J. Austral. Math. Soc. (*Series A*) **53** (1992), 377–384

HYPERSURFACES OF E^{n+1} SATISFYING $\Delta x = Ax + B$

TH. HASANIS and TH. VLACHOS

(Received 18 June 1990; revised 20 April 1991)

Communicated by K. C. H. Mackenzie

Abstract

We consider hypersurfaces of E^{n+1} whose position vector x satisfies $\Delta x = Ax + B$, where Δ is the induced Laplacian, and prove that these are open parts of minimal hypersurfaces, hyperspheres or generalized circular cylinders.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 53 C 40, 53 A 05.

0. Introduction

Let M be a connected n-dimensional submanifold of a Euclidean space E^m, equipped with the induced metric. Denote by Δ the Laplacian of M associated with the induced metric. Let x and H denote the position vector and the mean curvature vector of M in E^m respectively. Then we have

$$\Delta x = -nH. \tag{0.1}$$

In [3], T. Takahashi proved that the submanifolds for which

$$\Delta x = \lambda x, \tag{0.2}$$

that is, for which all coordinate functions are eigenfunctions of Δ with the same eigenvalue $\lambda \in R$ are either the minimal submanifolds of $E^m (\lambda = 0)$ or the minimal submanifolds of hyperspheres $S^{m-1} (\lambda \neq 0)$ in E^m.

O. Garay in [2] studied hypersurfaces in E^{n+1} for which

$$\Delta x = Ax, \tag{0.3}$$

where A is a constant diagonal matrix

$$A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{n+1} \end{pmatrix}, \quad \lambda_i \in \mathbb{R}$$

and proved that the only hypersurfaces which satisfy (0.3) are open portions of minimal hypersurfaces of E^{n+1}, ordinary hyperspheres and generalized circular cylinders.

It is easy to observe that condition (0.3) is not coordinate-invariant. With a change of the coordinate system of E^{n+1}, (0.3) becomes

$$\Delta x = Ax + B, \tag{0.4}$$

where $A = (\alpha_{ij})$ is a constant $(n+1) \times (n+1)$ matrix and $B = (\beta_i)$ a constant vector in E^{n+1}.

From this point of view it would be an interesting problem to determine those hypersurfaces which satisfy (0.4) with respect to a certain coordinate system of E^{n+1}. In [1] the problem was treated for surfaces in E^3 and it was proved that a surface of E^3 satisfies (0.4) if and only if it is an open part of a minimal surface, a sphere or a circular cylinder. The work of these authors has been brought to our attention by O. Garay and we thank him for it.

Our aim is to classify completely the hypersurfaces of E^{n+1} satisfying (0.4). The main result is given by the following

THEOREM. *A connected hypersurface of E^{n+1} which satisfies* (0.4) *is an open part of a minimal hypersurface, a hypersphere or a generalized circular cylinder.*

1. Some basic lemmas

Let M be a hypersurface in E^{n+1} $(n \geq 2)$, which satisfies (0.4). Without loss of generality we may assume that M is given locally as the graph of a smooth function $f: U \to E$, where U is an open subset of E^n. That is, M can be locally described as the set of points $(x_1, \dots, x_n, f(x_1, \dots, x_n))$.

Now it is obvious that the vector $(-f_{x_1}, -f_{x_2}, \dots, -f_{x_n}, 1)$ is normal to M. So we have

$$nH = \varphi(-f_{x_1}, -f_{x_2}, \dots, -f_{x_n}, 1),$$

where φ is a smooth function. Combining the last equation with (0.1), we get the following system of differential equations:

$$\Delta x_i = \varphi f_{x_i}, \quad i = 1, \dots, n \tag{1.1}$$

(1.2) $$\Delta f = -\varphi.$$

Taking account of (0.4) equations (1.1) and (1.2) become

(1.3) $$\varphi f_{x_i} = \sum_{m=1}^{n} a_{im} x_m + a_{in+1} f + \beta_i, \quad i = 1, \dots, n$$

(1.4) $$-\varphi = \sum_{m=1}^{n} a_{n+1m} x_m + a_{n+1n+1} f + \beta_{n+1}.$$

We also have

(1.5) $$\varphi f_{x_j} = \sum_{m=1}^{n} a_{jm} x_m + a_{jn+1} f + \beta_j.$$

Differentiating (1.3) with respect to x_j and (1.5) with respect to x_i we get

(1.6) $$\varphi_{x_j} f_{x_i} + \varphi f_{x_i x_j} = a_{ij} + a_{in+1} f_{x_j}$$

(1.7) $$\varphi_{x_i} f_{x_j} + \varphi f_{x_j x_i} = a_{ji} + a_{jn+1} f_{x_i}.$$

From (1.4) we get

(1.8) $$\varphi_{x_i} = -a_{n+1i} - a_{n+1n+1} f_{x_i}$$

(1.9) $$\varphi_{x_j} = -a_{n+1j} - a_{n+1n+1} f_{x_j}.$$

Subtracting (1.6) from (1.7) and substituting φ_{x_i} and φ_{x_j} from (1.8) and (1.9) we obtain

(1.10) $$(a_{jn+1} - a_{n+1j}) f_{x_i} + (a_{n+1i} - a_{in+1}) f_{x_j} = a_{ij} - a_{ji}, \qquad 1 \le i < j \le n.$$

The above equation shows that the vectors

$$\vec{c}_{ij} = (0, \dots, 0, a_{jn+1} - a_{n+1j}, 0, \dots, 0, a_{n+1i} - a_{in+1}, 0, \dots, 0, a_{ij} - a_{ji}), \quad 1 \le i < j \le n$$

where $a_{jn+1} - a_{n+1j}$, $a_{n+1i} - a_{in+1}$ appear in the i-th and j-th position respectively, are tangent to M. So we have proved the following

LEMMA 1. *Let M be a hypersurface in E^{n+1} for which* (0.4) *holds. Then the constant vectors*

$$\vec{c}_{ij} = (0, \dots, 0, a_{jn+1} - a_{n+1j}, 0, \dots, 0, a_{n+1i} - a_{in+1}, 0, \dots, 0, a_{ij} - a_{ji}), \quad 1 \le i < j \le n$$

are everywhere tangent to M.

There are $n(n-1)/2$ vectors $\vec{c}_{ij}$ $(1 \le i < j \le n)$. We denote by C the set of these and call $\operatorname{rank} C$ the dimension of the space generated by C.

It is obvious that $\operatorname{rank} C \leq n$. We need some more computations. Setting $a_i = a_{n+1i} - a_{in+1}$ and $\gamma_{ij} = a_{ij} - a_{ji}$, (1.10) becomes

$$-a_j f_{x_i} + a_i f_{x_j} = \gamma_{ij}, \quad 1 \leq i < j \leq n. \tag{1.10$'$}$$

Now, it is obvious that

$$-a_j \gamma_{ki} + a_i \gamma_{kj} = a_k \gamma_{ij}, \quad 1 \leq k < i < j \leq n \tag{1.11}$$

and since $\overrightarrow{c}_{ij} = (0, \dots, 0, -a_j, 0, \dots, 0, a_i, 0, \dots, 0, \gamma_{ij})$ we easily obtain

$$-a_j \overrightarrow{c}_{ki} + a_i \overrightarrow{c}_{kj} = a_k \overrightarrow{c}_{ij}, \quad 1 \leq k < i < j \leq n. \tag{1.12}$$

The following lemma is useful for the proof of the main result.

Lemma 2. *With the preceding notation we have* $\operatorname{rank} C \leq n-1$. *Moreover if* $\operatorname{rank} C < n-1$ *then* A *is a symmetric matrix.*

Proof. We first show that $\operatorname{rank} C \leq n-1$. In fact, if all a_i $(i = 1, \dots, n)$ are zero then $(1.10')$ ensures that A is symmetric. Hence $\operatorname{rank} C = 0$. Now, suppose there exist a k $(1 \leq k \leq n)$ such that $a_1 = a_2 = \cdots = a_{k-1} = 0$ and $a_k \neq 0$. Then, from the system (1.12), we deduce that all $\overrightarrow{c}_{ij}$ belong to the space generated by the vectors $\overrightarrow{c}_{1k}, \overrightarrow{c}_{2k}, \dots, \overrightarrow{c}_{k-1k}, \overrightarrow{c}_{kk+1}, \dots, \overrightarrow{c}_{kn}$, of which there are $n-1$. Thus $\operatorname{rank} C \leq n-1$. If $\operatorname{rank} C < n-1$ then the vectors $\overrightarrow{c}_{12}, \dots, \overrightarrow{c}_{1n}$ must be linearly dependent and so $a_1 = 0$. Similarly from $a_1 = 0$ and the fact that the vectors $\overrightarrow{c}_{12}, \overrightarrow{c}_{23}, \dots, \overrightarrow{c}_{2n}$ must be linearly dependent we obtain $a_2 = 0$. Proceeding in an analogous way we get $a_i = 0$, $i = 1, \dots, n-1$. Finally the vectors $\overrightarrow{c}_{1n}, \overrightarrow{c}_{2n}, \dots, \overrightarrow{c}_{n-1n}$ must be also linearly dependent, so $a_n = 0$. This implies that $a_{n+1i} = a_{in+1}$ $(i = 1, \dots, n)$ and equation $(1.10')$ implies that A is symmetric.

Moreover, we need the following.

Lemma 3. *Let* $g\colon E^{n+1} \to E$ *be a smooth function. The mean curvature vector of the level hypersurface*

$$M = \{(x_1, \dots, x_{n+1}) \in E^{n+1} \mid g(x_1, \dots, x_{n+1}) = c\}$$

is given by

$$nH = \left(\frac{\bar{\Delta} g}{|\bar{\nabla} g|^2} + \frac{\langle \bar{\nabla} |\bar{\nabla} g|^2, \bar{\nabla} g \rangle}{2 |\bar{\nabla} g|^4} \right) \bar{\nabla} g,$$

where $\bar{\Delta}$ and $\bar{\nabla}$ denote the Laplace and gradient operators of E^{n+1}, respectively.

2. The results

The case of plane curves ($n = 1$) is not covered by the analysis given in the first paragraph, so we consider it separately. The conclusion is the following result which has been proved in [1].

PROPOSITION. *Let $\gamma(s)$ be a unit speed curve of E^2 satisfying $\Delta\gamma = A\gamma + B$, where A is a constant 2×2 matrix and B is a constant vector in E^2. Then $\gamma(s)$ has constant curvature and so is a line segment or a portion of a plane circle.*

PROOF. We outline the proof in [1] for the sake of completeness. Using the Frenet frame (T, N) of γ the relation $\Delta\gamma = A\gamma + B$ becomes (since $\Delta = -d^2/ds^2$)

$$-T' = A\gamma + B$$

or, equivalently,

$$-kN = A\gamma + B.$$

Differentiating the last equation twice we compute the entries of the constant matrix A with respect to the base (T, N),

$$A \sim \begin{pmatrix} k^2 & -k' \\ 3k' & \frac{1}{k}(k^3 - k'') \end{pmatrix}.$$

From the constancy of $\det A$ and $\operatorname{trace} A$ we see that the curvature function k satisfies a system of two differential equations, whose solutions are just the constant functions.

The next examples illustrate, in some cases, the proof of the theorem.

EXAMPLE 1. Let M be the hyperplane of E^{n+1} with equation $x_{n+1} = 0$. We easily verify that

$$\Delta x = Ax, \text{ where } A = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \mathbf{0} & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

In this case we have $\vec{c}_{ij} = (0, \dots, 1, 0, \dots, -1, 0, \dots, 0)$, where 1 and -1 occur in the i-th and j-th positions respectively. The $n-1$ linearly independent vectors $\vec{c}_{12}, \dots, \vec{c}_{1n}$ generate all $\vec{c}_{ij}$. Hence $\operatorname{rank} C = n - 1$.

EXAMPLE 2. Let M be the circular cylinder of E^{n+1} described by $x_1^2 + x_{n+1}^2 = R^2$. Then

$$\Delta x = \Gamma x, \quad \text{where } \Gamma = \begin{pmatrix} 1/R^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/R^2 \end{pmatrix}.$$

Suppose that y is another coordinate system in E^{n+1}. Then $x = Py + D$, where P is an orthogonal matrix and D is a constant vector. With respect to the system y we obtain $\Delta y = Ay + B$, where $A = P^{-1}\Gamma P$ and $B = P^{-1}\Gamma D$. Obviously A is a symmetric matrix. So, in the case of the cylinder, we have $\operatorname{rank} C = 0$.

PROOF OF THE THEOREM. We distinguish two cases:

CASE I. Assume that $\operatorname{rank} C = n - 1$. Then, by using Lemma 1, we see that all tangent spaces of M are parallel to a constant space of dimension $n - 1$. So M is a cylinder erected over a plane curve γ.

We may assume, without loss of generality, that the position vector of M is given by $x(s, t_1, \dots, t_{n-1}) = \gamma(s) + \sum_{i=1}^{n-1} t_i\xi_i$ where $\gamma(s)$ is the arc length parametrization of γ and $\xi_i = (0, \dots, 1, \dots, 0)$, where 1 appears in the $(i+2)$-position, are normal to the plane of γ. Then (0.4) implies that

$$\begin{aligned} -\gamma_1'' &= a_{11}\gamma_1 + a_{12}\gamma_2 + a_{13}t_1 + \cdots + a_{1n+1}t_{n-1} + \beta_1 \\ -\gamma_2'' &= a_{21}\gamma_1 + a_{22}\gamma_2 + a_{23}t_1 + \cdots + a_{2n+1}t_{n-1} + \beta_2 \\ 0 &= a_{i1}\gamma_1 + a_{i2}\gamma_2 + \sum_{j=3}^{n+1} a_{ij}t_{j-2} + \beta_i, \quad 2 < i \le n+1. \end{aligned}$$

From the first two equations, after differentiating with respect to t_i, $(i = 1, \dots, n-1)$, we find that $a_{ij} = 0$, $i = 1, 2$, $j = 3, \dots, n+1$. Hence the first two equations imply

$$\Delta\gamma = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \gamma + \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix},$$

where $\Delta = -d^2/ds^2$. Now, by using Proposition 1, γ is a portion of a line or a circle. Hence M is a hyperplane or a circular cylinder. Since for a fixed coordinate system the matrix A in (0.4) is unique, (unless M is a hyperplane and because of Example 2), we conclude that the second case is not possible.

CASE II. Assume that $\operatorname{rank} C < n - 1$. Then A is symmetric. After a coordinate transformation we may suppose that

$$\Delta x = Ax + B \tag{2.1}$$

https://doi.org/10.1017/S1446788700036545 Published online by Cambridge University Press

where

$$A = \begin{pmatrix} \lambda_1 & & & & & 0 \\ & \ddots & & & & \\ & & \lambda_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_r \\ \vdots \\ \beta_{n+1} \end{pmatrix}$$

and r is the rank of A. If $r = 0$, then (2.1) becomes $\Delta x = B$ and from (0.1) we get $-nH = B$, which implies that $H = 0$ and thus M is a minimal hypersurface. From now on, we assume that $0 < r \le n+1$. By a parallel translation at the point $(\beta_1/\lambda_1, \dots, \beta_r/\lambda_r, 0, \dots, 0)$ we may suppose that $B^t = (0, 0, \dots, \beta_{r+1}, \dots, \beta_{n+1})$.

Let $(x_1(u_1, \dots, u_n), \dots, x_{n+1}(u_1, \dots, u_n))$ be a parametrization of this hypersurface. Then from (0.1) the vector $Ax+B = (\lambda_1 x_1, \dots, \lambda_r x_r, \beta_{r+1}, \dots, \beta_{n+1})$ is normal to M. So we have

$$\langle(\lambda_1 x_1, \dots, \lambda_r x_r, \beta_{r+1}, \dots, \beta_{n+1}), \left(\frac{\partial x_1}{\partial u_i}, \dots, \frac{\partial x_{n+1}}{\partial u_i}\right)\rangle = 0, \quad i = 1, \dots, n$$

or, equivalently,

$$\sum_{i=1}^{r} \lambda_i x_i^2 + 2\sum_{j=1}^{n-r+1} \beta_{r+j} x_{r+j} = c, \tag{2.2}$$

where c is a constant, from which we see that M should be a part of a quadratic hypersurface. Computing the mean curvature vector from Lemma 3 and taking into account that $-nH = Ax+B = (\lambda_1 x_1, \dots, \lambda_r x_r, \beta_{r+1}, \dots, \beta_{n+1})$ we obtain

$$-\frac{\sum_{i=1}^{r} \lambda_i}{\sum_{i=1}^{r} \lambda_i^2 x_i^2 + \sum_{j=1}^{n-r+1} \beta_{j+r}^2} + \frac{\sum_{i=1}^{r} \lambda_i^3 x_i^2}{\left(\sum_{i=1}^{r} \lambda_i^2 x_i^2 + \sum_{j=1}^{n-r+1} \beta_{j+r}^2\right)^2} = -1,$$

or, equivalently,

(2.3)

$$-\left(\sum_{i=1}^{r} \lambda_i\right)\left(\sum_{i=1}^{r} \lambda_i^2 x_i^2 + \sum_{j=1}^{n-r+1} \beta_{j+r}^2\right) + \sum_{i=1}^{r} \lambda_i^3 x_i^2 + \left(\sum_{i=1}^{r} \lambda_i^2 x_i^2 + \sum_{j=1}^{n-r+1} \beta_{j+r}^2\right)^2 = 0.$$

We are going to prove that $\beta_{r+1} = \dots = \beta_{n+1} = 0$. We proceed by contradiction. If one of $\beta_{r+1}, \dots, \beta_{n+1}$ is different from zero, equation (2.3) becomes a polynomial which is identically zero on some open set. Thus the coefficient λ_i^4 of x_i^4 $(i = 1, \dots, r)$ should be zero, which is a contradiction.

Now, the equation of the hypersurface becomes

$$\sum_{i=1}^{r} \lambda_i x_i^2 = c \tag{2.4}$$

and (2.3) also becomes

$$-\left(\sum_{i=1}^{r} \lambda_i\right)\left(\sum_{i=1}^{r} \lambda_i^2 x_i^2\right) + \sum_{i=1}^{r} \lambda_i^3 x_i^2 + \left(\sum_{i=1}^{r} \lambda_i^2 x_i^2\right)^2 = 0. \tag{2.5}$$

It is obvious from (2.4) that, for $r = 1$, M is a hyperplane. So, in the following, we assume $2 \le r \le n+1$. Eliminating x_1^2 from (2.4) and (2.5) we obtain the following polynomial

$$\begin{aligned} &\sum_{i=2}^{r} \lambda_i^2 (\lambda_i - \lambda_1)^2 x_i^4 + 2 \sum_{2 \le i < j \le r} \lambda_i \lambda_j (\lambda_i - \lambda_1)(\lambda_j - \lambda_1) x_i^2 x_j^2 \\ &\quad + \sum_{i=2}^{r} \lambda_i (\lambda_i - \lambda_1) \left(2\lambda_1 c + \lambda_1 + \lambda_i - \sum_{j=1}^{r} \lambda_j\right) x_i^2 \\ &\quad + \lambda_1^2 c^2 + \lambda_1^2 c - \lambda_1 c \left(\sum_{j=1}^{r} \lambda_j\right) = 0. \end{aligned}$$

The above polynomial is identically zero on some open subset. Hence the coefficient of x_i^4 must be zero, that is, $\lambda_i = \lambda_1$, $i = 2, \dots, r$. This shows that all the λ_i are equal to λ_1 and $c = r - 1$. Thus, if $r = n+1$, then M is a portion of a hypersphere with radius $\sqrt{n/\lambda_1}$ and if $2 \le r < n+1$ then M is the generalized circular cylinder $S_\rho^{r-1} \times E^{n-r+1}$ of radius $\rho = \sqrt{(r-1)/\lambda_1}$.

Added April 15, 1991. B.-Y. Chen and M. Petrovic independently obtained our main result in the paper "On spectral decomposition of immersions of finite type", which will appear in the Bulletin of the Australian Math. Society.

References

[1] F. Dillen, J. Pas and L. Verstraelen, 'On surfaces of finite type in Euclidean 3-space', *Kodai Math. J.* **13** (1990), 10–21.

[2] O. Garay, 'An extension of Takahashi's Theorem', *Geom. Dedicata* **34** (1990), 105–112.

[3] T. Takahashi, 'Minimal immersions of Riemannian manifolds', *J. Math. Soc. Japan* **18** (1966), 380–385.

University of Ioannina
45110 Ioannina
Greece