint

$$g(u) := \int_{\Omega} B(x, u) = \mu$$

for some $\mu \in \mathbb{R}$. We will assume that this minimizing problem is feasible, i.e. that μ is such that

$$\exists u \in Y \text{ with } f(u) < +\infty \text{ and } g(u) = \mu. \tag{5.5}$$

THEOREM 5.1. *Let $b(x, t)$ satisfy either (5.1) or (5.2), depending on whether M^* is defined or not. Assume the feasibility condition (5.5). Then the problem of minimizing $f(u)$ on Y under the constraint $g(u) = \mu$ has at least one solution.*

THEOREM 5.2. *Assume, as above, conditions (5.1) or (5.2), and (5.5). If u_0 is a minimizer of $f(u)$ on Y under the constraint $g(u) = \mu$ and if*

$$b(x, u_0) \not\equiv 0, \tag{5.6}$$

then $m(|\nabla u_0|) \in L_{\bar{M}}(\Omega)$ and u_0 solves (5.4) for some $\lambda \in \mathbb{R}$.

Examples will be given at the end of this section that illustrate the role of assumptions (5.5) and (5.6), as well as that of the parameter μ.

Some continuity properties of $b(x, u)$ and $B(x, u)$ will be needed in the proofs of theorems 5.1 and 5.2.

LEMMA 5.3. *If (5.1) holds, then the mapping $u \to b(\cdot, u(\cdot))$ (respectively, $u \to B(\cdot, u(\cdot))$) is sequentially continuous from Y endowed with $\sigma(Y, Z_0)$ into $L_{\bar{M}^*}(\Omega)$) (respectively, $L^1(\Omega)$). If (5.2) holds, then the mappings $u \to b(\cdot, u(\cdot))$ and $u \to B(\cdot, u(\cdot))$ are sequentially continuous from Y endowed with $\sigma(Y, Z_0)$ into $L^1(\Omega)$.*

Proof. We will only deal with the mapping $u \to B(\cdot, u(\cdot))$ under (5.1). The other parts of the lemma can be derived by similar arguments. Let $u_k \to u$ with respect to Y, $\sigma(Y, Z_0)$. The imbedding theorem implies that, for a subsequence, $u_k \to u$ in $E_P(\Omega)$ and a.e. in Ω. Consequently, by property (b), $P(u_k) \to P(u)$ and $P(cu_k) \to P(cu)$ in $L^1(\Omega)$. Here, P is the N-function and c is the constant appearing in (5.3). It follows that there exists $v, w \in L^1(\Omega)$ such that, for a further subsequence, $P(u_k) \leqslant v$ and $P(cu_k) \leqslant w$ a.e. in Ω. Inequality (5.3) then implies

$$|B(x, u_k)| \leqslant a(x)P^{-1}(v) + c\bar{P}^{-1}(w)P^{-1}(v), \tag{5.7}$$

where $P^{-1}(v) \in \mathcal{L}_P(\Omega) \subset L_P(\Omega)$ and $\bar{P}^{-1}(w) \in \mathcal{L}_{\bar{P}}(\Omega) \subset L_{\bar{P}}(\Omega)$. The conclusion that $B(x, u_k) \to B(x, u)$ in $L^1(\Omega)$ now follows from Lebesgue's theorem. □

Proof of theorem 5.1. The existence of a minimizer follows from the three facts (i), (ii) and (iii) appearing in the proof of theorem 4.1. Facts (i) and (iii) are verified exactly as before, and fact (ii) is now part of lemma 5.3. □

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Proof of theorem 5.2. In order to apply corollary 2.2, we have to see that $g : Y \to \mathbb{R}$ is C^1 and that condition (2.7) holds. The fact that g is C^1, with

$$\langle g'(u), v\rangle = \int_\Omega b(x,u)v \quad \text{for } u, v \in Y, \tag{5.8}$$

follows by applying the mean-value theorem as in the proof of theorem 4.2 (one should simply rely here on lemma 5.3 instead of property (a)). The verification of the first part of (2.7) (the second part is proved similarly) amounts to showing the existence of $u_1 \in \operatorname{dom} f$ such that

$$\int_\Omega b(x,u_0)u_1 > \int_\Omega b(x,u_0)u_0. \tag{5.9}$$

It is here that assumption (5.6) is used, in order to adapt the argument by regularization from the proof of theorem 4.2.

Corollary 2.2 thus provides the existence of $\lambda \in \mathbb{R}$ such that

$$\int_\Omega M(|\nabla v|) \geqslant \int_\Omega M(|\nabla u_0|) + \lambda \int_\Omega b(x,u_0)(v-u_0) \tag{5.10}$$

for all $v \in Y$. The rest of the proof of theorem 4.2 then carries over to the present situation without any change. □

Example 5.4. Let

$$b(x,t) = \rho(x)p(t),$$

where $\rho \in L^\infty(\Omega)$, $\rho^+ \not\equiv 0$, $p : \mathbb{R} \to \mathbb{R}$ is a function like the function m from the introduction, with its primitive $P(t) := \int_0^t p(s)\,\mathrm{d}s$ verifying $P \ll M^*$. (We are assuming here that we are in the case where M^* is defined; if M^* is not defined, then no growth condition on P is needed.) The feasibility condition (5.5) then holds for any $\mu \geqslant 0$. This can be verified by an argument of regularization as in the proof of theorem 4.1. The assumption (5.6) is fulfilled for any $\mu \neq 0$. This can be verified as in the proof of theorem 4.2. Theorems 5.1 and 5.2 thus apply for any $\mu > 0$. Note that the problem of the present example with $\rho \equiv 1$ and M, $\bar{M}$ verifying the Δ_2 condition at infinity was considered in [5].

Example 5.5. Let $b(x,t)$ satisfy (5.1) (or (5.2)) and assume that for some $\mu_0 > 0$,

$$\exists E \subset \Omega, \text{ with } \operatorname{cl} E \subset \Omega \text{ and } \operatorname{meas}(E) > 0, \text{ and } \exists s \subset \mathbb{R}, \text{ with } \int_E B(x,s) > \mu_0. \tag{5.11}$$

The feasibility condition (5.5) is then satisfied for any μ with $0 \leqslant \mu \leqslant \mu_0$. Indeed, taking such a μ, equation (5.5) will follow from lemma 5.3 if we show the existence of $v \in \mathcal{D}(\Omega)$ with $g(v) := \int_\Omega B(x,v) \leqslant \mu$ and of $w \in \mathcal{D}(\Omega)$ with $g(v) \geqslant \mu$. One can take $v \equiv 0$ and construct w as in the proof of theorem 4.1 by regularization of the function $s1_E$, where s and E are given by (5.11). The fact that $B(x,(s1_E)_\epsilon) \to B(x,s1_E)$ in $L^1(\Omega)$ is used here and follows by an easy adaptation of the proof of lemma 5.3. Let us now look at assumption (5.6). If we assume that, for a.e. $x \in \Omega$,

$$\left.\begin{array}{c} |b(x,t)| \text{ is non-decreasing (respectively, non-increasing)} \\ \text{for } t \geqslant 0 \text{ (respectively, } t \leqslant 0), \end{array}\right\} \tag{5.12}$$

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then $|B(x,t)| \leqslant |b(x,t)||t|$ for a.e. x and all t, and we deduce that assumption (5.6) is fulfilled if $\mu \neq 0$. Consequently, under (5.11) and (5.12), theorems 5.1 and 5.2 apply for any μ with $0 < \mu \leqslant \mu_0$.

REMARK 5.6. Assumptions (5.5) and (5.6) in theorems 5.1 and 5.2 are usually connected one with the other, as is seen in the preceding two examples.

6. Application to problem (1.5)

In this section we will see that the preceding approach can also be applied to problem (1.5). As before, Ω is a bounded open subset of $\mathbb{R}^N$ with the segment property.

Higher-order Orlicz–Sobolev spaces can be defined in a way similar to the first-order spaces considered in § 3. We will only recall here some notation related to the imbedding theorem of [3].

Let C_0 be an N-function and suppose first the dimension $N \geqslant 2$. Changing the values of C_0 on a bounded subset of $\mathbb{R}$, one can assume

$$\int_0^1 \frac{C_0^{-1}(t)}{t^{1+1/N}}\,\mathrm{d}t < +\infty.$$

If

$$\int_1^\infty \frac{C_0^{-1}(t)}{t^{1+1/N}}\,\mathrm{d}t = +\infty,$$

define a new N-function C_1 by

$$C_1^{-1}(s) := \int_0^s \frac{C_0^{-1}(t)}{t^{1+1/N}}\,\mathrm{d}t.$$

Repeating this process, one obtains a finite sequence of N-functions $C_0, C_1, \dots, C_q$, where $q = q(C_0)$ is such that

$$\int_1^\infty \frac{C_{q-1}^{-1}(t)}{t^{1+1/N}}\,\mathrm{d}t = +\infty,$$

but

$$\int_1^\infty \frac{C_q^{-1}(t)}{t^{1+1/N}}\,\mathrm{d}t < +\infty.$$

If $N = 1$, we put $q(C_0) = 0$.

The imbedding theorem then says the following. Let Ω be a bounded open subset of $\mathbb{R}^N$ and let M be an N-function. For $n - q(M) \leqslant |\alpha| \leqslant n$, write $M_\alpha = C_{n-|\alpha|}$, starting as above with $C_0 = M$. Then (i) for $n - q(M) \leqslant |\alpha| < n$, the mapping $u \in W_0^n L_M(\Omega) \to D^\alpha u \in L_{M_\alpha}(\Omega)$ is continuous and the mapping $u \in W_0^n L_M(\Omega) \to D^\alpha u \in E_P(\Omega)$ is compact if $P \ll M_\alpha$; (ii) for $|\alpha| < n - q(M)$, the mapping $u \in W_0^n L_M(\Omega) \to D^\alpha u \in C(\bar{\Omega})$ is compact.

Some more notation will be needed. We write

$$\xi = (\xi_\alpha)_{|\alpha| \leqslant n} \in \mathbb{R}^{N_n} \quad \text{and} \quad \eta = (\eta_\beta)_{|\beta| \leqslant n-1} \in \mathbb{R}^{N_{n-1}},$$

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where N_n (respectively, N_{n-1}) is the number of partial derivatives of order less than or equal to n (respectively, $n-1$) for a function on $\mathbb{R}^N$. For such a function u on $\mathbb{R}^N$, $\xi(u)$ (respectively, $\eta(u)$) denotes $(D^\alpha u)_{|\alpha|\leqslant n}$ (respectively, $(D^\beta u)_{|\beta|\leqslant n-1}$).

The following conditions will be imposed on the coefficients A_α and B_β of the differential operators $\mathcal{A}$, B in (1.5).

(A_1) $A_\alpha(x,\xi)$ and $B_\beta(x,\eta)$ are Carathéodory functions on $\Omega\times\mathbb{R}^{N_n}$ and $\Omega\times\mathbb{R}^{N_{n-1}}$, respectively.

(A_2) There exist an N-function M, N-functions M_α for $|\alpha|<n-q(M)$ (note that, for $n-q(M)\leqslant|\alpha|\leqslant n$, the N-functions M_α that appear below are provided by the imbedding theorem), $a_\alpha\in E_{\bar{M}_\alpha}$ for $|\alpha|\leqslant n$ and a constant c such that, for a.e. $x\in\Omega$, all $\xi\in\mathbb{R}^{N_n}$ and all $|\alpha|\leqslant n$,

$$|A_\alpha(x,\xi)|\leqslant a_\alpha(x)+c\sum_{|\gamma|\leqslant n}\bar{M}_\alpha^{-1}M_\gamma(c\xi_\gamma).$$

(A_3) There exist N-functions P_β with $P_\beta\ll M_\beta$ for $n-q(M)\leqslant|\beta|\leqslant n-1$ (the N-functions M and M_β are those provided by (A_2)), $d_\beta\in E_{\bar{P}_\beta}(\Omega)$, $d\in L^1(\Omega)$, $e\in C(\mathbb{R}^{N_{n-q(M)-1}})$ and a constant c such that, for a.e. $x\in\Omega$ and all $\eta\in\mathbb{R}^{N_{n-1}}$ with components in $\mathbb{R}^{N_{n-q(M)-1}}$ denoted by $\tilde{\eta}$, if $n-q(M)\leqslant|\beta|\leqslant n-1$,

$$|B_\beta(x,\eta)|\leqslant e(\tilde{\eta})\biggl[d_\beta(x)+c\sum_{n-q(M)\leqslant|\gamma|\leqslant n-1}\bar{P}_\beta^{-1}P_\gamma(c\eta_\gamma)\biggr],$$

and if $|\beta|<n-q(M)$,

$$|B_\beta(x,\eta)|\leqslant e(\tilde{\eta})\biggl[d(x)+c\sum_{n-q(M)\leqslant|\gamma|\leqslant n-1}P_\gamma(c\eta_\gamma)\biggr].$$

(A_4) There exist Carathéodory functions $A(x,\xi)$ on $\Omega\times\mathbb{R}^{N_n}$ and $B(x,\eta)$ on $\Omega\times\mathbb{R}^{N_{n-1}}$, derivable with respect to the components of ξ, η, such that

$$A_\alpha(x,\xi)=\frac{\partial A}{\partial\xi_\alpha}(x,\xi)\quad\text{and}\quad B_\beta(x,\eta)=\frac{\partial B}{\partial\eta_\beta}(x,\eta)$$

for a.e. $x\in\Omega$ and all $\xi\in\mathbb{R}^{N_n}$, $\eta\in\mathbb{R}^{N_{n-1}}$, $|\alpha|\leqslant n$ and $|\beta|\leqslant n-1$.

(A_5) For a.e. $x\in\Omega$ and all ξ, $\xi'\in\mathbb{R}^{N_n}$,

$$\sum_{|\alpha|\leqslant n}(A_\alpha(x,\xi)-A_\alpha(x,\xi'))(\xi_\alpha-\xi'_\alpha)\geqslant 0.$$

(A_6) There exist functions $b_\alpha\in E_{\bar{M}_\alpha}(\Omega)$ for $|\alpha|\leqslant n$, $b\in L^1(\Omega)$ and positive constants d_1, d_2 such that

$$\sum_{|\alpha|\leqslant n}A_\alpha(x,\xi)\xi_\alpha\geqslant d_1\sum_{|\alpha|=n}M(d_2\xi_\alpha)-\sum_{|\alpha|\leqslant n}b_\alpha(x)\xi_\alpha-b(x)$$

for a.e. $x\in\Omega$ and all $\xi\in\mathbb{R}^{N_n}$.

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Here are some comments on the above conditions. (A_2) and (A_3) are standard growth conditions in the study of quasilinear elliptic problems in Orlicz–Sobolev spaces, while (A_6) is the corresponding coercivity condition (cf., for example, [6,7]). The monotonicity condition (A_5) is equivalent to the requirement that $A(x,\xi)$ is convex with respect to ξ for a.e. $x \in \Omega$. Note here that we can assume, without loss of generality, $A(x,0) \equiv B(x,0) \equiv 0$, so that

$$A(x,\xi) = \int_0^1 \sum_{|\alpha|\leqslant n} A_\alpha(x,s\xi)\xi_\alpha \,\mathrm{d}s, \tag{6.1}$$

$$B(x,\eta) = \int_0^1 \sum_{|\beta|\leqslant n-1} B_\beta(x,s\eta)\eta_\beta \,\mathrm{d}s. \tag{6.2}$$

It then follows from conditions (A_2), (A_3) and equations (6.1), (6.2) that $A(x,\xi)$ and $B(x,\eta)$ satisfy

$$|A(x,\xi)| \leqslant \sum_{|\alpha|\leqslant n} a_\alpha(x)|\xi_\alpha| + c \sum_{\substack{|\gamma|\leqslant n\\|\alpha|\leqslant n}} \bar{M}_\alpha^{-1} M_\gamma(c\xi_\gamma)|\xi_\alpha|, \tag{6.3}$$

$$\begin{aligned}|B(x,\eta)| \leqslant e(\tilde{\eta})\biggl[&\sum_{n-q(M)\leqslant|\beta|\leqslant n-1} d_\beta(x)|\eta_\beta| + \sum_{\substack{n-q(M)\leqslant|\gamma|\leqslant n-1\\ n-q(M)\leqslant|\beta|\leqslant n-1}} \bar{P}_\beta^{-1} P_\gamma(c\eta_\gamma)|\eta_\beta| \\ &+ c \sum_{n-q(M)\leqslant|\gamma|\leqslant n-1} P_\gamma(c\eta_\gamma) + d(x)\biggr]\end{aligned} \tag{6.4}$$

(for another continuous function e on $\mathbb{R}^{N_{n-q(M)-1}}$).

We will work in the framework of the complementary system $(Y, Y_0; Z, Z_0)$, where

$$Y := W_0^n L_M(\Omega), \quad Y_0 := W_0^n E_M(\Omega), \quad Z := W^{-n} L_{\bar{M}}(\Omega), \quad Z_0 := W^{-n} E_{\bar{M}}(\Omega).$$

The following lemma is the analogue in the present setting of lemma 5.3. Its proof involves the same type of arguments as that of lemma 5.3 and we will not detail it.

Lemma 6.1. *Assume (A_1) and (A_3). Then, for each β with $n - q(M) \leqslant |\beta| \leqslant n-1$ (respectively, $|\beta| < n - q(M)$), the mapping $u \to B_\beta(\cdot,\eta(u(\cdot)))$ is sequentially continuous from Y endowed with $\sigma(Y, Z_0)$ into $L_{\bar{M}_\beta}(\Omega)$ (respectively, $L^1(\Omega)$). Assume (A_1), (A_3) and (A_4). Then the mapping $u \to B(\cdot,\eta(u(\cdot)))$ is sequentially continuous from Y endowed with $\sigma(Y, Z_0)$ into $L^1(\Omega)$.*

By a solution to (1.5), we mean a pair (λ, u), with $\lambda \in \mathbb{R}$ and $u \in W_0^n L_M(\Omega)$, such that $A_\alpha(\cdot,\xi(u(\cdot))) \in L_{\bar{M}_\alpha}(\Omega)$ for $|\alpha| \leqslant n$ and

$$\int_\Omega \sum_{|\alpha|\leqslant n} A_\alpha(x,\xi(u))D^\alpha v = \lambda \int_\Omega \sum_{|\beta|\leqslant n-1} B_\beta(x,\eta(u))D^\beta v \tag{6.5}$$

for all $v \in W_0^n L_M(\Omega)$. Note that, by the imbedding theorem and lemma 6.1, the integral on the right-hand side is well defined. The integral on the left-hand side is also well defined by the requirement that $A_\alpha(\cdot,\xi(u(\cdot)))$ belongs to $L_{\bar{M}_\alpha}(\Omega)$.

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As in the previous sections, our approach to problem (6.5) consists of minimizing

$$f(u) := \int_\Omega A(x, \xi(u))$$

on Y under the constraint

$$g(u) := \int_\Omega B(x, \eta(u)) = \mu$$

for some $\mu \in \mathbb{R}$. Note that, by the convexity of A,

$$A(x,\xi) \geqslant \sum_{|\alpha|\leqslant n} A_\alpha(x,0)\xi_\alpha, \tag{6.6}$$

which implies that the convex functional $f(u)$ is well defined on Y, with values in $\mathbb{R} \cup \{+\infty\}$.

We will also assume that our minimizing problem is feasible, i.e. that μ is such that

$$\exists u \in Y \text{ with } f(u) < +\infty \text{ and } g(u) = \mu. \tag{6.7}$$

THEOREM 6.2. *Assume (A_1)–(A_6) and (6.7). Then the problem of minimizing $f(u)$ on Y under the constraint $g(u) = \mu$ has at least one solution.*

THEOREM 6.3. *Assume again (A_1)–(A_6) and (6.7). If u_0 is a minimizer of f on Y under the constraint $g(u) = \mu$ and if*

$$\sum_{|\beta|\leqslant n-1} (-1)^{|\beta|} D^\beta B_\beta(x, \eta(u_0)) \not\equiv 0 \quad \textit{in } \mathcal{D}'(\Omega), \tag{6.8}$$

then $A_\alpha(\cdot, \xi(u_0(\cdot))) \in L_{\bar{M}_\alpha}(\Omega)$ for all $|\alpha| \leqslant m$ and u_0 solves (6.5) for some $\lambda \in \mathbb{R}$.

Proof of theorem 6.2. The existence of a minimizer follows from the three facts appearing in the proof of theorem 4.1. Fact (i) is proved by first applying theorem 3.4 of [2] to $f(u) - \sum_{|\alpha|\leqslant n} A_\alpha(x,0)D^\alpha u$ (which is greater than or equal to 0 by (6.6)) and then observing that, by (A_2), $\sum_{|\alpha|\leqslant n} A_\alpha(x,0)D^\alpha u$ is sequentially continuous on Y endowed with $\sigma(Y, Z_0)$. Fact (ii) is part of lemma 6.1. It remains to verify fact (iii), i.e. that any minimizing sequence u_k is bounded in Y.

Let $u_k \in Y$ be a sequence such that $f(u_k) \leqslant C$. Successively using lemma 6.4 below and (A_6), one deduces

$$d_1 \int_\Omega \sum_{|\alpha|=n} M(\tfrac{1}{2} d_2 D^\alpha u_k) - \int_\Omega \sum_{|\alpha|\leqslant n} (\tfrac{1}{2} b_\alpha - A_\alpha(x,0)) D^\alpha u_k - \int_\Omega b(x) \leqslant C.$$

This implies, by Young's inequality and Poincaré's inequality as given in lemma 5.7 of [6], that, for any $r > 0$,

$$\begin{aligned} d_1 \int_\Omega \sum_{|\alpha|=n} M(\tfrac{1}{2} d_2 D^\alpha u_k) &\leqslant \tilde{C} + \int_\Omega \sum_{|\alpha|\leqslant n} M_\alpha\left(\frac{D^\alpha u_k}{r}\right) \\ &\leqslant \tilde{C} + \int_\Omega \sum_{|\alpha|=n} M\left(\frac{C_1 D^\alpha u_k}{r}\right), \end{aligned}$$

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where $\tilde{C} = \tilde{C}(r)$ and C_1 is some constant. Choosing r sufficiently large then yields a bound on $\int_\Omega \sum_{|\alpha|=n} M(C_2 D^\alpha u_k)$ for some positive constant C_2. Applying once more Poincaré's inequality gives the conclusion that u_k remains bounded in Y. ☐

LEMMA 6.4. *Assume* (A_1), (A_4) *and* (A_5). *Then*

$$A(x,\xi) \geqslant \sum_{|\alpha|\leqslant n} \tfrac{1}{2} A_\alpha(x, \tfrac{1}{2}\xi)\xi_\alpha + \sum_{|\alpha|\leqslant n} A_\alpha(x,0)\xi_\alpha$$

for a.e. $x \in \Omega$ *and all* $\xi \in \mathbb{R}^{N_n}$.

Proof. Write

$$A(x,\xi) = \int_0^1 \sum_{|\alpha|\leqslant n} (A_\alpha(x,s\xi) - A_\alpha(x,0))\xi_\alpha \,\mathrm{d}s + \int_0^1 \sum_{|\alpha|\leqslant n} A_\alpha(x,0)\xi_\alpha \,\mathrm{d}s$$

and use (A_5). ☐

Proof of theorem 6.3. In order to apply corollary 2.2, we have to see that $g : Y \to \mathbb{R}$ is C^1 and that condition (2.7) holds. The fact that g is C^1 with

$$\langle g'(u), v\rangle = \int_\Omega \sum_{|\beta|\leqslant n-1} B_\beta(x,\eta(u)) D^\beta v$$

for $u, v \in Y$ follows by applying the mean-value theorem (and lemma 6.1), as in the proof of theorems 4.1 and 5.1. The verification of the first part of (2.7) (the second part is proved similarly) amounts to showing the existence of $u_1 \in \operatorname{dom} f$ such that

$$\int_\Omega \sum_{|\beta|\leqslant n-1} B_\beta(x,\eta(u_0)) D^\beta u_1 > \int_\Omega \sum_{|\beta|\leqslant n-1} B_\beta(x,\eta(u_0)) D^\beta u_0. \tag{6.9}$$

It is here that assumption (6.8) enters, in order to find $u_1 \in \mathcal{D}(\Omega)$ satisfying (6.9). Clearly, by (6.3), such a function u_1 belongs to $\operatorname{dom} f$.

Corollary 2.2 then provides the existence of $\lambda \in \mathbb{R}$ such that

$$\int_\Omega A(x,\xi(v)) \geqslant \int_\Omega A(x,\xi(u_0)) + \lambda \int_\Omega \sum_{|\beta|\leqslant n-1} B_\beta(x,\eta(u_0)) D^\beta (v - u_0) \tag{6.10}$$

for all $v \in Y$. We claim that u_0 belongs to the domain of our differential operator $\mathcal{A}$, i.e. that

$$A_\alpha(x,\xi(u_0)) \in L_{\bar{M}_\alpha}(\Omega) \tag{6.11}$$

for all $|\alpha| \leqslant n$ (cf. definition (6.5) of a solution to (1.5)). To prove this claim, one replaces v by $u_0 + \epsilon v$ in (6.10) and uses the mean-value theorem as in the proof of theorem 4.1 to obtain

$$\int_\Omega \sum_{|\alpha|\leqslant n} A_\alpha(x,\xi(u_0+\theta v)) D^\alpha v \geqslant \lambda \int_\Omega \sum_{|\beta|\leqslant n-1} B_\beta(x,\eta(u_0)) D^\beta v \tag{6.12}$$

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for some $\theta = \theta(u_0, v, \varepsilon, x)$, with $0 < \theta < \varepsilon$. One then takes $v = -u_0$ in (6.12) and uses (A_5) and the Fatou lemma as $\varepsilon \to 0$ to get

$$\int_\Omega \sum_{|\alpha| \leqslant n} A_\alpha(x, \xi(u_0)) D^\alpha u_0 \leqslant \lambda \int_\Omega \sum_{|\beta| \leqslant n-1} B_\beta(x, \eta(u_0)) D^\beta u_0 < +\infty. \tag{6.13}$$

The end of the argument to derive (6.11) is now adapted from [6, remark 4.2]. One denotes by $\xi(u_0)_k(x)$ the N_n-uple $[D^\alpha u_0(x)]_k$, where $[D^\alpha u_0(x)]_k = D^\alpha u_0(x)$ if $|D^\alpha u_0(x)| \leqslant k$ and $[D^\alpha u_0(x)]_k = 0$ if $|D^\alpha u_0(x)| > k$. Let $w = (w_\alpha) \in \Pi E_{M_\alpha}$. It follows from ($A_5$) that

$$\begin{aligned}\int_\Omega \sum_{|\alpha| \leqslant n} & A_\alpha(x, \xi(u_0)_k) w_\alpha \\ & \leqslant \int_\Omega \sum_{|\alpha| \leqslant n} A_\alpha(x, \xi(u_0)_k)[D^\alpha u_0]_k \\ & \qquad - \int_\Omega \sum_{|\alpha| \leqslant n} A_\alpha(x, w)[D^\alpha u_0]_k + \int_\Omega \sum_{|\alpha| \leqslant n} A_\alpha(x, w) w_\alpha.\end{aligned}$$

Using (6.13) and (A_5), one sees that the first integral in the right-hand side remains bounded from above independently of k; the second integral in the right-hand side also remains bounded independently of k, and the last one does not depend on k. Consequently, each $A_\alpha(\cdot, \xi(u_0(\cdot))_k)$ remains bounded in $L_{\bar{M}_\alpha}(\Omega)$ endowed with $\sigma(L_{\bar{M}_\alpha}, E_{M_\alpha})$. The Banach–Steinhaus theorem then implies that $A_\alpha(\cdot, \xi(u_0(\cdot))_k)$ remains bounded in $L_{\bar{M}_\alpha}(\Omega)$. Since $A_\alpha(\cdot, \xi(u_0(\cdot))_k) \to A_\alpha(\cdot, \xi(u_0(\cdot)))$ a.e. in Ω, we conclude from theorem 14.6 in [8] that $A_\alpha(\cdot, \xi(u_0(\cdot))) \in L_{\bar{M}_\alpha}(\Omega)$. Our claim (6.11) is thus proved.

We now return to (6.12) and take $v = w - u_0$ with $w \in Y_0$. By (A_5), the corresponding left-hand side,

$$\sum_{|\alpha| \leqslant n} A_\alpha(x, \xi(u_0 + \theta(w - u_0))) D^\alpha(w - u_0),$$

is

$$\text{greater than or equal to } \sum_{|\alpha| \leqslant n} A_\alpha(x, \xi(u_0)) D^\alpha(w - u_0)$$

(which belongs to $L^1(\Omega)$ by (6.11)) and is

$$\text{less than or equal to } \sum_{|\alpha| \leqslant n} A_\alpha(x \xi(u_0)) D^\alpha(w - u_0)$$

(which belongs to $L^1(\Omega)$ by (A_2)). Consequently, Lebesgue's theorem can be applied when $\varepsilon \to 0$, which yields

$$\int_\Omega \sum_{|\alpha| \leqslant n} A_\alpha(x, \xi(u_0)) D^\alpha(w - u_0) \geqslant \lambda \int_\Omega \sum_{|\beta| \leqslant n-1} B_\beta(x, \eta(u_0)) D^\beta(w - u_0)$$

for all $w \in Y_0$. The proof that u_0 solves (6.5) can now be completed as in the proof of theorem 4.2, by first deriving the above relation for all $w \in Y$ and then putting $w = u_0 \pm v$ with v arbitrary in Y. □

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