The Journal of Symbolic Logic
Volume 84, Number 2, June 2019

BEING LOW ALONG A SEQUENCE AND ELSEWHERE

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Abstract. Let an oracle be called low for prefix-free complexity on a set in case access to the oracle improves the prefix-free complexities of the members of the set at most by an additive constant. Let an oracle be called weakly low for prefix-free complexity on a set in case the oracle is low for prefix-free complexity on an infinite subset of the given set. Furthermore, let an oracle be called low and weakly for prefix-free complexity along a sequence in case the oracle is low and weakly low, respectively, for prefix-free complexity on the set of initial segments of the sequence. Our two main results are the following characterizations. An oracle is low for prefix-free complexity if and only if it is low for prefix-free complexity along some sequences if and only if it is low for prefix-free complexity along all sequences. An oracle is weakly low for prefix-free complexity if and only if it is weakly low for prefix-free complexity along some sequence if and only if it is weakly low for prefix-free complexity along almost all sequences. As a tool for proving these results, we show that prefix-free complexity differs from its expected value with respect to an oracle chosen uniformly at random at most by an additive constant, and that similar results hold for related notions such as a priori probability. Furthermore, we demonstrate that on every infinite set almost all oracles are weakly low but are not low for prefix-free complexity, while by Shoenfield absoluteness there is an infinite set on which uncountably many oracles are low for prefix-free complexity. Finally, we obtain no-gap results, introduce weakly low reducibility, or WLK-reducibility for short, and show that all its degrees except the greatest one are countable.

§**1. Introduction.** One of the main goals of algorithmic randomness is to come up with suitable formalizations of the intuitive concept of randomness of an infinite binary sequence. The main approaches to such formalizations is via effective compression, via effective prediction or betting, and via effective variants of null sets. The most relevant and most intensively studied formalization is the notion of a Martin-Löf random sequence, which can be equivalently characterized via all three approaches. In particular, by fundamental classical results of Levin and of Schnorr, a sequence is Martin-Löf random if and only if almost all of its initial segments cannot be compressed to less than their lengths in terms of an appropriate version of Kolmogorov complexity, where these versions most prominently include prefix-free complexity [5, Theorem 6.2.3].

In a fruitful line of research, the concepts and results mentioned in the last paragraph have been studied in a setting where the involved computations have access to additional information in the form of an infinite binary sequence—referred to as oracle—that may help to perform certain computational tasks. In general, one may ask if at all and, if yes, how concepts and results change with respect to various

Received July 15, 2017.
2010 *Mathematics Subject Classification.* 03D28, 03D30, 03D32, 68Q30.
Key words and phrases. Kolmogorov complexity, randomness, lowness.

0022-4812/19/8402-0004
DOI:10.1017/jsl.2018.63

types of oracles. In this direction, various notions of lowness are investigated. In general, a sequence X is low for a certain concept if this concept is essentially the same with and without access to X as an oracle. For example, an oracle X is LOW FOR MARTIN-LÖF RANDOMNESS if the set of Martin-Löf-random sequences is the same with and without access to X, and similarly the notion of LOW FOR $\mathbf{\Omega}$ can be defined as not changing the set of sequences that are left-c.e. and Martin-Löf random. Further lowness notions can be introduced in connection with prefix-free complexity K. Call an oracle LOW FOR **K** ON A SET OF WORDS in case prefix-free complexity with and without the oracle differ on all words in this set at most by an additive constant. Furthermore, an oracle is LOW FOR **K** and is WEAKLY LOW FOR **K** in case the oracle is low for K on the set of all words and on some infinite set of words, respectively. By celebrated results of Nies and of Miller, each of the two latter lowness notions is equivalent to a lowness notions defined in terms of Martin-Löf randomness.

THEOREM 1.1 (Nies [16]). *A sequence X is low for* K *if and only if X is low for Martin-Löf randomness.*

THEOREM 1.2 (Miller [14]). *A sequence X is weakly low for* K *if and only if X is low for* Ω.

With both results, lowness or weak lowness for K is asserted to be equivalent to a lowness property that informally can be stated as

> *for every sequence from a certain set, access to the oracle does not allow to compress the initial segments of the sequence significantly better than without the oracle.*

Here one considers the set of all and of all left-c.e., respectively, sequences that are Martin-Löf random, and for both results, significantly better compression means improving compression to an extent such that the incompressibility condition in the above-mentioned Levin-Schnorr characterization of Martin-Löf randomness becomes false.

We will introduce below the notions of an oracle being LOW and being WEAKLY LOW FOR **K** ALONG $\boldsymbol{A}$, which is defined by the condition that the oracle is low for K on the set of all and on an infinite set of, respectively, initial segments of A. We will then consider the properties that an oracle is low and is weakly low for K along all sequences in a given set. These lowness properties are again described by the informally formulated lowness property above. For example, in case the inability of significantly better compression means that compression is improved at most by an additive constant on all initial segments, this is a description of lowness for K along the considered sequences. In Section 3, similar to the mentioned results of Nies and of Miller, we will investigate into the relations between oracles that are low or weakly low for K, and oracles that are low or weakly low for K along certain sequences. Our main results are that an oracle is low for K if and only if it is low for K along every sequence, and an oracle is weakly low for K if and only if it is weakly low for K along almost all sequences. On the way to proving these equivalences, we obtain in Section 2 results about the expected values of K and related functions with respect to an oracle chosen at random according to Lebesgue measure. In particular, the value $\mathrm{K}(w)$ and its expected value differ at most by an additive constant. Furthermore, we demonstrate in Section 4 that on every infinite set almost all oracles are weakly low but are not low for K, while by Shoenfield absoluteness there is an infinite set on which uncountably many oracles are low for K. In Section 5, we obtain no-gap results. In Section 6, we introduce weakly low reducibility, or WLK-reducibility for short, and show that all its degrees except the greatest one are countable.

1.1. Notation. We assume the reader to be familiar with basic concepts of algorithmic randomness, for details and further explanation see the monographs by Downey and Hirschfeldt [5], Li and Vitányi [12], and Nies [17]. The terms word and sequence refer to finite and infinite, respectively, binary sequences. Recall that Lebesgue measure is the uniform measure on the set of all infinite binary sequences, which can be obtained by determining the bits of the sequence by independent tosses of a fair coin.

§2. Prefix-free complexity with respect to oracles chosen uniformly at random.

In this section, we consider prefix-free complexity relativized to an oracle that is chosen at random according to Lebesgue measure. We show that in this setting for any given word its expected (relativized) and its unrelativized prefix-free complexity differ at most by an additive constant. Similar results hold for related values, e.g., the expected and the unrelativized apriori probability of a word differ at most by a constant factor. We treat these results in more depth and detail than required for our purposes since we suppose that they have interest in their own and will be applicable elsewhere. For a start, we review some standard notation.

Definition 2.1. For an oracle Turing machine M, the Kolmogorov complexity of a word w with respect to oracle X is

$$\mathrm{C}_M^X(w) = \min\{|p|\colon M^X(p) = w\}, \text{ and we write } \mathrm{C}_M(w) \text{ for } \mathrm{C}_M^{\emptyset}(w).$$

An oracle Turing machine M is prefix-free if for every sequence X, the set of all p such that M terminates on oracle X and input p is prefix-free. As usual, in case M is prefix-free, we write $\mathrm{K}_M^X(w)$ and $\mathrm{K}_M(w)$ in place of $\mathrm{C}_M^X(w)$ and $\mathrm{C}_M(w)$.

An oracle Turing machine U is a universal prefix-free oracle Turing machine if U is prefix-free and for every prefix-free oracle Turing machine M there is a constant c_M such that for all oracles X and words w it holds that

$$\mathrm{K}_U^X(w) \leq \mathrm{K}_M^X(w) + c_M.$$

In order to obtain a universal prefix-free oracle Turing machine U it suffices to let $U^X(1^e0p) \cong M_e^X(p)$ where $M_0, M_1, \dots$ is an appropriate effective listing of all prefix-free oracle Turing machines.

Definition 2.2. We fix some universal prefix-free oracle Turing machine U and define the prefix-free complexity of a word w with respect to oracle X as $\mathrm{K}^X(w) = \mathrm{K}_{\mathrm{U}}^X(w)$.

We review the notion of an information content measure and some of its properties [5, Section 3.7].

Definition 2.3. An information content measure is a right-c.e. function $f\colon \{0,1\}^* \to \mathbb{R}$ such that the sum of $2^{-f(w)}$ over all words w is finite.

We leave it to the reader to show that any information content measure has an effective nonincreasing approximation with values in the natural numbers.

Lemma 2.4 (Chaitin). *Every information content measure f is an upper bound for prefix-free complexity up to an additive constant, i.e., it holds that* $\mathrm{K}(n) \leq^{+} f(n)$.

Remark 2.5. Downey and Hirschfeldt [5, Section 3.7] define information content measure, when restricted to total functions, as a right-c.e. function $f\colon \{0,1\}^* \to \mathbb{N}$ such that the sum of $2^{-f(w)}$ over all words w is at most 1. Given an information content measure f in the sense of Definition 2.3, it is easy to see that there is an information content measure f' in the sense of Downey and Hirschfeldt such that f and f' differ at most by an additive constant.

The following lemma states the Ample Excess Lemma due to Miller and Yu [15] in two essentially equivalent forms [5, Lemma 6.6.1 and Corollary 6.6.4]. The equivalence holds since the function $d(n)$ defined in the lemma is right-c.e. relative to X and because Lemma 2.4 remains valid when relativized to an oracle.

Lemma 2.6 (Miller and Yu). *Let X be a Martin-Löf random sequence and let $d(n)$ be equal to $\mathrm{K}(X \upharpoonright n) - n$. Then the sum $\sum_n 2^{-d(n)}$ is finite. Equivalently, it holds that* $\mathrm{K}^X(n) \leq^{+} d(n)$.

By the usual abuse of notation, we denote by f^X both, a binary function that receives as arguments a word plus a sequence but also the corresponding unary function obtained by fixing the sequence argument to a specific sequence X.

Definition 2.7. The expected value of a real-valued function f^X at place w with respect to Lebesgue measure μ is denoted by $\mathbf{E}[f^X(w)]$.

Given a binary function f^X as above, we denote $f^{\emptyset}$ by f, we call f^X a relativized version of f, and call $\mathbf{E}[f^X(w)]$ the expected value of $f(w)$.

Theorem 2.8. *Prefix-free complexity agrees with its expected values up to an additive constant, i.e.,* $\mathrm{K}(w) =^{+} \mathbf{E}[\mathrm{K}^X(w)]$.

Proof. Fix a constant c such that we have $2|w| + c \geq \mathrm{K}(w)$ and $\mathrm{K}(w) + c \geq \mathrm{K}^X(w)$ for all sequences X and words w. Applying the expectation operator on both sides of the latter inequality yields

$$\mathrm{K}(w) \geq^{+} \mathbf{E}[\mathrm{K}^X(w)].$$

In order to demonstrate the reverse inequality, it suffices to show that the mapping $w \mapsto \mathbf{E}[\mathrm{K}^X(w)]$ is an information content measure, i.e., is right-c.e. and the sum of $2^{-\mathbf{E}[\mathrm{K}^X(w)]}$ over all words w is finite.

First, we demonstrate that $\mathbf{E}[\mathrm{K}^X(w)]$ is right-c.e. by describing a corresponding approximation process. Fix an enumeration of all triples (p, w, σ) such that $\mathrm{U}^{\sigma}(p) = w$ in the sense that $\mathrm{U}^{\sigma 0^{\infty}}(p) = w$ while U accesses only the first $|\sigma|$ bits of its oracle. Moreover, for all X, w and $s \geq 0$, define $\mathrm{K}^X_s(w)$ as follows. In case for the considered word w some triple of the form (p, w, σ) where $|p| < 2|w| + 2c$ and σ is an initial segment of X is enumerated among the first s triples in the above enumeration, then let $\mathrm{K}^X_s(w)$ be equal to the minimum length of p that occurs among all triples with these properties. Otherwise, let $\mathrm{K}^X_s(w) = 2|w| + 2c$. By construction, for

https://doi.org/10.1017/jsl.2018.63 Published online by Cambridge University Press

all X and w, the values $\mathrm{K}_s^X(w)$ tend nonincreasingly to $\mathrm{K}^X(w)$. Fix a computable function ℓ such that $\ell(s)$ is larger than the length of the third components σ that occur in the first s triples of the enumeration. Then for any oracle X, the value of $\mathrm{K}_s^X(w)$ is already determined by the initial segment of X of length $\ell(s)$. Again by construction, for every word w we have that

$$\frac{1}{2^{\ell(s)}} \sum_{\sigma \in \{0,1\}^{\ell(s)}} \mathrm{K}_s^{\sigma 0^\infty}(w) \xrightarrow{s \to \infty} \mathbf{E}[\mathrm{K}^X(w)],$$

where the convergence is nonincreasing and the terms on the left-hand side can be computed from s and w.

It remains to show that the sum of $2^{-\mathbf{E}[\mathrm{K}^X(w)]}$ over all words w is finite. For any given word w and natural number k, let $p(k)$ be the probability that $\mathrm{K}^X(w)$ is equal to k. Then we obtain

$$2^{-\mathbf{E}[\mathrm{K}^X(w)]} = 2^{-\sum_{k=0}^{2|w|+2c} p(k)k} \leq \sum_{k=0}^{2|w|+2c} p(k)\, 2^{-k} = \mathbf{E}[2^{-\mathrm{K}^X(w)}]. \tag{1}$$

Here both equations hold because $\mathrm{K}^X(w)$ is bounded from above by $2|w|+2c$, hence for all k larger than the latter bound the probability $p(k)$ is 0. The inequality follows by applying Jensen's inequality to the convex function $x \mapsto 2^{-x}$. This concludes the proof because we obtain

$$\sum_{w \in \{0,1\}^*} 2^{-\mathbf{E}[\mathrm{K}^X(w)]} \leq \sum_{w \in \{0,1\}^*} \mathbf{E}[2^{-\mathrm{K}^X(w)}]$$
$$\leq \mathbf{E}\left[\sum_{w \in \{0,1\}^*} 2^{-\mathrm{K}^X(w)}\right] \leq \mathbf{E}\left[\sum_{\{p:\ \mathrm{U}^X(p)\downarrow\}} 2^{-|p|}\right] \leq 1.$$

The first two inequalities hold by (1) and by the monotone convergence theorem [19], respectively. The third inequality holds because for every word w there must be some p of length $\mathrm{K}^X(w)$ such that $\mathrm{U}^X(p) = w$. Concerning the last inequality, it suffices to observe that for each fixed oracle X, the set of all p such that U terminates on oracle X and input p is prefix-free. ⊣

THEOREM 2.9. *The values $2^{-\mathrm{K}(w)}$ differ from their expected values at most by a constant factor, that is, we have $\mathrm{K}(w) =^+ -\log \mathbf{E}[2^{-\mathrm{K}^X(w)}]$.*

PROOF. The proof is very similar to the one of Proposition 2.8. We obtain the asserted equality up to additive constants by showing that the corresponding relations $\geq^+$ and $\leq^+$ hold. In the case of $\geq^+$, we use again that $\mathrm{K}(w) + c \geq \mathrm{K}^X(w)$ for some c and all X and w, hence we have $2^{-\mathrm{K}(w)-c} \leq \mathbf{E}[2^{-\mathrm{K}^X(w)}]$, and the assertion follows by taking logarithms. For the case of $\leq^+$, it suffices to show that the mapping $w \mapsto -\log \mathbf{E}[2^{-\mathrm{K}^X(w)}]$ is an information content measure. The argument that the mapping is right-c.e. is very similar to the one in the proof of Proposition 2.8 and we omit the details. In the latter proof it has been shown that the sum of $\mathbf{E}[2^{-\mathrm{K}^X(w)}]$ over all words w is at most 1, so we are done. ⊣

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DEFINITION 2.10. The APRIORI PROBABILITY of a word w WITH RESPECT TO ORACLE X is

$$\Omega^X(w) = \mu([\{p \colon \mathrm{U}^X(p) = w\}]) = \sum_{\{p \colon \mathrm{U}^X(p)=w\}} 2^{-|p|}.$$

The APRIORI PROBABILITY of a word w is $\Omega(w) = \Omega^{\emptyset}(w)$.

THEOREM 2.11. *The apriori probabilities $\Omega(w)$ differ from their expected values at most by a constant factor, that is, we have*

$$\mathrm{K}(w) =^+ -\log \Omega(w) =^+ -\log \mathbf{E}[\Omega^X(w)].$$

PROOF. Recall that by the coding theorem $\mathrm{K}(w)$ is equal to $-\log m(w)$ up to an additive constant for every maximal left-c.e. discrete semi-measure m [5, Theorem 3.9.4]. Furthermore, recall the discussion of information content measures at the beginning of this section. The first equation in the statement of the theorem is a well-known variant of the coding theorem. The equation holds because $-\log \Omega$ is an information content measure, hence is an upper bound for K up to an additive constant, but is also a lower bound because the summation in the definition of $\Omega(w)$ contains at least one term of the form $2^{-|p|}$ for some code p for w of length $\mathrm{K}(w)$. The latter argument also works when relativized to an oracle X, hence for all X and w the value $\Omega^X(w)$ is at least as large as $2^{-\mathrm{K}^X(w)}$. Then a similar assertion holds for the corresponding expected values and by Theorem 2.9 we obtain

$$\mathrm{K}(w) \geq^+ -\log \mathbf{E}[2^{-\mathrm{K}^X(w)}] \geq -\log \mathbf{E}[\Omega^X(w)].$$

We conclude by showing that the mapping $f \colon w \mapsto -\log \mathbf{E}[\Omega^X(w)]$ is an information content measure. For all words w, τ, and p, let

$$\begin{aligned}
B_w &= \{(\tau, p) \colon \mathrm{U}^\tau(p) = w \text{ and } \mathrm{U}^\sigma(p)\uparrow \text{ for all proper prefixes } \sigma \text{ of } \tau\},\\
C_w &= \{(X, Y) \colon \mathrm{U}^X(p) = w \text{ for some word } p \preceq Y\},\\
R_{\tau,p} &= \{(X, Y) \colon \tau \preceq X \text{ and } p \preceq Y\}.
\end{aligned}$$

We identify sets of the form C_w and $R_{\tau,p}$ in the natural way with subsets of the unit square and by slight abuse of notation we denote the uniform measure on the unit square by μ. By construction and because U is prefix-free, we have

$$\mathbf{E}[\Omega^X(w)] = \mu(C_w) = \sum_{(\tau,p)\in B_w} \mu(R_{\tau,p}) = \sum_{(\tau,p)\in B_w} 2^{-|\tau|}\cdot 2^{-|p|}.$$

Consequently, since the sets B_w are uniformly c.e. in w, the function f is right-c.e. Moreover, the sum of $2^{-f(n)} = \mathbf{E}[\Omega^X(w)]$ over all words w is at most 1 because the sets C_w are mutually disjoint subsets of the unit square. ⊣

COROLLARY 2.12. *Let α be a right-c.e. real where $0 < \alpha < 1$ and let*

$$\mathrm{K}_\alpha(w) = \min\{t \in \mathbb{N} \colon \mu(\{X \colon \mathrm{K}^X(w) \leq t\}) > \alpha\}.$$

Then it holds that $\mathrm{K}(w) =^+ \mathrm{K}_\alpha(w)$.

PROOF. Fix some α as in the assumption of the corollary. Let c be a constant such that for all X and w the value $\mathrm{K}^X(w)$ is less than or equal to $\mathrm{K}(w) + c$. Then the latter value is also an upper bound for $\mathrm{K}_\alpha(w)$ by definition of K_α. So it remains to show that $d(w) = \mathrm{K}(w) - \mathrm{K}_\alpha(w)$ is bounded from above. For any word w we have

$$\mathbf{E}[\mathrm{K}^X(w)] \leq \alpha \mathrm{K}_\alpha(w) + (1-\alpha)(\mathrm{K}(w) + c) = \mathrm{K}(w) - \alpha d(w) + (1-\alpha)c.$$

So in case the values $d(w)$ were unbounded, also the differences between $\mathbf{E}[\mathrm{K}^X(w)]$ and $\mathrm{K}(w)$ were unbounded, which contradicts Theorem 2.8. ⊣

Definition 2.13. An ASSIGNMENT is a partial mapping from a subset of the natural numbers to $\{0, 1\}$. An assignment is FINITE if its domain is finite. For two assignments α and β, we call α a SUBASSIGNMENT of β, $\boldsymbol{\alpha} \preceq \boldsymbol{\beta}$ for short, in case the domain of α is a subset of the domain of β and the partial functions α and β agree on the domain of α. In the latter situation, we also say that β EXTENDS α.

Unless stated otherwise, we suppose that finite assignments are given in a form from which their domain and the function values on the domain can be computed, e.g., as a pair of canonical indices for the domain and the subset of the domain where the assignment attains the value 1. Given an oracle Turing machine M, an assignment α and a natural number x, we write $M^\alpha(x) = y$ in case M computes y on input x and given any oracle that has α as a subassignment without ever accessing the oracle outside of the domain of α.

A bounded request sequence, also called bounded request set or Kraft-Chaitin set, is a computable sequence $(\ell_0, w_0), (\ell_1, w_1), \dots$ of pairs of a natural number ℓ_i and a word w_i such that $\sum_i 2^{|\ell_i|} \leq 1$. By the Kraft-Chaitin theorem [5, Theorem 3.6.1], for every such sequence there is a prefix-free Turing machine M such that the domain of M is equal to $\{p_0, p_1, \dots\}$ and for all i we have that $|p_i| = \ell_i$ and $M(p_i) = w_i$, hence for some constant c_M we have $\mathrm{K}(w_i) \leq \ell_i + c_M$. Now consider the following relativized version of the Kraft-Chaitin theorem.

Definition 2.14. An ORACLE REQUEST is a triple (ℓ, w, τ) of a natural number ℓ, a word w, and a finite assignment τ. For such an oracle request, we refer to $2^{-\ell}$ as its WEIGHT and to $2^{-|\mathrm{dom}(\tau)|}$ as its ORACLE WEIGHT. A BOUNDED ORACLE REQUEST SEQUENCE is a computable sequence $(\ell_0, w_0, \tau_0), (\ell_1, w_1, \tau_1), \dots$ of oracle requests such that for every sequence X it holds that

$$\sum_{\{i \,:\, \tau_i \preceq X\}} 2^{-\ell_i} \leq 1. \tag{2}$$

Proposition 2.15. *Let $(\ell_0, w_0, \tau_0), (\ell_1, w_1, \tau_1), \dots$ be a bounded oracle request sequence. Then there is a prefix-free oracle Turing machine M such that for every sequence X and all i such that $\tau_i \preceq X$ it holds that $M^X(p) = w_i$ for some word p of length ℓ_i, hence $\mathrm{K}^X(w_i) \leq \ell_i + c_M$ for some appropriate constant c_M that depends only on M but neither on X nor on i.*

Proof. A prefix-free oracle Turing machine M as required can be obtained as follows. On input p and oracle X, the machine M considers the sequence of all pairs (ℓ_i, w_i) such that τ_i is a subassignment of the oracle X. Then M applies the construction in the proof of the Kraft-Chaitin theorem to the latter sequence in order to obtain a sequence $p_0, p_1, \dots$ as in the conclusion of the theorem. If the input p is equal to p_i for some i, then M outputs w_i. ⊣

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Note that in case Proposition 2.15 is applied to a bounded oracle request sequence $(\ell_0, w_0, \tau_0), (\ell_1, w_1, \tau_1), \dots$ where τ_i is a proper extension of τ_j for indices $i < j$, in general two different codes of length ℓ_i for w_i are assigned for the sake of the request (ℓ_j, w_j, τ_j) with respect to two oracles that both extend τ_j but where only one extends τ_i.

The lower bound asserted in Theorem 2.17 below is immediate from the following lemma. Recall that Ω is the halting probability of the universal Turing machine U that was used to define K, i.e., the sum of 2^{-p} over all p in the domain of U.

LEMMA 2.16. *There is a constant d such that for every natural number n and almost all words w it holds that*

$$2^{-(\mathrm{K}(\Omega\restriction n)+d)} \leq \mu(\{X : \mathrm{K}^X(w) \leq \mathrm{K}(w) - n \text{ for almost all words } w\}). \tag{3}$$

PROOF. Let $p_0, p_1, \dots$ an enumeration without repetition of the domain of the universal Turing machine U that was used to define K, and let $w_i = \mathrm{U}(p_i)$ and $n_i = |w_i|$. Let $0.w_i$ denote the rational number that has a finite binary expansion determined by w_i in the obvious way. For all t and i let

$$\Omega_t = \sum_{j=0}^{t} 2^{-|p_j|} \qquad \text{and} \qquad t_i = \min\,\{t \in \mathbb{N} : 0.w_i < \Omega_t\}$$

where t_i is undefined in case the minimization is over the empty set. The latter is equivalent to $0.w_i > \Omega$ because Ω is Martin-Löf random, hence is not rational.

Independently for each i, we define a bounded oracle request sequence that is empty in case t_i is undefined and, otherwise, contains all requests of the form

$$(|p_s| - n_i, w_s, p_i) \quad \text{such that} \quad t_i < s \quad \text{and} \quad \sum_{j=t_i+1}^{s} 2^{-(|p_j|-n_i)} < 1.$$

By construction, the oracle requests issued for the sake of i are uniformly enumerable in i and form a bounded oracle request sequence. Moreover, for each i all oracle words are of the form p_i and the set of the p_i is prefix-free. Consequently, the oracle requests issued for the various values of i can be combined into a single bounded oracle request sequence. By Proposition 2.15, let the coding lengths required by this sequence be realized by some prefix-free oracle Turing machine M with coding constant c.

Next fix any i such that p_i is a code of minimum length for the initial segment w_i of Ω, i.e., $\mathrm{K}(w_i) = |p_i|$ and $w_i = \Omega \restriction n_i$. Then t_i is defined and we have for all s

$$\sum_{j=t_i+1}^{s} 2^{-(|p_j|-n_i)} = 2^{n_i}(\Omega_s - \Omega_{t_i}) < 2^{n_i}(\Omega - 0.w_i) \leq 1.$$

Accordingly, the sequence of oracle requests issued for the sake of i is infinite and contains all oracle requests of the form $(|p_s| - n_i, w_s, p_i)$ where $s > t_i$. Now almost all words w have a code of minimum length of the form p_s where $s > t_i$. For each such w and s and for every sequence X with initial segment p_i we have

$$\mathrm{K}^X(w) \leq \mathrm{K}^X_M(w_s) + c = |p_s| - n_i + c = \mathrm{K}(w) - (n_i - c),$$

and the set of such sequences X has measure of $2^{-|p_i|}$. Now the lemma follows because for all n and i where p_i is a code of minimum length for $\Omega \restriction n + c$, the length of p_i is equal to $\mathrm{K}(\Omega \upharpoonright n + c)$, hence is at most $\mathrm{K}(\Omega \upharpoonright n) + d$ for some constant d that depends on neither n nor i. ⊣

THEOREM 2.17. *There is a constant d such that for every natural number t and almost all words w it holds that*

$$2^{-(t+\mathrm{K}(t)+d)} \leq \mu(\{X : \mathrm{K}^X(w) \leq \mathrm{K}(w) - t\}) \leq 2^{-(t-d)}. \tag{4}$$

PROOF. We demonstrate separately that the lower and upper bound in (4) each hold for an appropriate choice of the constant d, hence both bounds hold for all sufficiently large d and the theorem follows. For the lower bound, this is immediate by Lemma 2.16 and since it holds that $\mathrm{K}(w) \leq^+ n + \mathrm{K}(\mathrm{n})$ for all words w. Concerning the upper bound, by Theorem 2.9 let d be a constant such that for all words w the value $2^{-\mathrm{K}(w)}$ is exceeded by its expected value by at most a factor of 2^d. Fix any natural number t and any word w. In case $\mathrm{K}(w) < t$ there is nothing to prove, so we assume otherwise. For any natural number k, let $p(k)$ be the probability that $\mathrm{K}^X(w)$ is equal to k. In case the upper bound in (4) was false, the strict inequality in the following chain of relations would be true, and we would obtain the contradiction

$$\begin{aligned} 2^{-\mathrm{K}(w)+d} &\geq \mathbf{E}[2^{-\mathrm{K}^X(w)}] = \sum_{k=0}^{\infty} p(k)2^{-k} \geq \sum_{k=0}^{\mathrm{K}(w)-t} p(k)2^{-(\mathrm{K}(w)-t)} \\ &= 2^{-(\mathrm{K}(w)-t)} \sum_{k=0}^{\mathrm{K}(w)-t} p(k) > 2^{-(\mathrm{K}(w)-t)}2^{-(t-d)} = 2^{-\mathrm{K}(w)+d}. \end{aligned}$$

⊣

The following Lemma 2.18 will be used in the proof of Theorem 4.1 below. Using notation introduced only later, the lemma can be equivalently stated by saying that the set of oracles that are low on the given set E has measure 0.

LEMMA 2.18. *Let E be an infinite set of words, let c be a natural number and let*

$$\mathcal{D}_c = \{X : \text{there exist infinitely many } w \in E \text{ such that } \mathrm{K}^X(w) \leq \mathrm{K}(w) - c\}.$$

Then the set $\mathcal{D}_c$ has Lebesgue measure 1.

PROOF. For a given natural number r, we construct a prefix-free oracle Turing machine M that witnesses that $\mathcal{D}_c$ has Lebesgue measure of at least $1 - 2^{-r}$. Since r is arbitrary, the lemma follows. By the recursion theorem, we can assume that the coding constant c_M of M can be used during the construction.

Fix an enumeration $p_0, p_1, \dots$ without repetition of the domain of U. For all i, for certain oracles X and an appropriate initial segment τ_i^X of X we will issue oracle requests of the form

$$(\ell_i, w_i, \tau_i^X) \qquad \text{where} \qquad \ell_i = |p_i| - c - c_M \quad \text{and} \quad w_i = \mathrm{U}(p_i). \tag{5}$$

The issued oracle requests will form a bounded oracle request sequence, hence by Proposition 2.15 there is an oracle Turing machine M such that for all i and and all X where an oracle request of the form (5) was issued, we have

$$\mathrm{K}^X(w_i) \leq \mathrm{K}_M^X(w_i) + c_M = \ell_i + c_M = |p_i| - c. \tag{6}$$

In order to specify the oracle request sequence, partition the natural numbers into consecutive intervals

$$J_0, J_1, \dots \quad \text{of equal length} \quad t = r + c + c_m.$$

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For a given oracle X, let τ_i^X be equal to the restriction of X to the first i intervals, that is, to the initial segment of X of length it. For every i, issue exactly the oracle requests of the form (ℓ_i, w_i, τ_i^X) where i and X satisfy the following two conditions.

(i) The sequence X is identically zero on J_i.
(ii) The sum of the weights $2^{-\ell_j}$ over all $j \leq i$ where oracle X is identically zero on J_j is at most 1.

Condition (ii) ensures that for every oracle X the sum over the weights $2^{-\ell_i}$ of all issued oracle requests of the form (ℓ_j, w_j, τ) where τ is an initial segment of X can never exceed 1, that is, the constructed sequence satisfies the measure condition (2) in Definition 2.14, hence is indeed a bounded oracle request sequence.

Let $\alpha_i(X)$ be equal to $2^{-\ell_i}$ in case X and i satisfy condition (i) and, otherwise, let $\alpha_i(X)$ be equal to 0. Furthermore, let $\alpha(X)$ be equal to the sum of the values $\alpha_i(X)$ over all i. Then an oracle X does not satisfy Condition (ii) for some i if and only if $\alpha(X)$ exceeds 1. The probability of the latter is at most 2^{-r}, as follows from the Markov inequality and because the expectation of $\alpha(X)$ can be bounded from above as follows

$$\begin{aligned}\mathbf{E}[\alpha(X)] &= \mathbf{E}\left[\sum_{i=0}^{\infty}\alpha_i(X)\right] = \sum_{i=0}^{\infty}\mathbf{E}\left[\alpha_i(X)\right] = \sum_{i=0}^{\infty}2^{-t}2^{-\ell_i}\\ &= \sum_{i=0}^{\infty}2^{-r-c-c_m-|p|_i+c+c_M} = 2^{-r}\sum_{i=0}^{\infty}2^{-|p_i|} \leq 2^{-r},\end{aligned}$$

where the second and last relation hold by linearity of expectation and because the p_i are codes of a prefix-free Turing machine, respectively. Now there are infinitely many indices i where p_i is a prefix-free code of minimum length for some word in E. By the Borel-Cantelli lemma, the set of sequences X that satisfy Condition (i) for infinitely many such i has Lebesgue measure 1. By construction and (6), each X in this set is a member of $\mathcal{D}_c$ unless X does not satisfy Condition (ii) for some i. Since the probability for the latter is at most 2^{-r}, the set $\mathcal{D}_c$ has Lebesgue measure of at least $1 - 2^{-r}$. ⊣

§3. Lowness and weak lowness along a sequence.

Lowness for prefix-free complexity on a set as introduced in Definition 3.1 can be used to define further, more specific lowness notions, including the well-known concepts low for K and weakly low for K. In all contexts, the phrases low for prefix-free complexity and low for K will be used interchangeably.

DEFINITION 3.1. Let E be a set of words. A sequence X is LOW FOR **K** ON ***E***, in case access to X as an oracle improves the prefix-free complexities of the words in E at most by an additive constant, i.e., there is a natural number c such that for all words w in E we have

$$\mathrm{K}(w) \leq \mathrm{K}^X(w) + c. \tag{7}$$

A sequence is LOW FOR **K** in case the sequence is low for K on the set of all words. A sequence is WEAKLY LOW FOR **K** in case the sequence is low for K on an infinite set of words.

In the sequel, we consider lowness notions where condition (7) is required only for certain initial segments of some fixed sequence.

DEFINITION 3.2. Let A be a sequence and let I be a set of natural numbers. A sequence X is LOW FOR **K** ALONG ***A*** in case X is low for K on the set of initial segments of A. A sequence X is LOW FOR **K** ALONG ***A*** ON ***I*** in case X is low for K on the set of initial segments of A with length in I.

DEFINITION 3.3. Let A be a sequence. A sequence X is WEAKLY LOW FOR **K** ALONG ***A*** in case X is low for K on some infinite set of initial segments of A.

THEOREM 3.4. *For any sequence X, the following assertions are equivalent.*

(i) *X is low for* K.
(ii) *X is low for* K *along all sequences.*
(iii) *X is low for* K *along some sequence.*
(iv) *X is low for* K *along some sequence on some infinite computable set.*

PROOF. The forward implications, i.e., (i) to (ii), (ii) to (iii), and (iii) to (iv), are immediate. The remaining implication from (iv) to (i) is the contraposition of the assertion of the subsequent Lemma 3.5. ⊣

LEMMA 3.5. *Let the sequence X be not low for* K, *let A be a sequence, and let I be an infinite computable set. Then X is not low for* K *along A on I.*

PROOF. We first give the proof for the case where I is equal to the set of natural numbers and then argue that the proof extends to the general case. Recall that by a result of Chaitin stated as Lemma 2.4, every information content measure f is an upper bound of prefix-free complexity up to an additive constant, i.e., $\mathrm{K}(n) \leq^+ f(n)$, where we consider K and f as functions on natural numbers by the usual identification of natural numbers and words. In particular, we have

$$\mathrm{K}(n) \leq^+ -\log \sum_{\{p\colon |\mathrm{U}(p)|=n\}} 2^{-|p|}. \tag{8}$$

Observe that the function on the right-hand side of (8) is right-c.e in n, hence is an information content measure because we have

$$\sum_{n=0}^{\infty} 2^{\log \sum_{\{p\colon |\mathrm{U}(p)|=n\}} 2^{-|p|}} = \sum_{n=0}^{\infty} \sum_{\{p\colon |\mathrm{U}(p)|=n\}} 2^{-|p|} = \sum_{\{p\colon \mathrm{U}(p) \text{ is defined}\}} 2^{-|p|} \leq 1.$$

Let $d(n) = \mathrm{K}(n) - \mathrm{K}^X(n)$ and let c_0 be a constant as hidden in the notation $\leq^+$ in (8). Then we have

$$\sum_{n=0}^{\infty} \sum_{\{p\colon |\mathrm{U}(p)|=n\}} 2^{-(|p|-d(n)+c_0)} \leq \sum_{n=0}^{\infty} 2^{d(n)-c_0} 2^{-\mathrm{K}(n)+c_0} = \sum_{n=0}^{\infty} 2^{-\mathrm{K}^X(n)} \leq 1,$$

where the relations follow by elementary rearrangements and, from left to right, by (8), by definition of $d(n)$, and because K^X is defined via a prefix-free Turing machine. This shows that there is some prefix-free code that codes every word w by a code of length $\mathrm{K}(w) - d(|w|) + c_0$. A corresponding coding could be realized by a prefix-free oracle Turing machine with access to oracle X, similar to the construction in the relativized version of the Kraft-Chaitin theorem from Proposition 2.15, in

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case the function d were computable in X. Since we cannot assume the latter, we use a similar but slightly more involved construction that works by an approximation to the values of $d(n)$ that is effective in the oracle X. This way we obtain for some constant c_1 and for all n where $d(n) \geq 6$ that

$$\mathrm{K}^X(w) \leq \mathrm{K}(w) - d(n)/2 + c_1 \text{ for all words } w \text{ of length } n. \tag{9}$$

Since the values $d(n)$ are unbounded by assumption on X, it follows that K exceeds K^X on the initial segments of any sequence A by arbitrary constants, hence the lemma follows in the special case $I = \mathbb{N}$ currently considered.

We consider triples (n, i, d) where i and d are meant as guesses for the values of $\mathrm{K}^X(n)$ and of $d(n)$, and thus $i+d$ can be viewed as a guess for $\mathrm{K}(n)$. Accordingly, let

$$G = \{(n, i, d) \colon n \in \mathbb{N} \text{ and } \mathrm{K}^X(n) \leq i \text{ and } 6 \leq d\}.$$

For each triple in G, we specify a sequence of requests, where for the various triples the definitions of these sequences are mutually independent. For some fixed enumeration without repetitions of the domain of U and for all n, let $p_0^n, p_1^n, \dots$ be the not necessarily finite subsequence of all p such that $\mathrm{U}(p)$ has length n. The sequence of requests issued for a tripel (n, i, d) then consists of all requests of the form

$$(|p_r^n| - \lceil d/2 \rceil + c_0, \mathrm{U}(p_r^n)) \qquad \text{where} \qquad \sum_{j=0}^{r} 2^{-|p_j^n|} \leq 2^{-i-d+c_0}. \tag{10}$$

Note that in case the guess $i + d$ for $\mathrm{K}(n)$ is correct, such a request is issued for all codes of the form p_r^n since in this case by (8) and choice of c_0 the inequality in (10) holds for all r. Furthermore, the sum of the weights of the requests issued for a triple (n, i, d) in G can be bounded from above as follows

$$\sum_{\{r \colon r \text{ as in } (10)\}} 2^{-|p_r^n| + \lceil d/2 \rceil - c_0} \leq 2^{-i-d+c_0+\lceil d/2 \rceil - c_0} = 2^{-\lfloor d/2 \rfloor - i}.$$

Summing up these upper bounds over all triples in G yields

$$\sum_{(n,i,d) \in G} 2^{-\lfloor d/2 \rfloor - i} \leq \sum_{n \in \mathbb{N}} \sum_{\{i \colon \mathrm{K}^X(n) \leq i\}} 2^{-i} \underbrace{\sum_{d \geq 6} 2^{-\lfloor d/2 \rfloor}}_{=1/2} = \frac{1}{2} \sum_{n \in \mathbb{N}} 2^{-\mathrm{K}^X(n)+1} \leq 1.$$

The set G is computably enumerable in X, and given a triple in G, one can effectively in X enumerate the set of requests issued for this triple. Consequently, the sequences of requests issued for the various triples in G can be combined into a single sequence that is a bounded request sequence relative to the oracle X.

Next fix any n such that $d(n) \geq 6$. By construction and the discussion above, the triple $(n, \mathrm{K}^X(n), d(n))$ is in G, hence for all p where $\mathrm{U}(p)$ is equal to a word w of length n, a request of the form $(|p| - \lceil d(n)/2 \rceil + c_0, w)$ is issued. Since for all words w of length n there is such p of length $\mathrm{K}(w)$, we have for all such w

$$\mathrm{K}^X(w) \leq \mathrm{K}(w) - \frac{d(n)}{2} + c_0 + c_M,$$

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where c_M is the coding constant of a prefix-free oracle Turing machine M that realizes the coding given by the constructed sequence of requests. But the values $d(n)$ are unbounded, hence the lemma follows in the special case $I = \mathbb{N}$ currently considered.

It remains to extend the proof above to an arbitrary infinite computable set I. Given such a set I, let $n_0 < n_1 < \cdots$ be the members of I. Then for all i, each of the functions K^X, K and then also d differ on the arguments i and n_i at most by an additive constant. Since d is unbounded, also the values $d(n_i)$ are unbounded, and by the discussion in the preceding paragraph it follows for every sequence A that the oracle X is not low along A on I. ⊣

THEOREM 3.6. *For any sequence X, the following assertions are equivalent.*

(i) *X is weakly low for* K.

(ii) *X is weakly low for* K *along some sequence.*

(iii) *X is weakly low for* K *along almost all sequences.*

PROOF. The implications from (iii) to (ii) and from (ii) to (i) are immediate. We demonstrate the remaining implication from (i) to (iii). Let for all natural numbers n and d

$$C_{n,d} = \{w\colon |w| = n \text{ and } \mathrm{K}^X(w) \le \mathrm{K}(w) - d\},$$
$$L_{n,d} = \{w\colon |w| = n \text{ and } \mathrm{K}^X(w) \le n + \mathrm{K}^X(n) - d\}.$$

Fix some constant c_0 such that for all n and all words w of length n it holds that $\mathrm{K}(w) \le n + \mathrm{K}(n) + c_0$. Furthermore, since X is assumed to be weakly low for K, fix a constant c_1 such that there are infinitely many n such that $\mathrm{K}(n) \le \mathrm{K}^X(n) + c_1$. For all such n, we have for all d and all words w in $C_{n,d}$

$$\mathrm{K}^X(w) \le \mathrm{K}(w) - d \le n + \mathrm{K}(n) + c_0 - d \le n + \mathrm{K}^X(n) + c_0 + c_1 - d,$$

hence $C_{n,d}$ is a subset of $L_{n,d-c_0-c_1}$. Then for all such n, we have

$$|C_{n,d}| \le |L_{n,d-c_0-c_1}| \le 2^{n-d+c_0+c_1+c_2}, \tag{11}$$

where the last inequality holds for some appropriate constant c_2 because Chaitin's Counting Theorem and its proof [5, Section 3.7] relativizes. Let $d = c_0 + c_1 + c_2 + 1$. Then there are infinitely many n such that the open set $[C_{n,d}]$ has Lebesgue measure of at most $1/2$. Consider the set of sequences on which X is not weakly low. By definition of weakly low, each sequence in this set must be a member of $[C_{n,d}]$ for almost all n, thus the set has Lebesgue measure of at most $1/2$. But then the set has measure 0 since it is closed under finite variants, hence has measure equal to either 0 or 1 by the Kolmogorov 0-1-law. ⊣

§4. Lowness and weak lowness on infinite sets.

THEOREM 4.1. *Let E be an infinite set of words. The set of sequences that are low for* K *on E has measure* 0. *The set of sequences that are weakly low for* K *on E has measure* 1.

PROOF. Recall from Lemma 2.18 that the sets $\mathcal{D}_c$ defined there all have Lebesgue measure 1. The intersection of the sets $\mathcal{D}_c$ has again measure 1 by σ-additivity of Lebesgue measure and coincides by definition with the set of all sequences that are not low for K on E, hence the first assertion in the theorem follows.

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Concerning the second assertion in the theorem, observe that the set of sequences that are weakly low for K on E is Borel and closed under finite variations, hence has measure either 0 or 1 by the Kolmogorov 0-1-law. We assume for a proof by contradiction that the set has measure 0, i.e., almost all sequences X satisfy

$$\lim_{w \in E} \mathrm{K}(w) - \mathrm{K}^X(w) = +\infty, \tag{12}$$

where the limit is taken with respect to length-lexicographical ordering on E. Then for any constant c and for all sufficiently large w in E, the values $\mathrm{K}(w)$ and $\mathrm{K}^X(w)$ will differ by at least c for a set of sequences X of measure at least $1/4$. This contradicts Corollary 2.12, which asserts that the value $\mathrm{K}_{1/4}(w)$ defined there agrees with $\mathrm{K}(w)$ up to an additive constant. ⊣

For a given infinite set E of words, consider the set of sequences that are low for K on E. This set always has Lebesgue measure 0 according to Theorem 4.1. Furthermore, this set coincides with the countable set of sequences that are low for K in case E is any computable set or is the set of initial segments of any sequence, where the latter case follows by Theorem 3.4. Thus one may ask whether there is a set E for which this set is uncountable. The following theorem answers this question in the affirmative.

THEOREM 4.2. *There is an infinite set E such that the set of sequences that are low for* K *on E, i.e., the set*

$$R_E = \{X : \exists c \forall n \in E(\mathrm{K}^X(n) \geq \mathrm{K}(n) - c)\}$$

is uncountable.

The proof of Theorem 4.2 is by a set theoretical forcing argument, more precisely, by Mathias forcing, via Shoenfield absoluteness.[1]

We start with L, the constructible universe. We call a sequence L-random if it is random relative to X for any $X \in L$.

Let $\mathbb{M} = (\mathbf{M}, \leq)$ be Mathias forcing. So a condition $p \in \mathbf{M}$ has the form (σ, A) for a word $\sigma \in \{0,1\}^*$ and a set $A \subseteq \omega$ where $\max\{n : \sigma(n) = 1\} < \min A$. Furthermore, $(\sigma, A) \leq (\tau, B)$ holds if

$$\sigma \succeq \tau, \qquad A \subseteq B, \qquad \text{and} \qquad \{n : \sigma(n) = 1 \wedge n > |\tau|\} \subseteq B.$$

A set $G \subseteq \omega$ is a *Mathias set* if

$$G = \bigcup \{\sigma : \exists A((\sigma, A) \in \mathcal{G})\}$$

for some $\mathbb{M}$-generic filter $\mathcal{G}$. Recall from [8, Pages 524–529, Chapter 26] and [2, Sections 7.2.A and 7.4.A] the following basic facts.

PROPOSITION 4.3. *Suppose that $\mathbb{M} \in L$, and $\mathcal{G}$ is an $\mathbb{M}$-generic filter, then the corresponding Mathias set G has the following properties.*

(i) *Every infinite subset of G is a Mathias set corresponding to some $\mathbb{M}$-generic filter.*
(ii) *There is no L-random sequence in $L[G]$.*

[1] Roughly speaking, Shoenfield absoluteness says that if a "simple" mathematical statement (such as a Σ^1_2-statement) can be proved via some "mild" set theoretical argument such as forcing or the assumption that $V = L$, then it can be proved by ZFC alone.

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Proof of Theorem 4.2. We assume that V, the universe of set theory, is a forcing extension of L by Levy Collapse $Coll(\omega, (\omega_2)^L)$ (see [11, Section 7.8]). Since $(\omega_2)^L$ is countable in V, the set of L-random sequences has full measure and there is a Mathias set G in V. By applying Theorem 4.1 to the sets of words G, we have that the set

$$R = \{X \colon \underline{\lim}_{m \in G} \mathrm{K}(m) - \mathrm{K}^X(m) < +\infty\} \tag{13}$$

has measure 1. Since the set R defined in formula (13) has measure 1, we can pick some L-random $X \in R$. Let

$$E = \{m \in G \colon \mathrm{K}(m) - \mathrm{K}^X(m) < c\}$$

where the constant c is chosen so large that E is infinite. We conclude by arguing that E witnesses that the theorem is true since R_E is uncountable. Otherwise, R_E is a countable $\Delta^1_1(E)$ set such that $X \in R_E$. Then we have that $X \in L[E]$. By the first part of Proposition 4.3, E is a Mathias set. But this contradicts the second part of Proposition 4.3 since X is chosen to be L-random. ⊣

If we consider in place of the set R_E from the proof of Theorem 4.2 the intersection $R_E \cap \{Z \colon Z \text{ is random}\}$, then by the same argument as before, this intersection is uncountable. And since the intersection is still Borel, it must contain a perfect subset. Thus we have the following two corollaries, where the first one is immediate by the Ample Excess Lemma 2.6.

Corollary 4.4. *There is an infinite set E and a constant c such that the set $S_E = \{X \colon \forall n \in E(\mathrm{K}(X \upharpoonright n) \geq n + \mathrm{K}(n) - c)\}$ is uncountable.*

Recall that a sequence X is LK-reducible to a sequence Y, for short $X \leq_{\mathrm{LK}} Y$, in case it holds that $\mathrm{K}^Y(n) \leq^+ \mathrm{K}^X(n)$.

Corollary 4.5. *There are an infinite set E and a perfect tree T with $[T] \subseteq R_E$ such that any two different sequences $X, Y \in [T]$ are incomparable with respect to LK-reducibility, i.e., we have $X \not\leq_{\mathrm{LK}} Y$ and $Y \not\leq_{\mathrm{LK}} X$.*

Proof. By Theorem 4.2, there is an infinite set E and a perfect tree T_0 with $[T_0] \subseteq R_E$, hence every $X \in [T_0]$, X is weakly low for K. So there are at most countably many oracles that are LK-reducible to X. Now let $X \leq_P Y$ if either $X, Y \notin [T_0]$ or $X, Y \in [T_0]$ and $X \leq_{\mathrm{LK}} Y$. Then $\leq_P$ is a Borel partial ordering without a perfect chain and the theorem follows by Harrington et al. [7]. ⊣

Remark 4.6. We give an alternate proof of Corollary 4.5 in terms of the properties of LK-reducibility, see Section 6 for further explanations and references. As before, we obtain an infinite set E and a perfect tree T_0 such that $[T_0] \subseteq R_E$ and all $X \in [T_0]$ are weakly low for K. By a usual fusion argument, we obtain a perfect subtree T_1 of T_0 such that for any two different infinite branches X and Y in $[T_1]$, the sequences $X \oplus \emptyset'$ and $Y \oplus \emptyset'$, hence also X' and Y' are incomparable with respect to Turing reducibility. The latter follows since every weakly low sequence Z is generalized low, i.e., Z' and $Z \oplus \emptyset'$ are Turing equivalent [17, Fact 3.6.18]. The assertion follows because, as detailed in the proof of Corollary 6.4 below, in case $X \leq_{\mathrm{LK}} Y$ holds for weakly low Y, this implies $X \leq_{\mathrm{LK}} Y'$ and then also $X' \leq_{\mathrm{T}} Y'$, which contradicts the choice of T_1.

By methods similar to the ones used in the proof of Theorem 4.2, and by replacing randomness with genericity, one obtains the following result, which answers a question of Barmpalias [1].

COROLLARY 4.7. *Let f be a function that is unbounded on some set A. Then there is an infinite subset B of A and a constant c such that the set of sequences Y that satisfy for all n in B that $\mathrm{K}(Y \upharpoonright f(n)) < \mathrm{K}(f(n)) + n + c$ is uncountable.*

§5. Application to no-gap results. Recall that a sequence X is weakly low for K if there is a constant c such that we have $\mathrm{K}(n) - c \leq \mathrm{K}^X(n)$ for infinitely many n [14]. The following result, which may be called a no-gap result [3, 4], asserts that the defined concept is changed when replacing the constant c by an unbounded function f.

THEOREM 5.1. *Let f be an unbounded function. Then there is a sequence X that is not weakly low for* K *but there are infinitely many n where $\mathrm{K}(n) - f(n) \leq \mathrm{K}^X(n)$.*

PROOF. First, recall that any Π^0_1-class that contains a Martin-Löf random sequence must have nonzero measure. For a proof, observe that all members of a Π^0_1-class of measure 0 are covered by a Martin-Löf test $U_0, U_1, \dots$ where U_i is obtained as follows. Enumerate a set V_i of words σ such that $[V_i]$ is contained in the complement of the given Π^0_1-class and has Lebesgue measure of at least $1 - 2^{i+1}$, then let U_i be an open cover of the complement of $[V_i]$.

Second, recall from the usual proof of the hyperimmune-free basis theorem due to Jockusch and Soare [5, Theorem 2.19.11] that for every nonempty Π^0_1-class P and every index e there is a nonempty Π^0_1-subclass P' of P and a computable function g such that the oracle Turing machine with index e either is not total on every oracle X in P' or computes on every oracle X in P' a function that is dominated by g.

Third, we show that for every Π^0_1-class P of nonzero measure and for any natural number m there is a Π^0_1-subclass P'' of P that has nonzero measure and for some $n > m$ contains only sequences X where it holds that

$$\mathrm{K}(n) - f(n) \leq \mathrm{K}^X(n). \tag{14}$$

For a proof, fix such P and m. Let E be an infinite set of natural numbers $m_0 < m_1 < \cdots$ such that $m < m_0$ and $f(m_i) > i$. For all natural numbers c and i let

$$\begin{aligned} Q_{c,i} &= \{X : \mathrm{K}^X(m_i) \geq \mathrm{K}(m_i) - c\}, \\ D_c &= \{X : X \in Q_{c,i} \text{ for infinitely many } i\}. \end{aligned}$$

Observe that every $Q_{c,i}$ is a Π^0_1-class. The set of sequences that are weakly low for K on E has measure 1 by Theorem 4.1 and is by definition equal to the union of the sets D_c. By σ-additivity, we can fix c such that $P \cap D_c$ has nonzero measure. But D_c is a subset of the union of the sets $Q_{c,i}$ over all $i \geq c$, hence again by σ-additivity there must be an index $t \geq c$ such that the class $P \cap Q_{c,t}$ has nonzero measure. This class is a Π^0_1-class since it is the intersection of two such classes and, by choice of t and m_t, contains only sequences X that satisfy (14) with n replaced by m_t.

Now the theorem follows by a standard argument. We construct by a non-effective process Π^0_1-classes $P_0, P_1, \dots$ of nonzero measure and natural numbers $n_0 < n_1 < \cdots$ such that every sequence X in the nonempty intersection of these

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classes is not weakly low and satisfies for all i the inequality in the conclusion of the theorem with n replaced by n_i. Let n_{-1} be equal to 0, and let P_{-1} be a Π^0_1-class P_0 of nonzero measure that contains only ML-random sequences. For example, such a class can be obtained as the complement of any component of a universal ML-test. Inductively, the number n_{i+1} and the class P_{i+1} is obtained from P_i as follows. First apply the transformation from the proof of the hyperimmunne-free basis theorem as discussed in the second paragraph of this proof to the oracle Turing machine with index i and the class P_i in order to obtain P'_i. Then apply to the latter class and to n_i the transformation from the third paragraph of this proof in order to obtain

$$P_{i+1} = (P'_i)''.$$

By construction and the discussion in the second paragraph, the classes P_i are all Π^0_1-classes of nonzero measure, hence in particular their intersection is nonempty. Fix some sequence X in this intersection. We omit the straightforward proofs that by construction the sequence X satisfies (14) with n replaced by n_i for all $i \geq 0$ and is computably dominated. The latter property implies that X is not 2-random [5, Theorem 8.21.2], i.e., is not Martin-Löf random relative to Ω. But X is Martin-Löf random, hence it follows by van Lambalgen's theorem that Ω is not Martin-Löf random relative to X, i.e., X is not low for Ω, hence is not weakly low for K by Theorem 1.2 due to Miller. ⊣

By results of Miller [13] and of Nies, Stephan, and Terwijn [18], a sequence is 2-random if and only if it is Kolmogorov random, i.e., has infinitely many initial segments w that up to an additive constant have maximum prefix-free complexity $|w| + \mathrm{K}(|w|)$. By this equivalence, the following corollary can be seen as a no-gap result.

Corollary 5.2. *Let f be an unbounded function. Then there is a weakly-2-random set X that is not 2-random such that there are infinitely many n where up to an additive constant it holds that $n + \mathrm{K}(n) - f(n) \leq \mathrm{K}(X \restriction n)$.*

Proof. The sequence X constructed in the proof of Theorem 5.1 has the required properties. The sequence X satisfies (14) for infinitely many n. For each such n, we have

$$\mathrm{K}(n) - f(n) \leq \mathrm{K}^X(n) \leq^+ \mathrm{K}(X \restriction n) - n,$$

where the second inequality holds by the Ample Excess Lemma stated as Lemma 2.6. Furthermore, the sequence X cannot be 2-random because it is computably dominated [5, Theorem 8.21.2]. Finally, X is weakly 2-random because the latter property is equivalent to Martin-Löf randomness for all sequences that are computably dominated [5, Theorems 8.11.11 and 8.11.12]. ⊣

For every unbounded function f it can be shown that there is a $\emptyset'$-Schnorr random but not 2-random sequence X such that $\mathrm{K}^X(n) \geq \mathrm{K}(n) - f(n)$ holds for infinitely many n. The proof is similar to the proof of Theorem 5.1 but works with $\Pi^0_1(\emptyset')$-classes that have $\emptyset'$-computable nonzero measure, details are omitted.

§6. On weakly-low reducibility.

In this section, we introduce a partial preorder $\leq_{\mathrm{WLK}}$ and briefly discuss some of its properties. For any sequence X and constant c, let

$$\mathrm{L}_{X,c} = \{n : \mathrm{K}(n) - c \leq \mathrm{K}^X(n)\}.$$

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Definition 6.1. Let X and Y be sequences. Then X is weakly-low reducible to Y or $\boldsymbol{X \leq_{\mathrm{WLK}} Y}$, for short, if for any constant c, there is a constant d such that $\mathrm{L}_{Y,c} \subseteq \mathrm{L}_{X,d}$.

Obviously, $X \leq_{\mathrm{WLK}} Y$ is implied by $X \leq_{\mathrm{LK}} Y$, where by definition the latter is equivalent to $\mathrm{K}^Y(n) \leq^+ \mathrm{K}^X(n)$. It is easy to see that the relation $\leq_{\mathrm{WLK}}$ is reflexive and transitive, and has a least and an greatest degree. The least degree coincides with the set of sequences that are low for K because exactly for these sequences X there is a constant c such that $\mathrm{L}_{X,c}$ contains all natural numbers. The greatest degree coincides with the set of sequences that are not weakly low for K because exactly for these sequence Y for every c the set $\mathrm{L}_{Y,c}$ is finite, hence is contained in $\mathrm{L}_{X,d}$ for every sequence X for sufficiently large d. In particular, in case two sequences are in the same WLK-degree then either both are weakly low for K or both are not.

By the discussion in the preceding paragraph and Theorem 4.1, the greatest WLK-degree is uncountable. We will show that all other WLK-degrees are countable. Recall that 2-X-randomness is defined as Martin-Löf randomness relative to X'.

Lemma 6.2. *Let X be weakly low for* K *and let R be 2-X-random. Then $X \oplus R$ is weakly low for* K.

Proof. By Theorem 1.2, Ω is X-random. The 2-X-random sequence R is in particular $\Omega \oplus X$-random. By van-Lambalgen's theorem relative to X, we obtain that Ω is $R \oplus X$-random, from which again by Theorem 1.2 the conclusion of the lemma follows. ⊣

Proposition 6.3. *Let Y be weakly low for* K *and let $X \leq_{\mathrm{WLK}} Y$. Then it holds that $X' \leq_{\mathrm{LK}} Y'$.*

Proof. It has been shown by Kjos-Hanssen et al. [9] that LK-reducibility coincides with LR-reducibility [5, Theorem 10.5.9], hence it suffices to prove that $X' \leq_{\mathrm{LR}} Y'$, i.e., that every 2-Y-random sequences is 2-X-random.

Let R be a 2-Y-random sequence. By Lemma 6.2, $Y \oplus R$ is weakly low for K. So there exists some c_0 such that the set $\mathrm{L}_{Y\oplus R,c_0}$ contains infinitely many n. For each such n, we have

$$\mathrm{K}^Y(R \restriction n) \geq n + \mathrm{K}^{Y\oplus R}(n) - c_1 \geq n + \mathrm{K}(n) - c_0 - c_1 \geq \mathrm{K}(R \restriction n) - c_0 - c_1 - c_2,$$

where the inequalities hold, from left to right and for appropriate constants c_0, c_1, and c_2, by a relativized version of the Ample Excess Lemma, by the choice of n, and, finally, because $n + \mathrm{K}(n)$ is an upper bound for the prefix-free complexity of words of length n up to an additive constant.

Consequently, for any $n \in \mathrm{L}_{Y\oplus R,c_0}$, we have that $R \restriction n \in \mathrm{L}_{Y,c_0+c_1+c_2}$. Thus for some d_0 and any such n, we have that $R \restriction n$ is a member of L_{X,d_0}, hence

$$\begin{aligned}\mathrm{K}^X(R \restriction n) &\geq \mathrm{K}(R \restriction n) - d_0 \geq n + \mathrm{K}^R(n) - d_1 - d_0\\ &\geq n + \mathrm{K}^{Y\oplus R}(n) - d_2 - d_1 - d_0 \geq n + \mathrm{K}(n) - c_0 - d_2 - d_1 - d_0,\end{aligned}$$

where the inequalities follow, from left to right and for appropriate constants d_i, by the mentioned property of $R \restriction n$, by the Ample Excess Lemma, because up to an additive constant the optimum prefix-free codes relative to $Y \oplus R$ are not longer than the ones relative to R, and, finally, by choice of n. But then the sequence R is

https://doi.org/10.1017/jsl.2018.63 Published online by Cambridge University Press

2-X-random because its initial segments have infinitely often maximum prefix-free complexity relative to X. ⊣

Corollary 6.4. *Let Y be weakly low for* K *and let $X \equiv_{\mathrm{WLK}} Y$. Then $X' \equiv_{\mathrm{LK}} Y'$ and thus $X'' \equiv_{\mathrm{T}} Y''$.*

Proof. By the remarks at the beginning of this section, the sequences that are not weakly low for K form a WLK-degree, thus X is weakly low for K, too. By Proposition 6.3, we then have $X' \equiv_{\mathrm{LK}} Y'$, which implies $X' \leq_{\mathrm{T}} Y''$ and $Y' \leq_{\mathrm{T}} X''$ [5, Theorem 10.5.11]. The latter follows by the relativized version of Chaitin's result that every sequence that is low for K is computable in the halting problem [5, Theorem 11.1.1], since for example $X' \leq_{\mathrm{LK}} Y'$ means that X' is low for $\mathrm{K}^{Y'}$.

Now $X'' \equiv_{\mathrm{T}} Y''$ follows by the result of Nies [16] that low for K implies low, i.e., $A \leq_{\mathrm{LK}} \emptyset$ implies $A' \leq_{\mathrm{T}} \emptyset'$, which also holds in the relativized form that $A \leq_{\mathrm{LK}} B$ and $A \leq_{\mathrm{T}} B'$ together imply $A' \leq_{\mathrm{T}} B'$ [17, Exercise 5.6.10]. ⊣

Remark 6.5. For any $n > 0$, there is a sequence Y such that $\emptyset' \not\leq_{\mathrm{WLK}} Y$ but $\emptyset^{(n)} \leq_{\mathrm{LK}} Y'$. By the theorem of Gács and Kučera relativized to $\emptyset'$ [6, 10], let Y be a 2-random sequence such that $\emptyset^{(n)}$ is Turing-reducible to $Y \oplus \emptyset'$. Then Y is weakly low for K by Lemma 6.2, hence we have $Y \oplus \emptyset' \equiv_{\mathrm{T}} Y'$ [17, Fact 3.6.18]. In particular, every 2-Y-random sequence is random relative to $\emptyset^{(n)}$, i.e., we have $\emptyset^{(n)} \leq_{\mathrm{LR}} Y'$ and also $\emptyset^{(n)} \leq_{\mathrm{LK}} Y'$. Furthermore, we have $\emptyset' \not\leq_{\mathrm{WLK}} Y$ because Y is weakly low for K but $\emptyset'$ is not.

As already mentioned, $X \leq_{\mathrm{LK}} Y$ implies $X \leq_{\mathrm{WLK}} Y$. We don't know whether the converse is true or not, but note that $X \leq_{\mathrm{WLK}} \emptyset$ implies $X \leq_{\mathrm{LK}} \emptyset$.

Question 6.6. *Assuming that Y is weakly low for K, does $X \leq_{\mathrm{WLK}} Y$ imply $X \leq_{\mathrm{LK}} Y$?*

Acknowledgments. We are grateful to Alexander Shen for helpful discussion during a Dagstuhl workshop in February 2017 and to the anonymous referee of the *Journal of Symbolic Logic* for useful comments and corrections. Yu gratefully acknowledges support from the National Natural Science Fund of China 11671196 and by a Humboldt Research Fellowship for Experienced Researchers.

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