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Dehn twist exact sequences through Lagrangian cobordism

Cheuk Yu Mak and Weiwei Wu

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ABSTRACT

This paper introduces a new Lagrangian surgery construction that generalizes Lalonde–Sikorav and Polterovich’s well-known construction, and combines this with Biran and Cornea’s Lagrangian cobordism formalism. With these techniques, we build a framework which both recovers several known long exact sequences (Seidel’s exact sequence, including the fixed point version and Wehrheim and Woodward’s family version) in symplectic geometry in a uniform way, and yields a partial answer to a long-term open conjecture due to Huybrechts and Thomas; this also involved a new observation which relates projective twists with surgeries.

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1. Introduction

1.1 Motivations and overview

Lagrangian cobordisms were first introduced by Arnold [Arn80a, Arn80b] and subsequently studied by Eliashberg [Eli84], Audin [Aud85], Chekanov [Che97], etc. More recently, Biran and Cornea, in their celebrated series of papers [BC13, BC14, BC17], considered Floer theories on them, and have achieved great success encapsulating information of the triangulated structures of the derived Fukaya category. A particularly attractive application is that they establish the long-expected relation between Lagrangian surgeries [LS91, Pol91] and the mapping cones in derived Fukaya categories.

A primary purpose of this paper is to revisit such surgery-cobordism relations with an emphasis on applications to Dehn twists. The underlying philosophy of our approach is to understand the functors between Fukaya categories via Lagrangian cobordisms. This functor-level point of view has been exploited in several other contexts by many authors [WMW, WW16, AS15] etc.

We explore this direction through the eyes of Lagrangian correspondences. Intuitively, one may regard Lagrangian correspondences as symplectic mirrors of kernels of Fourier–Mukai transforms. The observation is that almost all exact sequences involving Lagrangian Dehn twists can be interpreted as cone relations between these ‘kernels’. Explicitly, Lagrangian cobordism constructions geometrically realize all these cones on the correspondence level and provide a completely analogous picture on the symplectic side, versus various twist constructions on derived categories. This point of view greatly simplifies the proof of several known exact sequences and leads to new cone relations in Floer theory such as Lagrangian $\mathbb{C}\mathbb{P}^n$ -twists, partially verifying a conjecture due to Huybrechts and Thomas.

To this end, much work needs to be done on the general geometric framework. We designed a new approach to Lagrangian surgeries called the *flow surgery*, which is coordinate-free and easy to compare with other constructions such as Dehn twists. The construction also allows many variants open for future exploration.

Another geometric observation is that Dehn twists along various projective spaces are equivalent to Lagrangian surgeries with certain immersed spheres. Using the aforementioned idea from the complex side, we package this information in product symplectic manifolds.

Readers who are cautious about technical conditions throughout the paper will find a list by the end of this introduction.

1.2 Flow surgeries and flow handles

Recall that for two Lagrangians $L_1 \pitchfork L_2 = \{x\}$, their Lagrangian surgery at x is given by adding an explicit Lagrangian handle in the Darboux chart [LS91, Pol91]. Then a Lagrangian cobordism

can be obtained by using ‘half’ of a Lagrangian handle of one dimension higher [BC13]. This line of thought has led to remarkable breakthroughs in both constructions of new examples of Lagrangian submanifolds [Pol91] and cobordism theory [BC13].

To implement this construction to Lagrangian ‘fiber sums’ (surgery along clean intersections), the patching of local models requires delicate consideration on the choice of connections on normal bundles. On top of that, in most of our applications, the main difficulty is to show that the resulting manifold is Hamiltonian isotopic to certain given Lagrangians, usually those obtained by Lagrangian Dehn twists.

Our basic idea to solve both problems at once is to use a reparametrized geodesic flow, mimicking the original construction of the symplectic Dehn twist by Seidel, to produce a new Lagrangian surgery operation called the *flow surgery* (see § 2.2). This flow surgery recovers the usual Lagrangian surgery when the auxiliary data is chosen appropriately, but has much better flexibility. For example, the resulting Lagrangian handle needs not be diffeomorphic to a punctured ball (or a bundle with punctured-ball fibers in the clean surgery case). Moreover, Biran and Cornea’s cobordism construction via surgeries is easily seen to fit into this framework.

The main examples we have are the following (see § 3 for relevant definitions).

THEOREM 1.1. *Consider the following embedded submanifolds in M :*

- (1) $S^n \hookrightarrow M$ is a Lagrangian sphere embedding;
- (2) $C \hookrightarrow M$ is a spherically coisotropic submanifold embedding;
- (3) $S \hookrightarrow M$ is a Lagrangian embedding where S is diffeomorphic to either $\mathbb{R}P^n$, $\mathbb{C}P^n$ or $\mathbb{H}P^n$;
- (4) $C_P \hookrightarrow M$ is a projectively coisotropic submanifold embedding.

Let τ_{S^n} , τ_C , τ_S and τ_{C_P} denote the corresponding Dehn twists. One has the following surgery equalities up to Hamiltonian isotopies in $M \times M^-$ (see § 2.2 for relevant notation).

- (1) The graph of a Dehn twist along a sphere is the surgery of $S^n \times S^n$ with the diagonal, that is,

$$(S^n \times (S^n)^-)\#_{\Delta_{S^n}, E_2} \Delta_M = \text{Graph}(\tau_{S^n}^{-1}). \quad (1.1)$$

- (2) Let $\tilde{C} \hookrightarrow M \times M^-$ be the Lagrangian submanifold associated to C . Then the graph of family Dehn twist along C is the surgery of an associated Lagrangian \tilde{C} and the diagonal, that is,

$$\tilde{C}\#_{\mathcal{D}, E_2} \Delta_M = \text{Graph}(\tau_C^{-1}). \quad (1.2)$$

- (3) The graph of projective twist along S is the surgery of two copies of $S \times S^-$ along with the diagonal, that is,

$$(S \times S^-)\#_{D^{\text{op}}, E_2} (S \times S^-)\#_{\Delta_S, E_2} \Delta_M = \text{Graph}(\tau_S^{-1}). \quad (1.3)$$

- (4) Let $\tilde{C}_P \hookrightarrow M \times M^-$ be the Lagrangian submanifold associated to C_P . Then the graph of family Dehn twist along C_P is the surgery of two copies of the associated Lagrangian \tilde{C}_P and the diagonal, that is,

$$\tilde{C}_P\#_{D^{\text{op}}, E_2} \tilde{C}_P\#_{\mathcal{D}, E_2} \Delta_M = \text{Graph}(\tau_{C_P}^{-1}). \quad (1.4)$$

The surgery equalities immediately lead to the existence of corresponding Lagrangian cobordisms. Note that in case (1), a similar cobordism construction was established in [AS15] using Lefschetz fibrations independently.

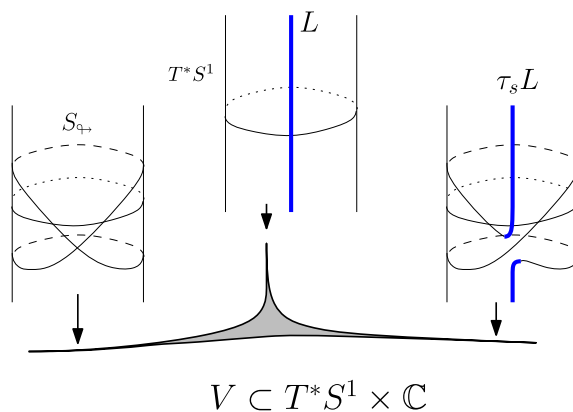


FIGURE 1. A surgery with an immersed S^1 .

To motivate our construction, we point out a direct point of view on projective twists via immersed Lagrangian submanifolds as follows: for each Lagrangian projective space S , there is a naturally associated immersed Lagrangian sphere $S_{\uparrow\downarrow}$. Then the Dehn twist of L along S is equivalent to performing a surgery with a copy of $S_{\uparrow\downarrow}$ for each intersection $L \cap S$.

Example 1.2. The simplest instance of a projective twist can be demonstrated concretely in $M = T^*S^1$, see Figure 1. Here we consider $S_{\uparrow\downarrow} \subset M$ to be an immersed circle with a unique transverse immersed point. Here L is given by the cotangent fiber at a point, and we assume it passes through the unique immersed point of $S_{\uparrow\downarrow}$. While the base is regarded as an $\mathbb{R}P^1$, the surgery Lagrangian $S_{\uparrow\downarrow} \# L$ is $\tau_{\mathbb{R}P^1} L$. Here the surgery is performed through one of the branches of $S_{\uparrow\downarrow}$ at the immersed point.

This surgery can be recast into a Lagrangian cobordism in $T^*S^1 \times \mathbb{C}$. The cobordism can be constructed so that it naively satisfies Biran and Cornea’s definition, i.e., outside $T^*S^1 \times K$ for some compact set K , it is a union of products between rays and immersed Lagrangian submanifolds in T^*S^1 . However, it is evident that the self-intersection cannot be clean since they form a ray. In general, any surgery process involving resolution of an immersed point will suffer from the same shortcoming. Therefore, we will need a modification for the Floer theory to be well defined.

For the general case, the above approach becomes rather technical. Our key novelty is to apply the surgery construction to Lagrangian submanifolds in product manifolds, such as the *graph of projective twists*, to package the same information in a way better suited for doing Floer theory, hence case (3). More precisely, one should imagine that the two copies of $S \times S^-$ in case (3) are obtained from breaking an immersed Lagrangian relevant to the Dehn twist, similar to the fact that $S_{\uparrow\downarrow}$ in Example 1.2 can be obtained as a Lagrangian surgery between two copies of zero section.

Remark 1.3. Formal proofs will only be given in the case of S^n and $\mathbb{C}P^n$, since the $\mathbb{H}P^n$ and $\mathbb{R}P^n$ cases will follow from the proof of $\mathbb{C}P^n$ word by word. The common feature for these manifolds we used is the existence of a metric g_S with the following property: for any point $x \in S$, the injectivity radius at x equals π , and $S \setminus B_x(\pi)$ is a smooth closed submanifold, where $B_x(\pi)$ is the open ball of radius π around x in the round metric on S .

We include a detailed discussion on gradings involved in Lagrangian surgeries (see § 4), which allows us to compute the connecting maps later on (Appendix A). But we emphasize that the grading is a vital part of the foundation of Lagrangian surgeries for an intrinsic reason. Consider the simple case when all the involved Lagrangians (including those considered in general surgery constructions) are \mathbb{Z} -graded and embedded. According to the cone relation proved in [BC13], the algebra instructs that a surgery happen only at *degree-zero* cocycles. This principle was noticed first by Paul Seidel [Sei00].

Such a principle interprets several known phenomena in a uniform way. First of all there are two different surgeries at a single point (see [FOOO09, ch. 10]) and they should be viewed as two different cones $\text{Cone}(L_0 \xrightarrow{c} L_1)$ and $\text{Cone}(L_1[-n] \xrightarrow{c^\vee[-n]} L_0)$, which are *a priori* very different. When the resolved intersections involve generators with different degrees, in many cases this leads to obstructions in Floer theory, as exemplified in [FOOO09, ch. 10]. In better situations when resolved intersections have zero degree modulo N , the surgery at least collapses \mathbb{Z} -gradings to \mathbb{Z}/N -gradings. This can also be checked directly on the Maslov classes of the surgered Lagrangians.

For our applications, we extend this principle to clean surgeries. The upshot is that, for two graded Lagrangians L_0, L_1 with $L_0 \cap L_1 = D$ being a clean intersection with zero Maslov index, L_1 and $L_0[\dim(D) + 1]$ can be surgered to produce a graded Lagrangian. This matches well with predictions from homological algebra dictated by Lagrangian Floer theory with clean intersections [FOOO09, ch. 10]. It also extends the surgery exact sequence to the clean intersection case.

1.3 Cone relations in functor categories via Lagrangian cobordisms

From the surgery equalities in Theorem 1.1 and the corresponding cobordisms, we immediately recover Seidel’s exact sequence and Wehrheim and Woodward’s family Dehn twist sequence on the functor level, assuming all monotonicity conditions discussed in § 6, as follows.

THEOREM 1.4 (See [Sei03, WW16, BC17], also Theorems 6.4 and 6.6). *Let M be a monotone symplectic manifold. When $S^n \subset M$ is a monotone Lagrangian sphere, there is a cone in $\text{End}(\text{TwFuk}(M))$.*

$$\begin{array}{ccc}
 \text{hom}(S^n, -) \otimes S^n & \longrightarrow & \text{id} \\
 & \searrow [1] & \downarrow \tau_{S^n}
 \end{array} \tag{1.5}$$

When $C \subset M$ is a spherically coisotropic submanifold with appropriate monotonicity assumptions (see Theorem 6.6), there is a cone in $\text{End}(\text{TwFuk}(M))$.

$$\begin{array}{ccc}
 C^t \circ C & \longrightarrow & \text{id} \\
 & \searrow [1] & \downarrow \tau_C
 \end{array} \tag{1.6}$$

(Here, C^t is the transpose of a Lagrangian correspondence, and \circ denotes the composition. See § 5 for the precise definitions.)

Besides invoking Biran and Cornea’s general cobordism formalism and Theorem 1.1, we derived natural sufficient conditions for Lagrangian cobordisms to be monotone/exact in Lemma 6.3 and 6.2, which are particularly adapted to the surgery setting.

Through our methods, we can cover the case when $C \subset M$ is of codimension one, under the assumption that $\pi_1(M)$ is torsion. For more general symplectic manifolds, our method reduces the problem of proving exact sequences to only checking the monotonicity conditions as in Theorem 6.6 for codimension one spherically coisotropic manifolds.

In another direction, since our construction holds for arbitrary symplectic manifolds, when combined with the general framework due to Fukaya *et al.* [FOOO09], it yields a proof for Seidel’s exact sequence in an arbitrary symplectic manifold.

Remark 1.5. There are interesting consequences of these cones of endofunctors of Fukaya categories. For example, one may prove that for the Milnor fibers of any *ADE*-singularities, the auto-equivalences of the Fukaya category induced by compactly supported symplectomorphisms are split generated by those induced by compositions of Dehn twists along vanishing cycles, modulo aligning some technical ingredients (such as establishing the cobordism theory in the wrapped context, as pointed out to us by Sheel Ganatra). This is a generalization (though in a weaker categorical sense) of a result in dimension 4, which says that for surface A_n -Milnor fibers, any symplectomorphism is Hamiltonian isotopic to certain compositions of Dehn twists along vanishing cycles [Eva11, Wu14]. This part will appear separately soon.

1.4 The Huybrechts–Thomas conjecture and projective twists

There is a natural extension of the Dehn twist along spheres to arbitrary rank-one symmetric spaces, which has been known for a long time. Seidel’s long exact sequence associated to a Dehn twist along a sphere should be viewed as the mirror of a spherical twist in derived categories [ST01]. Also, such a cone relation on the *A*-side has become a foundational tool in the study of homological mirror symmetry, especially in the symplectic Picard–Lefschetz theory [Sei08a].

Since then, how to describe the effect of the Dehn twists along a rank-one symmetric space in Floer theory has remained a mystery.

Fortunately, one could at least formulate a conjectural algebraic expression from homological mirror symmetry in this case. On the *B*-side, Huybrechts and Thomas [HT06] first defined \mathbb{P}^n -objects in the derived category. Recall that an object $\mathcal{E} \in D^b(X)$ for a smooth projective variety is called a \mathbb{P}^n -object if $\mathcal{E} \otimes \omega_X \cong \mathcal{E}$ and $\text{Ext}^*(\mathcal{E}, \mathcal{E})$ is isomorphic as a graded ring to $H^*(\mathbb{P}^n, \mathbb{C})$. Then they constructed an auto-equivalence of $D^b(X)$ called the \mathbb{P}^n -twist associated to \mathcal{E} , which is the Fourier–Mukai transform with kernel

$$\text{Cone}(\text{Cone}(\mathcal{E}^\vee \boxtimes \mathcal{E}[-2] \xrightarrow{\bar{h}^\vee \times \text{id} - \text{id} \times \bar{h}} \mathcal{E}^\vee \boxtimes \mathcal{E}) \xrightarrow{ev} \mathcal{O}_\Delta). \tag{1.7}$$

Here $\bar{h} \in \text{hom}^2(\mathcal{E}, \mathcal{E})$ is a representative of the generator in cohomology. They then conjectured the \mathbb{P}^n -twist is exactly the mirror auto-equivalence of the one induced by a Dehn twist along Lagrangian $\mathbb{C}\mathbb{P}^n$ on the derived Fukaya categories. Written explicitly in the derived Fukaya category, the conjecture reads as follows.

CONJECTURE 1.6 ([HT06], see also [Har11]). Given a monotone Lagrangian $\mathbb{C}\mathbb{P}^n$ in M^{4n} , denoted by S , and a compact monotone Lagrangian L , then in $D^\pi \mathcal{Fuk}(M)$

$$\tau_{\mathbb{C}\mathbb{P}^n}(L) \cong \text{Cone}(\text{Cone}(\text{hom}(S, L) \otimes S[-2] \xrightarrow{\mu^2(-, h) \times \text{id} - \text{id} \times \mu^2(h, -)} \text{hom}(S, L) \otimes S) \xrightarrow{ev} L). \tag{1.8}$$

Here the right-hand side is an iterated mapping cone, $D^\pi \mathcal{Fuk}(M)$ denotes the Karoubi completion of the derived Fukaya category generated by compact Lagrangian branes, and $h \in \text{hom}^2(S, S)$ is the Floer cochain in degree 2 representing the dual of the hyperplane class in cohomology.

Richard Harris studied the conjecture in A_∞ context: he formulated a projective twist as an A_∞ -autoequivalence of an A_∞ category [Har11]. His construction spelled out a conjectural algebraic expression about $\tau_S L$ in the twisted complexes of a Fukaya category (up to quasi-isomorphisms, of course).

As an application of the surgery equalities in Theorem 1.1, we show the following cone relations.

THEOREM 1.7 (See Theorem 6.7). *For any given monotone Lagrangian submanifolds L_0, L_1 and S in M with minimal Maslov numbers at least two such that S is diffeomorphic to $\mathbb{C}\mathbb{P}^n$, there is a quasi-isomorphism of cochain complexes*

$$\begin{aligned} & CF^*(L_0, \tau_S L_1) \\ & \cong \text{Cone}(CF^*(S, L_1) \otimes CF^{*-2}(L_0, S) \rightarrow CF^*(S, L_1) \otimes CF^*(L_0, S) \rightarrow CF^*(L_0, L_1)). \end{aligned} \quad (1.9)$$

Here, the right-hand side is an iterated mapping cone. That is, we used $\text{Cone}(A \rightarrow B \rightarrow C)$ as a shorthand for $\text{Cone}(\text{Cone}(A \rightarrow B) \rightarrow C)$ for chain complexes A, B and C . On the categorical level, Biran and Cornea's cobordism framework and (1.3) imply a cone relation between product Lagrangians, and this translates, using the M'au–Wehrheim–Woodward functor [WMW], into a cone relation between endofunctors. This allows one to verify the (A_∞) Fukaya categorical version of the conjecture in the monotone case.

THEOREM 1.8 (See Theorem 6.10). *Huybrechts–Thomas conjecture 1.6 is true modulo determination of connecting maps.*

The proof of Theorem 1.8 follows from the construction of a cobordism representing an iterated cone on the functor level, see Theorem 1.1 and Lemma 4.18. With some extra work, our proof implies that, on the A_∞ level, our geometric expression matches Harris's construction up to quasi-isomorphisms. Our method applies well on $\mathbb{R}\mathbb{P}^n$ or $\mathbb{H}\mathbb{P}^n$, as well as their family versions (see Theorem 6.11). However, even if all the geodesics on a Lagrangian are closed, our method cannot apply when the geodesics do not have a common period. Examples of these manifolds include free quotient of spheres by finite groups of order greater than two, and their Dehn twists are studied in a forthcoming paper by the authors.

We should mention that the A_∞ version should also hold for Fukaya categories of exact symplectic manifolds: however, at the time of writing, the construction of infinitesimal Fukaya categories is not fully carried out, and the wrapped version of Lagrangian cobordism theory in the present situation requires independent efforts (see [Gao17]).

Remark 1.9. While it is not difficult to find examples of Lagrangian $\mathbb{R}\mathbb{P}^n$ in problems in symplectic topology ([She15], [Wu14] etc.), the search of interesting examples of Lagrangian $\mathbb{C}\mathbb{P}^n$ is more intriguing. In [HT06] the authors suggested several sources of \mathbb{P}^n -objects in derived categories. An interesting instance is given by sheaves pulled back from the zero section of a holomorphic Lagrangian fibration on a hyperkähler manifold. From the Strominger–Yau–Zaslow (SYZ) point of view, this should correspond to a Lagrangian $\mathbb{C}\mathbb{P}^n$ section on the SYZ mirror. While the role of \mathbb{P}^n objects on either side of mirror symmetry remains widely open so far, it is interesting to know whether such objects split generate either side of mirror symmetry.

Another significant source of Lagrangian projective spaces is due to Manolescu [Man07]. In particular, he constructed a family of exact symplectic manifolds which admit certain symplectic fibrations. The monodromies of such fibrations are closely related to projective twists.

Conjecturally, the Khovanov–Rozansky homology of a link can be recovered by $HF(L, L')$, where L is certain Lagrangian submanifold on the fiber, and L' is the parallel transport of L back to the same fiber over a path on the base. Therefore, it seems that the projective twist formula should imply interesting information on Khovanov–Rozansky homology, which will be explored in the future.

Remark 1.10. In a different direction, the \mathbb{P}^n -cone relation should be interesting in understanding some basic problems in symplectic topology, such as mapping class groups of a symplectic manifold and the search of exotic Lagrangian submanifolds. For instance, while a Lagrangian $\mathbb{C}\mathbb{P}^n$ -twist is always smoothly isotopic to identity (see [Sei00, Proposition 4.6]), it is usually not *Hamiltonian* isotopic to identity. A simplest model result along this line is to generalize Seidel’s twisted Lagrangian sphere construction [Sei99]: in the plumbing of three $T^*\mathbb{C}\mathbb{P}^n$, the iterated Dehn twists along $\mathbb{C}\mathbb{P}^n$ in the middle should generate an infinite subgroup of the symplectic mapping class group. It is not as clear how to obtain a free group in the mapping class group from two projective twists, though, as exhibited by Keating in the spherical case [Kea14].

Remark 1.11. With Theorem 1.1 the projective twist cone formula easily generalizes to $\mathbb{R}\mathbb{P}^n$ and $\mathbb{H}\mathbb{P}^n$. The only difference between the formulas is the grading shift of the first term, as specified in Theorem 6.10.

$\mathbb{R}\mathbb{P}^n$ also gains a special feature: in this case the associated sphere S_{\pm} is equivalent to $\mathbb{R}\mathbb{P}^n$ equipped with a non-trivial \mathbb{Z}_2 -local system in the Fukaya category (see [Dam12, AB14, She15]). Therefore, the iterated cone relation can be packaged directly into a long exact sequence without invoking the iterated cones.

Structure of the paper

The geometric construction on Lagrangian surgeries via flow handles is part of the technical heart of the paper, and will occupy the first three sections: §2 describes the basic constructions, §3 explains how flow surgeries could be compared to Dehn twists, and §4 investigates the grading issues in Lagrangian surgeries in general. This proves Theorem 1.1, including the consideration of gradings. After briefly recalling the Wehrheim–Woodward quilted theory in §5, we give proofs of all claimed long exact sequences in §6. Appendix A is devoted to some computations of the connecting maps.

Conventions

- Throughout the paper, we assume any Lagrangian submanifold L of a symplectic manifold (M, ω) under consideration to be *exact* or *monotone*, which means the following.
 - (exactness) We have $\omega = d\alpha$ for some $\alpha \in \Omega^1(M)$, and $\alpha|_L = df$ for some smooth function f on L .
 - (monotonicity) For any $\beta \in \pi_2(M, L)$, $\omega(\beta) = \lambda\mu(\beta)$. Here $\lambda > 0$ and μ denotes the Maslov class, the minimal Maslov number satisfies $\mu_{\min} \geq 2$, and M is compact without boundary.

More precisely, results in §§2–4 do not involve Floer theory and are valid without these assumptions.

- All Lagrangian embeddings are assumed to be proper unless specified otherwise. In most situations, Lagrangian embeddings to M are compact even in the exact setting, while Lagrangian embeddings to $M \times \mathbb{C}$ are usually non-compact.

- The Hamiltonian vector field X_h of a Hamiltonian function h is defined by $\iota_{X_h}\omega = \omega(X_h, -) = -dh$ and the time- t flow under X_h is denoted as ϕ_t^h .
- We will also denote by $M^- = (M, -\omega)$ the *negation of the symplectic manifold* (M, ω) .

1.5 A note on coefficients

Throughout this paper we will use coefficient rings $\mathbb{Z}/2$ or $\Lambda_{\mathbb{Z}/2}$. Using characteristic zero coefficients is possible up to checking orientations for the general framework on Lagrangian cobordisms for [BC13, BC14]. As far as the coefficients are concerned, however, the Huybrechts–Thomas conjecture might not hold in general for \mathbb{C} -coefficients: for example, $\mathbb{C}\mathbb{P}^{2k}$ are not spin thus will be constrained to Fukaya categories defined over $\mathbb{Z}/2$ in many cases.

2. Dehn twist and Lagrangian surgeries

2.1 Dehn twist

Let S be a connected closed manifold equipped with a Riemannian metric $g(\cdot, \cdot)$ such that every geodesic is closed of length 2π (i.e., the shortest period of every unit-speed geodesic is 2π). We identify T^*S with TS by g and switch freely between the two. The following lemma is well known.

LEMMA 2.1. *The Hamiltonian $\sigma : T^*S \rightarrow \mathbb{R}$ defined by*

$$\sigma(\xi) = \|\xi\|$$

for all $q \in S$ and $\xi \in T_q^*S$ has the property that its Hamiltonian vector field X_σ generates the normalized geodesic flow on $T^*S \setminus \{0_{\text{section}}\}$ (this is the flow that parallel transports every tangent vector at unit-speed along the geodesic starting from it).

To define the Dehn twist, we need to introduce an auxiliary function. We first consider the case when S is not diffeomorphic to a sphere. For $\epsilon > 0$ small, we define a *Dehn twist profile* to be a smooth function $\nu_\epsilon^{\text{Dehn}} : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that (see Figure 2):

- (1) $\nu_\epsilon^{\text{Dehn}}(r) = 2\pi - r$ for $r \ll \epsilon$;
- (2) $0 < \nu_\epsilon^{\text{Dehn}}(r) < 2\pi$ for all $r < \epsilon$; and
- (3) $\nu_\epsilon^{\text{Dehn}}(r) = 0$ for $r \geq \epsilon$.

DEFINITION 2.2. If S is not diffeomorphic to a sphere, the model Dehn twist $(\tau_S, \nu_\epsilon^{\text{Dehn}})$ on T^*S is given by

$$\tau_S(\xi) = \phi_{\nu_\epsilon^{\text{Dehn}}(\|\xi\|)}^\sigma(\xi)$$

on $T^*S - \{0_{\text{section}}\}$ and identity on the zero section.

We will simply write τ_S instead of $(\tau_S, \nu_\epsilon^{\text{Dehn}})$.

When S is diffeomorphic to a sphere, the *spherical Dehn twist profile* $\nu_\epsilon^{\text{Dehn,sp}}$ is picked with (1), (2) above replaced by:

- (1') $\nu_\epsilon^{\text{Dehn,sp}}(r) = \pi - r$ for $r \ll \epsilon$; and
- (2') $0 < \nu_\epsilon^{\text{Dehn,sp}}(r) < \pi$ for all $r < \epsilon$.

In this case, the Dehn twist $(\tau_S, \nu_\epsilon^{\text{Dehn,sp}})$ is defined analogously but the antipodal map is used to extend smoothly along the zero section instead of the identity map.

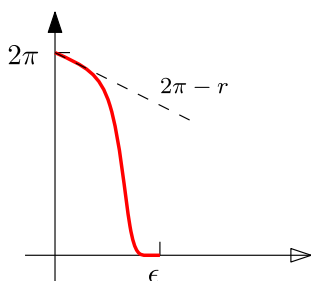


FIGURE 2. A Dehn twist profile $\nu_\epsilon^{\text{Dehn}}$ when $S \neq S^n$.

Example 2.3. Let $T^*S^1 = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$ with coordinates $(q, p) \in \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$ be equipped with the standard symplectic form $\omega_{S^1} = dp \wedge dq$. For a spherical profile $\nu_\epsilon^{\text{Dehn,sp}}$, $(\tau_{S^1}, \nu_\epsilon^{\text{Dehn,sp}})$ is defined by

$$\tau_{S^1}(q, p) = \begin{cases} \left(q + \nu_\epsilon^{\text{Dehn,sp}}(\|p\|) \frac{p}{\|p\|}, p \right) & \text{for } p \neq 0, \\ (q + \pi, 0) & \text{for } p = 0. \end{cases}$$

Consider the double cover $\iota_{\text{double}} : T^*S^1 \rightarrow T^*\mathbb{R}P^1 = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$ given by

$$\iota_{\text{double}}(q, p) = (2q, \frac{1}{2}p) = (\tilde{q}, \tilde{p}). \tag{2.1}$$

For $(\tilde{q}, \tilde{p}) = \iota_{\text{double}}(q, p) \in T^*\mathbb{R}P^1$, we define

$$T(\tilde{q}, \tilde{p}) = \iota_{\text{double}} \circ \tau_{S^1}(q, p)$$

which is independent of the choice of the preimage (q, p) of (\tilde{q}, \tilde{p}) . It is an easy exercise to show that T is Hamiltonian isotopic to $\tau_{\mathbb{R}P^1}$ for the new Dehn twist profile under the change of coordinates (2.1). Also, if we identify $T^*\mathbb{R}P^1$ with T^*S^1 so that τ_{S^1} is well defined on $T^*\mathbb{R}P^1$, then T is also Hamiltonian isotopic to $\tau_{S^1}^2$ for an appropriate choice of spherical profile.

This example has the following well-known immediate generalizations.

LEMMA 2.4. *Let $\iota_{\text{double}} : T^*S^n \rightarrow T^*\mathbb{R}P^n$ be the symplectic double cover obtained from the double cover of the zero section. For $(\tilde{q}, \tilde{p}) = \iota_{\text{double}}(q, p) \in T^*\mathbb{R}P^n$,*

$$T(\tilde{q}, \tilde{p}) = \iota_{\text{double}} \circ \tau_{S^n}(q, p)$$

is well defined and T is Hamiltonian isotopic to $\tau_{\mathbb{R}P^n}$ for an appropriate choice of auxiliary function defining $\tau_{\mathbb{R}P^n}$.

If $n > 1$, the choice of auxiliary function defining $\tau_{\mathbb{R}P^n}$ is irrelevant up to Hamiltonian isotopy.

Proof. Since the Hamiltonian function is radial and invariant under the antipodal map on S^n , the time- t flow on $T^*\mathbb{R}P^n$ lifts to a time- t flow on T^*S^n . More precisely, the function $\tilde{\sigma} := \sigma \circ \iota_{\text{double}} : T^*S^n \rightarrow \mathbb{R}$ is given by $\tilde{\sigma}(\xi) = \|\xi\|$ with respect to the pull-back bundle metric from $T^*\mathbb{R}P^n$ by ι_{double} so we have $\iota_{\text{double}} \circ \phi_t^{\tilde{\sigma}} = \phi_t^\sigma \circ \iota_{\text{double}}$. In particular, we have

$$\iota_{\text{double}} \circ \phi_{\nu_\epsilon^{\text{Dehn}}(\|\xi\|)}^{\tilde{\sigma}}(\xi) = \phi_{\nu_\epsilon^{\text{Dehn}}(\|\iota_{\text{double}}(\xi)\|)}^\sigma \circ \iota_{\text{double}}(\xi) \tag{2.2}$$

for any Dehn twist profile $\nu_\epsilon^{\text{Dehn}}$. Since the shortest period of geodesics of S^n with respect to the pull-back Riemannian metric by ι_{double} is 4π instead of 2π , the left-hand side of (2.2) equals $\iota_{\text{double}} \circ \tau_{S^n}(\xi)$ when $\|\xi\| \neq 0$. The right-hand side of (2.2) equals $\tau_{\mathbb{R}P^n} \circ \iota_{\text{double}}(\xi)$ when $\|\xi\| \neq 0$, so the result follows. \square

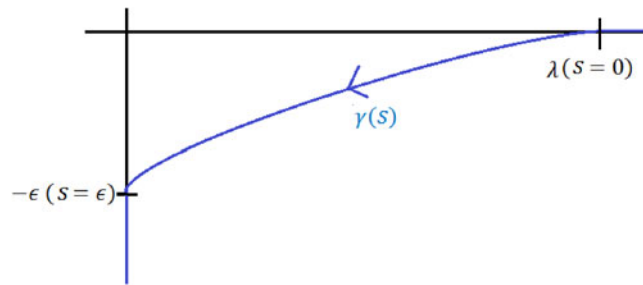


FIGURE 3. Picture of an admissible curve.

LEMMA 2.5 (See [Sei00] or [Har11]). For $T^*S^2 = T^*\mathbb{C}\mathbb{P}^1$, $\tau_{S^2}^2$ is Hamiltonian isotopic to $\tau_{\mathbb{C}\mathbb{P}^1}$.

As usual, one may globalize the model Dehn twist.

DEFINITION 2.6. A Dehn twist along S in M is a compactly supported symplectomorphism defined by the model Dehn twist as above in a Weinstein neighborhood of S and extended by identity outside.

For more details and the dependence of choices used to define τ_S , see [Sei99] and [Sei03].

2.2 Lagrangian surgery through flow handles

2.2.1 *Surgery at a point.* We first recall the definition of a Lagrangian surgery at a transversal intersection point from [LS91], [Pol91] and [BC13].

DEFINITION 2.7. Let $a(s), b(s) \in \mathbb{R}$. A smooth curve $\gamma(s) = a(s) + ib(s) \in \mathbb{C}$ is called λ -admissible if (see Figure 3):

- $(a(s), b(s)) = (-s + \lambda, 0)$ for $s \leq 0$;
- $a'(s), b'(s) < 0$ for $s \in (0, \epsilon)$; and
- $(a(s), b(s)) = (0, -s)$ for $s \geq \epsilon$ (note that $b(\epsilon) = -\epsilon$).

The part of a λ -admissible curve with $s \in [0, \epsilon]$ can be uniquely captured by $\nu_\lambda(r) := a(b^{-1}(-r)) \in [0, \lambda]$. The main property of an admissible curve can be translated into properties of ν_λ as follows:

- (1) $\nu_\lambda(0) = \lambda > 0$, and $\nu'_\lambda(r) < 0$ for $r \in (0, \epsilon)$;
- (2) $\nu_\lambda^{-1}(r)$ and $\nu_\lambda(r)$ have vanishing derivatives of all orders at $r = \lambda$ and $r = \epsilon$, respectively.

Such a function will also be called λ -admissible. We will frequently use the two equivalent descriptions of admissibility interchangeably.

We also define a class of *semi-admissible functions*, by relaxing (2) to:

- (2') $\nu'_\lambda(0) = -\alpha \in [-\infty, 0]$. Here $\alpha = \infty$ if ν_λ is admissible.

See an example of admissible and semi-admissible function in Figure 4.

Note that in all definitions of (semi-)admissibility there is an extra variable ϵ . We will see that the dependence on ϵ is not significant in this paper: we fix ϵ for each pair of Lagrangian submanifolds (L_1, L_2) once and for all. In any surgery constructions appearing later, the resulting surgery manifold yields a smooth family of Lagrangian isotopic submanifolds as ϵ varies. In the



(a) graph of an admissible function ν_λ (b) graph of a semi-admissible function ν_λ^α

FIGURE 4. Admissible and semi-admissible functions.

context of Dehn twists, this family becomes Hamiltonian isotopic. As a result, we will suppress the dependence of ϵ unless necessary.

Given a λ -admissible curve γ , define the handle

$$H_\gamma = \left\{ (\gamma(s)x_1, \dots, \gamma(s)x_n) \mid s, x_i \in \mathbb{R}, \sum x_i^2 = 1 \right\} \subset \mathbb{C}^n.$$

LEMMA 2.8. For a λ -admissible γ , H_γ is a Lagrangian submanifold of $(\mathbb{C}^n, \sum dx_i \wedge dy_i)$.

Proof. Let J be the standard complex structure on \mathbb{C}^n and $g(\cdot, \cdot) := \omega(\cdot, J\cdot)$ be the standard flat metric. Observe that $T_{\gamma(s)x}H_\gamma = \text{Span}_{\mathbb{R}}\{\gamma'(s)x\} \oplus \gamma(s)T_xS^{n-1}$, for $x = (x_1, \dots, x_n) \in S^{n-1} \subset \mathbb{R}^n$. A tangent vector in the second summand can be represented by $\gamma(s)y = \gamma(s)(y_1, \dots, y_n)$, where $y \in \mathbb{R}^n$ and $g(x, y) = 0$. We have

$$\begin{aligned} \omega(\gamma'(s)x, \gamma(s)y) &= -g(\gamma'(s)x, J(\gamma(s)y)) \\ &= -g(a'(s)x + b'(s)J(x), -b(s)y + a(s)J(y)) \\ &= a'(s)b(s)g(x, y) - b'(s)a(s)g(J(x), J(y)) = 0, \end{aligned}$$

where $\gamma(s) = (a(s), b(s))$. The fact that $\omega|_{\gamma(s)T_xS^{n-1}} = 0$ is obvious, so the result follows. \square

As a consequence, we have the following corollary.

COROLLARY 2.9. Let $L_1, L_2 \subset (M, \omega)$ be two Lagrangians transversely intersecting at p . Let $\iota : U \rightarrow M$ be the Darboux chart around p with a standard complex structure so that $\iota^{-1}(L_1) \subset \mathbb{R}^n$ and $\iota^{-1}(L_2) \subset i\mathbb{R}^n$ are disks centered at the origin. Then one can obtain a Lagrangian $L_1 \#_{p, st} L_2$ by attaching a Lagrangian handle $\iota(H_\gamma)$ to $(L_1 \cup L_2) \setminus \iota(U)$.

The Lagrangian $L_1 \#_{p, st} L_2$ is called a *Lagrangian surgery* from L_1 to L_2 following [LS91, Pol91]. Note that, the Lagrangian $L_2 \#_{p, st} L_1$ obtained by performing Lagrangian surgery from L_2 to L_1 is in general not even smoothly isotopic to $L_1 \#_{p, st} L_2$.

Now, we present an new approach to performing Lagrangian surgery which also motivates the definition of Lagrangian surgery along clean intersections.

DEFINITION 2.10. Given the zero section $L \subset T^*L$, a Riemannian metric g on L (hence inducing a bundle metric on T^*L), a point $x \in L$, and a choice of σ as in Lemma 2.1, we define the *flow handle* H_ν with respect to a λ -admissible function ν to be (see Figure 5)

$$H_\nu = \{ \phi_{\nu(\|p\|)}^\sigma(p) \in T^*L : p \in (T_x^*L)_\epsilon \setminus \{x\} \},$$

where $(T_x^*L)_\epsilon$ denotes the cotangent vectors at $x \in L$ with length $\leq \epsilon$.

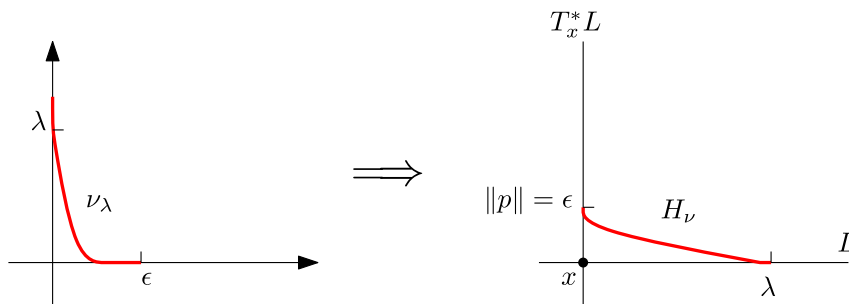


FIGURE 5. A flow handle.

The following elementary fact about geodesics will be repeatedly used.

LEMMA 2.11. *Let L , x and H_ν be as above. Let $\gamma : [0, c] \rightarrow L$ be a unit-speed geodesic starting at $\gamma(0) = x$. Then there is an induced symplectic embedding $T^*[0, c] \rightarrow T^*L$ and*

$$T^*[0, c] \cap H_{\nu^\alpha} = \left\{ \left(\frac{\nu(\|p\|)p}{\|p\|}, p \right) \in T^*[0, c] \mid p \in (0, \epsilon], \nu(p) \leq c \right\}.$$

Proof. We have an embedding $\gamma_* : T[0, c] \rightarrow TL$. The tangent bundles can be dualized to an embedding $T^*[0, c] \rightarrow T^*L$ using the standard metric on $T[0, 1]$ and TL . To see that this embedding is symplectic when γ is a geodesic, we can pick a geodesic ball U with a normal coordinate system $q = (q_1, \dots, q_n)$ centered at x . We can assume $\gamma(t) = (t, 0, \dots, 0)$. Let $0 \neq v \in T[0, c]$ and $\gamma_*v \in TL$. By definition, the dual of $\gamma_*v \in TL$, denoted by $(\gamma_*v)^* := g(-, \gamma_*v) \in T^*L$, vanishes when it is evaluated at vectors perpendicular to γ_*v . Since U is a geodesic ball, vectors perpendicular to $\mathbb{R}\gamma_*v = \mathbb{R}\partial_{q_1}$ are spanned by vectors $\partial_{q_2}, \dots, \partial_{q_n}$. Therefore, $(\gamma_*v)^* = |v|dq_1$ where $|v|$ is the norm of v measured in standard metric in $[0, c]$. As a result, the embedding $T^*[0, c] \rightarrow T^*L$ is given by $(q', p') \mapsto (q, p) = ((q', 0, \dots, 0), (p', \dots, 0))$ which is clearly symplectic. By definition, we have $T^*[0, c] \cap H_{\nu^\alpha} = \{(\nu(\|p\|)p/\|p\|, p) \in T^*[0, c] \mid p \in (0, \epsilon], \nu(p) \leq c\}$, so the result follows. \square

Remark 2.12. The time-1 Hamiltonian flow of $\tilde{\nu}(\|p\|)$, where $\tilde{\nu}'(s) = \nu(s)$, is $\phi_{\nu(\|p\|)}^\sigma$. For this reason, the reader should keep in mind that H_ν is automatically Lagrangian for any choice of admissible ν . For our purposes, the discussion on ν will be more flexible so we suppress the role of the actual Hamiltonian function $\tilde{\nu}$ unless otherwise specified.

LEMMA 2.13. *Let $S_\lambda(T_x^*L)$ be the radius λ -sphere in the cotangent plane of x . We use the exponential map on the cotangent bundle via the identification of T^*L and TL . If $\exp : S_\lambda(T_x^*L) \rightarrow L$ is an embedding, and $\partial H_\nu \cap L = \exp(S_\lambda(T_x^*L)) \subset L$ divides L into two components, then H_ν glues with exactly one of the components, as well as $T_x^*L \setminus (T_x^*L)_\epsilon$, to form a smooth Lagrangian submanifold coinciding with T_x^*L outside a compact set for a λ -admissible ν .*

Proof. The only thing left to prove is the smoothness of gluing on $\partial H_\nu = \exp(S_\lambda(T_x^*L))$. Note that from the assumptions, λ is not a critical radius of the exponential map, hence the exponential map is a diffeomorphism near $S_\lambda(T_x^*L)$. The flow $\phi_{\nu(\|p\|)}^\sigma$, followed by the projection to L , is the same as $\exp(\nu(\|p\|) \cdot (p/\|p\|))$, which is a diffeomorphism. Therefore, near ∂H_ν , the handle is a smooth

Lagrangian section of T^*L over the open shell $\exp(B_\lambda(T_x^*L) \setminus \overline{B_{\lambda-\delta}(T_x^*L)})$, which is a smooth open submanifold of L . Here, $B_r(T_x^*L)$ is the radius- r ball of T_x^*L centered at the origin. Moreover, the section has vanishing derivatives for all orders on the boundary component $S_\lambda(T_x^*L)$ due to the assumption of admissibility on $\nu(r)$ near $r = 0$. The conclusion follows. \square

Example 2.14. One may match the Lagrangian handle H_γ and flow handle H_ν for an admissible γ and its corresponding admissible $\nu(r) = a(b^{-1}(-r))$ (see Definition 2.7 and the paragraph after it) via the identification between $T^*\mathbb{R}^n$ and \mathbb{C}^n .

To see this, take an admissible curve $\gamma(s) = (a(s), b(s))$ and its flow handle H_γ . We consider the flow handle

$$H_\nu = \left\{ \phi_{\nu(\|p\|)}^\sigma(0, p) = \left(0 + \frac{p}{\|p\|} \cdot \nu(\|p\|), p \right) : p \in (T_0^*\mathbb{R}^n)_\epsilon \right\}.$$

We now identify $T^*\mathbb{R}^n$ with \mathbb{C}^n by sending $(q, p) \mapsto q - ip$, which matches the symplectic form $dp \wedge dq$ and $(1/-2i) dz \wedge d\bar{z} = dx \wedge dy$. Then by definition

$$\begin{aligned} \left(\nu(\|p\|) \frac{p}{\|p\|}, p \right) &\mapsto \nu(\|p\|) \frac{p}{\|p\|} - ip \\ &= (\nu(\|p\|) - i\|p\|) \frac{p}{\|p\|} \\ &= (a(b^{-1}(-\|p\|)) + ib(b^{-1}(-\|p\|))) \frac{p}{\|p\|} \\ &= (a(s) + ib(s))x \end{aligned}$$

by a change of variable $s = b^{-1}(-\|p\|)$ and $x = p/\|p\|$. By this identification, we will simply use H_ν to denote both handles.

COROLLARY 2.15. *Let $L_1, L_2 \subset (M, \omega)$ be two Lagrangians transversely intersecting at p . Under the assumption in Lemma 2.13, one can obtain a Lagrangian $L_1 \#_p^\nu L_2$ by gluing: (1) $L_2 \setminus U$, for an ϵ -neighborhood U of p ; (2) the Lagrangian flow handle H_ν ; and (3) an open set in L_1 that glues with H_ν given by Lemma 2.13. For appropriately chosen ν , $L_1 \#_p^\nu L_2$ coincides with $L_1 \#_{p, st} L_2$ defined in Corollary 2.9.*

Proof. Clearly H_ν glues smoothly with $L_2 \setminus U$ and the open set in L_1 by Lemma 2.13, and the result is a smooth Lagrangian submanifold. Let $\nu(x) = a(b^{-1}(-x))$ be chosen as in Example 2.14 for an admissible curve. The flow handle is then identified with the standard Lagrangian handle defined by Lalonde and Sikorav, and Polterovich. \square

The following lemma addresses the independence of the surgery Lagrangians on the choice of profile functions.

LEMMA 2.16. *If L_1 is simply-connected, and the surgery profile and the injectivity radius of the exponential map at x satisfies $\nu(0) < \text{inj}_g(x)$, then the Hamiltonian isotopy type of the surgery Lagrangian is independent of the choice of $\nu(r)$.*

Proof. For any two flow handles with different surgery profiles, there is a Lagrangian isotopy between them by a family of flow handles. We want to show that this Lagrangian isotopy can be extended to a symplectic isotopy using (a minor modification of) Banyaga’s isotopy extension theorem [MS98, Theorem 3.19].

Let W be a Weinstein neighborhood of L_1 and \mathcal{N} be a neighborhood of a flow handle in W . Following the proof of [MS98, Theorem 3.19], it is sufficient to show that the relative cohomology $H^2(W, \partial W \cup \mathcal{N}; \mathbb{R})$ vanishes. But this cohomology is nothing but the second cohomology of the following space: take the Thom space of T^*L_1 (whose second cohomology vanishes when $\dim L_1 > 2$), then collapse an additional cycle obtained from the connected sum of a fiber and the zero section. Therefore, the vanishing of $H^2(W, \partial W \cup \mathcal{N}; \mathbb{R})$ can be obtained by the long exact sequence of the relative cohomologies when L_1 is simply connected and $\dim L_1 > 2$. When $\dim L_1 = 2$, the long exact sequence reads

$$\begin{aligned} H^1(L_1; \mathbb{R}) \cong H^1(D^2 \# L_1, S^1; \mathbb{R}) &\rightarrow H^2(W, \partial W \cup D^2 \# L_1; \mathbb{R}) \rightarrow H^2(W, \partial W) \\ &\rightarrow H^2(D^2 \# L_1, S^1; \mathbb{R}). \end{aligned}$$

The last arrow is an isomorphism by restriction of the Thom form, hence the first arrow is again surjective. The rest of the argument will then go through. To argue the symplectic isotopy obtained by Banyaga's argument is Hamiltonian, use the assumption of simply-connectedness again. \square

We give two examples of flow handles that are different from standard handles.

Example 2.17. Let $r(p)$ be the injectivity radius of p . For different choices of $\nu(r)$ with $\nu(0) < r(p)$, these handles will define a family of different Lagrangian surgeries which are all Lagrangian isotopic to each other.

The situation becomes more interesting when $\nu(0) > r(p)$. Some simple instances are given by $S = \mathbb{R}P^n$, $\mathbb{C}P^n$ or any finite cover of rank-one symmetric space. Take $\mathbb{C}P^n$ and its standard Fubini–Study metric as an example, for any $p \in S$, the flow surgery can be performed for $kr(p) < \nu(0) < (k+1)r(p)$ for any $k \in \mathbb{Z}$. Later we will see that such surgeries are indeed iterated surgeries in the ordinary sense in Lemma 3.3 (although surgeries along clean intersections will be involved).

Example 2.18. A less standard example is essentially given by exotic spheres in [Sei14]. Given any $f \in \text{Diff}^+(S^{n-1})$, one may form an exotic sphere $S_f = B_- \cup_f B_+$ by gluing two copies B_{\pm} of B^n via f . There is a Riemannian metric on S_f so that all geodesics starting from the origins $0_{\pm} \in B_{\pm}$ are closed, passing through both 0_{\pm} , and of the same length [Sei14, Lemma 2.1]. Take $p = 0_- \in B_-$. When λ is below the injectivity radius, the corresponding flow handle surgery is the original one considered in [Pol91]. When $\nu(0) > r(p)$, the generalized surgery defined above is identified with an iterated surgery along p and $q = 0_+ \in B_+$ in a successive order, which is exactly the family constructed in [Sei14] by the geodesic flow.

The following lemma can be found in [Sei99], but we feel that it is instructive to sketch its proof from the point of view of flow handles to make our discussion complete.

LEMMA 2.19 [Sei99]. *Let $x \in S^n$ be a point and consider $L = \tau_{S^n}(T_x^*S^n) \subset T^*S^n$. Then $S^n \#_x^{\nu} T_x^*S^n$ is Hamiltonian isotopic to L through a compactly supported Hamiltonian, where ν is an admissible function such that $\nu(0) < r(x)$, the injectivity radius of x under the round metric.*

Proof. Let S^n be equipped with the round metric such that every embedded closed geodesic has length 2π . This induces a metric on T^*S^n which we will use throughout. Denote by $A : S^n \rightarrow S^n$ the antipodal map.

We consider two embeddings of open geodesic balls $B_\pi(x) \hookrightarrow S^n$ and $B_\pi(A(x)) \hookrightarrow S^n$ into the zero section, which are of radius π , and centered at x and $A(x)$, respectively. Also fix a normal coordinate system on each of these balls, which induces a trivialization (and a chart) on $T^*B_\pi(x)$ and $T^*B_\pi(A(x))$.

The two embeddings give two symplectomorphisms $f_x : T^*B_\pi(x) \rightarrow T^*S^n \setminus T^*_{A(x)}S^n$ and $f_{A(x)} : T^*B_\pi(A(x)) \rightarrow T^*S^n \setminus T^*_xS^n$. Indeed, these symplectomorphisms come from the definition of the canonical symplectic form on a cotangent bundle, which patches the forms $dp \wedge dq$ for the two local charts on the zero section as above. Recall the notation of spherical Dehn twist profile from the paragraph following Lemma 2.1, and ϵ is an arbitrary small positive constant. Under these symplectomorphisms we have by Lemma 2.11

$$f_x^{-1}(L) = \left\{ \left(\nu_\epsilon^{\text{Dehn,sp}}(\|p\|) \frac{p}{\|p\|}, p \right) \in T^*B_\pi(x) : p \in \mathbb{R}^n \setminus \{0\} \right\}$$

and

$$f_{A(x)}^{-1}(L) = \left\{ \left(\left(\pi - \nu_\epsilon^{\text{Dehn,sp}}(\|p\|) \right) \frac{p}{\|p\|}, p \right) \in T^*B_\pi(A(x)) : p \in B_\epsilon(0) \right\}.$$

Here, we use the properties that $B_\pi(x) \hookrightarrow S^n$ are geodesic ball embeddings.

On the other hand, suppose $\nu = \nu_\lambda$ is such that $\nu_\lambda(0) = \lambda < \pi = r(x)$ (see § 2.2.1). Then $f_x^{-1}(H_{\nu_\lambda})$ is given by

$$\begin{aligned} f_x^{-1}(H_{\nu_\lambda}) = & \left\{ \left(\nu_\lambda(\|p\|) \frac{p}{\|p\|}, p \right) \in T^*B_\pi(x) : p \in \mathbb{R}^n \setminus \{0\} \right\} \\ & \cup \{ (q, 0) \in T^*B_\pi(A(x)) : q \in B_\pi(0) \setminus B_\lambda(0) \}. \end{aligned}$$

Let $\delta > 0$ be such that $\nu_\epsilon^{\text{Dehn,sp}}(r) = \pi - r$ for $r < \delta$. We can pick ν_λ such that $\nu_\lambda(r) = \nu_\epsilon^{\text{Dehn,sp}}(r)$ for $r \geq \delta$. The resulting $S^n \#_x^{\nu_\lambda} T^*S^n$ hence coincides with L outside $T^*B_\delta(A(x))$. Inside $T^*B_\delta(A(x))$, even though $\nu_\epsilon^{\text{Dehn,sp}}$ is not an admissible function, both $S^n \#_x^{\nu_\lambda} T^*S^n$ and L are graphs of exact 1-forms (although the primitive function does not vanish near the boundary). Therefore, $S^n \#_x^{\nu_\lambda} T^*S^n$ is Lagrangian isotopic to L by varying the primitive function in $T^*B_\delta(A(x))$ but fixing them near the boundary, and hence Hamiltonian isotopic to L by a compactly supported Hamiltonian due to the simply-connectedness of T^*S^n and L . (Extending an exact Lagrangian isotopy to a Hamiltonian isotopy is well known when the Lagrangian is compact, but when the exact Lagrangian has cylindrical ends, then it requires a relative version of Banyaga extension and the simply-connectedness assumption. See a completely parallel argument in Lemma 2.16.) □

Remark 2.20. Consider the setting as in Corollary 2.15. For semi-admissible ν^α that is not admissible, the gluing with L_1 cannot be smooth in general (for example, consider a semi-admissible profile where $\nu(0) < \pi$ in the case when L_1 is a sphere with radius 1, then we will have a corner locus which is the circle that we glue along). Lemma 2.19 is an instance when a surgery using a semi-admissible profile $\nu_\epsilon^{\text{Dehn}}$ yields a smooth Lagrangian submanifold. Intuitively, the lemma regards $\nu_\epsilon^{\text{Dehn}}$ as a degenerate case of an admissible function. The point is that, when $\lambda = r(p)$, we only need to glue $\text{Cl}(H_\nu)$ with $L_2 \setminus U$ (compare with Lemma 2.13, which exemplifies the more common case where $\text{Cl}(H_\nu)$ glues with an open subset of L_2), where $\text{Cl}(\cdot)$ denotes the closure and U is an ϵ -neighborhood of p .

In the case when a semi-admissible function defines a smooth Lagrangian surgery manifold, we will continue to denote it as $L_1 \#_p^{\nu^\alpha} L_2$. This applies to other surgeries along clean intersections and will be used several more times in a parametrized version in the paper.

2.2.2 *Surgery along clean intersection.* Let L_1 and L_2 be two Lagrangians in (M, ω) which intersect cleanly at a submanifold D . In other words, we have $T_p D = T_p L_1 \cap T_p L_2$ for all $p \in D$. The following well-known local proposition due to Pozniak allows us to extend the definition of flow handles to this case.

PROPOSITION 2.21 [Poz99]. *Let $L_1, L_2 \subset (M, \omega)$ be two closed embedded Lagrangians with clean intersection at $L_1 \cap L_2 = D$. Then there is a symplectic embedding φ from a neighborhood U of $0_{\text{section}} \subset T^*L_1$ to M such that $\varphi(0_{\text{section}}) = L_1$ and $\varphi^{-1}(L_2) \subset N_D^*$, where 0_{section} is the zero section and N_D^* is the conormal bundle of D in L_1 .*

DEFINITION 2.22. Fix a metric g on L . We define the *flow handle* for $D \subset L$ with respect to an admissible function ν to be

$$H_\nu^D = \{\phi_{\nu(\|\xi\|)}^\sigma(\xi) \in T^*L : \xi \in (N_D^*)_\epsilon \setminus D\},$$

where $(N_D^*)_\epsilon$ consists of covectors in the conormal bundle of D in L with length $\leq \epsilon$.

LEMMA 2.23. *Let $S_\lambda(N_D^*)$ be the radius- λ sphere bundle in the conormal bundle of D . If $\exp : S_\lambda(N_D^*) \rightarrow L$ is an embedding, and $\partial H_{\nu_\lambda}^D \cap L \subset L$ divides L into two components, then H_ν^D glues with exactly one of the components to form a smooth Lagrangian submanifold coinciding with N_D^* outside a compact set.*

The proof is exactly the same as Lemma 2.13 and we omit it. Below, we call $r(D) > 0$ the *injectivity radius* of D , which is the supremum of $r > 0$ such that the (dual) exponential map of $(N_D^*)_{\leq r}$ is an embedding. It is positive since it is easy to check to be positive in a chart, and we only consider the case when D is compact. As in the transversal intersection case, the surgery is always well defined when we choose $\nu(0) = \lambda < r(D)$. Using Proposition 2.21, we globalize the construction as follows.

COROLLARY 2.24. *Let $L_1, L_2 \subset (M, \omega)$ be two Lagrangians intersecting cleanly at D . By choosing a metric on L_1 , a symplectic embedding $\iota : (T^*L_1)_\epsilon \rightarrow M$ such that $\iota(0_{\text{section}}) = L_1$ and $\iota^{-1}(L_2) \subset N_D^*$, one can obtain a Lagrangian $L_1 \#_D^\nu L_2$ by attaching a Lagrangian flow handle $\iota(H_\nu^D)$ to $(L_1 \setminus U_1) \cup (L_2 \setminus U_2)$, with $U_i \subset L_i$ appropriate open neighborhoods of D , and ϵ being sufficiently small.*

As in Example 2.17, we denote $L_1 \#_D^\nu L_2$ by $L_1 \#_D L_2$ if $\lambda < r(D)$.

2.3 E_2 -flow surgery and its family version

We will introduce a generalization of flow handle which will be useful when performing surgeries on Lagrangian submanifolds in product symplectic manifolds later. Heuristically, our previous constructions have taken advantage of the fact that $\|p\|$ has a well-defined Hamiltonian flow on the whole cotangent bundle except for the zero section. More crucially, the resulting flow handle should have an embedded boundary in L_1 . Indeed, any Hamiltonian function with such properties will suffice for defining a meaningful Lagrangian handle.

A variant of the flow handle can therefore be defined as follows. Let $L = K_1^{n-m} \times K_2^m$ be a product manifold equipped with a product Riemannian metric. Then there is an orthogonal decomposition $T^*L = E_1 \oplus E_2$ given by the two factors respectively. Let $D \subset L$ be of codimension m and transverse to $\{p\} \times K_2$ for all $p \in K_1$. Let $\pi_2 : T^*L \rightarrow E_2$ be the projection to E_2 . One may then use the function $\sigma_\pi(\cdot) = \|\pi_2(\cdot)\|_g : T^*L \rightarrow \mathbb{R}$ to define a new flow handle. Note that $\sigma_\pi = \|\pi_2(\cdot)\|_g$ is smooth on $T^*L \setminus E_1$.

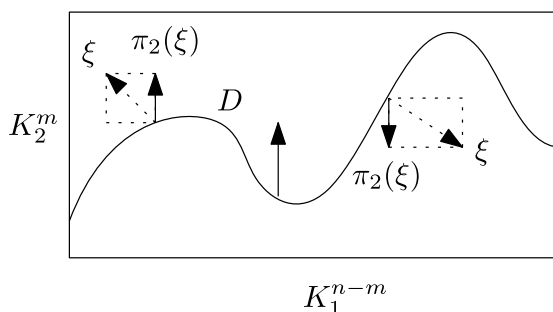


FIGURE 6. An E_2 -flow handle.

DEFINITION 2.25. In the situation above, we define the E_2 -flow handle for D (or flow handle along the E_2 -direction) with respect to an λ -admissible ν_λ to be (see Figure 6)

$$H_{\nu_\lambda}^{D,E_2} = \{ \phi_{\nu_\lambda(\|\pi_2(\xi)\|)}^{\sigma\pi}(\xi) \subset T^*L : \xi \in (N_D^*)_{\epsilon,E_2} \setminus D \},$$

where $(N_D^*)_{\epsilon,E_2}$ consists of covectors ξ in the conormal bundle of D in L such that $\|\pi_2(\xi)\| \leq \epsilon$.

We note that for any point $\xi = (\xi_1, \xi_2) \in E_1 \oplus E_2$, $\phi_t^{\sigma\pi}(\xi) = (\xi_1, \phi_t^\sigma(\xi_2))$ so E_2 -flow is the normalized (co)geodesic flow on the second factor and is trivial on the first factor.

Let $S_\lambda(E_2|_D)$ be the radius- λ sphere bundle of E_2 over D . We consider $\exp_\lambda^{E_2} : S_\lambda(E_2|_D) \rightarrow L$, which is the exponential map restricted on $S_\lambda(E_2|_D)$ along the leaves of the foliation given by second factor. We define the E_2 -injectivity radius $r^{E_2}(D)$ of D as the supremum of λ such that $\exp_s^{E_2}$ is an embedding for all $s < \lambda$.

LEMMA 2.26. Let $D \subset L = K_1 \times K_2$ be of dimension $n - m$ and transversal to $\{p\} \times K_2$ for all $p \in K_1$. If $\exp_\lambda^{E_2} : S_\lambda(E_2|_D) \rightarrow L$ is an embedding and $\partial H_\nu^{D,E_2} \subset L$ divides L into two components, then H_ν^{D,E_2} glues with exactly one of the components of L to form a smooth Lagrangian submanifold coinciding with N_D^* outside a compact set.

Proof. The proof is again similar to that of Lemma 2.23. □

Similarly to the cases we considered before, if $L_1 = K_1 \times K_2$ and L_2 are Lagrangians cleanly intersecting at D as above, we can add an E_2 -flow handle to $L_1 \cup L_2$ outside a tubular neighborhood of D to get a new Lagrangian submanifold for $\lambda < r^{E_2}(D)$. We will denote the resulting Lagrangian submanifold by $L_1 \#_{D,E_2} L_2$, called the E_2 -flow surgery from L_1 to L_2 along D .

A family version of E_2 -flow surgery

We now consider the E_2 -flow surgery for a family over a symplectic base. Assume that we have a smooth manifold pair $(L = K_1 \times K_2, D)$, a decomposition $T^*L = E_1 \oplus E_2$ and a Lagrangian handle H_ν^{D,E_2} as above.

Let $(\mathcal{P}, \omega_\mathcal{P})$ be a symplectic manifold equipped with a symplectic fiber bundle structure $\pi : \mathcal{P} \rightarrow \mathcal{B}$, such that it has a symplectic base $(\mathcal{B}, \omega_\mathcal{B})$ and fibers symplectomorphic to T^*L . Let $i : B \hookrightarrow \mathcal{B}$ be a Lagrangian submanifold such that the structure group G of $i^*\mathcal{P} \rightarrow B$ is a subgroup of $\text{Isom}(L) \hookrightarrow \text{Symp}(T^*L)$. In particular, G preserves L and we make the following further assumptions on G .

- (i) G preserves D , the subbundles E_1 and E_2 and H_ν^{D,E_2} .
- (ii) For any loop $\gamma : \mathbb{R}/\mathbb{Z} \rightarrow B$, the symplectic monodromy around γ for the fiber bundle $\pi : \mathcal{P} \rightarrow B$ lies in G .
- (iii) Let $x \in \mathcal{P}$, $b := \pi(x) \in B$ and $\mathcal{P}_b := \pi^{-1}(b)$. Assume Ker_x is the symplectic orthogonal complement of the fiber $T_x\mathcal{P}_b \subset T_x\mathcal{P}$ at x , then we ask $\omega_{\mathcal{P}}(v_1, v_2) = 0$ for any $v_1, v_2 \in \text{Ker}_x$ such that $\pi_*v_i \in TB \subset TB$ for both i .

Since L and D are invariant under G , we can use symplectic parallel transport to obtain a smooth manifold pair $(\mathcal{L}, \mathcal{D})$ in \mathcal{P} . More precisely, let $\mathcal{P}_b = T^*L$ be a reference fiber with based point $b \in B$. We define $\mathcal{L} := \bigcup_{\gamma \in I_B} \Gamma_\gamma(L)$, $\mathcal{D} := \bigcup_{\gamma \in I_B} \Gamma_\gamma(D)$, where I_B consists of all paths γ from b to another point $b' \in B$ and Γ_γ is the symplectic parallel transport along γ . As a result, $(\mathcal{L}, \mathcal{D})$ has a compatible fiber bundle structure over the base B , that is,

$$\begin{array}{ccc} L & \longrightarrow & \mathcal{L} \\ & & \downarrow \\ & & B \end{array} \quad \text{and} \quad \begin{array}{ccc} D & \longrightarrow & \mathcal{D} \\ & & \downarrow \\ & & B \end{array}$$

where the two bundle structures are compatible with the inclusions $\mathcal{D} \hookrightarrow \mathcal{L} \hookrightarrow i^*\mathcal{P}$.

All previous symplectic constructions on T^*L are G -invariant, by assumption (i), hence can be glued over B . For example, N_D^*L glues into $N_{\mathcal{D}}^*\mathcal{L} := \bigcup_{\gamma \in I_B} \Gamma_\gamma(N_D^*L)$. When $i^*\mathcal{P}$ is regarded as a vector bundle over \mathcal{L} , it comes with a natural splitting $i^*\mathcal{P} = \mathcal{E}_1 \oplus \mathcal{E}_2$, where $\mathcal{E}_i := \bigcup_{\gamma \in I_B} (\Gamma_\gamma)_*(E_i)$ for $i = 1, 2$. Moreover, the E_2 -handle H_ν^{D,E_2} on fibers can be glued together, which gives a smooth handle $\mathcal{H}_\nu := \bigcup_{\gamma \in I_B} \Gamma_\gamma(H_\nu^{D,E_2}) \subset i^*\mathcal{P}$. The fact that $\mathcal{H}_\nu \hookrightarrow \mathcal{P} \supset i^*\mathcal{P}$ is indeed a Lagrangian embedding follows from the assumptions (ii) and (iii) on G .

LEMMA 2.27. *For two cleanly intersecting Lagrangians $\mathcal{L}_0, \mathcal{L}_1 \subset (M^{2n}, \omega)$, if there is a neighborhood of \mathcal{L}_0 which can be identified with $(\mathcal{P}, \omega_{\mathcal{P}})$ above (together with all the bundle structures and assumptions on G) such that $(\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_0 \cap \mathcal{L}_1)$ is identified with $(\mathcal{L}, N_{\mathcal{D}}^*\mathcal{L}, \mathcal{D})$, then the family E_2 -surgery between \mathcal{L}_0 and \mathcal{L}_1 can be performed and gives a Lagrangian submanifold $\mathcal{L}_0 \#_{\mathcal{D}, E_2}^{\nu} \mathcal{L}_1$ of (M, ω) .*

Despite the fact that many assumptions are imposed on $(\mathcal{P}, \omega_{\mathcal{P}})$ and G , there are practical examples where Lemma 2.27 applies (see § 3.3).

Remark 2.28. It is easy to see that our construction works word by word as long as there is a decomposition of the vector bundle $T^*L = E_1 \oplus E_2$. However, one needs to impose technical conditions to make $\exp_{\lambda}^{E_2} : S_{\lambda} \rightarrow L$ an embedding even for small λ . An easy condition is to assume that E_2 is integrable at least near D , but it should also work in some cases when E_2 is completely non-integrable near D but integrable outside a small neighborhood. Considerations along this line might result in delicate constructions of new Lagrangian submanifolds.

3. Isotopies: from surgeries to Dehn twists

This section contains the construction which relates Lagrangian surgeries to various kinds of Dehn twists. The general idea is the same as in Lemma 2.19, which may also be interpreted as deforming an admissible profile to a semi-admissible one. The deformation from an admissible profile to a semi-admissible profile will correspond to a Lagrangian isotopy between appropriate Lagrangians.

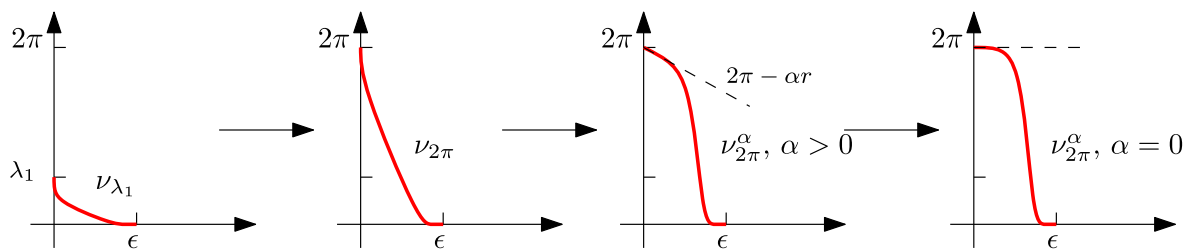


FIGURE 7. Isotopy from ν_{λ_1} to $\nu_{2\pi}$ to $\nu_{2\pi}^\alpha$ (Dehn twist profile).

We first explain how this works in the $\mathbb{C}\mathbb{P}^n$ case, then give a proof of Theorem 1.1(1), (2), (3), (4) using family versions of this observation.

3.1 Fiber version

In this section, we are interested in L being $\mathbb{R}\mathbb{P}^n$, $\mathbb{C}\mathbb{P}^{m/2}$ or $\mathbb{H}\mathbb{P}^n$ equipped with the Riemannian metric such that every geodesic is closed of length 2π . All actual proofs will be given only in the case of $\mathbb{C}\mathbb{P}^n$ but are easily generalized. Let $x \in L$ be a point and $F_x = T_x^*L$. We also let $D_x = \{y \in L \mid \text{dist}(x, y) = \pi\}$ be the *submanifold opposite to x* . In the case of $\mathbb{C}\mathbb{P}^{m/2}$, $D_x \simeq \mathbb{C}\mathbb{P}^{m/2-1}$.

LEMMA 3.1. *Let $x \in L$ be a point and ν_{λ_i} , $i = 1, 2$, be λ_i -admissible functions such that $(k-1)\pi < \lambda_i < k\pi$ for some positive integer k for both $i = 1, 2$. Then $L\#_x^{\nu_{\lambda_i}}F_x$ for $i = 1, 2$ are isotopic by a compactly supported Hamiltonian.*

Moreover, if we choose a semi-admissible function $\nu_{k\pi}^\alpha : (0, \infty) \rightarrow [0, k\pi)$ that is monotonic decreasing and all orders of derivatives vanish at $r = \epsilon$ such that $\nu_{k\pi}^\alpha(r) = k\pi - \alpha r$ near $r = 0$ ($\alpha \geq 0$), then $L\#_x^{\nu_{k\pi}^\alpha}F_x$ (see Remark 2.20 for the definition of $L\#_x^{\nu_{k\pi}^\alpha}F_x$ when $\nu_{k\pi}^\alpha$ is semi-admissible) is a smooth Lagrangian that is isotopic to $L\#_x^{\nu_{\lambda_i}}F_x$ by a compactly supported Hamiltonian.

Furthermore, these Hamiltonian isotopies can be chosen to be invariant under the action of the group of isometries of L that fix x .

COROLLARY 3.2. *For $\pi < \lambda < 2\pi$ and L being $\mathbb{R}\mathbb{P}^n$, $\mathbb{C}\mathbb{P}^{m/2}$ or $\mathbb{H}\mathbb{P}^n$, $L\#_x^{\nu_\lambda}F_x$ is Hamiltonian isotopic to $\tau_L(F_x)$ for an admissible ν_λ .*

Proof of Corollary 3.2. Observe that when $\alpha = 1$ and $k = 2$, $\nu_{k\pi}^\alpha(r)$ is a Dehn twist profile (see Figure 7). The Corollary follows from Lemma 3.1. \square

Proof of Lemma 3.1. For the first statement, we observe that the space of λ -admissible functions for $(k-1)\pi < \lambda < k\pi$ is connected. A smooth isotopy $\{\nu_t\}$ from ν_{λ_1} to ν_{λ_2} in this space results in a smooth Lagrangian isotopy from $L\#_x^{\nu_{\lambda_1}}F_x$ to $L\#_x^{\nu_{\lambda_2}}F_x$ since ∂H_{ν_t} does not pass any critical locus. This is a Hamiltonian isotopy because $H^1(L\#_x^{\nu_t}F_x, \partial^\infty(L\#_x^{\nu_t}F_x); \mathbb{R}) = 0$ (cf. Example 2.17). Here, $\partial^\infty(L\#_x^{\nu_t}F_x)$ is the infinite end of $L\#_x^{\nu_t}F_x$.

For the second statement, we only consider the case that $k = 1$ and $L = \mathbb{C}\mathbb{P}^{m/2}$, and the remaining cases are similar. In this case, denote $\nu^\alpha = \nu_\pi^\alpha$. Then for a handle H_{ν^α} at x , $\text{Cl}(H_{\nu^\alpha}) \setminus H_{\nu^\alpha} = D_x = \mathbb{C}\mathbb{P}^{m/2-1}$. We pick a local chart $U \subset L$ with local coordinates (q_1, \dots, q_m) adapted to D_x in the sense that $U \cap D_x = \{q_1 = q_2 = 0\}$ and $c(t) = (tq_1, tq_2, q_3, \dots, q_m)$ are unit-speed geodesics perpendicular to D_x at $t = 0$, for any (q_1, \dots, q_m) such that $|q_1|^2 + |q_2|^2 = 1$.

It induces canonically a Darboux chart T^*U in T^*L . We write a point in T^*U as (q_a, q_b, p_a, p_b) , where $q_a = (q_1, q_2)$, $q_b = (q_3, \dots, q_m)$ and similarly for p_a and p_b . Since H_{ν^α} is defined by the geodesic flow, we have

$$T^*U \cap H_{\nu^\alpha} = \{(q_a, q_b, p_a, 0) \mid q_a = -\alpha p_a \neq 0\} \tag{3.1}$$

$$T^*U \cap D_x = \{(0, q_b, 0, 0)\} \tag{3.2}$$

when U is sufficiently small. To see that it is true, we consider a unit-speed geodesic $\gamma : [0, \pi] \rightarrow L$ from x to a point $x' \in D_x$. By Lemma 2.11, we have a symplectic embedding $T^*[0, \pi] \rightarrow T^*L$ and

$$T^*[0, \pi] \cap H_{\nu^\alpha} = \left\{ \left(\frac{\nu^\alpha(\|p\|)p}{\|p\|}, p \right) \in T^*[0, \pi] \mid p \in (0, \epsilon] \right\}. \tag{3.3}$$

On the other hand, the symplectic embedding $T^*[0, \pi] \cap T^*U \rightarrow T^*U$ is given by

$$(q, p) \mapsto (q_a, q_b, p_a, p_b) = ((\pi - q)c_1, c_2, -pc_1, 0) \tag{3.4}$$

for some c_1, c_2 such that $|c_1| = 1$ because U is adapted to D_x (by the same reasoning as in the proof of Lemma 2.11). When $p > 0$ close to zero, $q = \nu^\alpha(\|p\|)p/\|p\| = \pi - \alpha p$ so we have $(\pi - q) = \alpha p$ and hence $(\pi - q)c_1 = \alpha pc_1$. Therefore, (3.3) and (3.4) imply (3.1) when we consider all possible unit-speed geodesics from x to points on D_x .

From the local description, it is clear that H_{ν^α} and D_x can be glued smoothly to become $\text{Cl}(H_{\nu^\alpha})$. The gluing from H_{ν^α} to $F_x - B_\epsilon$ is the same as in the admissible case. It results in a smooth Lagrangian $L\#_x^{\nu^\alpha} F_x$.

Finally, we want to show that $L\#_x^{\nu^\alpha} F_x$ is Hamiltonian isotopic to $L\#_x^{\nu_\lambda} F_x$. We can assume $\alpha \neq 0$, by a Hamiltonian perturbation if necessary. Locally near D_x , we have

$$T^*U \cap (H_{\nu^\alpha} \cup D_x) = \{(-\alpha p_a, q_b, p_a, 0)\} = \left\{ \left(q_a, q_b, -\frac{1}{\alpha}q_a, 0 \right) \right\} \tag{3.5}$$

$$T^*U \cap L = \{(q_a, q_b, 0, 0)\}. \tag{3.6}$$

Notice that, by the choice of U , we have $\text{dist}^2(\cdot, D_x) = |q_a|^2 := |q_1|^2 + |q_2|^2$ in U . It implies that $H_{\nu^\alpha} \cup D_x$ is the graph of $d(-(1/2\alpha)\text{dist}^2(\cdot, D_x))$ in T^*U . Since the definition of the graph of $d(-(1/2\alpha)\text{dist}^2(\cdot, D_x))$ is coordinate-free, by gluing charts that are adapted to D_x , we know that there is a small $\delta > 0$ such that $(H_{\nu^\alpha} \cup D_x) \cap T^*B_\delta(D_x)$ is the graph of $d(-(1/2\alpha)\text{dist}^2(\cdot, D_x))$ over $B_\delta(D_x)$, where $B_\delta(D_x)$ is the δ neighborhood of D_x in L . Take a smooth decreasing function $f(r) : [0, \delta] \rightarrow \mathbb{R}$ so that $f = 0$ near $r = 0$ and $f(r) = -(1/2\alpha)r$ near $r = \delta$. Denote $f_t(r) = tf(r) - (1-t)(1/2\alpha)r$.

Then the graph of $d(f_t \circ \text{dist}^2(\cdot, D_x))$ can be patched with $H_{\nu^\alpha} \setminus T^*B_\delta(D_x)$ to give a Hamiltonian isotopy from $L\#_x^{\nu^\alpha} F_x$ to $L\#_x^{\nu_\lambda} F_x$ for some admissible ν_λ with $0 < \lambda < \pi$. We remark that the Hamiltonian isotopy is invariant under $\text{Isom}(L)_x$, the isometry group of L fixing x . This concludes the proof.

The following could be helpful to understand the construction. An alternative way to describe the isotopy $d(f_t \circ \text{dist}^2(\cdot, D_x))$ is that we use a family of semi-admissible function $\{\nu^{\alpha_t}\}_{t \in [0,1]}$ ($\alpha_t > 0$) such that $\alpha_0 = \alpha$ and ν^{α_t} approaches to ν_λ when t goes to 1. Then we will have $L\#_x^{\nu^\alpha} F_x = d(f_t \circ \text{dist}^2(\cdot, D_x))$ near D_x . □

Later we will see that, when the surgery profile ν_λ has λ exceeding the injectivity radius, there is no cobordism directly associated to such a surgery. To fit such a surgery into the cobordism framework, in general we need to decompose the surgery into several steps. The following lemma shows how this works in the case of $\mathbb{C}\mathbb{P}^n$ (which easily generalizes to $\mathbb{R}\mathbb{P}^n$ and $\mathbb{H}\mathbb{P}^n$).

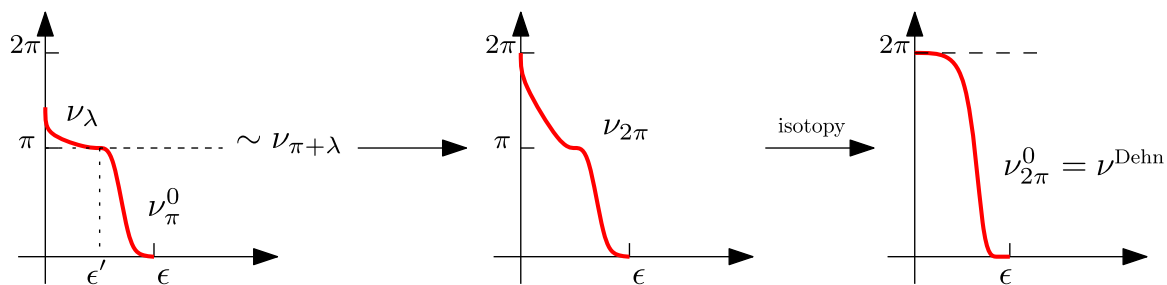


FIGURE 8. Left and middle: identification of $\mathbb{C}\mathbb{P}^{m/2} \#_{D_x}^{\nu_\lambda} Q_x$ and $\mathbb{C}\mathbb{P}^{m/2} \#_x^{\nu_{\lambda+\pi}} F_x$. Right: isotopy from an admissible function to a Dehn twist profile.

LEMMA 3.3. Let $x \in \mathbb{C}\mathbb{P}^{m/2}$ be a point and $F_x = T_x^* \mathbb{C}\mathbb{P}^{m/2}$. Let $D_x = \{y \in \mathbb{C}\mathbb{P}^{m/2} \mid \text{dist}(x, y) = \pi\}$ be the submanifold opposite to x . Then there is an embedded Lagrangian $Q_x \subset T^* \mathbb{C}\mathbb{P}^{m/2}$ such that:

- (1) $Q_x = F_x$ away from a neighborhood of zero section;
- (2) Q_x is Hamiltonian isotopic to $\mathbb{C}\mathbb{P}^{m/2} \#_{x, \text{std}} F_x$;
- (3) Q_x intersects cleanly with $\mathbb{C}\mathbb{P}^{m/2}$ at D_x ;
- (4) $\mathbb{C}\mathbb{P}^{m/2} \#_{D_x} Q_x$ is Hamiltonian isotopic to $\tau_{\mathbb{C}\mathbb{P}^{m/2}}(F_x)$.

As a result, as far as Hamiltonian isotopy class is concerned, we have $\mathbb{C}\mathbb{P}^{m/2} \#_{D_x}(\mathbb{C}\mathbb{P}^{m/2} \#_x F_x) = \tau_{\mathbb{C}\mathbb{P}^{m/2}}(F_x)$.

Proof. Choose a semi-admissible profile ν_π^0 such that $\nu_\pi^0 = \pi$ near $r = 0$ and let $Q_x = \mathbb{C}\mathbb{P}^{m/2} \#_x^{\nu_\pi^0} F_x$. Then (1), (3) follows from definition, and (2) is a consequence of Lemma 3.1 and Corollary 3.2.

To see (4), note that near D_x , Q_x coincides with the ϵ' -disk conormal bundle at D_x for some $\epsilon' \ll \epsilon$. Therefore, $\mathbb{C}\mathbb{P}^{m/2} \#_{D_x}^{\nu_\lambda} Q_x$ coincides with $\mathbb{C}\mathbb{P}^{m/2} \#_x^{\nu_{\lambda+\pi}} F_x$ for any $0 < \lambda < \pi$ and an appropriate choice of $\nu_{\lambda+\pi}$ (see Figure 8 for the demonstration). The latter is then Hamiltonian isotopic to $\mathbb{C}\mathbb{P}^{m/2} \#_x^{\nu_{2\pi}^0} F_x = \tau_{\mathbb{C}\mathbb{P}^{m/2}}(F_x)$ by Corollary 3.2. \square

3.2 Product version

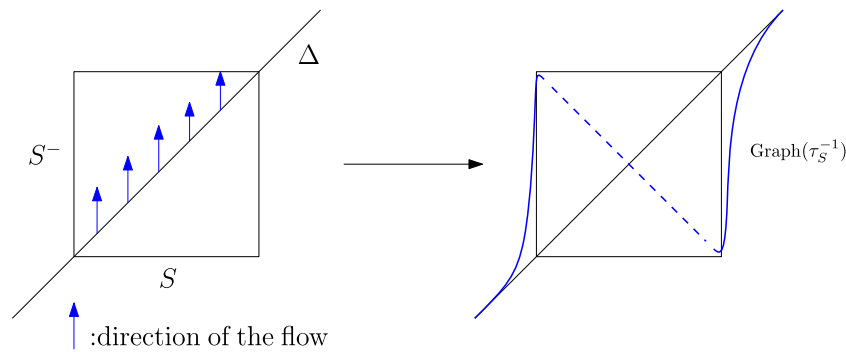
In this section we prove Theorem 1.1(1), (3). The proofs here are similar to that in the last section, and should be considered as family versions of it. In this subsection, we use S to denote S^n , $\mathbb{R}\mathbb{P}^n$, $\mathbb{C}\mathbb{P}^{m/2}$ or $\mathbb{H}\mathbb{P}^n$ equipped with the Riemannian metric such that every geodesic is closed of length 2π .

For a symplectomorphism $\tau : (M, \omega) \rightarrow (M, \omega)$, we define the graph of τ as

$$\text{Graph}(\tau) := \{(p, \tau(p)) \in M \times M \mid p \in M\}.$$

In particular, $\text{Graph}(\tau)$ is a Lagrangian submanifold if we equip $M \times M$ with the symplectic form $\omega \oplus -\omega$.

For the moment, let $S \subset (M, \omega)$ be a Lagrangian sphere and $S^- := S \subset M^-$. One may consider the clean surgery of $L_1 = S \times S^-$ and $L_2 = \Delta$ in $M \times M^-$. In this case, they cleanly intersect along $D = \Delta_S \subset S \times S^-$. In Definition 2.25, take $E_2 = S \times (T^*S)^- \subset T^*S \times (T^*S)^-$, $E_1 = T^*S \times S^- \subset T^*S \times (T^*S)^-$ and a π -admissible function ν_π .


 FIGURE 9. Obtaining the graph of a Dehn twist by E_2 -flow surgery.

Let $U \subset M$ be a Weinstein neighborhood of S which can be identified with T_ϵ^*S , the open set of T^*S consisting of covectors with length less than ϵ . It induces an identification between $U \times U$, a Weinstein neighborhood of L_1 , and $T_\epsilon^*S \times T_\epsilon^*S$. Now consider a point $(p, p) \in \Delta_U := \Delta \cap (U \times U)$, where p can be considered as a point in T_ϵ^*S . The flow in Definition 2.25 defines a symplectomorphism fixing the first coordinate in $(T^*S \times (T^*S)^-) \setminus E_1$; when restricted to Δ_U , the E_2 -flow sends $(p, p) \mapsto (p, \phi_{\nu_\pi(\|p\|)}^\sigma(p))$. Therefore, the image of $\Delta_U \setminus \Delta_S$ under the flow is an open subset of the graph of τ_S^{-1} (the inverse owes to the negation of symplectic form on M^-), except that we have used an admissible profile for the handle which is not a Dehn twist profile. Lemma 3.4 below ensures that this could be compensated by a local Hamiltonian perturbation. Hence modulo Lemma 3.4, this shows that $(S \times S^-) \#_{\Delta_S, E_2}^{\nu_\pi} \Delta = \text{Graph}(\tau_S^{-1})$ (see Figure 9). The whole construction applies when S is $\mathbb{R}P^n$, $\mathbb{C}P^{m/2}$ or $\mathbb{H}P^n$, except that the admissible profile has $\nu(0) = 2\pi$.

LEMMA 3.4. Let S be S^n , $\mathbb{R}P^n$, $\mathbb{C}P^{m/2}$ or $\mathbb{H}P^n$. Let ν_{λ_i} be λ_i -admissible functions such that $(k-1)\pi < \lambda_i < k\pi$ for some positive integer k for both $i = 1, 2$. Then the E_2 -flow surgered Lagrangian manifolds $(S \times S^-) \#_{\Delta_S, E_2}^{\nu_{\lambda_i}} \Delta$ above with surgery profiles ν_{λ_i} are Hamiltonian isotopic.

Moreover, if we choose a semi-admissible function $\nu_{k\pi}^\alpha$ such that $\nu_{k\pi}^\alpha(r) = k\pi - \alpha r$ near $r = 0$ ($\alpha \geq 0$), then $(S \times S^-) \#_{\Delta_S, E_2}^{\nu_{k\pi}^\alpha} \Delta$ is a smooth Lagrangian that is Hamiltonian isotopic to $(S \times S^-) \#_{\Delta_S, E_2}^{\nu_{\lambda_i}} \Delta$.

Furthermore, these Hamiltonian isotopies can be chosen to be $\text{Isom}_\Delta(S)$ invariant, where $\text{Isom}_\Delta(S)$ is the diagonal isometry group in $\text{Isom}(S) \times \text{Isom}(S)$ acting on $T^*S \times (T^*S)^-$.

We have the following corollary whose proof is similar to Corollary 3.2

COROLLARY 3.5 (Cf. Theorem 1.1(1)). For $\pi < \lambda < 2\pi$ and S being $\mathbb{R}P^n$, $\mathbb{C}P^{m/2}$ or $\mathbb{H}P^n$ (respectively $0 < \lambda < \pi$ and $S = S^n$), $(S \times S^-) \#_{\Delta_S, E_2}^{\nu_\lambda} \Delta$ is Hamiltonian isotopic to $\text{Graph}(\tau_S^{-1})$.

Proof of Corollary 3.5. When $\alpha = 1$ and $k = 2$ (respectively $k = 1$), $(S \times S^-) \#_{\Delta_S, E_2}^{\nu_{k\pi}^\alpha} \Delta$ coincides with $\text{Graph}(\tau_S^{-1})$. Therefore, the result follows from Lemma 3.4. \square

Proof of Lemma 3.4. The proof of the first statement is exactly the same as Lemma 3.1. For the second statement, we again only consider the case that $k = 1$ and $S = \mathbb{C}P^{m/2}$ and the remaining cases are similar.

Define $D^{\text{op}} = \{(x, y) \in S \times S \mid \text{dist}(x, y) = \pi\}$. Projection to the first factor in $S \times S$ equips D^{op} with a $\mathbb{C}\mathbb{P}^{m/2-1}$ -bundle structure over $S = \mathbb{C}\mathbb{P}^{m/2}$. Therefore, a tubular neighborhood of D^{op} in $S \times S^-$ is the total space $\tilde{\mathcal{V}}$ of a fiber bundle $\pi_{\tilde{\mathcal{V}}} : \tilde{\mathcal{V}} \rightarrow S$, whose fibers \mathcal{V} are total spaces of topological $\mathcal{O}(1)$ -bundles over $\mathbb{C}\mathbb{P}^{m/2-1}$. We pick an open subset $U^B \subset S$ and a local trivialization $U^B \times U^F \simeq \pi_{\tilde{\mathcal{V}}}^{-1}(U^B)$ (so $U^F = \mathcal{V}$). Readers should note that the product structure of $U^B \times U^F$ is not compatible with the product structure of $L = S \times S^-$, but $\{q\} \times U^F$ is an open set of $\{q\} \times S^-$ for any $q \in U^B$. In other words, we should regard U^B as a parameter space and points in U^B are parameterizing some open subsets of S^- .

Pick a chart $U^B \times U^{F'}$ on $U^B \times U^F$ with local coordinates $(q^B, q^F) = (q_1^B, \dots, q_m^B, q_1^F, \dots, q_m^F)$ adapted to D^{op} in the sense that:

- $(U^B \times U^{F'}) \cap D^{\text{op}} = \{q_1^F = q_2^F = 0\}$;
- $c(t) = (q^B, tq_1^F, tq_2^F, q_3^F, \dots, q_m^F)$ are unit-speed geodesics for all (q^B, q^F) such that $|q_1^F|^2 + |q_2^F|^2 = 1$; and
- $c(t) \in \{q^B\} \times S$ is perpendicular to $\{q^B\} \times D_{q^B} \subset \{q^B\} \times S$ at $t = 0$ with respect to the metric on $\{q^B\} \times S = S$, where D_{q^B} is the submanifold opposite to $q^B \in S$.

Note that $c(0) \in D^{\text{op}}$ and the projection of $c(0)$ to the first factor is q^B so $c(0) \in \{q^B\} \times D_{q^B}$.

The inclusions $U^B \times U^{F'} \subset U^B \times U^F \subset S \times S^-$ induces canonical inclusions $T^*(U^B \times U^{F'}) \subset T^*(U^B \times U^F) \subset T^*(S \times S^-)$. We write a point in $T^*(U^B \times U^{F'})$ as $(q^B, p^B, q_a^F, q_b^F, p_a^F, p_b^F)$, where $q_a^F = (q_1^F, q_2^F)$, $q_b^F = (q_3^F, \dots, q_m^F)$ and similarly for p_a^F and p_b^F . We consider points $(q^B, p^B) \in T^*U^B$ as points in T^*S (because $U^B \subset S$).

For each point $(q^B, p^B) \in T^*S$, there is a corresponding point $\phi_{\nu_\pi(\|p^B\|)}^\sigma(q^B, p^B) \in T^*S^-$ such that $((q^B, p^B), \phi_{\nu_\pi(\|p^B\|)}^\sigma(q^B, p^B)) \in H_{\nu^\alpha} \subset T^*S \times T^*S^-$. There are also corresponding open sets $\{(q^B, p^B)\} \times T^*U^{F'} \subset \{(q^B, p^B)\} \times T^*S^-$ and $\{q^B\} \times U^{F'} \subset \{q^B\} \times S^-$ such that $\{q^B\} \times U^{F'}$ is adapted to $\{q^B\} \times D_{q^B}$ in the sense that:

- $\{q^B\} \times U^{F'} \cap \{q^B\} \times D_{q^B} = \{q^B\} \times \{q_a^F = 0\}$; and
- $c(t) = (q^B, tq_1^F, tq_2^F, q_3^F, \dots, q_m^F)$ are unit-speed geodesics and perpendicular to D_{q^B} at $t = 0$, for any $q^F \in U^F$ such that $|q_1^F|^2 + |q_2^F|^2 = 1$.

Since $\phi_{\nu_\pi(\|\cdot\|)}^\sigma(\cdot)$ is defined by the geodesic flow on $\{(q^B, p^B)\} \times T^*S^- = T^*S^-$ and $\{q^B\} \times U^{F'}$ is adapted to $\{q^B\} \times D_{q^B}$, we know that $\phi_{\nu^\alpha(\|p^B\|)}^\sigma(q^B, p^B) = (-\alpha p_a^F, q_b^F, p_a^F, 0)$ when $p^B \neq 0$ small (as in the proof of Lemma 3.1). Now, we have a parametrized version of (3.5)

$$T^*(U^B \times U^{F'}) \cap H_{\nu^\alpha} = \{(q^B, p^B, -\alpha p_a^F, q_b^F, p_a^F, 0) \mid p^B \neq 0, \phi_{\nu^\alpha(\|p^B\|)}^\sigma(q^B, p^B) = (-\alpha p_a^F, q_b^F, p_a^F, 0)\} \quad (3.7)$$

when U^B and $U^{F'}$ are sufficiently small.

Here, both $\phi_{\nu_\pi(\|p^B\|)}^\sigma(q^B, p^B)$ and $(-\alpha p_a^F, q_b^F, p_a^F, 0)$ are considered as points in T_ϵ^*S although they belong to different factors of $T^*(S \times S^-)$. Therefore, in $T^*(U^B \times U^{F'}) \cap H_{\nu^\alpha}$, fixing q^B and letting p^B go to 0 linearly leads to fixing q_b^F and letting p_a^F go to zero linearly.

Since H_{ν^α} is globally defined, the above discussion is true for any charts $U^B \times U^{F'}$ on $U^B \times U^F$ adapted to D^{op} . From the discussion using local charts on $U^B \times U^F$ and the fact that p_a^F goes to zero linearly as p^B goes to 0 linearly, we can see that H_{ν^α} and D^{op} can be glued smoothly (linearity is not necessary for H_{ν^α} and D^{op} to be glued smoothly but it is sufficient). The fact that H_{ν^α} can be glued smoothly with Δ is because all orders of derivatives of ν^α vanish at $r = \epsilon$. It results in a smooth Lagrangian, which we denote by $(S \times S^-) \#_{\Delta_S, E_2}^{\nu^\alpha} \Delta$.

Finally, we show that $(S \times S^-) \#_{\Delta_S, E_2}^{\nu^\alpha} \Delta$ is Hamiltonian isotopic to $(S \times S^-) \#_{\Delta_S, E_2}^{\nu_{\lambda_1}} \Delta$. We can choose $\{\nu^{\alpha t}\}_{t \in [0,1]}$ interpolating ν^α and ν_{λ_1} as in the proof of Lemma 3.1. This is a smooth Lagrangian isotopy which is invariant under the diagonal $\text{Isom}(S)$ action. \square

In parallel to Lemma 3.3, we have the following.

LEMMA 3.6 (Theorem 1.1(3)). *Let S be $\mathbb{R}P^n, \mathbb{C}P^{m/2}$ or $\mathbb{H}P^n$ and*

$$D^{\text{op}} = \{(x, y) \in S \times S^- \mid \text{dist}(x, y) = \pi\}.$$

*Up to Hamiltonian isotopy in $T^*S \times (T^*S)^-$, we have*

$$(S \times S^-) \#_{D^{\text{op}}, E_2} ((S \times S^-) \#_{\Delta_S, E_2} \Delta) = \text{Graph}(\tau_S^{-1}).$$

Proof. The proof is similar to that of Lemma 3.3 and we again assume $S = \mathbb{C}P^{m/2}$. We use the function ν_π^0 in Lemma 3.3 to define $\mathcal{S} = (S \times S^-) \#_{\Delta_S, E_2}^{\nu_\pi^0} \Delta$, which is Hamiltonian isotopic to $(S \times S^-) \#_{\Delta_S, E_2} \Delta$ by Lemma 3.4. Now, \mathcal{S} intersects $S \times S^-$ cleanly along D^{op} . We can perform another E_2 -flow surgery from $S \times S^-$ to \mathcal{S} . Then $S \times S^- \#_{D^{\text{op}}, E_2} \mathcal{S}$ is identical to $S \times S^- \#_{\Delta_S, E_2}^{\nu_{\pi+\lambda}} \Delta$ for some admissible function $\nu_{\pi+\lambda}$. By Lemmas 3.4 and 3.3, $S \times S^- \#_{\Delta_S, E_2}^{\nu_{\pi+\lambda}} \Delta$ is Hamiltonian isotopic to $S \times S^- \#_{\Delta_S, E_2}^{\nu_{2\pi}^0} \Delta = \text{Graph}(\tau_S^{-1})$ so we obtain the result. \square

3.3 Family versions

One may also generalize the above example to the case of family Dehn twists [WW16] which we now recall. Let G be $\text{SO}(l+1)$ with Lie algebra \mathfrak{g} . A connection one form on a principal G bundle $\pi : P \rightarrow B$ is a one form $\alpha \in \Omega^1(P, \mathfrak{g})$ satisfying the following two conditions:

- $\alpha(\xi_P) = \xi$ for any $\xi \in \mathfrak{g}$, where ξ_P is the vector field generating the action of ξ ;
- $g^* \alpha = \text{Ad}(g)^{-1} \alpha$ for any $g \in G$, where the adjoint action is on the values of α .

We have a splitting

$$TP = \ker(\alpha) \oplus \ker(D\pi)$$

which is invariant under the group action.

Suppose B admits a symplectic structure ω_B . We equip $(T^*S^l, \omega_{T^*S^l})$ with the Hamiltonian G -action induced by the isometry of the zero section $G = \text{Isom}(S^l) \subset \text{Symp}(T^*S^l)$, where $\omega_{T^*S^l}$ is the canonical symplectic structure on T^*S^l . We denote the moment map by $\Phi : T^*S^l \rightarrow \mathfrak{g}^\vee$. The *minimally coupling form* on $P \times T^*S^l$ is defined by

$$\omega_{P \times T^*S^l, \alpha} := \pi_P^* \pi^* \omega_B + \pi_S^* \omega_{T^*S^l} + d\langle \pi_P^* \alpha, \pi_S^* \Phi \rangle \in \Omega^2(P \times T^*S^l), \tag{3.8}$$

where $\pi_P : P \times T^*S^l \rightarrow P$ and $\pi_S : P \times T^*S^l \rightarrow T^*S^l$ are projections to the first and second factors, respectively. The two form $\omega_{P \times T^*S^l, \alpha}$ has the property that $\iota_{\xi_{P \times F}} \omega_{P \times F, \alpha} = 0$ and it descends to a symplectic form $\omega_{P(T^*S^l)}$ on $P \times_G T_\epsilon^* S^l$ for some $\epsilon > 0$, where $\xi_{P \times F}$ is the vector field in $P \times F$ generating the diagonal action of ξ and $T_\epsilon^* S^l$ consists of the cotangent vectors with norm less than ϵ . Moreover, $\pi \circ \pi_P : P \times T^*S^l \rightarrow B$ descends to a symplectic fiber bundle map $\pi_{P(T^*S^l)} : P \times_G T_\epsilon^* S^l \rightarrow B$. An important feature of $\omega_{P(T^*S^l)}$ is the following.

LEMMA 3.7. *We have $\omega_{P \times T^*S^l, \alpha}(v_1, v_2) = 0$ for any $v_1 \in \ker(\alpha)$ and $v_2 \in T(T^*S^l)$. Therefore, the symplectic orthogonal complement of fibers of $\pi_{P(T^*S^l)}$ in $P \times_G T_\epsilon^* S^l$ is the image of $\ker(\alpha)$ under the quotient map $P \times T_\epsilon^* S^l \rightarrow P \times_G T_\epsilon^* S^l$.*

Proof. We have a decomposition $T(P \times T^*S^l) = \ker(\alpha) \oplus \ker(D\pi) \oplus T(T^*S^l)$. Since $v_1 \in \ker(\alpha)$, we have $v_1 \in \ker(\pi_S^* \omega_{T^*S^l})$ and $v_1 \in \ker(\pi_S^* d\Phi)$. Since $v_2 \in T(T^*S^l)$, we have $v_2 \in \ker(\pi_P^* \pi^* \omega_B)$ and $v_2 \in \ker(\pi_P^* d\alpha)$. The result follows from (3.8). \square

COROLLARY 3.8. *For the symplectic fibration $\pi_{P(T^*S^l)} : P \times_G T_\epsilon^* S^l \rightarrow B$, the monodromy of each loop in B by symplectic parallel transport lies in G .*

Proof. By definition, those monodromies of parallel transports with respect to α in P lie in G . Therefore, so is $P \times T_\epsilon^* S^l$ and hence $P \times_G T_\epsilon^* S^l$. The result follows because symplectic orthogonal complement of fibers coincides with the image of $\ker(\alpha)$, by Lemma 3.7. \square

Recall that a *spherically fibered coisotropic manifold* $i : C^{2n-l} \hookrightarrow M^{2n}$ is a coisotropic submanifold so that there is a fibration $\rho : C \rightarrow B^{2n-2l}$ over a symplectic base, whose fibers are leaves of the characteristic foliation (also called null-leaves) and are diffeomorphic to S^l . In other words, $\rho^* \omega_B = i^* \omega_M$. Moreover, we equip the fibers with round metrics such that all their geodesics are closed of length 2π and ask that the structure group of ρ lie in $SO(l+1)$.

A neighborhood U of C can be symplectically identified with $P \times_{SO(l+1)} T_\epsilon^* S^l$, where P is the principal $SO(l+1)$ -bundle associated to C . The *family Dehn twist* τ_C can then be defined fiberwise as the fiberwise Hamiltonian function $\tilde{v}_\epsilon^{\text{Dehn}}(\|p\|)$ (see Remark 2.12) is preserved by the structure group. With respect to the fiberwise metric g^v , the function $h(\cdot) = \tilde{v}(\|\cdot\|_{g^v})$ defines a flow along fibers whose time-1 map is the desired Dehn twist (with a continuation over C defined by the fiberwise antipodal map on C).

Equivalently, we can choose a reference fiber $F = T^*S^l$ of the symplectic fibration $\pi_{P(T^*S^l)}$ and the family Dehn twist restricted to any fiber F' is defined to be $\tau_C|_{F'} := \Gamma_{\gamma^{-1}} \circ \tau_{S^l} \circ \Gamma_\gamma$, where γ is a path from the base point of F' to the base point of F , Γ_γ is symplectic parallel transport along γ and τ_{S^l} is the Dehn twist on F . By Corollary 3.8, $\Gamma_{\gamma^{-1}} \circ \tau_{S^l} \circ \Gamma_\gamma$ is independent of γ and gives a symplectomorphism of F' . The fiberwise symplectomorphisms patch together to a diffeomorphism τ_C . To see that τ_C is a symplectomorphism of $P \times_{SO(l+1)} T_\epsilon^* S^l$, it suffices to check that

$$\omega_{P(T^*S^l)}((\tau_C)_* v_1, (\tau_C)_* v_2) = \omega_{P(T^*S^l)}(v_1, v_2) \tag{3.9}$$

for any $v_1, v_2 \in \ker(\alpha)$ because $T(P \times_{SO(l+1)} T_\epsilon^* S^l) = \ker(\alpha) \oplus T(T_\epsilon^* S^l)$, $(\ker(\alpha))^{\omega_{P(T^*S^l)}} = T(T_\epsilon^* S^l)$ by Lemma 3.7 and we already know that τ_C is fiberwise symplectic. By construction, $(\tau_C)_* v_i \in \ker(\alpha)$ and $(\pi_{P(T^*S^l)} \circ \tau_C)_* v_i = (\pi_{P(T^*S^l)})_* v_i$ for $i = 1, 2$ so (3.9) follows from (3.8).

Now consider the natural Lagrangian embedding $\tilde{C} := C \times_B C \hookrightarrow M \times M$, where M is a symplectic manifold such that we have an inclusion $i : C^{2n-l} \hookrightarrow M^{2n}$ making C a spherically fibered coisotropic manifold. Explicitly, the image of this map is

$$\tilde{C} = \{(x, y) \in C \times C \subset M \times M : \pi(x) = \pi(y)\},$$

where $\pi : C \rightarrow B$ is the S^l -bundle projection. Indeed, $\tilde{C} = C^t \circ C$ is a composition Lagrangian in the sense of (5.2). Here we have abused the notation by identifying C with its Lagrangian image in $B \times M$ defined by

$$\{(x, y) \in B \times C \subset B \times M : \pi(y) = x\}.$$

Note that \tilde{C} is a fiber bundle over B with fiber $S^l \times S^l$ and structure group the diagonal $SO(l+1)$.

We continue to use U to denote a neighborhood of C in M which can be identified with $P \times_{SO(l+1)} T_\epsilon^* S^l$. Consider a symplectic trivialization $U_0 := B_0 \times T_\epsilon^* S^l$ of the symplectic fiber

bundle $\pi_{P(T^*S^l)} : U = P \times_{\text{SO}(l+1)} T_\epsilon^* S^l \rightarrow B$ for some contractible open set $B_0 \subset B$. Any point contained in $\Delta \cap (U_0 \times U_0^-) \subset M \times M^-$ thus takes the form $((x, p), (x, p))$, where $x \in B_0$ and $p \in T_\epsilon^* S^l$. In this setting, the graph of τ_C^{-1} in $U_0 \times U_0^-$ consist of points

$$\text{Graph}(\tau_C^{-1}) = \{((x, p), (x, \phi_{\nu_{\text{Dehn}}(\|p\|)}^\sigma(p))) \mid x \in B_0, p \in T_\epsilon^* S^l\}.$$

This is true for any contractible open subset $B_0 \subset B$ and $\text{Graph}(\tau_C^{-1})$ coincides with Δ outside $U \times U^-$.

As before, we want to realize $\text{Graph}(\tau_C^{-1})$ as a surgery from \tilde{C} to Δ . In this case, we want to perform a family E_2 -surgery.

In the notation of § 2.3, let $L = S^l \times (S^l)^-, D = \Delta_{S^l} \subset L, \mathcal{P} = U \times U^-, \mathcal{B} = B \times B^-, i : B \rightarrow \Delta_B \subset \mathcal{B}$ be the diagonal embedding, $\mathcal{L} = \tilde{C}, N_{\mathcal{D}}^* \mathcal{L} = \Delta_U, \mathcal{D} = \Delta_U \cap \tilde{C}$, and $G = \text{SO}(l+1) = \text{Isom}_\Delta(L) \subset \text{SO}(l+1) \times \text{SO}(l+1)$ is the diagonal isometry group. It is clear how to define the fiber bundle structures

$$\begin{array}{ccc} L & \longrightarrow & \mathcal{L} \\ & & \downarrow \\ & & B \end{array} \quad \text{and} \quad \begin{array}{ccc} D & \longrightarrow & \mathcal{D} \\ & & \downarrow \\ & & B \end{array}$$

compatible with the inclusions $\mathcal{D} \hookrightarrow \mathcal{L} \hookrightarrow i^* \mathcal{P}$. As in the calculation for Lemma 3.7 and Corollary 3.8, we know that the symplectic orthogonal complement of the fibers of $\pi : \mathcal{P} \rightarrow \mathcal{B}$ is given by the product horizontal distribution $\ker(\alpha) \oplus (\ker(\alpha))^-$ and the symplectic monodromy for loops in B lies in the diagonal isometry group G . We leave it as an exercise for readers to check that G satisfies all the other assumptions in § 2.3. As a result, we can define a global Lagrangian handle $\mathcal{H}_{\nu_\pi} \subset \mathcal{P}$.

By the same token, we can define a *projectively fibered coisotropic manifold* which is a coisotropic manifold with null-leaves complex (or real, quaternionic) projective spaces. Family Dehn twists for these spaces are defined similarly.

LEMMA 3.9. *Let $C \subset (M, \omega)$ be a spherically (respectively projectively) coisotropic submanifold with base B . Let ν_{λ_i} be λ_i -admissible functions such that $(k-1)\pi < \lambda_i < k\pi$ for some positive integer k for both $i = 1, 2$. Then the family E_2 -flow surgered Lagrangian manifolds $\tilde{C} \#_{\mathcal{D}, E_2}^{\nu_{\lambda_i}} \Delta$ for $i = 1, 2$ are Hamiltonian isotopic.*

Moreover, if we choose a semi-admissible function $\nu_{k\pi}^\alpha : (0, \infty) \rightarrow [0, k\pi)$ such that $\nu_{k\pi}^\alpha(r) = k\pi - \alpha r$ near $r = 0$ ($\alpha \geq 0$), then $\tilde{C} \#_{\mathcal{D}, E_2}^{\nu_{k\pi}^\alpha} \Delta$ is a smooth Lagrangian that is Hamiltonian isotopic to $\tilde{C} \#_{\mathcal{D}, E_2}^{\nu_{\lambda_i}} \Delta$.

COROLLARY 3.10 (Theorem 1.1(2), (4)). *For spherically (respectively projectively) coisotropic submanifold C , the family E_2 -flow clean surgery $\tilde{C} \#_{\mathcal{D}, E_2} \Delta$ (respectively $\tilde{C} \#_{\mathcal{D}^{\text{op}}, E_2} \tilde{C} \#_{\mathcal{D}, E_2} \Delta$) is Hamiltonian isotopic to $\text{Graph}(\tau_C^{-1})$. Here \mathcal{D}^{op} is a D^{op} -bundle over the base B and D^{op} is as in Lemma 3.6.*

Proof of Lemma 3.9. We give the proof for the spherical case and the other cases are similar. Since the construction in Lemma 3.4 is $\text{SO}(l+1)$ invariant, we can apply Lemma 3.4 to \tilde{C} and Δ_U inside $\mathcal{P} = U \times U^-$ fiberwise to obtain the desired Lagrangian isotopy from $\tilde{C} \#_{\mathcal{D}, E_2}^{\nu_{\lambda_1}} \Delta$ to $\tilde{C} \#_{\mathcal{D}, E_2}^{\nu_{\lambda_2}} \Delta$ and from $\tilde{C} \#_{\mathcal{D}, E_2}^{\nu_{\lambda_2}} \Delta$ to $\tilde{C} \#_{\mathcal{D}, E_2}^{\nu_{\lambda_1}} \Delta$.

What remains to show is that the Lagrangian isotopies are Hamiltonian isotopies. We prove the case where the Lagrangian isotopy is from $\tilde{C}\#_{\mathcal{D},E_2}^{\nu^\alpha}\Delta$ to $\tilde{C}\#_{\mathcal{D},E_2}^{\nu_{\lambda_i}}\Delta$. The other case is similar. Let $\iota_{K,t} : K \rightarrow T^*S^l \times (T^*S^l)^{-1}$ be the Lagrangian isotopy from $(S \times S^-)\#_{\Delta_S,E_2}^{\nu_{k\pi}}\Delta$ to $(S \times S^-)\#_{\Delta_S,E_2}^{\nu_{\lambda_i}}\Delta$ in Lemma 3.4, which is $SO(l+1)$ invariant. Let $b \in B$ be a reference based point. We define the Lagrangian isotopy $\iota_{\mathcal{K},t} : \mathcal{K} \rightarrow M \times M^-$ by

$$\iota_{\mathcal{K},t}(x) = \Gamma_{\gamma^{-1}} \circ \iota_{K,t} \circ \Gamma_{\gamma}(x),$$

where $t \in [0, 1]$, $\gamma : [0, 1] \rightarrow B$ is a path from the based point of x (i.e., projection of x to B) to b and $\mathcal{K} = \tilde{C}\#_{\mathcal{D},E_2}^{\nu^\alpha}\Delta$. It is well defined because the monodromy of $\mathcal{K} \rightarrow B$ is inherited from $i^*\mathcal{P} \rightarrow B$.

Since $\iota_{K,t} : L \rightarrow T^*S^l \times (T^*S^l)^{-1}$ is a Hamiltonian isotopy, it is an exact isotopy (i.e. $\theta_0 := \iota_{K,t}^*(\omega_{\text{can}} \oplus -\omega_{\text{can}})(\partial\iota_{K,t}/\partial t, \cdot)$ is exact). Since the fiberwise symplectic form and the isotopy are $SO(l+1)$ -invariant, so is θ_0 and its primitive. These primitives on fibers can be patched together to form a function $f : \mathcal{K} \rightarrow \mathbb{R}$ such that $\theta - df$ vanishes on fibers, where $\theta := \iota_{\mathcal{K},t}^*(\omega_M \oplus -\omega_M)(\partial\iota_{\mathcal{K},t}/\partial t, \cdot)$. As a consequence of Lemma 3.7, the distribution of symplectic orthogonal complements of fibers of $\mathcal{P} \rightarrow \mathcal{B}$ coincides with $\ker(\alpha) \oplus (\ker(\alpha))^-$. Let $\text{Ker} \subset T(\mathcal{K})$ be the horizontal distribution with respect to the fiber bundle $\mathcal{K} \rightarrow B$ that is mapped into (and actually also onto) the diagonal horizontal distribution $\ker(\alpha)_\Delta \subset \ker(\alpha) \oplus (\ker(\alpha))^-$ under the inclusion $\mathcal{K} \rightarrow \mathcal{P}$. By definition, θ vanishes on Ker . Since f is $SO(l+1)$ -invariant, df also vanishes on Ker and hence $\theta = df$. This implies that $\iota_{\mathcal{K},t}$ is an exact, thus a Hamiltonian, isotopy. \square

Corollary 3.10 is now an immediate consequence of Lemma 3.9 by setting $k = 1$ for spherical case and $k = 2$ for the projective cases.

4. Gradings and energy

In this section we discuss the gradings in Lagrangian surgeries. We follow mostly the exposition in [AB14] to review the definition of gradings in §4.1. The subsequent subsections provide computations for sufficient and necessary criteria to perform graded surgeries. Starting from §5, all surgeries between graded Lagrangians will be graded surgeries. Our discussion stays in the \mathbb{Z} -graded and exact case but the corresponding results for \mathbb{Z}/N -gradings in the monotone setting can be obtained by modifying our argument using the setting in [Sei00] and the statements will be a modulo- N reduction of what we have here.

4.1 Basic notions

We assume $2c_1(M) = 0$ and fix once and for all a nowhere-vanishing section Ω^2 of $(\Lambda_C^{\text{top}}(T^*M, J))^{\otimes 2}$.

Let $\iota_L : L \rightarrow M$ be an exact Lagrangian immersion (i.e., $\iota_L^*\alpha$ is exact). A *grading* on (L, ι_L) (sometimes simply denoted by ι_L) is defined as a continuous function $\theta_L : L \rightarrow \mathbb{R}$ such that $e^{2\pi i\theta_L} = \text{Det}_\Omega^2(\text{Im}(D\iota_L))$, where $\text{Im}(D\iota_L)$ is the image of $D\iota_L$ and Det_Ω^2 is defined as

$$\text{Det}_\Omega^2(\Lambda_p) = \text{Det}_\Omega^2(X_1, \dots, X_n) = \frac{\Omega^2(X_1, \dots, X_n)}{\|\Omega^2(X_1, \dots, X_n)\|} \in S^1$$

for any Lagrangian plane $\Lambda_p \subset T_pM$ at a point p and any choice of a basis $\{X_1, \dots, X_n\}$ for Λ_p .

Given two transversal Lagrangian planes Λ_0, Λ_1 (of dimension n) at the same point with a choice of θ_0, θ_1 such that $e^{2\pi i\theta_j} = \text{Det}_{\Omega}^2(\Lambda_j)$ for both j , we can identify them as graded Lagrangian vector subspaces of \mathbb{C}^n . The index of the pair (Λ_0, θ_0) and (Λ_1, θ_1) is defined as

$$\text{Ind}((\Lambda_0, \theta_0), (\Lambda_1, \theta_1)) = n + \theta_1 - \theta_0 - 2\text{Angle}(\Lambda_0, \Lambda_1) \tag{4.1}$$

where $\text{Angle}(\Lambda_0, \Lambda_1) = \sum_{j=1}^n \beta_j$ and $\beta_j \in (0, 1/2)$ are such that there is a unitary basis $\{u_1, \dots, u_n\}$ of Λ_0 satisfying $\Lambda_1 = \text{Span}_{\mathbb{R}}\{e^{2\pi i\beta_j} u_j\}_{j=1}^n$.

In general, when $\Lambda_0 \cap \Lambda_1 = \Lambda \neq \{0\}$, the definition of index for the pair (Λ_0, θ_0) and (Λ_1, θ_1) is the same as above with the definition of $\text{Angle}(\Lambda_0, \Lambda_1)$ modified as follows. Pick a path of Lagrangian planes Λ_t from Λ_0 to Λ_1 such that both the following hold.

- We have $\Lambda \subset \Lambda_t \subset \Lambda_0 + \Lambda_1$ for all $t \in [0, 1]$.
- The image $\overline{\Lambda}_t$ of Λ_t inside the symplectic vector space $(\Lambda_0 + \Lambda_1)/\Lambda$ is a positive definite path from $\overline{\Lambda}_0$ to $\overline{\Lambda}_1$.

Let β_t be a continuous path of real numbers such that $e^{2\pi i\beta_t} = \text{Det}_{\Omega}^2(\Lambda_t)$. Then, the Lagrangian angle is defined as

$$2\text{Angle}(\Lambda_0, \Lambda_1) = \beta_1 - \beta_0.$$

Equivalently, we can define $2\text{Angle}(\Lambda_0, \Lambda_1) := 2\text{Angle}(\overline{\Lambda}_0, \overline{\Lambda}_1)$.

DEFINITION 4.1. For two graded Lagrangian immersions $(\iota_{L_1}, \theta_1), (\iota_{L_2}, \theta_2)$ (not necessarily distinct), and points $p_j \in L_j$ for $j = 1, 2$ such that $\iota_{L_1}(p_1) = \iota_{L_2}(p_2) = p$, the index for the ordered pair (p_1, p_2) is

$$\text{Ind}_{(p_1, p_2)}(\iota_{L_1}, \iota_{L_2}) = \text{Ind}(((\iota_{L_1})_* T_{p_1} L_1, \theta_1(p_1)), ((\iota_{L_2})_* T_{p_2} L_2, \theta_2(p_2))).$$

We also use the notation $\text{Ind}_p(L_1, L_2)$ to denote $\text{Ind}_{(p_1, p_2)}(\iota_{L_1}, \iota_{L_2})$ if $\iota_{L_1}^{-1}(p) = \{p_1\}$ and $\iota_{L_2}^{-1}(p) = \{p_2\}$. Note that if L_1 and L_2 are two Lagrangian embeddings such that L_1 intersects L_2 cleanly along a connected submanifold $D \subset M$, then $\text{Ind}_p(L_1, L_2) = \text{Ind}_q(L_1, L_2)$ for all $p, q \in D$. In this case, we denote the index as $\text{Ind}_D(L_1, L_2)$.

Example 4.2. For a graded Lagrangian immersion (ι_L, θ) and an integer k , $\iota_L[k]$ is defined as $\iota_L[k] = (\iota_L, \theta - k)$. In particular, we have

$$\text{Ind}_D(\iota_{L_1}[k], \iota_{L_2}[k']) = \text{Ind}_D(\iota_{L_1}, \iota_{L_2}) + k - k'.$$

Example 4.3. Let $M = \mathbb{C}^n$ be equipped with the standard symplectic form, complex structure and complex volume form. Let $L_1 = \mathbb{R}^n = \{y_1 = \dots = y_n = 0\}$ and $L_2 = \{x_1 = \dots = x_{n-k} = y_{n-k+1} = \dots = y_n = 0\}$ be two Lagrangian planes for some $0 \leq k \leq n$. We have $\text{Det}_{\Omega}^2(L_1) = 1$ and $\text{Det}_{\Omega}^2(L_2) = (-1)^{n-k}$. Let $\theta_{L_1} = n - k - 1$ and $\theta_{L_2} = (n - k)/2$ be the gradings of L_1 and L_2 . Then, we have $\text{Ind}_0(L_1, L_2) = (n) + (n - k)/2 - (n - k - 1) - 2(n - k)(1/4) = k + 1$.

DEFINITION 4.4. For a Lagrangian isotopy $\Phi = (\Phi_t)_{t \in [0, 1]} : L \times [0, 1] \rightarrow (M, \omega)$, if Φ_0 is equipped with grading θ_0 , then the induced grading on Φ_1 is defined as follows. There is a unique way to extend $\theta_0 : L \times \{0\} \rightarrow \mathbb{R}$ continuously to $\theta : L \times [0, 1] \rightarrow \mathbb{R}$ such that $e^{2\pi i\theta(\cdot, t)} = \text{Det}_{\Omega}^2(\text{Im}(D\Phi_t(\cdot)))$ and the induced grading on Φ_1 is defined by $\theta(\cdot, 1)$.

Example 4.5. Let $L = \mathbb{R} \subset (\mathbb{R}^2, dx \wedge dy)$ and identify the latter with \mathbb{C} equipped with the standard complex volume form. Consider $h : \mathbb{R} \rightarrow \mathbb{R}$ given by $h(q) = c(q^2/2)$ for some constant c . The graph of dh , $\text{Graph}(dh)$, is given by $\{(q, cq) \in T^*L \mid q \in L\}$. By setting $q = x$ and $p = -y$ to identify \mathbb{C} with T^*L , $\text{Graph}(dh)$ is given by $\{(x, -cx) \in \mathbb{C}\}$. Using our sign convention, $\text{Graph}(dh)$ is the time-1 Hamiltonian flow of L generated by the Hamiltonian $-h \circ \pi : T^*L \rightarrow \mathbb{R}$, where $\pi : T^*L \rightarrow L$ is the projection. If we give a grading to L and induce from it a grading on $\text{Graph}(dh)$ by the Hamiltonian isotopy, then

$$\text{Ind}_0(L, \text{Graph}(dh)) = \begin{cases} 1 & \text{if } c \leq 0, \\ 0 & \text{if } c > 0. \end{cases}$$

In short, the index equals the Morse index of h at $q = 0$ if $c \neq 0$. We call the induced grading on $\text{Graph}(dh)$ by the Hamiltonian isotopy generated by $-h \circ \pi$ the *canonical induced grading on $\text{Graph}(dh)$* .

Example 4.6. Let $L = \mathbb{R}^n \subset (\mathbb{C}^n, \sum_{i=1}^n dx_i \wedge dy_i)$. Consider $h : L \rightarrow \mathbb{R}$ given by $h(q) = c \sum_{j=1}^k (q_j^2/2)$. If we let $q_i = x_i$ and $p_i = -y_i$ to identify \mathbb{C}^n with T^*L and equip $\text{Graph}(dh)$ with the canonical induced grading, then

$$\text{Ind}_{\mathbb{R}^{n-k}}(L, \text{Graph}(dh)) = \begin{cases} n & \text{if } c \leq 0, \\ n - k & \text{if } c > 0, \end{cases}$$

where \mathbb{R}^{n-k} is the last $n - k$ q_i coordinates.

COROLLARY 4.7. *Let $h : L \rightarrow \mathbb{R}$ be a Morse–Bott function with Morse–Bott maximum at critical submanifold D_1 of dimension k_1 and minimum at D_2 of dimension k_2 . If the zero section $L \subset T^*L$ is graded and $\text{Graph}(dh)$ is equipped with the canonical induced grading, then $\text{Ind}_{D_1}(L, \text{Graph}(dh)) = n$ and $\text{Ind}_{D_2}(L, \text{Graph}(dh)) = n - k_2$.*

4.2 Local computation for surgery at a point

The grading of a Lagrangian surgery in the local model was considered by Seidel [Sei00], and we include an account for completeness. Let H_γ be a Lagrangian handle. We equip \mathbb{C}^n with the standard complex volume form $\Omega = dz_1 \wedge \cdots \wedge dz_n$.

LEMMA 4.8 [Sei00]. *Let \mathbb{R}^n and $i\mathbb{R}^n$ be equipped with gradings θ_r and θ_i , respectively. Then, there is a grading θ_H on H_γ and a unique integer m such that θ_H can be patched with $\theta_r + m$ and θ_i to give a grading on $\mathbb{R}^n \#_0 i\mathbb{R}^n$. If $\text{Ind}_0((\mathbb{R}^n, \theta_r), (i\mathbb{R}^n, \theta_i)) = 1$, we have $m = 0$.*

Proof. As shown in Example 2.14, $H_\gamma = H_\nu$ for some flow handle H_ν . Since H_ν is obtained by Hamiltonian flow from $i\mathbb{R}^n \setminus \{0\}$, H_ν is canonically graded by θ_i using the Hamiltonian isotopy. We call this grading θ_H and continuously extend it on $\text{Cl}(H_\nu)$. Since $\mathbb{R}^n \cap \text{Cl}(H_\nu)$ has one grading induced from θ_r and one induced from θ_H , $\theta_H|_{\mathbb{R}^n \cap \text{Cl}(H_\nu)} - \theta_r|_{\mathbb{R}^n \cap \text{Cl}(H_\nu)}$ is a locally constant integer-valued function. If $\mathbb{R}^n \cap \text{Cl}(H_\nu)$ is connected, then there is a unique integer m such that $\theta_H|_{\mathbb{R}^n \cap \text{Cl}(H_\nu)} = \theta_r|_{\mathbb{R}^n \cap \text{Cl}(H_\nu)} + m$. If $\mathbb{R}^n \cap \text{Cl}(H_\nu)$ is not connected, then $n = 1$ and one can check directly that the same conclusion holds. As a result, this m is the unique integer such that θ_H can be patched with $\theta_r + m$ and θ_i to give a grading on $\mathbb{R}^n \#_0 i\mathbb{R}^n$. In what follows, we want to show that $m = 0$ if $\text{Ind}_0((\mathbb{R}^n, \theta_r), (i\mathbb{R}^n, \theta_i)) = 1$.

Pick a point $x = (x_1, \dots, x_n) \in S^{n-1}$. Let $c(s) = \gamma(s)x \in H_\gamma$, where γ is an admissible curve (see Definition 2.7), and denote the image curve by $\text{Im}(c)$. The Lagrangian plane Λ_s at $c(s)$ is

spanned by $\{\gamma'(s)x\} \cup \{\gamma(s)v_j\}_{j=2}^n$, where $v_j \in T_x S^{n-1}$ forms an orthonormal basis. (See also the proof of Lemma 2.8.) Therefore, we have

$$\text{Det}_\Omega^2(\Lambda_s) = e^{i2(\arg(\gamma'(s)) + (n-1)\arg(\gamma(s)))}$$

for all s . There is a unique continuous function $\theta_c : \text{Im}(c) \rightarrow \mathbb{R}$ such that:

- $\theta_c(c(s)) = n - 1$ for $s < 0$;
- $\theta_c(c(s)) = n/2$ for $s > \epsilon$; and
- $e^{2\pi i\theta_c(c(s))} = \text{Det}_\Omega^2(\Lambda_s)$ for all s .

Therefore, we have $\theta_c - \theta_H|_{\text{Im}(c)} \in \mathbb{Z}$ and θ_c describes the change of Lagrangian planes from \mathbb{R}^n to $i\mathbb{R}^n$ along the handle. By comparing with Example 4.3 (for $k = 0$), we can see that if the graded Lagrangians \mathbb{R}^n and $i\mathbb{R}^n$ inside \mathbb{C}^n intersect at the origin with index 1, then $m = 0$. In other words, θ_c determines a grading $(\theta_c)_H$ on H_ν when we consider all possible $c = \gamma x$ for $x \in S^{n-1}$. The graded Lagrangians $(H_\nu, (\theta_c)_H)$, $(\mathbb{R}^n, n - 1)$ and $(i\mathbb{R}^n, n/2)$ can be glued continuously. If the graded Lagrangians \mathbb{R}^n and $i\mathbb{R}^n$ inside \mathbb{C}^n intersect at the origin with index 1, we can equip \mathbb{R}^n and $i\mathbb{R}^n$ with gradings $n - 1$ and $n/2$ by Example 4.3. This finishes the proof. \square

COROLLARY 4.9 [Sei00]. *Let $\iota_i : L_i \rightarrow (M, \omega)$ for $i = 1, 2$ be two graded Lagrangian immersions with gradings θ_1 and θ_2 , respectively, intersecting transversally at a point p . If $\text{Ind}_p((L_1, \theta_1), (L_2, \theta_2)) = 1$, then $\iota : L_1 \#_p L_2 \rightarrow (M, \omega)$ can be equipped with a grading θ_{12} extending θ_1 and θ_2 . In this case, we call $L_1 \#_p L_2$ together with its grading a graded surgery from L_1 to L_2 .*

4.3 Local computation for surgery along clean intersection

This subsection discusses the grading for Lagrangian surgery along a clean intersection. We start with ordinary clean surgery (see §2.2.2).

LEMMA 4.10. *Let $L_1, N_D^* \subset T^*L_1$ be equipped with gradings θ_r and θ_i , respectively. For any λ -admissible function ν such that $\lambda < r(D)$, there is a grading θ_H on H_ν^D and a unique integer m such that θ_H can be patched with $\theta_i, \theta_r + m$ to become a grading on $L_1 \#_D^\nu N_D^*$.*

Moreover, $m = 0$ if and only if $\text{Ind}_D((L_1, \theta_r), (N_D^, \theta_i)) = \dim(D) + 1$.*

Immediately from Lemma 4.10, we have the following.

COROLLARY 4.11. *Let $L_1, L_2 \subset (M, \omega)$ be graded Lagrangians cleanly intersecting at D . We can perform a graded surgery $L_1 \#_D L_2$ from L_1 to L_2 along D if and only if $\text{Ind}_D(L_1, L_2) = \dim(D) + 1$.*

Proof of Lemma 4.10. The first statement of the lemma follows as in the first paragraph of the proof of Lemma 4.8. Therefore, we just need to prove that $m = 0$ if and only if $\text{Ind}_D((L_1, \theta_r), (N_D^*, \theta_i)) = \dim(D) + 1$. Let $\dim(D) = k$.

Pick a Darboux chart such that in local coordinates N_D^* is represented by points of the form $(q, p) = (q_b, 0, 0, p_f)$. Here $(q, p) = (q_b, q_f, p_b, p_f)$, $q_b = (q_1, \dots, q_k)$, $q_f = (q_{k+1}, \dots, q_n)$, $p_b = (p_1, \dots, p_k)$ and $p_f = (p_{k+1}, \dots, p_n)$. We also require (q_b, tq_f) are unit-speed geodesics on L_1 as t varies and perpendicular to D at $t = 0$, for any q such that $|q_f|^2 = 1$. As a result, the handle H_ν^D in local coordinates is given by (here, we suppose that the surgery is supported in a sufficiently small region relative to the Darboux chart)

$$\left\{ \phi_{\nu(\|p_f\|)}^\sigma(q_b, 0, 0, p_f) = \left(q_b, \nu(\|p_f\|) \frac{p_f}{\|p_f\|}, 0, p_f \right) \right\},$$

where $q_b \in B^k \subset \mathbb{R}^k, p_f \in B^{n-k} \subset \mathbb{R}^{n-k}$ and B^k, B^{n-k} are some small open balls centered at the origin.

We consider the standard complex volume form $\Omega = dz_1 \wedge \cdots \wedge dz_n$ in the chart. Let $e_{\pi_2} \in S^{n-k-1} \subset \mathbb{R}^{n-k}$ be a vector in the unit sphere of last $n - k$ p_i -coordinates. Let

$$c(r) = \left(0, \nu(\|re_{\pi_2}\|) \frac{re_{\pi_2}}{\|re_{\pi_2}\|}, 0, re_{\pi_2} \right) = (0, \nu(r)e_{\pi_2}, 0, re_{\pi_2})$$

be a smooth curve on H_ν^D for $r \in (0, \epsilon]$. We define $c(0) = \lim_{r \rightarrow 0^+} c(r)$.

We want to understand how the Lagrangian planes change from L_1 to N_D^* along the handle and it suffices to look at how the Lagrangian planes change along $c(r)$. The Lagrangian plane Λ_r at $c(r)$ is spanned by

$$\{(e_j, 0, 0, 0)\}_{j=1}^k \cup \{(0, \nu'(r)e_{\pi_2}, 0, e_{\pi_2})\} \cup \left\{ \left(0, \nu(r) \frac{e_j^\perp}{r}, 0, e_j^\perp \right) \right\}_{j=2}^{n-k},$$

where $e_j \in \mathbb{R}^k$ are coordinate vectors and e_j^\perp form an orthonormal basis for orthogonal complement of e_{π_2} in \mathbb{R}^{n-k} .

Then we have

$$\text{Det}_\Omega^2(\Lambda_r) = e^{i2(\arg(\nu'(r)-\sqrt{-1})+(n-k-1)\arg(\nu(r)/r-\sqrt{-1}))}$$

for all r . Here, the convention we use is still $z_i = q_i - \sqrt{-1}p_i$. Observe that $\nu'(\epsilon) = 0$ and $\nu(\epsilon)/\epsilon = 0$. When r goes to 0, $\nu'(r)$ decreases monotonically to $-\infty$. Similarly, $\nu(r)/r$ increases monotonically to infinity when r goes to zero.

In particular, $\arg(\nu'(r)-\sqrt{-1})$ increases from π to $3\pi/2$ as r increases and $\arg(\nu(r)/r-\sqrt{-1})$ decreases from 2π to $3\pi/2$ as r increases. Therefore, there is a unique continuous function $\theta_c : \text{Im}(c) \rightarrow \mathbb{R}$ such that:

- $\theta_c(c(r)) = n - k - 1$ for $r = 0$;
- $\theta_c(c(r)) = (n - k)/2$ for $r = \epsilon$; and
- $e^{2\pi i \theta_c(c(r))} = \text{Det}_\Omega^2(\Lambda_r)$ for all $r \in [0, \epsilon]$.

By Example 4.3, we have $\text{Ind}_{\mathbb{R}^k}((\mathbb{R}^n, n - k - 1), (N^*(\mathbb{R}^k), (n - k)/2)) = k + 1$. Hence, $m = 0$ if and only if $\text{Ind}_D((L_1, \theta_r), (N_D^*, \theta_i)) = k + 1$. □

For the E_2 -flow surgery, we use the setting in § 2.3 and we have the following.

LEMMA 4.12. *Suppose $D \subset L = K_1 \times K_2$ is a smooth submanifold of dimension k which is transversal to $\{p\} \times K_2$ for all $p \in K_1$. Let $L, N_D^* \subset T^*L$ be equipped with gradings θ_r and θ_i , respectively. For any λ -admissible function ν such that $\lambda < r^{E_2}(D)$, there is a grading θ_H on H_ν^{D, E_2} and a unique integer m such that θ_H can be patched with $\theta_r + m$ and θ_i to become a grading on $L \#_{D, E_2}^{\nu} N_D^*$.*

Moreover, we have $m = 0$ if and only if $\text{Ind}_D((L, \theta_r), (N_D^*, \theta_i)) = \dim(D) + 1$.

COROLLARY 4.13. *Let $L_1 = K_1 \times K_2, L_2 \subset (M, \omega)$ be graded Lagrangians cleanly intersecting at D . Suppose D is transversal to $\{p\} \times K_2$ for all $p \in K_1$. Then we can perform a graded E_2 -flow surgery $L_1 \#_{D, E_2} L_2$ from L_1 to L_2 along D if and only if $\text{Ind}_D(L_1, L_2) = \dim(D) + 1$.*

Proof of Lemma 4.12. As explained before (cf. Lemmas 4.8 and 4.10), we just need to show that $m = 0$ if and only if $\text{Ind}_D((L, \theta_r), (N_D^*, \theta_i)) = \dim(D) + 1$. Again denote $k = \dim(D)$.

Pick a product chart $U = U_1 \times U_2$ for $L = K_1 \times K_2$ (i.e., $U_i \subset K_i$ for $i = 1, 2$). Points in U are denoted by $q = (q_b, q_f)$, where $q_b \in U_1$ and $q_f \in U_2$. We also want that (q_b, tq_f) is a unit-speed geodesic as t varies and perpendicular to D at $t = 0$, for any q_b, q_f such that $|q_f| = 1$. We can also assume the origin belongs to D and denote a basis of the tangent space T_0D of D at the origin by $\{w^1, \dots, w^k\}$ and $w^j = w_b^j + w_f^j$, where w_b^j and w_f^j are the q_b and q_f components of w^j , respectively. Since D is transversal to the second factor, we can assume that w_b^j are the unit coordinate vectors in the q_b -plane for $1 \leq j \leq k$. Moreover, there is a U_2 -valued function $q_f^D(q_b)$ of q_b near the origin such that $(q_b, q_f^D(q_b)) \in D$.

This chart gives a corresponding Darboux chart on T^*L and we define p_b^D as a function of q_b, p_f near the origin such that $(q_b, q_f^D(q_b), p_b^D(q_b, p_f), p_f) \in N_D^*$. Note that $p_b^D(\cdot, \cdot)$ is linear on the second factor. Near the origin (close enough to the origin so that $q_f^D(q_b)$ is well defined), the handle H_ν^{D, E_2} is given in local coordinates by

$$\left\{ \phi_{\nu(\|p_b\|)}^{\sigma_\pi}(q_b, q_f^D(q_b), p_b^D(q_b, p_f), p_f) = \left(q_b, q_f^D(q_b) + \nu(\|p_f\|) \frac{p_f}{\|p_f\|}, p_b^D(q_b, p_f), p_f \right) \right\},$$

where $q_b \in B^k \subset \mathbb{R}^k, p_f \in B^{n-k} \subset \mathbb{R}^{n-k}$ and B^k, B^{n-k} are small open balls centered at the origin.

We consider the standard complex volume form $\Omega = dz_1 \wedge \dots \wedge dz_n$ in the chart. Let $e_{\pi_2} \in S^{n-k-1} \subset \mathbb{R}^{n-k}$ be a vector in the unit sphere in the p_f coordinates. Let

$$\begin{aligned} c(r) &= \phi_{\nu(\|re_{\pi_2}\|)}^{\sigma_\pi}(0, 0, p_b^D(0, re_{\pi_2}), re_{\pi_2}) \\ &= \left(0, \nu(\|re_{\pi_2}\|) \frac{re_{\pi_2}}{\|re_{\pi_2}\|}, p_b^D(0, re_{\pi_2}), re_{\pi_2} \right) \\ &= (0, \nu(r)e_{\pi_2}, p_b^D(0, re_{\pi_2}), re_{\pi_2}) \end{aligned}$$

be a smooth curve in H_ν^{D, E_2} for $r \in (0, \epsilon]$. We define $c(0) = \lim_{r \rightarrow 0^+} c(r)$.

The Lagrangian plane Λ_r of H_ν^{D, E_2} at $c(r)$ is spanned by

$$\{(w_b^j, w_f^j, \kappa(r, w^j), 0)\}_{j=1}^k \cup \{(0, \nu'(r)e_{\pi_2}, p_b^D(0, e_{\pi_2}), e_{\pi_2})\} \cup \left\{ \left(0, \frac{\nu(r)}{r} e_j^\perp, p_b^D(0, e_j^\perp), e_j^\perp \right) \right\}_{j=2}^{n-k},$$

where $\kappa(r, w^j) = \partial_{q_j} p_b^D(0, re_{\pi_2}) = r(\partial_{q_j} p_b^D(0, e_{\pi_2}))$ is linear in r and e_j^\perp form an orthonormal basis for orthogonal complement of e_{π_2} in \mathbb{R}^{n-k} . We note that $(0, \nu'(r)e_{\pi_2}, p_b^D(0, e_{\pi_2}), e_{\pi_2}) = c'(r)$ and the computation uses the fact that $p_b^D(\cdot, \cdot)$ is linear on the second factor.

Let $\kappa_j(r, w^j)$ be the coefficient of w_b^j -component of $\kappa(r, w^j)$ (here, we identify the q_b -plane and the p_b -plane). Notice that

$$\text{Det}_\Omega^2(\Lambda_r) = e^{i2(\sum_{j=1}^k \arg(1 - \kappa_j(r, w^j)\sqrt{-1}) + \arg(\nu'(r) - \sqrt{-1}) + (n-k-1) \arg(\nu(r)/r - \sqrt{-1}))}$$

for all r (here, we use the fact that w_b^j are unit coordinates vectors and we use the convention $z_i = q_i - \sqrt{-1}p_i$). Let $K(r) = \sum_{j=1}^k \arg(1 - \kappa_j(r, w^j)\sqrt{-1})$.

As in the proof of Lemma 4.10, $\arg(\nu'(r) - \sqrt{-1})$ increases from π to $3\pi/2$ as r increases and $\arg(\nu(r)/r - \sqrt{-1})$ decreases from 2π to $3\pi/2$ as r increases. Therefore, there is a unique continuous function $\theta_c : \text{Im}(c) \rightarrow \mathbb{R}$ such that:

- $\theta_c(c(r)) = n - k - 1 + K(0)/\pi = n - k - 1$ for $r = 0$;
- $\theta_c(c(r)) = (n - k)/2 + K(\epsilon)/\pi$ for $r = \epsilon$; and
- $e^{2\pi i \theta_c(c(r))} = \text{Det}_\Omega^2(\Lambda_r)$ for all $r \in [0, \epsilon]$.

On the other hand, we can lift a path of Lagrangian planes Λ_r^N of N_D^* over the path $c_2(r) = (0, 0, p_b^D(0, re_{\pi_2}), re_{\pi_2})$ connecting the origin and $c(\epsilon)$. The Lagrangian plane Λ_r^N is spanned by

$$\{(w_b^j, w_r^j, \kappa(r, w^j), 0)\}_{j=1}^k \cup \{(0, 0, p_b^D(0, e_{\pi_2}), e_{\pi_2})\} \cup \{(0, 0, p_b^D(0, e_j^\perp), e_j^\perp)\}_{j=2}^{n-k}.$$

Therefore, the grading of N_D^* at the origin is the grading of N_D^* at $c(\epsilon)$ subtracted by $K(\epsilon)/\pi$. If we extend θ_c continuously over $\text{Im}(c_2)$ (note: $\text{Im}(c) \cap \text{Im}(c_2) = \{c(\epsilon)\}$), then $\theta_c(c_2(0)) = (n - k)/2$.

By an analogous calculation to Example 4.3, we have

$$\text{Ind}_D \left((L_1, n - k - 1), \left(N_D^*, \frac{n - k}{2} \right) \right) = k + 1$$

and by comparing it with θ_c the result follows. □

The following is a family version whose proof is similar.

COROLLARY 4.14. *Let $\mathcal{L}_0, \mathcal{L}_1 \subset (M^{2n}, \omega)$ as in Lemma 2.27 and let the dimension of \mathcal{D} be k . Assume $\mathcal{L}_0, \mathcal{L}_1$ are graded with grading θ_r and θ_i . Then $\text{Ind}_{\mathcal{D}}((\mathcal{L}_1, \theta_r), (N^*\mathcal{D}, \theta_i)) = k + 1$ if and only if $\mathcal{L}_0 \#_{\mathcal{D}, E_2}^{\nu} \mathcal{L}_1$ has a grading such that the grading restricted to $\mathcal{L}_0, \mathcal{L}_1$ coincides with θ_r and θ_i , respectively.*

4.4 Diagonal in product

We recall from [WW10b, Remark 3.0.5] how to associate a canonical grading to the diagonal in $M \times M^-$.

Let $(\mathbb{R}^{2n}, \omega_{std})$ be the standard symplectic vector space and $\text{Lag}(\mathbb{R}^{2n})$ be the Lagrangian Grassmannian. An N -fold Maslov cover $\widetilde{\text{Lag}}^N(\mathbb{R}^{2n})$ is a \mathbb{Z}_N covering $\widetilde{\text{Lag}}^N(\mathbb{R}^{2n}) \rightarrow \text{Lag}(\mathbb{R}^{2n})$ associated to the Maslov class in $\text{Hom}(\pi_1(\text{Lag}^N(\mathbb{R}^{2n})), \mathbb{Z})$. More precisely, the modulo- N reduction of the Maslov class defines a representation $\pi_1(\text{Lag}^N(\mathbb{R}^{2n})) \rightarrow \mathbb{Z}_N$ and the N -fold Maslov cover is given as $\widetilde{\text{Lag}}^N(\mathbb{R}^{2n}) := \widetilde{\text{Lag}}(\mathbb{R}^{2n}) \times_{\pi_1(\text{Lag}^N(\mathbb{R}^{2n}))} \mathbb{Z}_N \rightarrow \text{Lag}(\mathbb{R}^{2n})$, where $\widetilde{\text{Lag}}(\mathbb{R}^{2n})$ is the universal cover of $\text{Lag}(\mathbb{R}^{2n})$.

More explicitly, for $\Lambda_0 \in \text{Lag}(\mathbb{R}^{2n})$, the N -fold Maslov cover $\widetilde{\text{Lag}}^N(\mathbb{R}^{2n}, \Lambda_0)$ of $\text{Lag}(\mathbb{R}^{2n})$ based at Λ_0 consists of homotopy classes of paths (relative to end points) $\tilde{\Lambda} : [0, 1] \rightarrow \text{Lag}(\mathbb{R}^{2n})$ such that $\tilde{\Lambda}(0) = \Lambda_0$, modulo loops whose Maslov index is a multiple of N . If we identify \mathbb{R}^{2n} with $\mathbb{C}^{n,-}$ (see the identification in Lemma 4.15), then we can equip \mathbb{R}^{2n} with the standard complex volume form Ω . A lift of a Lagrangian subspace $\Lambda \in \text{Lag}(\mathbb{R}^{2n})$ to $\widetilde{\text{Lag}}(\mathbb{R}^{2n}, \Lambda_0)$ determines a grading $\theta_\Lambda : \Lambda \rightarrow \mathbb{R}$ (see § 4.1) which is a constant function. Conversely, a constant θ_Λ on Λ such that $e^{2\pi i \theta_\Lambda} = \text{Det}_{\mathbb{Q}}^2(\Lambda)$ determines a lift of Λ to $\widetilde{\text{Lag}}(\mathbb{R}^{2n}, \Lambda_0)$. Therefore, we also call of lift of Λ to $\widetilde{\text{Lag}}(\mathbb{R}^{2n}, \Lambda_0)$ (respectively $\widetilde{\text{Lag}}^N(\mathbb{R}^{2n}, \Lambda_0)$) a grading (respectively \mathbb{Z}_N -grading) of Λ .

Similarly, $(\mathbb{R}^{2n,-} \times \mathbb{R}^{2n}, -\omega_{std} \oplus \omega_{std})$ is a symplectic vector space and $\Lambda_0^- \times \Lambda_0$ is a Lagrangian subspace so we can define the N -fold Maslov cover $\widetilde{\text{Lag}}^N(\mathbb{R}^{2n,-} \times \mathbb{R}^{2n}, \Lambda_0^- \times \Lambda_0)$ based at $\Lambda_0^- \times \Lambda_0$. For any Lagrangian subspace $\Lambda \subset \mathbb{R}^{2n}$ and a path $\gamma : [0, 1] \rightarrow \text{Lag}(\mathbb{R}^{2n})$ from $\gamma(0) = \Lambda$ to $\gamma(1) = \Lambda_0$, the induced path $(\gamma^- \times \gamma)(t) := (\gamma(t))^- \times \gamma(t)$ from $\Lambda^- \times \Lambda$ to $\Lambda_0^- \times \Lambda_0$ gives an identification between $\widetilde{\text{Lag}}^N(\mathbb{R}^{2n,-} \times \mathbb{R}^{2n}, \Lambda_0^- \times \Lambda_0)$ and $\widetilde{\text{Lag}}^N(\mathbb{R}^{2n,-} \times \mathbb{R}^{2n}, \Lambda^- \times \Lambda)$. If $\gamma_2 : [0, 1] \rightarrow \text{Lag}(\mathbb{R}^{2n})$ is another path from $\gamma_2(0) = \Lambda$ to $\gamma_2(1) = \Lambda_0$, then $(\gamma^- \times \gamma) * ((\gamma_2^-)^- \times \gamma_2^-)$, where γ_2^- is the inverse path of γ_2 and $*$ is the concatenation of path, has Maslov index 0 so the identification between $\widetilde{\text{Lag}}^N(\mathbb{R}^{2n,-} \times \mathbb{R}^{2n}, \Lambda_0^- \times \Lambda_0)$ and $\widetilde{\text{Lag}}^N(\mathbb{R}^{2n,-} \times \mathbb{R}^{2n}, \Lambda^- \times \Lambda)$ is independent of the choice of γ . Therefore, $\Lambda^- \times \Lambda$ has a canonical lift to $\widetilde{\text{Lag}}^N(\mathbb{R}^{2n,-} \times \mathbb{R}^{2n}, \Lambda_0^- \times \Lambda_0)$, and hence a canonical (\mathbb{Z}_N) -grading.

To give a canonical grading to the diagonal $\Delta \subset \mathbb{R}^{2n,-} \times \mathbb{R}^{2n}$, it suffices to give once and for all an identification between $\text{Lag}^N(\mathbb{R}^{2n,-} \times \mathbb{R}^{2n}, \Lambda^- \times \Lambda)$ and $\text{Lag}^N(\mathbb{R}^{2n,-} \times \mathbb{R}^{2n}, \Delta)$. This is given by the concatenation of the two paths

$$(e^{Jt}\Lambda^- \times \Lambda)_{t \in [0, \pi/2]}, \quad (\{(tx + Jy, x + tJy) \mid x, y \in \Lambda\})_{t \in [0, 1]},$$

where J is an ω_{std} -compatible complex structure on \mathbb{R}^{2n} . This canonical grading induces a canonical grading on $\Delta_M \subset M^- \times M$ for any symplectic manifold M .

In the following lemma, we consider the symplectic manifold $M = \mathbb{C}^{n,-}$ and compute the index between a product Lagrangian with the diagonal Δ_M .

LEMMA 4.15 (Cf. [WW10b, § 3]). *For any graded Lagrangian subspace $\Lambda \subset \mathbb{C}^{n,-}$, we have*

$$\text{Ind}_{\Delta_\Lambda}(\Lambda^- \times \Lambda, \Delta_{\mathbb{C}^{n,-}}) = n,$$

where $\Lambda^- \times \Lambda$ and $\Delta_{\mathbb{C}^{n,-}}$ are equipped with their canonical gradings in $\mathbb{C}^n \times \mathbb{C}^{n,-}$.

Proof. It suffices to consider $\Lambda = \mathbb{R}^n \subset \mathbb{C}^{n,-}$ and $J = -J_{std} = -\sqrt{-1}$. Let $z_i = x_i + \sqrt{-1}y_i$ be the coordinates of \mathbb{C}^n and $w_i = u_i + \sqrt{-1}v_i$ be the coordinates of $\mathbb{C}^{n,-}$. We consider the standard complex volume form $\Omega = dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{w}_1 \wedge \cdots \wedge d\bar{w}_n$ on $\mathbb{C}^n \times \mathbb{C}^{n,-}$ and equip $\Lambda^- \times \Lambda$ with grading 0. We have $\text{Det}^2(e^{Jt}\Lambda^- \times \Lambda) = e^{-i2nt}$, which induces a grading of $-n/2$ on $e^{J(\pi/2)}\Lambda^- \times \Lambda$. We also have $\text{Det}^2(\{(tx + Jy, x + tJy) \mid x, y \in \Lambda\}) = e^{-in\pi}$ for all t so the canonical grading on Δ is $-n/2$.

To calculate $\text{Angle}(\Lambda^- \times \Lambda, \Delta)$, we observe that $(\Lambda^- \times \Lambda) \cap \Delta = \text{Span}\{(\partial_{x_i} + \partial_{u_i})\}_{i=1}^n$. We can use $\Lambda_t = (\Lambda^- \times \Lambda) \cap \Delta + \text{Span}\{(t(\partial_{y_i} + \partial_{v_i}) + (1-t)(-\partial_{x_i} + \partial_{u_i}))\}$ from $\Lambda^- \times \Lambda$ to Δ for the calculation of $\text{Angle}(\Lambda^- \times \Lambda, \Delta)$. As a result, we have $2\text{Angle}(\Lambda^- \times \Lambda, \Delta) = n/2$ and hence

$$\text{Ind}_{\Delta_\Lambda}(\Lambda^- \times \Lambda, \Delta_{\mathbb{C}^{n,-}}) = 2n + \left(-\frac{n}{2}\right) - 0 - \frac{n}{2} = n. \quad \square$$

COROLLARY 4.16. *Let L be a Lagrangian in M . With the canonical gradings of $L \times L \subset M \times M^-$ and $\Delta \subset M \times M^-$, one can perform graded clean surgery to obtain $(L \times L)[1] \#_{\Delta_L, E_2} \Delta$.*

Proof. This is a direct consequence of Lemmas 4.12 and 4.15 because $\text{Ind}_{\Delta_L}(L \times L[1], \Delta) = \text{Ind}_{\Delta_L}(L \times L, \Delta) + 1 = \dim(\Delta_L) + 1$ (see Example 4.2). □

COROLLARY 4.17 (Cf. Theorem 1.1(1)). *There is a graded clean surgery identity*

$$(S^n \times S^n)[1] \#_{\Delta_{S^n}, E_2} \Delta = \text{Graph}(\tau_{S^n}^{-1}).$$

Proof. This is a direct consequence of Corollaries 3.5 and 4.16. □

LEMMA 4.18 (Cf. Theorem 1.1(3)). *There is a graded clean surgery identity*

$$\mathbb{C}\mathbb{P}^{m/2} \times \mathbb{C}\mathbb{P}^{m/2} \#_{D^{\text{op}}, E_2} ((\mathbb{C}\mathbb{P}^{m/2} \times \mathbb{C}\mathbb{P}^{m/2})[1] \#_{\Delta_{\mathbb{C}\mathbb{P}^{m/2}, E_2} \Delta} \Delta) = \text{Graph}(\tau_{\mathbb{C}\mathbb{P}^{m/2}}^{-1}).$$

Proof. By Corollary 4.16, we can obtain a graded Lagrangian

$$L = (\mathbb{C}\mathbb{P}^{m/2} \times \mathbb{C}\mathbb{P}^{m/2})[1] \#_{\Delta_{\mathbb{C}\mathbb{P}^{m/2}, E_2} \Delta} \Delta.$$

As explained in the proof of Lemma 3.4 and Lemma 3.6, L is Hamiltonian isotopic to a Lagrangian Q cleanly intersecting with $\mathbb{C}\mathbb{P}^{m/2} \times \mathbb{C}\mathbb{P}^{m/2}$ along D^{op} such that Q coincides with the graph of a Morse–Bott function with maximum at D^{op} near D^{op} . Therefore, we have $\text{Ind}_{D^{\text{op}}}(\mathbb{C}\mathbb{P}^{m/2} \times \mathbb{C}\mathbb{P}^{m/2}, Q) = 2m - 1$. Here the first term $2m$ follows by Corollary 4.7 and the second term -1 comes from the grading shift of the first factor of L . Since D^{op} is of dimension $2m - 2$, we get the result by applying Lemma 4.12. □

The cases for $\mathbb{R}\mathbb{P}^n$ and $\mathbb{H}\mathbb{P}^n$ can be computed analogously.

LEMMA 4.19 (Cf. Theorem 1.1(3)). *There are graded clean surgery identities*

$$\mathbb{R}P^n \times \mathbb{R}P^n [1] \#_{D^{\text{op}}, E_2} ((\mathbb{R}P^n \times \mathbb{R}P^n) [1] \#_{\Delta_{\mathbb{R}P^n}, E_2} \Delta) = \text{Graph}(\tau_{\mathbb{R}P^n}^{-1})$$

and

$$\mathbb{H}P^n \times \mathbb{H}P^n [-2] \#_{D^{\text{op}}, E_2} ((\mathbb{H}P^n \times \mathbb{H}P^n) [1] \#_{\Delta_{\mathbb{H}P^n}, E_2} \Delta) = \text{Graph}(\tau_{\mathbb{H}P^n}^{-1}),$$

where D^{op} are defined similar to Lemma 4.18.

For the family Dehn twist, we have the following (see Corollary 3.10).

LEMMA 4.20 (Cf. Theorem 1.1(2), (4)). *There are graded clean surgery identities*

$$\begin{aligned} \tilde{C}_S [1] \#_{\mathcal{D}, E_2} \Delta &= \text{Graph}(\tau_{C_S}^{-1}), \\ \tilde{C}_R [1] \#_{\mathcal{D}^{\text{op}}, E_2} \tilde{C}_R [1] \#_{\mathcal{D}, E_2} \Delta &= \text{Graph}(\tau_{C_R}^{-1}), \\ \tilde{C}_C \#_{\mathcal{D}^{\text{op}}, E_2} \tilde{C}_C [1] \#_{\mathcal{D}, E_2} \Delta &= \text{Graph}(\tau_{C_C}^{-1}), \\ \tilde{C}_H [-2] \#_{\mathcal{D}^{\text{op}}, E_2} \tilde{C}_H [1] \#_{\mathcal{D}, E_2} \Delta &= \text{Graph}(\tau_{C_H}^{-1}), \end{aligned}$$

where C_S (respectively C_R, C_C, C_H) is a spherically (respectively real projectively, complex projectively, quaternionic projectively) coisotropic submanifold.

5. Review of Lagrangian Floer theory, Lagrangian cobordisms and quilted Floer theory

We first fix conventions for Lagrangian Floer theory for the rest of the paper, which follows that of [Sei08a]. Note that this is different from the homology convention of [BC13].

Let $L_0, L_1 \subset (M, \omega)$ be a pair of transversally intersecting graded compact Lagrangian submanifolds. For a generic one-parameter family of ω -compatible almost complex structure $\mathbf{J} = \{J_t\}_{t \in [0,1]}$, let

$$\begin{aligned} \mathcal{M}(p_-, p_+) &= \left\{ u : \mathbb{R} \times [0, 1] \rightarrow M : u_s(s, t) + J_t(u(s, t))u_t(s, t) = 0, u(s, 0) \in L_0 \right. \\ &\quad \left. \text{and } u(s, 1) \in L_1 \lim_{s \rightarrow +\infty} u(s, t) = p_+, \lim_{s \rightarrow -\infty} u(s, t) = p_- \right\} / \mathbb{R}. \end{aligned} \tag{5.1}$$

Then the Floer cochain complex $CF^*(L_0, L_1)$ is generated by $L_0 \cap L_1$ as a graded vector space and equipped with a differential by counting rigid elements from $\mathcal{M}(p_-, p_+)$, i.e.,

$$dp_+ = \sum_{p_- \in L_0 \cap L_1} \# \mathcal{M}^0(p_-, p_+) p_-.$$

We refer to [Sei08a] for the definition of Fukaya category $\mathcal{Fuk}(M)$ and the derived Fukaya category $D^\pi \mathcal{Fuk}(M)$ that involves higher operations defined using pseudo-holomorphic polygons.

DEFINITION 5.1. Let $L_i, L'_j \subset (M, \omega)$, $1 \leq i \leq k$, $1 \leq j \leq k'$ be a collection of Lagrangian submanifolds. A Lagrangian cobordism V from (L_1, \dots, L_k) to $(L'_1, \dots, L'_{k'})$ is an embedded Lagrangian submanifold in $M \times \mathbb{C}$ so that the following condition holds (see Figure 10).

- There is a compact set $K \subset \mathbb{C}$ such that $V \setminus (M \times K) = (\bigsqcup_{i=1}^k L_i \times \gamma_i) \sqcup (\bigsqcup_{j=1}^{k'} L'_j \times \gamma'_j)$, where $\gamma_i = (-\infty, x_i) \times \{a_i\}$ and $\gamma'_j = (x'_j, \infty) \times \{b'_j\}$ for some x_i, a_i, x'_j, b'_j such that $a_1 < \dots < a_k$ and $b'_1 > \dots > b'_{k'}$.

Each $L_i \times \gamma_i$ or $L'_j \times \gamma'_j$ is called an *end* of the Lagrangian cobordism V .

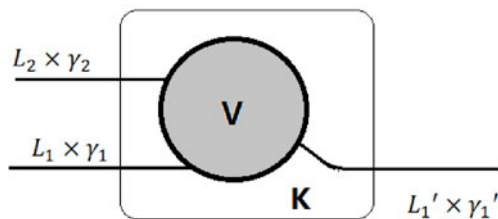


FIGURE 10. Projection of a Lagrangian cobordism.

The counterclockwise order for the ends starting from L'_1 is $L'_1, \dots, L'_{k'}, L_1, \dots, L_k$. For the grading we always choose the quadratic complex volume form on $M \times \mathbb{C}$ to be the quadratic complex volume form on M times the standard one on \mathbb{C} . When V is graded, the restriction induces a grading on each end. On an end, say $L_i \times \gamma_i$, we denote the induced grading as θ_{ii} . Since an end is a product Lagrangian, we can associate a grading θ_i to L_i by requiring $\theta_i(p) = \theta_{ii}(p \times z)$ for all $p \in L_i$ and $z \in \gamma_i$. The same rule applies to $L'_j \times \gamma'_j$. We use this grading convention between a cobordism and its fiber Lagrangians over its ends throughout.

The main result we will utilize from Biran and Cornea's Lagrangian cobordism formalism reads as follows.

THEOREM 5.2 [BC14]. *If there exists a graded monotone (or exact) Lagrangian cobordism from monotone (or exact) Lagrangians $(L_1[k-1], L_2[k-2], \dots, L_k)$ to $(L'_1[k'-1], L'_2[k'-2], \dots, L'_{k'})$, then there is an isomorphism between iterated cones in $\mathcal{D}^\pi \mathcal{Fuk}(M)$,*

$$\text{Cone}(L_1 \rightarrow L_2 \rightarrow \dots \rightarrow L_k) \cong \text{Cone}(L'_1 \rightarrow L'_2 \rightarrow \dots \rightarrow L'_{k'}).$$

Here $\text{Cone}(L_1 \rightarrow L_2 \rightarrow \dots \rightarrow L_k) = \text{Cone}(\dots \text{Cone}(\text{Cone}(L_1 \rightarrow L_2) \rightarrow L_3) \rightarrow \dots \rightarrow L_k)$ and the maps in the cones are given by counting appropriate pseudo-holomorphic polygons.

Note that $CF^*(K, \text{Cone}(L_1 \rightarrow L_2 \rightarrow \dots \rightarrow L_k)) = CF^*(K, L_1[k-1]) \oplus \dots \oplus CF^*(K, L_k)$ as a graded vector space for any graded Lagrangian K transversally intersecting each L_i . It explains the seemingly weird grading shift of the Lagrangians L_i, L'_j for the cobordism.

We dedicate the rest of this section to quilted Floer theory developed in [WW10b, WW10a, WW12, WMW].

DEFINITION 5.3 [WW10b, Definition 2.1.3]. Given a sequence of symplectic manifolds M_0, \dots, M_{r+1} , a generalized Lagrangian correspondence $\underline{L} = (L_{01}, \dots, L_{r(r+1)})$ is a sequence of Lagrangian submanifolds such that $L_{i(i+1)} \subset M_i^- \times M_{i+1}$ are compact Lagrangian submanifolds for all i . A cyclic generalized Lagrangian correspondence is one such that $M_0 = M_{r+1}$.

For a Lagrangian correspondence $L_{01} \subset M_0^- \times M_1$, $L^t_{01} \subset M_1^- \times M_0$ is defined to be $L^t_{01} = \{(x, y) \mid (y, x) \in L_{01}\}$. Given two Lagrangian correspondences $L_{01} \subset M_0^- \times M_1$ and $L_{12} \subset M_1^- \times M_2$, their geometric composition is defined as

$$L_{01} \circ L_{12} = \{(x, z) \mid \exists y \text{ such that } (x, y) \in L_{01} \text{ and } (y, z) \in L_{12}\}. \quad (5.2)$$

For the composition to work nicely, we require the following.

- $L_{01} \times L_{12}$ intersects $M_0^- \times \Delta \times M_2$ transversally in $M_0^- \times M_1^- \times M_1^- \times M_2$, where $\Delta \subset M_1 \times M_1^-$ is the diagonal.

- The projection $\pi_{02} : L_{01} \times_{M_1} L_{12} \rightarrow L_{01} \circ L_{12} \subset M_0^- \times M_2$ is an embedding, where $L_{01} \times_{M_1} L_{12}$ is the fiber product between $\pi_{01,1} : L_{01} \rightarrow M_1$ and $\pi_{12,1} : L_{12} \rightarrow M_1$, and π_{02} is the projection forgetting the M_1 factor.

In this case, the composition $L_{01} \circ L_{12}$ is called *embedded*. We refer to § 3.3 for a non-trivial example of Lagrangian correspondence and composition coming from coisotropic embeddings.

For a cyclic generalized Lagrangian correspondence \underline{L} , the *quilted Floer cohomology* is defined to be

$$HF^*(\underline{L}) := HF^*(L_{01} \times L_{23} \cdots L_{(r-1)r}, (L_{12} \times L_{34} \cdots L_{r(r+1)})^T)$$

in $M_0^- \times M_1 \times \cdots \times M_{r-1}^- \times M_r$ when r is odd, where $(-)^T : M_1^- \times M_2 \times \cdots \times M_r^- \times M_0 \rightarrow M_0^- \times M_1 \times \cdots \times M_{r-1}^- \times M_r$ is given by transposition of the last factor to the first factor, combined with an overall sign change in the symplectic form. When r is even, we have

$$HF^*(\underline{L}) := HF^*(L_{01} \times L_{23} \cdots L_{r(r+1)}, (L_{12} \times L_{34} \cdots L_{(r-1)r} \times \Delta_{M_0})^T)$$

in $M_0^- \times M_1 \times \cdots \times M_r^- \times M_{r+1}$, where $(-)^T$ is defined analogously.

It is worth pointing out that for the quilted Floer cohomology to be well defined, \underline{L} needs to satisfy a monotonicity condition [WW10b, Definition 4.1.2(b)] stronger than having all $L_{i,(i+1)}$ to be monotone. For monotone Lagrangian submanifolds $L_0 \subset \{\text{point}\}^- \times M_0$ and $L_1 \subset M_1^- \times \{\text{point}\}$, a sufficient condition for this stronger monotonicity to hold for $\underline{L} = (L_0, L_{01}, L_1)$ is when $\pi_1(L_{01}) = 1$ [WW10b, Lemma 4.1.3]. We refer readers to [WW10b] for further details on monotonicity, as well as orientation, grading and so forth for a generalized Lagrangian correspondence. The following theorems summarize the main properties that will concern us.

THEOREM 5.4 [WW10b, Theorem 5.2.6]. *For a cyclic generalized Lagrangian correspondence \underline{L} such that:*

- M_i are all compact monotone with the same positive monotonicity constant, or are all exact;
- $L_{i,(i+1)}$ are all compact, oriented and monotone (or all exact) with minimal Maslov number at least three;
- $M_0 = M_{r+1}$ is a point;
- $L_{i,(i+1)} = L_i \times L_{i+1}$ for Lagrangians $L_i \subset M_i$ and $L_{i+1} \subset M_{i+1}$ for some $1 \leq i < r$,

there is a canonical isomorphism

$$HF^*(\underline{L}) = HF^*(L_{01}, L_{12}, \dots, L_{(i-1)i}, L_i) \otimes HF^*(L_{i+1}, L_{(i+1)(i+2)}, \dots, L_{r(r+1)})$$

with coefficients in a field.

THEOREM 5.5 ([WW10b, Theorem 5.4.1] and [LL13, Theorems 1, 2]). *For a cyclic generalized Lagrangian correspondence \underline{L} such that:*

- M_i are all compact monotone with the same positive monotonicity constant, or are all exact;
- $L_{i,(i+1)}$ are all compact, oriented and monotone (or all exact) with minimal Maslov number at least three;
- \underline{L} is monotone, relatively spin and graded in the sense of [WW10b, § 4.3]; and
- $L_{(i-1)i} \circ L_{i,(i+1)}$ is embedded in the sense above,

there is a canonical isomorphism

$$HF^*(\underline{L}) = HF^*(L_{01}, L_{12}, \dots, L_{(i-1)i} \circ L_{i,(i+1)}, \dots, L_{r(r+1)})$$

where the orientation and grading on the right are induced by those on \underline{L} .

Remark 5.6. In [LL13], Theorem 5.5 was extended to greater generality than stated here, which should be useful for extending our results to negatively monotone cases.

For a graded symplectomorphism $\phi \in \text{Symp}(M)$, the fixed point Floer cohomology can be defined as

$$HF^*(\phi) = HF^*(\Delta, \text{Graph}(\phi)) = HF^*(\text{Graph}(\phi^{-1}), \Delta),$$

where the Lagrangian Floer cohomologies take place in $M \times M^-$ (see [Sei00], [WW10b, Definition 3.0.6 and Remark 3.0.7] for the details of graded symplectomorphisms and the associated gradings on the graphs).

Remark 5.7. We follow the convention in [WW10b], where $HF^*(\phi) = HF^*(\text{Graph}(\phi), \Delta)$ in $M^- \times M$. Therefore, we have $HF^*(\phi) = HF^*(\Delta, \text{Graph}(\phi))$ in $M \times M^-$.

6. Proof of the long exact sequences

6.1 Exactness and monotonicity of surgery cobordisms

We construct Lagrangian cobordisms associated to the surgery identities in Theorem 1.1(1), (2), (3), (4) and deduce the long exact sequences in this section. Throughout the whole section, we assume all Lagrangians in M to be \mathbb{Z} or \mathbb{Z}/N -graded.

LEMMA 6.1. *Let $L = L_1 \#_D L_2, L_1 \#_{D, E_2} L_2$ or $\mathcal{L}_1 \#_{D, E_2} \mathcal{L}_2$ as surgeries of graded Lagrangians. Then there is a graded Lagrangian cobordism V from L_1 and L_2 (or \mathcal{L}_1 and \mathcal{L}_2) to L .*

Proof. We give the proof for $L = L_1 \#_{D, E_2} L_2$; the proofs for $L_1 \#_D L_2$ and $\mathcal{L}_1 \#_{D, E_2} \mathcal{L}_2$ are similar. It suffices to consider $M = T^*L_1$ and $L_2 = N_D^*$ is the conormal bundle of D in L_1 . As usual, we assume that a product metric on $L_1 = K_1 \times K_2$ is chosen and $D \pitchfork (\{p\} \times K_2)$ for all $p \in K_1$ so that the E_2 -flow clean surgery can be performed.

First note that $L_1 \times \mathbb{R}$ intersects cleanly with $L_2 \times i\mathbb{R}$ at $D \times \{0\}$. Let the grading of L_i be θ_i . We give a grading θ_{1r} to $L_1 \times \mathbb{R}$ by requiring $\theta_{1r}(p, z) = \theta_1(p)$ for all $p \in L_1$ and $z \in \mathbb{R}$. On the other hand, we equip $L_2 \times i\mathbb{R}$ with grading θ_{2i} such that $\theta_{2i}(p, z) = \theta_1(p) - 1/2$ for all $p \in L_1$ and $z \in \mathbb{R}$. Then we have $\text{Ind}_{D \times \{0\}}(L_1 \times \mathbb{R}, L_2 \times i\mathbb{R}) = \text{Ind}_D(L_1, L_2) + \text{Ind}_0(\mathbb{R}, i\mathbb{R}) = \text{Ind}_D(L_1, L_2)$. Moreover, we also have $\text{Ind}_D(L_1, L_2) = \dim(D) + 1$ by the assumption that graded E_2 -flow surgery from L_1 to L_2 can be performed and Lemma 4.12.

Pick the standard metric on \mathbb{R} . By Lemma 4.12, we can perform the graded Lagrangian surgery from $L_1 \times \mathbb{R}$ to $L_2 \times i\mathbb{R}$ resolving the clean intersection by a $(E_2 \oplus \mathbb{R})$ -flow handle $H_{\nu}^{D, E_2 \oplus \mathbb{R}}$, where we canonically identify $T^*(L_1 \times \mathbb{R})$ with an $E_1 \oplus E_2 \oplus \mathbb{R}$ bundle over $L_1 \times \mathbb{R}$. We note that $E_2 \oplus \mathbb{R}$ -flow is well defined to give a smooth Lagrangian manifold because we stayed inside the injectivity radius (Lemma 2.26). Hence we have a graded embedded Lagrangian cobordism with four ends in $M \times \mathbb{C}$.

Let $\pi : M \times \mathbb{C} \rightarrow \mathbb{C}$ be the projection to the second factor and $\pi_H = \pi|_{H_{\nu}^{D, E_2 \oplus \mathbb{R}}}$. We define $S_+ = \{(x, y) \in \mathbb{R}^2 \mid y \geq x\}$ and $W = \pi_H^{-1}(S_+)$. A direct check shows that W is a smooth manifold with boundary $\pi_H^{-1}(0) = L$. Let $W_0 = W \cap \pi^{-1}([-3\epsilon, 0] \times [0, 3\epsilon])$. It has three boundary components, namely $L_1 \times \{(-3\epsilon, 0)\}$, $L_2 \times (0, 3\epsilon)$ and $L \times \{(0, 0)\}$, while $L \times \{(0, 0)\}$ is the only boundary component that is not cylindrical. One then applies a trick due to Biran and Cornea (see [BC13, § 6]). This yields a Hamiltonian perturbation φ supported on $\pi^{-1}([-\epsilon, \epsilon] \times [-\epsilon, \epsilon])$, so that $\varphi(W)$ has all three cylindrical ends. By extending $\varphi(\pi_H^{-1}(0))$ to infinity and bending the

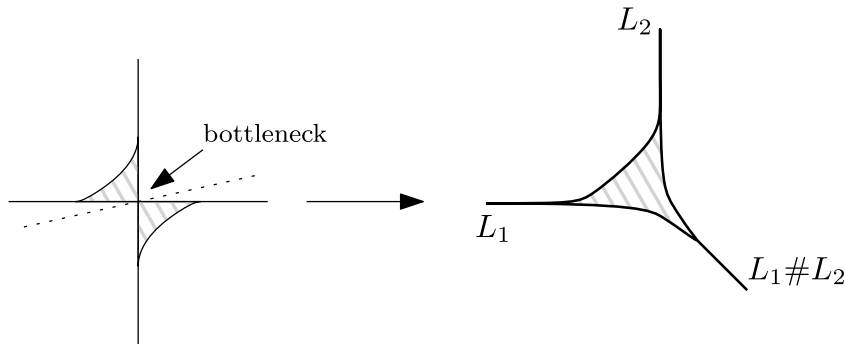


FIGURE 11. Construction of a surgery cobordism.

cylindrical end corresponding to L_2 to the left, we get the desired Lagrangian cobordism V (see Figure 11).

Finally, by the identification of gradings from ends to fiber Lagrangians, we conclude that it is a cobordism from L_1 and L_2 to L . \square

We call a cobordism obtained by Lemma 6.1 a *surgery cobordism*. When D is a single point, it reduces to the usual Lagrangian surgery and Lemma 6.1 was discussed in [BC13, § 6] in detail.

LEMMA 6.2. *Let V be a surgery cobordism from L_1, L_2 to L and D is connected. If L_1 and L_2 are exact Lagrangians, then L is exact and V is also exact.*

Proof. We give the proof for $L = L_1 \#_{D, E_2} L_2$. Without loss of generality, we can assume $M = T^*L_1$, $L_2 = N_D^*$ is the conormal bundle. We first assume $\text{codim}_{L_i}(D) \geq 2$.

Since the E_2 -flow handle H_ν^{D, E_2} is obtained by a Hamiltonian flow of $N_D^* \setminus D$, it is immediate that H_ν^{D, E_2} is an exact Lagrangian. Let f_1, f_2 and f_H be primitives of α restricted on L_1, L_2 and H_ν^{D, E_2} , respectively. Since we assume that D is of codimension two or higher, $(f_i - f_H)|_{L_i \cap \overline{H_\nu^{D, E_2}}}$ are locally constant and hence constant for $i = 1, 2$, where $\overline{H_\nu^{D, E_2}}$ denotes the closure of the handle. By possibly adding a constant to f_1 and f_2 , we can assume f_1, f_2 and f_H are chosen in such a way that they match together to give a primitive on L .

Now we drop the codimension assumption and only assume $\text{codim}_{L_i}(D) \geq 1$. We recall that in the proof of Lemma 6.1, the first step for constructing V is to resolve $L_1 \times \mathbb{R}$ and $L_2 \times i\mathbb{R}$ along $D \times \{(0, 0)\}$, which has now $\text{codim}_{L_i \times \mathbb{R}}(D) \geq 2$. This process preserves exactness by what we just proved. Then we cut the cobordism into a half, do Hamiltonian perturbation near $L \times \{(0, 0)\}$ and extend the cylindrical end. All of these steps preserve the exactness of the Lagrangian and hence V is exact. The restriction of V to the fiber over $\{(0, 0)\}$ is precisely L , proving the exactness of the surgery. \square

LEMMA 6.3. *Let V be a surgery cobordism from L_1, L_2 to L . If L_1 and L_2 are monotone Lagrangians such that either:*

- (1) $\pi_1(L_1, D) = 1$ or $\pi_1(L_2, D) = 1$; or
- (2) the image of $\pi_1(L_i)$ in $\pi_1(M)$ is torsion for either $i = 1, 2$,

then L is monotone and V is also monotone.

Proof. Again we give the proof for $L = L_1 \#_{D, E_2} L_2$ and we first assume that $\text{codim}_{L_i}(D) \geq 2$. For convenience we decompose $L = \mathring{L}_1 \cup \mathring{L}_2$. Here \mathring{L}_2 is the closure of the image of $L_2 \setminus D$ under the E_2 -flow defining the surgery, and \mathring{L}_1 is the closure of the complement of \mathring{L}_2 .

In case (1) it suffices to prove the lemma when $\pi_1(L_2, D) = 1$, since the slight asymmetry of L_1 and L_2 will be irrelevant. First it is easy to see that $\pi_1(U(D), U(D) \setminus D) = \pi_1(N_D^*, N_D^* \setminus D) = 1$ by our assumption on D , where $U(D)$ is a tubular neighborhood of D in L_2 . Since the flow handle H_ν^{D, E_2} is obtained by applying an E_2 -flow to $N_D^* \setminus D$, any path in \mathring{L}_2 with ends at H_ν^{D, E_2} can be homotoped to a path in H_ν^{D, E_2} , while H_ν^{D, E_2} in turn retracts to its boundary component that lies on \mathring{L}_1 .

The upshot is, we can find for any element in $\pi_2(M, L)$ a representative $u : \mathbb{D}^2 \rightarrow M$ with boundary completely lying in \mathring{L}_1 . Since L_1 is monotone, it finishes the proof for L .

Case (2) is similar. Without loss of generality, assume the image of $\pi_1(L_2) \rightarrow \pi_1(M)$ is torsion. Take again any disk $u : \mathbb{D}^2 \rightarrow M$ with boundary on L , and assume ∂u intersects $\partial \mathring{L}_2$ transversally. For any segment $I \subset \partial u$ contained in \mathring{L}_2 satisfying $\partial I \subset \partial \mathring{L}_2$, one connects the two endpoints of ∂I by $I' \subset \partial \mathring{L}_2$ (the relevant boundary is connected due to the assumption of connectedness and codimension of D). By assumption, we can take a disk $v : \mathbb{D}^2 \rightarrow M$ with $\partial v = m[I \cup I']$ for some integer m . Then one may decompose mu so that $m[u] = [mu - v] + [v]$, so that $\partial v \subset \mathring{L}_2$. By performing such a cutting iteratively, one may assume $\partial(mu - v) \subset \mathring{L}_1$. Since ∂v retracts to $L_2 \cap \mathring{L}_2$, the monotonicity follows from that of L_1 and L_2 with such a decomposition.

Now in either case the monotonicity of V is argued in a similar way as Lemma 6.2 because all processes involved preserve monotonicity. The restriction to the fiber over the origin again removes the assumption of $\text{codim}_{L_i} D \geq 2$ as in Lemma 6.2. □

6.2 Proof of long exact sequences

THEOREM 6.4 [Sei03, WW16, BC17]. *Let (M, ω) be a monotone symplectic manifold and S^n ($n > 1$) a graded embedded Lagrangian sphere. For any graded monotone Lagrangians L_1 and L_2 , there is a long exact sequence*

$$\cdots \rightarrow HF^*(S^n, L_2) \otimes HF^*(L_1, S^n) \rightarrow HF^*(L_1, L_2) \rightarrow HF^*(L_1, \tau_{S^n}(L_2)) \rightarrow \cdots .$$

Proof. By Corollaries 3.5, 4.17 and Lemma 6.1, there is a Lagrangian cobordism V from $S^n \times S^n[1]$ and the diagonal Δ to $\text{Graph}(\tau_{S^n}^{-1})$ in $M \times M^-$, where $M^- = (M, -\omega)$. By Lemma 6.3, the monotonicity of (M, ω) implies the same property for $S^n \times S^n[1]$, $\Delta \subset M \times M^-$ and the corresponding cobordism V .

In either case, $\text{Graph}(\tau_{S^n}^{-1})$ is a cone from $S^n \times S^n$ to Δ in the derived Fukaya category of $M \times M^-$ by Theorem 5.2. In particular, we have a long exact sequence

$$\cdots \rightarrow HF^*(L_1 \times L_2, S^n \times S^n) \rightarrow HF^*(L_1 \times L_2, \Delta) \rightarrow HF^*(L_1 \times L_2, \text{Graph}(\tau_{S^n}^{-1})) \rightarrow \cdots .$$

From the Kunneth formula, $HF^*(L_1 \times L_2, S^n \times S^n) \cong HF^*(L_1, S^n) \times HF^*(S^n, L_2)$ (recall that there is a negation on the symplectic form of the second factor). The identity $HF^*(L_1 \times L_2, \Delta) \cong HF^*(L_1, L_2)$ is also well known: in view of Lagrangian correspondence,

$$HF^*(L_1 \times L_2, \Delta) = HF^*(L_1, \Delta, L_2) = HF^*(L_1, \Delta \circ L_2) = HF^*(L_1, L_2)$$

by Theorem 5.5. □

COROLLARY 6.5 [Sei08b, WW16]. *In the same situation as Theorem 6.4, $f \in \text{Symp}(M)$, then*

$$\cdots \rightarrow HF^*(\tau \circ f) \rightarrow HF^*(f) \rightarrow HF^*(f(S^n), S^n) \rightarrow \cdots. \tag{6.1}$$

Proof. The exact sequence follows from applying the cohomological functor $HF^*(-, \text{Graph}(f))$ to the cone given by the cobordism. \square

The above result is predicted by Seidel [Sei08b, Remark 2.11] in a slightly different form from here. This is solely due to the cohomological convention we took. In the following theorem, we assume all involved symplectic manifolds and Lagrangians have the same monotonicity constant with minimal Maslov number at least three.

THEOREM 6.6 (Theorem 1.4, see also [WW16]). *Let C be a spherically fibered coisotropic manifold over the base (B, ω_B) in (M, ω) . Given Lagrangians L_1 and L_2 and assume both the following monotonicity conditions.*

- (i) *The generalized Lagrangian correspondence (L_1, C^t, C, L_2) is monotone in the sense of [WW10b].*
- (ii) *The surgery cobordism corresponding to the surgery in Corollary 3.10 is monotone.*

Then there is a long exact sequence

$$\cdots \rightarrow HF^*(L_1 \times C, C^t \times L_2) \rightarrow HF^*(L_1, L_2) \rightarrow HF^*(L_1, \tau_C(L_2)) \rightarrow \cdots.$$

In particular if the spherical fiber of C has dimension > 1 or $\pi_1(M)$ is torsion, (ii) is automatic.

Proof. The proof is analogous to Theorem 6.4 with Corollaries 3.5 and 4.17 replaced by Corollary 3.10 and Lemma 4.20. Here we give a sketch. First, (L_1, C^t, C, L_2) being monotone implies $\tilde{C} = C^t \circ C$ being monotone (see [WW10b, Remark 5.2.3]). The Lagrangian cobordism in Corollary 3.10 is monotone by Lemma 6.3. It is not hard to verify $\pi_1(\tilde{C}, \tilde{C} \cap \Delta) = 1$ when $\text{codim}_M C \geq 2$. Hence, Theorem 5.2 applies either in this case or when $\pi_1(M)$ is torsion, and we obtain the long exact sequence

$$\cdots \rightarrow HF^*(L_1 \times L_2, \tilde{C}) \rightarrow HF^*(L_1 \times L_2, \Delta) \rightarrow HF^*(L_1 \times L_2, \text{Graph}(\tau_C^{-1})) \rightarrow \cdots.$$

With our assumption on the monotonicity of (L_1, C^t, C, L_2) , we apply Theorem 5.5 to obtain the desired result. \square

There is a similar result on the fixed point version of family Dehn twist, and we will not state it explicitly here.

The new proofs for Theorems 6.4 and 6.6 go through for projective Dehn twists, by using Lemma 4.19. The family versions and (family) fixed point versions for projective Dehn twists can be obtained similarly (cf. Lemma 4.20).

THEOREM 6.7. *Let (M, ω) be a closed monotone symplectic manifold. Let S be a graded Lagrangian submanifold diffeomorphic to a complex (or real, quaternionic) projective space. For graded monotone Lagrangians L_0 and L_1 , there is a quasi-isomorphism of cochain complexes*

$$\begin{aligned} & CF^*(L_0, \tau_S L_1) \\ & \cong \text{Cone}(CF^*(S, L_1) \otimes CF^{*-1}(L_0, S) \rightarrow CF^*(S, L_1) \otimes CF^*(L_0, S) \rightarrow CF^*(L_0, L_1)). \end{aligned} \tag{6.2}$$

for some maps in the mapping cones, where $\dagger = 2, 1, 4$, respectively, for complex, real and quaternionic projective space.

Similarly, let C_P be a projectively coisotropic submanifold which satisfies the same monotonicity conditions as in Theorem 6.6,

$$CF^*(L_0, \tau_{C_P} L_1) \cong \text{Cone}(CF^{*-\dagger}(L_0 \times C_P, C_P^t \times L_1) \rightarrow CF^*(L_0 \times C_P, C_P^t \times L_1) \rightarrow CF^*(L_0, L_1)). \quad (6.3)$$

Proof. This is an immediate consequence of the cobordism of Theorem 1.1(3), (4), and [BC13, Theorem 2.2.1] by plugging in $N = L_0 \times L_1$ as the testing Lagrangian. \square

Remark 6.8. From elementary homological algebra, (6.2) leads to the following long exact sequences (cf. Theorem 6.4),

$$\dots \rightarrow HF^*(S, L_1) \otimes HF^{*-\dagger}(L_0, S) \rightarrow HF^*(S, L_1) \otimes HF^*(L_0, S) \rightarrow H^*(C) \dots$$

and

$$\dots \rightarrow H^*(C) \rightarrow HF^*(L_0, L_1) \rightarrow HF^*(L_0, \tau_S(L_1)) \rightarrow \dots$$

for some cochain complex C . Geometrically, C is the Floer complex between certain immersed Lagrangian submanifold and $L_0 \times L_1$ in the ambient symplectic manifold $M \times M$. The case is similar for projectively coisotropic submanifolds C_P .

We have focused on the monotone case so far for concreteness of the exposition. For exact Lagrangian submanifolds, we have the following.

THEOREM 6.9. *Theorem 6.4 and the first half of Theorem 6.7 holds true for compact exact Lagrangian submanifolds, and Corollary 6.5 holds for exact symplectic manifolds.*

The proof for the exact cases are almost completely identical to the monotone cases with Lemma 6.2 ensuring the exactness of the surgery cobordism involved, except for one instance, that the identity

$$HF^*(L_1 \times L_2, \Delta) \cong HF^*(L_1, L_2)$$

does not fit into the general machinery of Lagrangian correspondences directly.

However, this statement is again well known to experts, and we give a rough argument here. The chain group has a natural bijection sending $L_1 \times L_2 \cap \Delta \ni (x, x) \mapsto x \in L_1 \cap L_2$. For a (J, J^-) -holomorphic strip $u = (u^1, u^2) : \mathbb{R} \times [0, 1] \rightarrow M \times M^-$, where $J = (J_t)_{t \in [0, 1]}$ is a family of ω -compatible almost complex structures and $J^- = -J$, one has an $J' = (J'_t)_{t \in [0, 1]}$ -holomorphic strip

$$u'(s, t) = \begin{cases} u^2(2s, 1 - 2t) & \text{when } 0 \leq t < 1/2, \\ u^1(2s, 2t - 1) & \text{when } 1/2 \leq t \leq 1, \end{cases} \quad (6.4)$$

where $J'_t = J_{1-2t}$ for $t \in [0, 1/2]$ and $J'_t = J_{2t-1}$ for $t \in [1/2, 1]$. Conversely, given a J' -holomorphic strip u' , we can define $u = (u^2, u^1)$, $u^1(s, t) = u'(s/2, (1 + t)/2)$ and $u^2(s, t) = u'(s/2, (1 - t)/2)$.

This gives a bijection between $\mathcal{M}((x, x), (y, y); J \oplus J^-)$, the moduli space of $J \oplus J^-$ -holomorphic strips connecting (x, x) and (y, y) , and $\mathcal{M}(x, y; J)$, the J -holomorphic strips connecting x and y , for $x, y \in L_1 \cap L_2$. This concludes our discussion on exact cases for the long exact sequences.

6.3 Cones in Fukaya categories

We may recapitulate results from §6 on the functor level using results in [BC14, WMW]. We continue to use the monotone setting in this subsection.

The case of spherical twists and fibered Dehn twists are straightforward given the M'au–Wehrheim–Woodward A_∞ -functor Φ , which is

$$\Phi : \mathcal{Fuk}(M \times M) \longrightarrow \text{fun}(\mathcal{Fuk}^\#(M), \mathcal{Fuk}^\#(M)), \tag{6.5}$$

where $\mathcal{Fuk}^\#(M)$ is the A_∞ -category of generalized Lagrangians defined in [WMW]. When G_ϕ is a graph of a symplectomorphism ϕ , the functor $\Phi_\phi := \Phi(G_\phi)$ in $\text{fun}(\mathcal{Fuk}(M), \mathcal{Fuk}(M))$ is the functor induced by ϕ .

Given Corollaries 3.5, 4.17 and Lemma 6.1, we constructed a Lagrangian cobordism from $S^n \times S^n$ and Δ to $\text{Graph}(\tau_{S^n}^{-1})$. The main result from [BC14] then gives a cone in $\mathcal{Fuk}(M \times M)$

$$S^n \times (S^n)^- \rightarrow \Delta \rightarrow \text{Graph}(\tau_{S^n}^{-1}) \xrightarrow{[1]}. \tag{6.6}$$

Hence, under Φ this turns into a cone of functors

$$\text{hom}(S^n, -) \otimes S^n \rightarrow \text{Id}_{T_w \mathcal{Fuk}(M)} \rightarrow \Phi_{\tau_{S^n}} \xrightarrow{[1]} \tag{6.7}$$

or, if $\phi \times \text{id}$ is applied to the cobordism, the resulting cone reads

$$\text{hom}(\phi(S^n), -) \otimes S^n \rightarrow \Phi_\phi \rightarrow \Phi_{\tau_{S^n} \circ \phi} \xrightarrow{[1]}. \tag{6.8}$$

Evaluating (6.7) at any object $L \in \mathcal{Fuk}(M)$ hence recovers Seidel's cone relation [Sei08a]

$$\rightarrow \text{hom}(S^n, L) \otimes S^n \xrightarrow{ev} L \rightarrow \tau_{S^n} L \xrightarrow{[1]} \tag{6.9}$$

while further evaluating at another object gives the cohomological version: Theorem 6.4. Corollary 6.5 follows from (6.8) by considering the morphisms to the identity functor in simple cases. For the family Dehn twist Corollary 3.10, we may also interpret it as a cone of functors, but we need to go to the general Lagrangian correspondence framework; by the time of writing, it is not clear that the functor induced by \tilde{C} in $\text{fun}(\mathcal{Fuk}^\#(M), \mathcal{Fuk}^\#(M))$ can descend to a functor in $\text{fun}(\mathcal{Fuk}(M), \mathcal{Fuk}(M))$ even for spherically coisotropic manifolds.

For the case of projective twists, the same argument shows the following.

THEOREM 6.10. *Let $S \subset M$ be a monotone Lagrangian $\mathbb{C}P^n$ (respectively $\mathbb{R}P^n, \mathbb{H}P^n$). Then the auto-equivalence induced by Lagrangian S -twist is equivalent to the following iterated cone in $\text{fun}(\mathcal{Fuk}(M), \mathcal{Fuk}(M))$*

$$\text{Cone}(\text{hom}(S, -) \otimes S[-\dagger] \rightarrow \text{hom}(S, -) \otimes S \rightarrow \text{id}_{\mathcal{Fuk}(M)}),$$

where $\dagger = 2, 1, 4$, respectively, for $\mathbb{C}P^n, \mathbb{R}P^n$ and $\mathbb{H}P^n$. Evaluating an object L on these functors recovers Theorem 1.8.

Proof. The construction from Lemma 3.6 gives a Lagrangian cobordism from $S \times S[-\dagger]$, $S \times S$ and Δ to the graph of τ_S^{-1} , and grading considerations from Lemmas 4.18, 4.19 endow the cobordism with a well-defined grading that matches those on the ends. The main theorem in [BC14] gives a quasi-isomorphism of iterated cones in $\mathcal{Fuk}(M \times M^-)$

$$\text{Cone}((S \times S)[- \dagger] \rightarrow S \times S \rightarrow \Delta_M) \cong \text{Graph}(\tau_S^{-1}).$$

The desired assertion is simply the image of this equality under the M’au–Wehrheim–Woodward functor Φ . □

Similarly, by replacing Lemma 3.6 with Corollary 3.10, and Lemmas 4.18 and 4.19 with Lemma 4.20, we have the following.

THEOREM 6.11. *Given a projectively coisotropic manifold $C \subset M$ satisfying the monotonicity assumptions in Theorem 6.6, the auto-equivalence induced by the family projective twist is equivalent to the following iterated cone in the category $\text{fun}(\mathcal{Fuk}^\#(M), \mathcal{Fuk}^\#(M))$,*

$$\text{Cone}(\tilde{C}[-\dagger] \rightarrow \tilde{C} \rightarrow \text{id}_{\mathcal{Fuk}^\#(M)}),$$

where $\dagger = 2, 1, 4$, respectively, if the projective fiber is $\mathbb{C}P^l$, $\mathbb{R}P^l$ and $\mathbb{H}P^l$.

We remark that the functor \tilde{C} should be regarded as the composite of the functors $C^t : \mathcal{Fuk}^\#(M) \rightarrow \mathcal{Fuk}^\#(B)$ and $C : \mathcal{Fuk}^\#(B) \rightarrow \mathcal{Fuk}^\#(M)$.

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Appendix A. Computations of connecting maps for the surgery formula

In this appendix, we briefly explain how to ‘compute’ some connecting maps involved in the surgery exact triangle along a clean intersection through the following simple algebraic fact.

LEMMA A.1. *Given \mathbb{Z} -graded cochain complexes A, B over a field \mathbb{K} and $c, c' \in \text{hom}^0(A, B)$ which are closed, assume that $0 \neq t \in \mathbb{K}$ and $[c] = t[c']$. Then $\text{cone}(c)$ is quasi-isomorphic to $\text{cone}(c')$.*

Proof. This is a straightforward verification by sending $(a, b) \in A[1] \oplus B$ to $(a, tb + \eta(a))$, where η is a chain homotopy between c and tc' . □

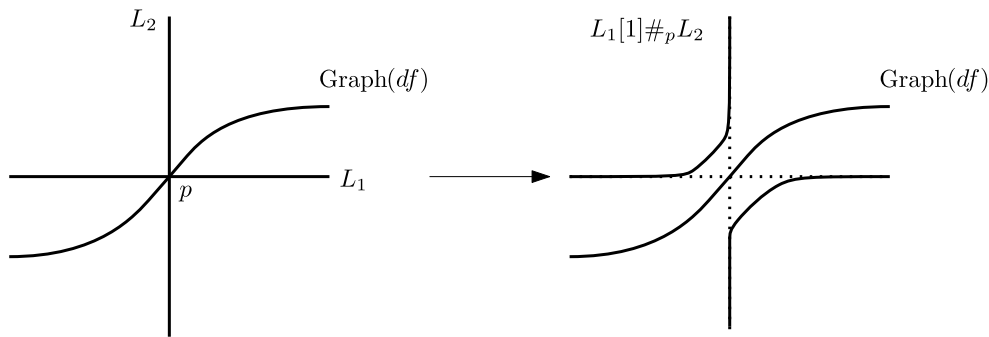


FIGURE A.1. Resolving the degree-zero intersection by surgery.

Lemma A.1 can be upgraded to a categorical level, for example, using the Yoneda lemma. This means that the quasi-isomorphism type of a non-trivial mapping cone is determined by the choice in $\mathbb{P}\text{Hom}^0(A, B)$, where $\text{Hom}^0(A, B) = H^0(\text{hom}(A, B))$. Hence, it suffices to compute the connecting morphisms up to a rescaling factor when only the quasi-isomorphism type of the mapping cone is concerned. In particular, when $\text{rank}(\text{Hom}^0(A, B)) = 1$, the mapping cone from A to B can have only one quasi-isomorphism type that is not the direct sum. The following perturbation lemma will be useful for excluding the direct sum.

LEMMA A.2. *Let $L_1, L_2 \subset M$ be a pair of \mathbb{Z} -graded exact Lagrangian submanifolds. Assume $L_1 \cap L_2 = D$ with index $\text{Ind}_D(L_1, L_2) = \dim(D) = k$ and the intersection is clean. Let $f : L_1 \rightarrow \mathbb{R}$ (respectively $f : L_2 \rightarrow \mathbb{R}$) be a Morse–Bott function which attains maximum (respectively minimum) at D and Morse elsewhere. Then the graph of df as a perturbation \tilde{L}_1 of L_1 (respectively \tilde{L}_2 of L_2) in a Weinstein neighborhood satisfies*

$$\tilde{L}_1 \pitchfork (L_1[1] \#_D L_2) = (\tilde{L}_1 \pitchfork L_1[1]) \setminus \{D\},$$

and respectively,

$$\tilde{L}_2 \pitchfork (L_1[1] \#_D L_2) = (\tilde{L}_2 \pitchfork L_2) \setminus \{D\},$$

as correspondences of intersection points preserving degrees.

Proof. Pick a Weinstein neighborhood W of L_1 such that L_2 can be identified as a conormal bundle (Proposition 2.21). Let \tilde{L}_1 be the graph of df and identify \tilde{L}_1 as a Lagrangian in W and hence in M . Pick a Darboux chart $U \subset W$ centered at a point $p \in D$ such that L_1 is identified with \mathbb{R}^n and L_2 is identified with $N_{\mathbb{R}^k}^*$. By choosing U to be a neighborhood of the zero section of the cotangent bundle of a Morse–Bott chart on L_1 near p , we can assume $f = c \sum_{i=1}^{n-k} x_i^2$ for some small negative constant c in U . In particular, \mathbb{R}^k is the only critical submanifold of f in U .

Let $L_3 = L_1[1] \#_D L_2$ and restrict our attention to $(L_3 \cap U) \subset U$. We have

$$\text{Graph}(df) = \{(\vec{x}, 2c\vec{x}) \mid \vec{x} \in \mathbb{R}^n\}$$

in $(T^*B(r))_r \subset U$ for some small $r > 0$. On the other hand, the flow handle is given by $H_\nu^D = \{(\exp(\nu(\|v\|) \cdot v/\|v\|), v) \mid v \in N_{\mathbb{R}^k}^*\}$, where \exp denotes the exponential map. Since $c < 0$, one sees that the two Lagrangians do not intersect in this Darboux chart U by checks on signs (cf. Example 4.5, 4.6 for our conventions). Since $p \in D$ is arbitrary, the flow handle does not intersect $\text{Graph}(df)$ (see Figure A.1).

The perturbation \tilde{L}_2 is constructed similarly, except f is taken to have a critical minimum submanifold along D on L_2 . We leave the details to the reader. \square

We exploit consequences of this simple fact. In the rest of this section all Lagrangians will be assumed to be \mathbb{Z} -graded and exact.

COROLLARY A.3 (Surgery exact triangle). *Let L_1, L_2 be graded exact closed embedded Lagrangians. Assume $L_1 \cap L_2 = D$ is connected such that $\text{Ind}_D(L_1, L_2) = \dim(D) = k$ and the intersection is clean. Let $L_3 = L_1[1] \#_D L_2$. Suppose also that there is a Morse–Bott function $f : L_1 \rightarrow \mathbb{R}$ (or $f : L_2 \rightarrow \mathbb{R}$) such that f attains local maximum (respectively minimum) exactly at D (i.e., no points other than D attains a local maximum). Then there is an exact triangle*

$$L_1 \xrightarrow{[D]} L_2 \rightarrow L_3 \rightarrow L_1[1],$$

where $[D]$ is the fundamental class of D regarded as an element in $HF^0(L_1, L_2)$ using Morse–Bott model.

Proof. When D is a point, the exact triangle is known to Fukaya *et al.* [FOOO09] in its cohomological version, and is a direct consequence of Biran and Cornea’s cobordism theory in the categorical version. When D is not just a point, we still have an exact triangle $L_1 \rightarrow L_2 \rightarrow L_3 \rightarrow L_1[1]$ by Lemma 6.1 and Biran and Cornea’s cobordism theory. We focus on the derivation of the connecting map $c_s : L_1 \rightarrow L_2$.

We assume $f : L_1 \rightarrow \mathbb{R}$ attains local maximum exactly at D . The case for L_2 is similar. Since $L_1 \cap L_2 = D$ and $\text{Ind}_D(L_1, L_2) = \dim(D) = k$, there is a Hamiltonian perturbation L'_1 of L_1 such that $CF^0(L'_1, L_2)$ is rank one. By standard Lagrangian Floer theory, we have a quasi-isomorphism $\Phi : CF(L_1, L_2) \rightarrow CF(L'_1, L_2)$ and $\Phi_*[D]$ is a generator of $HF^0(L'_1, L_2)$, which is at most rank one. By Lemma A.1, it suffices to show that the first connecting map is non-zero.

By Lemma A.2 there is no degree-zero element in $CF(\tilde{L}_1[1], L_3)$ (note: $\text{Ind}_D(\tilde{L}_1, L_1) = n - \text{Ind}_D(L_1, \tilde{L}_1) = 0$ by Example 4.6). If the connecting map is zero, $HF^0(\tilde{L}_1[1], L_3) = HF^0(\tilde{L}_1[1], L_1[1]) \oplus HF^0(\tilde{L}_1[1], L_2)$, which is at least rank one, so we arrive at a contradiction. \square

Remark A.4. One may recover connecting maps for exact sequences in Seidel’s exact sequence or Wehrheim and Woodward’s family Dehn twist by the same trick. The idea is to exploit the fact that $HF^0(S^n, S^n) \xrightarrow{\sim} HF^0(S^n \times S^n, \Delta)$ has rank one. One then needs to understand the above isomorphism explicitly and compare the image of identity with the evaluation map in Seidel’s exact sequence. One obvious possibility is to study the quilt unfolding [WMW].

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