APPENDIX.

The following paper on *Proportion* in its original form was remitted to a Committee of the Edinburgh Mathematical Society for consideration and report. After discussion in Committee it was reported to the Society, and at the meeting held on 12th January 1900 the following motion was unanimously adopted :—

> "The Edinburgh Mathematical Society resolves that Professor GIBSON'S Paper on Proportion be printed in its *Proceedings*, and recommends it to Mathematical Teachers as a suitable and sufficient substitute for Euclid's Fifth Book."

https://doi.org/10.1017/S0013091500029448 Published online by Cambridge University Press

Proportion: A Substitute for the Fifth Book of Euclid's "Elements."

By Prof. GEORGE A. GIBSON.

INTRODUCTION.

The following proposed substitute for the fifth book of Euclid's *Elements* has been drawn up with the object of filling a gap that undoubtedly exists in the current methods of teaching elementary geometry. It is a matter of common knowledge that Euclid's fifth book is rarely read in schools, and, in spite of its intrinsic excellence, it is not at all likely to be reintroduced into elementary teaching.

The chief reason—and it is a strong reason—for retaining Euclid's treatment of proportion is that it puts commensurable and incommensurable magnitudes on the same level; but this reason loses all its force if, as is usually the case, only the definitions and not the propositions of the fifth book are read. Besides, even when the fifth book has been studied, it is necessary to go further and to show that ratio as defined by Euclid is a quantity that can be used in calculations like ordinary numbers before the properties of ratios established in the fifth and sixth books of the *Elements* can logically be applied as is universally done in such a subject as trigonometry. In other words, since Euclid does not define a ratio as a number, proof ought to be given that it can be compared with numbers and that it has the same laws of combination as numbers before it can be legitimately subjected to the operations of algebra as its use in trigonometry, and applied geometry in general, requires. Such proof is rarely, if ever, attempted, not to say effected.

In whatever way a ratio may be defined to begin with, any treatment that is to be of the slightest use in practice must at some stage or other show that a ratio is a quantity subject to the laws of algebra. The common observation that the fifth book of Euclid is not geometrical, tells in favour of an arithmetical theory of proportion rather than against it, for it shows that Euclid could not develop

https://doi.org/10.1017/S0013091500029448 Published online by Cambridge University Press

his system on geometrical grounds alone. It would, indeed, be strange if a system of geometry which, like Euclid's, is essentially metrical, were independent of considerations of number. The difficulties attaching to the ratios of incommensurable magnitudes are identical with those of irrational numbers, and are not at all geometrical in their nature. The objection sometimes urged against an arithmetical treatment of proportion, that geometrical theorems should be established by geometrical methods alone, is without force as an argument in favour of Euclid's treatment, since his theorems on proportion are quite independent of geometry. The fifth book of Euclid is in reality a magnificent treatise on abstract number, not at all a treatise on geometry; but for that very reason it is quite unsuitable, as experience has proved, for elementary teaching.

The most natural method seems to be to begin with commensurable magnitudes, just as arithmetic begins with integers. When the pupil has in this way acquired some familiarity with the properties of ratios and their use in geometry, he can then have the difficulties of incommensurable magnitudes brought before him. In whatever way the subject be treated, the difficulties of the ratios of incommensurable magnitudes, like those of irrational numbers, are considerable, and a full discussion would probably be beyond the capacity of the average school-boy. In the following articles an effort has been made to render the conception of the ratio of two incommensurable magnitudes as clear as possible, and to emphasise the connection with the fundamental conception of a limit, but there is no attempt to prove the combining laws of such ratios. There is the less need for such proof, as in the forthcoming second edition of the second volume of Professor Chrystal's *Algebra* there is a thorough discussion of the irrational number. It is hoped that sufficient has been done to give the pupil clear and accurate ideas of the nature of an irrational number or of the ratio of two incommensurable magnitudes, and thus to prepare him for the fuller treatment that will be found in such an exposition as that referred to, as well as for the applications of the method of exhaustions or of limits in his more advanced studies.

It is therefore strongly recommended that the pupil should for a first reading confine himself to §§ 1–10, omitting §§ 11–16, and going on to the proof of the theorems in proportion that are required in geometry.

https://doi.org/10.1017/S0013091500029448 Published online by Cambridge University Press

Some propositions from Euclid's sixth book are appended, to show how they may be proved when Euclid's definition of proportion is replaced by that given in this paper.

Ratio and Proportion of Magnitudes.

1. If A and B denote two like magnitudes,* that is, two magnitudes of the same kind, *e.g.*, two straight lines or two rectangles, the sum of A and B will be denoted by $A+B$ and the difference by $A-B$, when A is greater than B, but by $B-A$ when A is less than B.

The sum of n magnitudes, each of which is equal to A, will be denoted by nA, and the magnitude nA will be said to contain A n times, or n times exactly.

2. *Definition.* If one magnitude contain another magnitude a certain number of times exactly, the greater is called a *multiple* of the less and the less is called a *sub-multiple* or a *measure* or an *aliquot part* of the greater.

If A contains B n times, A is called the nth multiple of B, and B the nth submultiple or the nth part of A.

These relations may be expressed by the equations

$$A = nB, \quad B = \frac{A}{n}.$$

3. If A, B be like magnitudes and m, n be integers, it follows from the first principles of arithmetic that

$$m(A \pm B) = mA \pm mB$$

$$(m \pm n)A = mA \pm nA$$

$$n \,.\, mA = nmA = mnA = m \,.\, nA$$

$$\frac{A \pm B}{n} = \frac{A}{n} \pm \frac{B}{n}$$

* In the fifth book of Euclid, magnitudes are usually represented by straight lines, and this procedure is very convenient, as it helps to make the somewhat abstract reasoning more definite. In class teaching, of course, the conclusions stated for integers m, n, etc., should be illustrated by the use of definite numbers, 2, 3, etc. On the comparison of magnitudes in general, see § 15.

Again, m times the nth part of A, that is $m\frac{A}{n}$, is the same magnitude as the nth part of m times A, that is $\frac{mA}{n}$. Each of these expressions $m\frac{A}{n}$ and $\frac{mA}{n}$ may therefore be represented by $\frac{m}{n}A$, so that the expression pA has a perfectly definite meaning even when p is a fraction. The arithmetic of fractions then shows that the above equations hold true even when m, n are fractional numbers.

4. *Definition.* When each of two like magnitudes is a multiple of a third magnitude, the third magnitude is called a *common measure* of the two magnitudes, and the two magnitudes are said to be *commensurable;* if the two magnitudes have no common measure, they are said to be *incommensurable.*

It will be proved in §11 that there are incommensurable magnitudes; §§5–10 deal with commensurable magnitudes only.

5. *Theorem.* If M be a common measure of A and B, every measure of M will also be a common measure of A and B. For a magnitude that contains M a certain number of times will contain $\frac{1}{2}$M, $\frac{1}{3}$M, etc., twice, thrice, etc., that number of times.

This theorem has the following converse: if G be the greatest common measure of A and B, every common measure of A and B will also be a measure of G.

For, let $A = aG$, $B = bG$ and let M be any other common measure of A and B.

Since G is the greatest common measure of A and B, the numbers a and b will be prime to each other; for if a and b had a common measure c, then cG would be a measure both of A and of B, and therefore G would not be the greatest common measure.

Now suppose $$A = mM, \quad B = nM.$$

Therefore, $$mM = aG, \quad nM = bG$$

or $$M = \frac{a}{m}G, \quad M = \frac{b}{n}G.$$

Hence $$\frac{a}{m} = \frac{b}{n} \quad \text{or} \quad \frac{m}{n} = \frac{a}{b}.$$

https://doi.org/10.1017/S0013091500029448 Published online by Cambridge University Press

But $\frac{a}{b}$ is a fraction in its lowest terms, and therefore $m = ra$, $n = rb$ where r is the greatest common measure of m and n.

Hence $M = \frac{a}{m}G = \frac{1}{r}G$, and therefore M is a measure of G.

6. From what has just been said about common measures it appears that any two commensurable magnitudes have an unlimited number of common measures; the smaller the measure is, the greater is the number that expresses the multiple which the magnitude is of its measure. But it has been proved that if M be any common measure of A and B, and if $A = mM$, $B = nM$, the fraction $\frac{m}{n}$ has the same value whatever common measure M be taken, being equal to the fraction $\frac{a}{b}$ where a, b are the number of times A, B respectively contain their greatest common measure.

Definition. If A and B be two like magnitudes having a common measure M, so that

$$A = mM, \quad B = nM, \quad \text{and therefore} \quad A = \frac{m}{n}B$$

the ratio of A to B is defined to be the fraction $\frac{m}{n}$.*

If A be greater than $\frac{m}{n}B$, the ratio of A to B is defined to be greater than the fraction $\frac{m}{n}$; and if less, less.

From what has just been stated the ratio of A to B is a number which is independent of the size of the common measure M.

It is stated in the definition that A and B are magnitudes of the same kind, and when the ratio of two magnitudes is spoken of, it is always to be understood that they are of the same kind.

---

* Instead of saying that "the ratio is equal to $\frac{m}{n}$," we sometimes say that "the ratio is measured by the fraction $\frac{m}{n}$"; but as a mathematical quantity it is the numerical value of the ratio which we always have in view.

https://doi.org/10.1017/S0013091500029448 Published online by Cambridge University Press

Again, in arithmetic the ratio of two integers m and n is defined to be the fraction $\frac{m}{n}$, so that the definition of ratio given above is equivalent to the following:—"When $A = \frac{m}{n}B$, the ratio of A to B is equal to the ratio of m to n.

The ratio of A to B is often expressed by the notation A : B. A and B are called the *terms* of the ratio; the term which comes first is called the *antecedent* and the other the *consequent*.

NOTE.—If A be equal to B, the ratio A : B is unity, and the ratio is in this case sometimes spoken of as *a ratio of equality*. If A be greater than B, the ratio A : B is an improper fraction and the ratio is then spoken of as a *ratio of greater inequality* or a *ratio of majority*, while if A be less than B, the ratio A : B is a proper fraction and is then spoken of as a *ratio of less inequality* or a *ratio of minority*.

7. *Definition.* If A and B be two like magnitudes, and if C and D be two other like magnitudes, though not necessarily of the same kind as A and B, the four magnitudes A, B, C, D are defined to be *proportionals* or to be *in proportion* when the ratio of A to B is equal to the ratio of C to D

The proportion is often expressed by the notation

$$A : B = C : D \text{ *}$$

or, in words, "A is to B as C is to D."

The magnitudes A, B, C, D are called the *terms* of the proportion; the first and fourth terms are called the *extremes*, the second and third the *means*. Thus A and D are the extremes, B and C the means.

When four magnitudes are proportionals, the first and third terms are said to be *homologous* to each other, and the second and fourth are also said to be *homologous* to each other. Instead of *homologous terms* the expression *corresponding terms* may be used, and the first and third may then be said to correspond.

When four magnitudes are proportionals, the fourth magnitude is sometimes called *the fourth proportional* to the other three.

It is obvious that C is greater than, equal to, or less than D according as A is greater than, equal to, or less than B.

* The notation A : B : : C : D is sometimes used instead of A : B = C : D.

8. *Definition.* If there be any number of like magnitudes greater than two, of which the first has to the second the same ratio that the second has to the third, and the second to the third the same ratio that the third has to the fourth, and so on, the magnitudes are said to be *continual proportionals* or *in continued proportion.*

When three magnitudes are in continued proportion, the second is said to be *the mean proportional* between the other two, and the third is said to be *the third proportional* to the other two.

Magnitudes in continued proportion are sometimes said to be in *geometrical progression*, and when there are three magnitudes in continued proportion the second is then called *the geometric mean* between the other two.

9. *Definition.* The ratio of B to A is defined to be *the reciprocal* or *the inverse* of the ratio of A to B.

From the definition of proportion it follows that

$$\text{if} \qquad A : B = C : D$$

$$\text{then} \qquad B : A = D : C$$

for the ratios $A : B$ and $C : D$ are each equal to the fraction $\frac{m}{n}$, and their reciprocals are each equal to the fraction $\frac{n}{m}$.

Again, the first of these two proportions may evidently be written in the form

$$C : D = A : B$$

and the second in the form

$$D : C = B : A$$

so that the truth of any one of these proportions implies the truth of the other three.

10. When the ratio of A to B is greater than the ratio of C to D, the relation between the magnitudes may be expressed by the notation $A : B > C : D$,
while if the ratio of A to B is less than that of C to D, the relation may be expressed by the notation

$$A : B < C : D.$$

It is clear that

$$\text{if} \qquad A : B > C : D$$

$$\text{then} \qquad B : A < D : C.$$

https://doi.org/10.1017/S0013091500029448 Published online by Cambridge University Press

Again, the first of these inequalities may be written in the form

$$C : D < A : B$$

and the second in the form

$$D : C > B : A$$

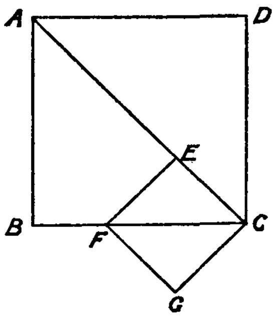
so that from any one of them the other three follow.

11. The magnitudes which have been considered up to this point have been supposed to be commensurable; it will now be proved that there are pairs of incommensurable magnitudes.

In proving that incommensurable magnitudes exist, and more generally in seeking to define the ratio of such magnitudes, the following axiom or postulate, called the axiom of Archimedes, is assumed :—

Given two unequal magnitudes of the same kind, there always exists a multiple of the less that is greater than the greater magnitude, or—which amounts to the same thing—there always exists a submultiple of the greater that is less than the less of the two given magnitudes.

To prove that the diagonal and the side of a square are incommensurable.



From the diagonal AC, cut off AE equal to AB or BC, and draw EF perpendicular to AC to meet BC at F.

Then BF = EF = EC.

Hence EF, EC are two sides of a square EFGC, of which FC is a diagonal, and the side EC consequently less than half BC.

Now $AC - BC = EC$ (1)

$BC - EC = FC$ (2)

Equation (1) shows that every common measure of AC and BC is a measure of EC, while equation (2) shows that every common measure of BC and EC is a measure of FC.

Hence every common measure of AC and BC is also a common measure of FC and EC, and EC has been shown to be less than half BC; that is, every common measure of the diagonal and the side of a given square is also a common measure of the diagonal and the side of another square, the side of which is less than half the side of the given square.

https://doi.org/10.1017/S0013091500029448 Published online by Cambridge University Press

The theorem can now be applied to the square EFGC, and so on indefinitely. If the theorem be applied n times in all, the conclusion is that every common measure of the diagonal and the side of a given square is also a common measure of the diagonal and the side of another square, the side of which is less than $1/2^n$ of the side of the given square.

If there be a common measure of AC and BC, let it be the line X. Now by the axiom of Archimedes we can take n so large that $BC/2^n$ shall be less than X. But the common measure sought for must be a measure of a line that is less than $BC/2^n$, no matter how large n may be. X therefore can not be a common measure of AB and BC since X is greater than $BC/2^n$. Hence the diagonal and the side of a square can have no common measure.

It has therefore been proved that there is one pair of incommensurable magnitudes;* as a matter of fact, incommensurable magnitudes are not exceptions of rare occurrence.

12. The general method of treating incommensurable magnitudes may be illustrated by considering the diagonal and the side of a square.

Denote AC by D and BC by S; then clearly

$$D > 1\,S \quad \text{but} \quad D < 2\,S.$$

Divide BC into 10 equal parts; then BC^2 contains 100 squares, the side of each being S/10, while AC^2 contains more than 196 but less than 225 such squares. Hence

$$D > 1{\cdot}4\,S \quad \text{but} \quad D < 1{\cdot}5\,S.$$

Proceeding in this way, it may be shown that

$$D > 1{\cdot}41\,S \quad \text{but} \quad D < 1{\cdot}42\,S$$

$$D > 1{\cdot}414\,S \quad \text{but} \quad D < 1{\cdot}415\,S$$

$$D > 1{\cdot}4142\,S \quad \text{but} \quad D < 1{\cdot}4143\,S$$

and so on.

In this way two sets of approximations to D are obtained, the one set being in defect, the other in excess; these may be called the lower and the upper sets respectively.

* As another instance of a pair of incommensurable lines, the segments of a straight line divided in medial section may be taken. See Mackay's *Euclid*, Book II., Prop. 11, Cor. 1.

https://doi.org/10.1017/S0013091500029448 Published online by Cambridge University Press

It is important to notice that the difference between corresponding members of the two sets gets less and less as the work proceeds, and further if we assume the axiom of Archimedes we can go so far as to make the difference between D and any approximation less than any given line. For if the given line be X, we can choose n so large that $S/10^n$ shall be less than X; if the work be carried out to n decimals, then the difference between the lower and the upper approximations is $S/10^n$, and therefore the difference between D and either of them less than $S/10^n$ and *a fortiori* less than X.

Hence although we can find no rational number a such that $D = aS$, we can find two rational numbers a, b such that

$$D > aS \quad \text{but} \quad D < bS$$

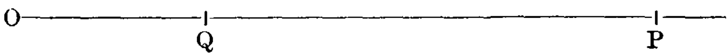
and at the same time $(b-a)S$ less than any given line,

In arithmetic the irrational number $\sqrt{2}$ is defined as the number which is greater than any of the numbers belonging to the lower set of approximations to D, namely, 1, 1.4, 1.41, etc., and less than any of the numbers belonging to the upper set, 2, 1.5, 1.42, etc.

Hence we may write $$D = \sqrt{2}S.$$

This symbol $\sqrt{2}$ is subject to the laws of algebra, and is, for that reason if for no other, called a *number;* it is, however, an *irrational* number, and the numbers 1, 1.4, 1.41, etc., are called rational approximations to it.

13. We may now take the general case of two incommensurable straight lines; let these be OP, OQ and suppose OP greater than OQ



We form the two sets of approximations to OP by considering (1) how often OP contains OQ, with a remainder less than OQ, (2) OQ/10 with remainder less than OQ/10, (3) $OQ/10^2$ with remainder less than $OQ/10^2$, and so on.

The lower set of approximations will consist of the multiples of OQ, OQ/10, $OQ/10^2$, etc., thus obtained, while the upper set will be these multiples increased by unity. Denote the lower set of approximations by a_1OQ, a_2OQ, a_3OQ, etc., and the corresponding

https://doi.org/10.1017/S0013091500029448 Published online by Cambridge University Press

members of the upper set by b_1OQ, b_2OQ, b_3OQ, etc.; then we get the following scheme. The first approximation gives

$$OP > a_1 OQ \quad \text{but} \quad OP < b_1 OQ, \quad (b_1 - a_1)OQ = OQ$$

the second gives

$$OP > a_2 OQ \quad \text{but} \quad OP < b_2 OQ, \quad (b_2 - a_2)OQ = OQ/10$$

the third gives

$$OP > a_3 OQ \quad \text{but} \quad OP < b_3 OQ, \quad (b_3 - a_3)OQ = OQ/10^2$$

and generally, the nth gives

$$OP > a_n OQ \quad \text{but} \quad OP < b_n OQ, \quad (b_n - a_n)OQ = OQ/10^{n-1}.$$

Now, by the axiom of Archimedes, we can choose n so large that $OQ/10^{n-1}$ shall be less than any given line. Hence we can find a rational number a_n or b_n such that OP shall differ from a_nOQ or b_nOQ by less than any given line.

But further, suppose the two sets of numbers a_1, a_2, etc., b_1, b_2, etc., are all known; then there cannot be two different lines which are both greater than every one of the approximations in defect and both less than every one of those in excess. For, if possible, let OP, OP′ be two such lines, and let OP′ be greater than OP, say OP′ − OP = D.

We can choose n so large that $OQ/10^{n-1}$ shall be less than D;

but $\qquad OP' < b_n OQ, \quad OP > a_n OQ.$

Therefore, $\qquad OP' - OP < (b_n - a_n)OQ.$

But $\qquad (b_n - a_n)OQ = OQ/10^{n-1} < D.$

Hence OP′ − OP is less than D, which contradicts the supposition that OP′ − OP = D. In the same way it may be shown that OP is not greater than OP′. Hence OP′ = OP.

Thus the two sets of approximations determine the line OP uniquely when OQ is given. We therefore introduce a symbol a and write OP = aOQ, where a is defined by this property that it is greater than every one of the numbers a_1, a_2, etc., of the lower set and less than every one b_1, b_2, etc., of the upper set. a is an irrational number, and the a's and b's are rational approximations to it.

These rational approximations are expressed as decimals; but this has been done for convenience merely. The reasoning would hold equally well if, for example, instead of OQ, OQ/10, OQ/10^2, etc., we had chosen OQ, OQ/2, OQ/3, etc. The essential point is that

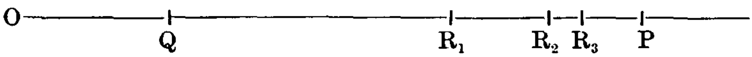
(1) the a's should give approximations in defect and the b's approximations in excess, and (2) that by taking n sufficiently large we should be able to make $(b_n - a_n)$OQ less than any given line, or $(b_n - a_n)$ less than any given fraction.

The symbol a has been called an irrational number, and it is shown in treatises on algebra * that it may be used according to the same rules that apply to rational numbers. In practice, approximations are always sufficient, and in place of the irrational number a rational approximation may be substituted according to the degree of accuracy required, so that even though the full theory of irrational numbers be not presupposed, there is, by the use of the approximations, an amount of control that is sufficient for all practical purposes.

14. There is another way of looking at the matter that is instructive. Consider the series of lines

$$a_1\text{OQ}, \quad a_2\text{OQ}, \ldots\ldots \qquad a_n\text{OQ}, \ldots\ldots$$

If we measure off these lines from O in the direction OP, we get for their second extremities, say, the points R_1, R_2, etc. As we pass from R_1 to R_2, then to R_3, and so on, we get nearer and nearer to P,



and though we never in this way quite reach P, we can come nearer to P than by any given distance; P is a *boundary* or a *limit* to our advance. Hence it is usual to speak of OP as the *limit* of the variable line a_nOQ, the variation being made by supposing n to increase indefinitely; and in the same way a is called the limit of the numbers a_n. In this case the limit is greater than each of the approximations: but clearly we might equally well regard OP as the limit of the lines b_nOQ where OP is less than each of the approximations.

It is also clear from this way of considering the matter that the knowledge of *one only* of the two sets of approximations would be sufficient to determine OP. In the general theory of limits it is not necessary that the approximations should be either always in defect

* See Chrystal's *Algebra*, Vol. II. (second edition).

https://doi.org/10.1017/S0013091500029448 Published online by Cambridge University Press

or always in excess; the essential thing is that we should be able to find n so that for that value and all greater values the difference between OP and a_nOQ shall be less than any given line.

We may express the above result in the notation

$$\mathrm{OP} = \underset{n=\infty}{\text{limit}} (a_n \mathrm{OQ}), \qquad a = \underset{n=\infty}{\text{limit}}\, a_n\,;$$

in words, OP is the limit for n increasing indefinitely of a_nOQ.

15. The considerations that have been applied to the comparison of straight lines hold for many of the other magnitudes of elementary geometry. Rectilineal areas can be supposed converted into rectangles of equal altitude, and one of these can be divided into equal areas by first dividing the base into equal parts. The comparison is then of exactly the same nature as in the case of straight lines. Angles, and arcs, and sectors of equal circles can be compared by superposition, but we must assume the possibility of subdivision into equal parts. Certain solids—for example, prisms—can be treated in a similar way; but, as a rule, outside this range, definitions are necessary to bring the magnitudes within the scope of mathematical treatment. Thus, before an arc of a circle can be compared with a straight line, some definition is required of the phrase "arc of a circle." In such cases recourse is usually had to the consideration of limits; for example, the circumference of a circle is considered as the limit of the perimeter of an inscribed (or circumscribed) polygon when the number of its sides is increased indefinitely, the length of each side at the same time diminishing indefinitely.

16. We may now define the ratio of two incommensurable magnitudes as follows:—

If A, B are two like incommensurable magnitudes, and if B be divided into any number n of equal parts of which A contains more than m but less than $m+1$, so that

$$\mathrm{A} > \frac{m}{n}\mathrm{B} \quad \text{but} \quad \mathrm{A} < \frac{m+1}{n}\mathrm{B}$$

then the ratio of A to B is defined to be the irrational number which is greater than every number of the set m/n and less than every number of the set $(m+1)/n$.

https://doi.org/10.1017/S0013091500029448 Published online by Cambridge University Press

It is clear that the ratio of two incommensurable magnitudes can not be equal to the ratio of two commensurable magnitudes, for that would mean that an irrational number can be equal to a rational number.

The definition of proportion is the same as before, but it can also be put into a form that is sometimes convenient in practice, namely,

If A, B be two like magnitudes, and if C, D be two other like magnitudes, though not necessarily of the same kind as A, B, then A, B, C, D will form a proportion if when

$$A > \frac{m}{n}B \quad \text{but} \quad < \frac{m+1}{n}B$$

at the same time

$$C > \frac{m}{n}D \quad \text{but} \quad < \frac{m+1}{n}D$$

and that for every value of n.

That this definition is equivalent to the first follows from the fact that the ratios of A to B and C to D are both equal to the irrational number determined by the sets m/n and $(m+1)/n$.

The proofs given of theorems in proportion for commensurable magnitudes hold equally for incommensurables, for the proofs depend on the hypothesis that the ratio of two magnitudes may be written in the form $\quad A : B = k : 1$
where k is a symbol that is subject to the rules of algebra.

17. The theorems in proportion that are required in elementary geometry will now be stated. The proofs are only given in a few cases, as they all run on the same lines and are very similar to those found in text-books of algebra. Capital letters are used throughout to denote magnitudes, and small letters to denote numbers.

Theorem 1

$$pA : pB = A : B$$

where p is any number.

Theorem 2

If $\quad A : B = C : D, \quad$ then $\quad pA : qB = pC : qD$

where p, q are any numbers.

https://doi.org/10.1017/S0013091500029448 Published online by Cambridge University Press

For, let each of the equal ratios A : B, C : D be equal to the ratio $k : 1$; then

$$A = kB, \quad C = kD.$$

$$\therefore \quad pA : qB = pkB : qB = pk : q$$

$$pC : qD = pkD : qD = pk :$$

$$\therefore \quad pA : qB = pC : qD$$

since each of these ratios is equal to $pk : q$.

THEOREM 3

If $A = B$, then $A : C = B : C$.

Conversely, if $A : C = B : C$, then $A = B$.

THEOREM 4

If $A > B$, then $A : C > B : C$

Conversely, if $A : C > B : C$, then $A > B$.

If $A < B$, then $A : C < B : C$

Conversely, if $A : C < B : C$, then $A < B$.

THEOREM 5

If $A : B = C : D$, then $B : A = D : C$

Proved in § 9.

This inference is referred to as *inversely*, or, *by inversion*.

THEOREM 6

If the magnitudes A, B, C, D be all of the same kind, and if

$$A : B = C : D, \text{ then } A : C = B : D.$$

For let each of the equal ratios A : B, C : D be equal to the ratio $k : 1$; then $A = kB, \quad C = kD$

$$\therefore \quad A : C = kB : kD = B : D \quad \text{by Th. 1.}$$

This inference is referred to as *alternately*, or, *by alternation*.

https://doi.org/10.1017/S0013091500029448 Published online by Cambridge University Press

Theorem 7

If $A:B=C:D$, then $A+B:B=C+D:D$.

This inference is referred to as *by addition*, or, *by composition*.

Theorem 8

If $A:B=C:D$, then $A-B:B=C-D:D$, when $A>B$, and therefore $C>D$, but $B-A:B=D-C:D$, when $A<B$, and therefore $C<D$.

This inference is referred to as *by subtraction*, or *by division*.

Theorem 9

If $A:B=C:D$, then $A+B:A-B=C+D:C-D$,

or $A+B:B-A=C+D:D-C$,

according as A is greater or less than B, and therefore C greater or less than D.

This inference is referred to as *by addition and subtraction*, or *by composition and division*.

For the proof, take the case of Theorem 9 when A is greater than B. As before, let each of the equal ratios $A:B$, $C:D$ be equal to the ratio $k:1$; then

$$A=kB;\quad\therefore\quad A+B=(k+1)B,\quad A-B=(k-1)B$$

$$\therefore\quad A+B:A-B=(k+1)B:(k-1)B=k+1:k-1.$$

Similarly, $C+D:C-D=k+1:k-1$

$$\therefore\quad A+B:A-B=C+D:C-D.$$

Theorem 10

If the magnitudes A, B, C, D be all of the same kind, and if $A:B=C:D$, then each ratio is equal to the ratio $A+C:B+D$ or to the ratio $A-C:B-D$, when $A>C$ and therefore $B>D$, but to the ratio $C-A:D-B$ when $A<C$ and therefore $B<D$.

Or, *in words*, as one antecedent is to its consequent, so is the sum of the antecedents to the sum of the consequents, or the difference of the antecedents to the difference of the consequents.

The proof is obvious, since in the same notation as before.

$$A + C = kB + kD = k(B + D)$$

and therefore

$$A + C : B + D = k : 1 = A : B = C : D.$$

This theorem is a particular case of a more general theorem, which may be stated as follows and may be proved in the same way :—

If the magnitudes A, B, C, D...R, S be all of the same kind, and if $A : B = C : D = \dots = R : S$, then each ratio is equal to the ratio

$$mA \pm nC \pm \dots \pm pR : mB \pm nD \pm \dots \pm pS$$

where $m, n, \dots p$ are any integers. It is, of course, to be remembered that in taking the difference of magnitudes, negative magnitudes are not considered.

Theorem 11

If the magnitudes A, B, C, D, E, F,...R, S be all of the same kind, and if the ratios A : B, C : D, E : F,...R : S be not all equal, then the ratio

$$A + C + E + \dots + R : B + D + F + \dots + S$$

is less than the greatest but greater than the least of the ratios

$$A : B, \quad C : D, \quad E : F, \dots R : S.$$

For suppose A : B to be the greatest and R : S the least of these ratios, and let R : S be equal to the ratio of $k : 1$; then

$$A : B > k : 1 \quad \therefore \quad A > kB,$$

Similarly, $\qquad C > kD, \;\; E > kF, \dots R = kS$

$$\therefore \quad A + C + E + \dots + R > k(B + D + F + \dots + S)$$

$$\therefore \quad A + C + E + \dots + R : B + D + F + \dots + S > k : 1 > R : S.$$

In the same way it may be proved that

$$A + C + E + \dots + R : B + D + F + \dots + S < A : B.$$

There is an obvious extension of this theorem corresponding to the general case of Theorem 10, but in this case only sums and not differences are to be taken.

18. Since a ratio is a number, ratios may be multiplied or divided. It is to be observed, however, that it is the *ratios* that are multiplied or divided; we may multiply the *ratio* of A to B by the *ratio* of C to D, but we do not multiply the *magnitude* A by the *magnitude* C, or the *magnitude* B by the *magnitude* D.

The product of the ratios A : B and C : D will be represented by the notation

$$(A : B) \times (C : D) \quad \text{or} \quad (A : B)\ (C : D).$$

The quotient of the ratio A : B by the ratio C : D may be represented by the notation

$$(A : B)/(C : D) \quad \text{or} \quad (A : B) \div (C : D) \quad \text{or} \quad (A : B) : (C : D)$$

or, in short, in any of the ways for expressing a quotient in arithmetic.

Euclid employs the phrase "to compound ratios" in the same sense as "to multiply ratios." Hence the

Definition. To compound two or more ratios means to take the product of the ratios.

It follows at once from the definition that if there be any number of like magnitudes A, B, C, D, the ratio A : D is the ratio compounded of the ratios A : B, B : C, C : D.

For, let $A : B = k : 1,\ B : C = l : 1,\ C : D = m : 1$;

then $A = kB,\ B = lC,\ C = mD$

$\therefore\ A = kB = klC = klmD$

$\therefore\ A : D = klm : 1.$

But $(A : B)(B : C)(C : D) = (k : 1)(l : 1)(m : 1) = klm : 1$

$\therefore\ A : D = (A : B)(B : C)(C : D).$

Again, since a ratio is merely a number, if

$$P : Q = A : B,\ R : S = B : C,\ T : V = C : D,$$

then $(P : Q)(R : S)(T : V) = (A : B)(B : C)(C : D)$

and $\therefore$ $(P : Q)(R : S)(T : V) = A : D.$

The ratio compounded of two or more ratios can therefore be expressed as the ratio of two magnitudes.

Euclid employs special names to denote the ratio compounded of two or more equal ratios.

https://doi.org/10.1017/S0013091500029448 Published online by Cambridge University Press

Definitions. The ratio compounded of two equal ratios is called the *duplicate* of either; the ratio compounded of three equal ratios is called the *triplicate* of any one of them; and so on.

The duplicate of $A:B$ may be written $(A:B)(A:B)$ or $(A:B)^2$; the triplicate may be written $(A:B)^3$; and so on.

In other words, the duplicate of a ratio is the square of it; the triplicate of a ratio is the cube of it; and so on. It would be much better to discard the phrases *duplicate ratio*, *triplicate ratio*, and the terminology of *compound ratios* altogether; the use of *duplicate ratio* is specially objectionable.

In geometrical work it is often convenient to express the square (or the duplicate) of the ratio of two magnitudes as the simple ratio of the two magnitudes. This may be done as follows:—

Let A, B be the two given magnitudes; find C the third proportional to A, B so that

$$A:B=B:C.$$

Then $A:C$ is the square (or duplicate) of $A:B$; for

$$A:C=(A:B)(B:C)=(A:B)(A:B)=(A:B)^2.$$

In the same way, to find the cube (or the triplicate) of $A:B$, find C, D so that

$$A:B=B:C \quad \text{and} \quad B:C=C:D$$

then $$A:D=(A:B)^3.$$

Higher powers of a ratio may be dealt with in the same way.

19. The following theorems involving the products of ratios are often required.

THEOREM 12.

If two ratios are equal, their squares (duplicates), their cubes (triplicates), etc., are equal.

Conversely, if the squares (duplicates), the cubes (triplicates), etc., of two ratios are equal, the ratios themselves are equal.

The proof is mere arithmetic since negative and imaginary numbers are not in question; for let $x:1$, $y:1$ be the two ratios, then if $x:1=y:1$, it follows that $(x:1)^n=(y:1)^n$ when n is any positive integer.

Conversely, if $(x:1)^n=(y:1)^n$, it follows that $x:1=y:1$.

THEOREM 13.

If A, B, C be three like magnitudes, and if P, Q, R be three other like magnitudes, though not necessarily of the same kind as A, B, C, such that

$$A : B = P : Q$$

and $$B : C = Q : R$$

then $$A : C = P : R.$$

For $$A : C = (A : B)(B : C)$$
$$= (P : Q)(Q : R)$$
$$= P : R.$$

More generally, if there be two sets of magnitudes

A, B, C,...H, K, L and P, Q, R,...X, Y, Z

such that

$$A : B = P : Q$$
$$B : C = Q : R$$
$$\dots\dots \quad \dots\dots$$
$$H : K = X : Y$$
$$K : L = Y : Z$$

then $$A : L = P : Z$$

The proof is as before,

This inference is referred to as "by equality."

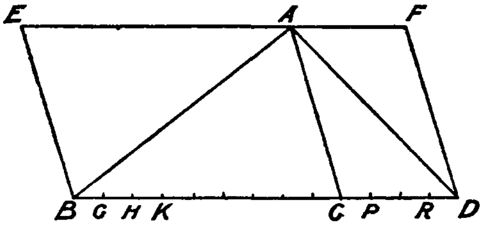
20. When Euclid's treatment of proportion is not adopted, the proofs of certain propositions in the sixth book require to be altered. The alteration is absolutely necessary only in the cases of the first and thirty third propositions, but alternative proofs are given for one or two others. It seems more natural to prove such fundamental propositions as the sixteenth and nineteenth without the use of reciprocal proportionals. In fact, the only use to which Euclid puts the conception of reciprocal proportionals in plane geometry is to establish Propositions 14 and 15 as stepping stones to Propositions 16, 17, and 19. Although much use was made in the ancient geometry of reciprocal proportionals, there would be little if any loss in discarding reciprocal proportionals from elementary geometry. The demands of examinations rather than the necessities of geometry seem, however, to compel the retention of Propositions 14 and 15, and the definition of reciprocal proportionals.

https://doi.org/10.1017/S0013091500029448 Published online by Cambridge University Press

EUCLID, VI. 1.

Triangles and parallelograms of the same altitude are to one another as their bases.

Let the triangles ABC, ACD and the parallelograms EC, CF have the same altitude, namely, the perpendicular from A to BD or BD produced:



it is required to prove that

triangle ABC : triangle ACD = BC : CD

and parallelogram EC : parallelogram CF = BC : CD.

First, let the bases BC, CD be commensurable, and let BG be a common measure of BC, CD.

Mark off on BC, CD the lines GH, HK......, CP,......RD each equal to BG and suppose A joined to the points G, H,......R.

Because BG, GH, HK, ... CP, ... RD are all equal, therefore,

the triangles ABG, AGH, AHK, ... ACP, ... ARD are all equal.

Hence,

if BC contain BG m times, △ABC will contain △ABG m times,

and if CD contain BG n times, △ACD will contain △ABG n times. Therefore, the two ratios BC : CD and the △ABC : △ACD are each equal to the fraction $\frac{m}{n}$, and are therefore equal to one another.

Hence, △ABC : △ACD = BC : CD.

Second, let the bases BC, CD be incommensurable.

Suppose that CD contains BG n times; then BC will not contain BG any number of times exactly. Let BC be greater than mBG but less than $(m+1)$BG.

As before, △ACD will contain △ABG n times and △ABC will be greater than m times but less than $(m+1)$ times △ABG.

∴ the two ratios BC : CD and △ABC : △ACD are each greater than the fraction $\frac{m}{n}$ but less than the fraction $\frac{m+1}{n}$, and that

https://doi.org/10.1017/S0013091500029448 Published online by Cambridge University Press

no matter how great n may be. Hence the two ratios are equal to the same irrational number, and are therefore equal to one another. Therefore, $\triangle ABC : \triangle ACD = BC : CD.$

Again, the parallelograms EC, CF are double of the triangles ABC, ACD respectively

$\therefore$ parallelogram EC : parallelogram CF = BC : CD.

Cor. 1. Triangles and parallelograms that have equal altitudes are to one another as their bases.

Cor. 2. Triangles and parallelograms that have equal bases are to one another as their altitudes.

Cor. 3. If BC, CD be any two lines and XY any other line,

$$BC : CD = BC . XY : CD . XY$$

for BC . XY, CD . XY are rectangles of the same altitude XY.

[In the same way Euclid VI., 33, may be proved.]

EUCLID VI. 16, 17.

If four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means.

Conversely: If the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportional.

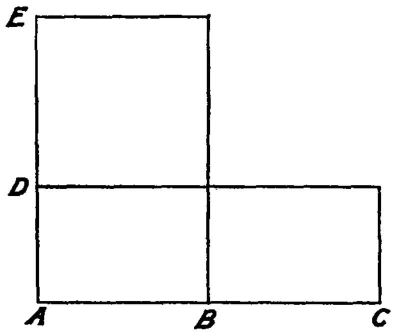
(1) Let the four straight lines AB, AC, AD, AE be proportional, so that

$$AB : AC = AD : AE :$$

it is required to prove

$$AB . AE = AC . AD.$$

Let AE be perpendicular to AC and complete the rectangles CD, BE.



$$AB : AC = \text{rectangle BD} : \text{rectangle CD}$$

and $\quad AD : AE = \text{rectangle BD} : \text{rectangle BE}.$

https://doi.org/10.1017/S0013091500029448 Published online by Cambridge University Press

But $AB : AC = AD : AE$

therefore, rectangle BD : rectangle CD = rectangle BD : rectangle BE

therefore, rectangle BE = rectangle CD

that is $AB . AE = AC . AD.$

(2) Let $AB . AE = AC . AD$:

it is required to prove $AB : AC = AD : AE.$

Make the same construction as before.

$AB : AC$ = rectangle BD : rectangle CD

and $AD : AE$ = rectangle BD : rectangle BE.

But rectangle BE = rectangle CD

therefore, $AB : AC = AD : AE.$

Cor. If AC be equal to AD, the proposition may be stated: –

If three straight lines be proportional, the rectangle contained by the extremes is equal to the square on the mean.

Conversely: if the rectangle contained by the extremes be equal to the square on the mean, the three straight lines are proportional.

Euclid VI. 19.

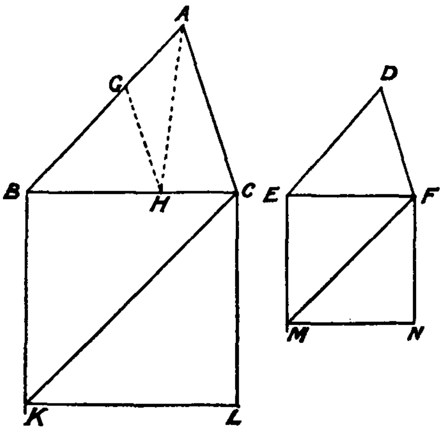
Similar triangles are to one another in the duplicate ratio of their homologous sides.

Let ABC, DEF be two similar triangles, having the angles at B, C equal to the angles at E, F respectively:

it is required to prove that

$$\triangle ABC : \triangle DEF = (BC : EF)^2$$

From BA, BC or from BA, BC produced, cut off BG, BH equal to ED, EF respectively, and join GH, AH.



https://doi.org/10.1017/S0013091500029448 Published online by Cambridge University Press

Then the triangle GBH is congruent with the triangle DEF, and therefore GH is parallel to AC.

$$\triangle ABC : \triangle ABH = BC : BH$$

and $$\triangle ABH : \triangle GBH = BA : BG$$

$$= BC : BH$$

therefore, $(\triangle ABC : \triangle ABH)(\triangle ABH : \triangle GBH) = (BC : BH)(BC : BH)$

that is, $$\triangle ABC : \triangle GBH = (BC : BH)^2$$

or, $$\triangle ABC : \triangle DEF = (BC : EF)^2.$$

Cor. If on BC, EF the squres BL, EN be drawn, the squares are double of the triangles KBC, MEF respectively, and these triangles are similar.

$$\therefore \quad BC^2 : EF^2 = \triangle KBC : \triangle MEF$$

$$= (BC : EF)^2.$$

Hence, the duplicate ratio of two lines is equal to the ratio of the squares on the lines, and therefore similar triangles are to one another as the squares on their homologous sides.

Euclid VI., 22.

See figures in *Mackay's Euclid*, or, *Todhunter's Euclid.*

(1) $$KAB : LCD = (AB : CD)^2$$

$$MF : NH = (EF : GH)^2$$

But $$AB : CD = EF : GH$$

$$\therefore \quad (AB : CD)^2 = (EF : GH)^2$$

$$\therefore \quad KAB : LCD = MF : NH$$

(2) $$KAB : LCD = (AB : CD)^2$$

$$MF : NH = (EF : GH)^2$$

But $$KAB : LCD = MF : NH$$

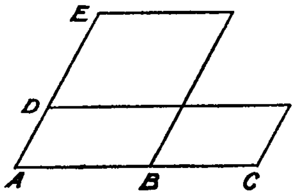
$$\therefore \quad (AB : CD)^2 = (EF : GH)^2$$

$$\therefore \quad AB : CD = EF : GH$$

https://doi.org/10.1017/S0013091500029448 Published online by Cambridge University Press

Euclid VI. 23.

Mutually equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides.



Let parallelogram BE be equiangular to parallelogram CD, and let

$$\angle EAB = \angle DAC;$$

to prove

$$\|^{m}BE : \|^{m}CD = (AB : AC)(AE : AD).$$

Because $\qquad \|^{m}BE : \|^{m}BD = AE : AD$

and $\qquad \|^{m}BD : \|^{m}CD = AB : AC$

therefore $\qquad (\|^{m}BE : \|^{m}BD)(\|^{m}BD : \|^{m}CD) = (AE : AD)(AB : AC)$

But $\qquad (\|^{m}BE : \|^{m}BD)(\|^{m}BD : \|^{m}CD) = \|^{m}BE : \|^{m}CD$

therefore $\qquad \|^{m}BE : \|^{m}CD = (AB : AC)(AE : AD).$

Cor. $\qquad \|^{m}BE : \|^{m}CD = AB . AE : AC . AD$

Suppose the sides AB, AE and AC, AD to remain constant and the angle A to vary; then the areas of the parallelograms BE and CD will vary, but the ratio of their areas will not vary, since this ratio is always equal to (AB : AC)(AE : AD).

If the angle A become a right angle, the parallelograms BE and CD will become the rectangles AB . AE and AC . AD.

Hence $\qquad AB . AE : AC . AD = (AB : AC)(AE : AD)$

and therefore $\qquad \|^{m}BE : \|^{m}CD = AB . AE : AC . AD.$

This corollary establishes the theorem that "the ratio compounded of the ratios of two pairs of lines is equal to the ratio of the rectangle contained by the antecedents to the rectangle contained by the consequents."

Cor. 2. If BE and CD be joined, the triangles EAB, DAC are halves of the parallelograms BE, CD. Hence,

Two triangles which have one angle of the one equal to one angle of the other have to each other the same ratio as the rectangles contained by the sides about the equal angles.