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ILL-POSEDNESS FOR THE COMPRESSIBLE NAVIER–STOKES EQUATIONS WITH THE VELOCITY IN L^6 FRAMEWORK‡

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Abstract Ill-posedness for the compressible Navier–Stokes equations has been proved by Chen *et al.* [On the ill-posedness of the compressible Navier–Stokes equations in the critical Besov spaces, *Revista Mat. Iberoam.* **31** (2015), 1375–1402] in critical Besov space L^p ($p > 6$) framework. In this paper, we prove ill-posedness with the initial data satisfying

$$\|\rho_0 - \bar{\rho}\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \leqslant \delta, \quad \|u_0\|_{\dot{B}^{-\frac{1}{2}}_{6,1}} \leqslant \delta.$$

To accomplish this goal, we require a norm inflation coming from the coupling term $L(a)\Delta u$ instead of $u \cdot \nabla u$ and construct a new decomposition of the density.

Keywords: Navier–Stokes equations; ill-posedness; Besov space

2010 *Mathematics subject classification*: Primary 35Q35; 35K55

1. Introduction

The purpose of this article is to study the Cauchy problem for the barotropic Navier–Stokes equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u + \nabla P = 0, \\ (\rho(0,x), u(0,x)) = (\rho_0(x), u_0(x)), \end{cases} \tag{1.1}$$

where $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$, $\rho \in \mathbb{R}$, $u = (u^1, u^2, u^3) \in \mathbb{R}^3$ stand for the density and the velocity field, respectively, $P = P(\rho)$ represents the scalar pressure. The constants μ and λ are viscosity coefficients satisfying

$$\mu > 0, \quad \lambda + 2\mu > 0.$$

It is easy to check that the solution (ρ, u) is scaling invariant under the transformation

$$(\rho_\lambda, u_\lambda) = (\rho(\lambda^2 t, \lambda x), \lambda u(\lambda^2 t, \lambda x)). \tag{1.2}$$

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We say a function space is critical means that the corresponding norm is scaling invariant under the transformation (1.2).

In [22], Nash proved the local existence and uniqueness of (1.1) for the smooth initial data without vacuum. Matsumura and Nishida [21] showed the global existence with the small initial data near equilibrium. It is Xin's work [28] that showed any non-zero smooth solutions of the full compressible Navier–Stokes (i.e., (1.1) coupling with temperature equation) will blow up in finite time provided the initial density admits the compact support. We also refer to [23] and [24] for the blow-up criteria. For the existence of global weak solutions, we refer to the books [14], [20] and the recent breakthrough paper [25] and references therein.

In a seminal paper [9], Danchin obtained the global well-posedness for (1.1) in the critical hybrid Besov space by applying Fourier analysis method, which is motivated by the work of Fujita and Kato [15] on the incompressible Navier–Stokes. One can see [3, 5, 6, 10–12, 18, 27, 30] for the local and global well-posedness. More precisely, Chen *et al.* [6] and Charve and Danchin [3] proved the global well-posedness in the hybrid Besov space, the high frequence of which is in L^p $(p > 3)$ framework allowing a class of highly oscillating initial velocity like $\sin(\frac{\pi}{\epsilon})\phi(x)$, while the authors in [18] and [30] obtained the associated results with large oscillating initial density. Later, Wang *et al.* [27] got the global well-posedness for a new class of large initial data allowing both highly oscillating initial density and velocity.

Seemingly, (1.1) is locally well-posed for the initial data satisfying

$$(\rho_0 - \bar{\rho}, u_0) \in \dot{B}^{\frac{3}{p}}_{p,1} \times \dot{B}^{\frac{3}{p}-1}_{p,1}, \quad p < 6,$$

see e.g., [8, 11, 13]. So we want to know whether ill-posedness can be proved under $p \geqslant 6$. Thanks to the work [7] by Chen–Miao–Zhang, one can see that a norm inflation happens under the initial data satisfying

$$\|\rho - \bar{\rho}\|_{\dot{B}^{\frac{3}{p}}_{p,1}} + \|u_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \ll 1.$$

However, their way does not suit the initial velocity in L^6 framework. Motivated by this, here we prove the ill-posedness under $u_0 \in \dot{B}^{-\frac{1}{2}}_{6,1}$. Our main result reads as follows:

Theorem 1.1. *Let $\bar{\rho}$ be a positive constant and $p > 6$. For any $\delta > 0$, there exists initial data (ρ_0, u_0) satisfying*

$$\|\rho_0 - \bar{\rho}\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \leqslant \delta, \quad \|u_0\|_{\dot{B}^{-\frac{1}{2}}_{6,1}} \leqslant \delta$$

such that a solution (ρ, u) to the system (1.1) satisfies

$$\|u(t)\|_{\dot{B}^{-\frac{1}{2}}_{6,1}} \geqslant \frac{1}{\delta}$$

for some $0 < t < \delta$.

Remark 1.2. In general, the larger space one consider, the easier one proves the ill-posedness. Comparing with the incompressible Navier–Stokes equations in L^∞

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framework (see, e.g., [2, 17, 26, 29]), here we can get ill-posedness for the compressible Navier–Stokes equations in smaller Besov space due to the more complex structure.

Remark 1.3. We obtain norm inflation with the initial velocity in L^6 framework, the proof of which is different from the previous work [7] (see the following comments). However, we do not know whether the initial density can be extended to L^6 framework. This question remains open.

Now, let us make some comments on the idea and the difficulty. The proofs of the main result in the present work and [7] are based on the idea of Bourgain and Pavlović [2] (see p. 2235), that is, the norm inflation comes from the first approximate, while the associate norm of other terms are sufficiently small.

(1) In [7], the norm inflation comes from the analysis of nonlinear term $u \cdot \nabla u$ which seems so hard to adapt to the L^6 framework. Our idea is taking into account the coupling term $L(a)\Delta u$ (see the § 3 for the definition of $L(a)$) and then norm inflation happens with $u_0 \in \dot{B}^{-\frac{1}{2}}_{6,1}$ and $a_0 \in \dot{B}^{\frac{3}{p}}_{p,1}$.

(2) To get the norm inflation, except we use a decomposition of the velocity like [7], we shall construct a new decomposition of density, which makes our proof more complex especially when we show the corresponding estimate of $K(a)\nabla a$ (see the § 3 for the definition of $K(a)$), see Step 4 in § 3.3.

This paper is organized as follows:
In § 2, we provide some lemmas and the definitions of some spaces. In § 3, we prove Theorem 1.1. Precisely, we give firstly a new form (i.e., (3.2)) of (1.1), and then in § 3.1, we choose the initial data and give some estimates of the corresponding norm. In § 3.2, we obtain norm inflation by the analysis of U_1. § 3.3 devotes to the analysis of U_2, while we close the proof of Theorem 1.1 in § 3.4. In the Appendix, we give the proof of Lemma 2.8.

Let us complete this section by describing the notations we shall use in this paper.

Notations. For A, B two operator, we denote $[A, B] = AB - BA$, the commutator between A and B. In some places of this paper, we may use L^p and $\dot{B}^s_{p,r}$ to stand for $L^p(\mathbb{R}^3)$ and $\dot{B}^s_{p,r}(\mathbb{R}^3)$, respectively. The uniform constant C, which may be different on different lines, while the constant $C(\cdot)$ means a constant depends on the element(s) in bracket. $(c_j)_{j\in\mathbb{Z}}$ will be a generic element of $l^1(\mathbb{Z})$ of norm $\leqslant 1$, and we also write $\sum_{j\in\mathbb{Z}} |c_j| \leqslant 1$.

2. Preliminaries

In this Section, we give some necessary definitions, propositions and lemmas.

The fractional Laplacian operator $|D|^\alpha = (-\Delta)^{\frac{\alpha}{2}}$ is defined through the Fourier transform, namely,

$$\widehat{|D|^\alpha f}(\xi) := |\xi|^\alpha \widehat{f}(\xi),$$

where the Fourier transform is given by

$$\widehat{f}(\xi) := \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)\, dx, \quad \text{or} \quad \mathcal{F}(f)(\xi) := \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)\, dx.$$

Let $\mathfrak{C} = \{\xi \in \mathbb{R}^3, \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$. Choose a nonnegative smooth radial function φ supported in $\mathfrak{C}$ such that

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^3\backslash\{0\}.$$

We denote $\varphi_j = \varphi(2^{-j}\xi)$, $h = \mathcal{F}^{-1}\varphi$, where $\mathcal{F}^{-1}$ stands for the inverse Fourier transform. Then the dyadic blocks Δ_j and S_j can be defined as follows

$$\Delta_j f = \varphi(2^{-j}D)f = 2^{3j}\int_{\mathbb{R}^3} h(2^j y) f(x-y)\,dy, \quad S_j f = \sum_{k\leqslant j-1} \Delta_k f.$$

One easily verifies that with our choice of φ

$$\Delta_j\Delta_k f = 0 \quad \text{if } |j-k| \geqslant 2 \quad \text{and} \quad \Delta_j(S_{k-1} f \Delta_k f) = 0 \quad \text{if } |j-k| \geqslant 5.$$

Let us recall the definitions of the Besov space and Chemin–Lerner type space [4].

Definition 2.1. Let $s \in \mathbb{R}$, $(p,q) \in [1,\infty]^2$, the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^3)$ is defined by

$$\dot{B}^s_{p,q}(\mathbb{R}^3) = \{f \in \mathfrak{S}'(\mathbb{R}^3);\ \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^3)} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^3)} = \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{sqj}\|\Delta_j f\|^q_{L^p(\mathbb{R}^3)}\right)^{\frac{1}{q}}, & \text{for } 1 \leqslant q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{sj}\|\Delta_j f\|_{L^p(\mathbb{R}^3)}, & \text{for } q = \infty, \end{cases}$$

and $\mathfrak{S}'(\mathbb{R}^3)$ denotes the dual space of $\mathfrak{S}(\mathbb{R}^3) = \{f \in \mathcal{S}(\mathbb{R}^3); \partial^\alpha \hat{f}(0) = 0; \forall \alpha \in \mathbb{N}^3$ multi-index$\}$ and can be identified by the quotient space of $\mathcal{S}'/\mathcal{P}$ with the polynomials space $\mathcal{P}$.

For the definition of $\mathfrak{S}'(\mathbb{R}^3)$, see [16, Appendix A.1].

Definition 2.2. Let $s \in \mathbb{R}$, $(p,q,r) \in [1,\infty]^3$, $0 < T \leqslant \infty$. The Chemin–Lerner type space $\tilde{L}^r_T \dot{B}^s_{p,q}(\mathbb{R}^3)$ is defined by

$$\tilde{L}^r_T \dot{B}^s_{p,q}(\mathbb{R}^3) = \{f \in \mathfrak{S}'(\mathbb{R}^3);\ \|f\|_{\tilde{L}^r_T \dot{B}^s_{p,q}(\mathbb{R}^3)} < \infty\},$$

where

$$\|f\|_{\tilde{L}^r_T \dot{B}^s_{p,q}(\mathbb{R}^3)} = \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{sqj}\|\Delta_j f\|^q_{L^r_T L^p(\mathbb{R}^3)}\right)^{\frac{1}{q}}, & \text{for } 1 \leqslant q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{sj}\|\Delta_j f\|_{L^r_T L^p(\mathbb{R}^3)}, & \text{for } q = \infty. \end{cases}$$

It is clear that $\tilde{L}^r_T \dot{B}^s_{p,r} = L^r_T \dot{B}^s_{p,r}$.

Let us introduce the homogeneous Bony's decomposition.

$$uv = T_u v + T_v u + R(u, v),$$

where

$$T_u v = \sum_{j\in\mathbb{Z}} S_{j-1}u\Delta_j v, \quad T_v u = \sum_{j\in\mathbb{Z}} \Delta_j u S_{j-1} v, \quad R(u, v) = \sum_{j\in\mathbb{Z}} \Delta_j u \tilde{\Delta}_j v,$$

here $\tilde{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}$.

The following propositions provide Bernstein type inequalities and the standard estimate of heat equation in order.

Proposition 2.3. *Let $1 \leqslant p \leqslant q \leqslant \infty$. Then for any $\beta, \gamma \in \mathbb{N}^3$, there exists a constant C independent of f, j such that*

(1) *If f satisfies*

$$supp\ \widehat{f} \subset \{\xi \in \mathbb{R}^3 : |\xi| \leqslant \mathcal{K}2^j\},$$

then

$$\|\partial^\gamma f\|_{L^q(\mathbb{R}^3)} \leqslant C2^{j|\gamma|+j(\frac{3}{p}-\frac{3}{q})}\|f\|_{L^p(\mathbb{R}^3)}.$$

(2) *If f satisfies*

$$supp\ \widehat{f} \subset \{\xi \in \mathbb{R}^3 : \mathcal{K}_1 2^j \leqslant |\xi| \leqslant \mathcal{K}_2 2^j\}$$

then

$$\|f\|_{L^p(\mathbb{R}^3)} \leqslant C2^{-j|\gamma|} \sup_{|\beta|=|\gamma|} \|\partial^\beta f\|_{L^p(\mathbb{R}^3)}.$$

The standard estimate of heat equation reads:

Proposition 2.4. *[1] Let $T > 0$, $s \in \mathbb{R}$ and $1 \leqslant r \leqslant \infty$. Assume that $u_0 \in \dot{B}^s_{r,1}$ and $f \in \tilde{L}^\rho_T \dot{B}^{s-2+\frac{2}{\rho}}_{r,1}$. If u is the solution of the heat equation*

$$\begin{cases} \partial_t u - \mu\Delta u = f, \\ u(0, x) = u_0(x), \end{cases}$$

with $\mu > 0$, then $\forall \rho_1 \in [\rho, \infty]$, we have

$$\mu^{\frac{1}{\rho_1}} \|u\|_{\tilde{L}^{\rho_1}_T \dot{B}^{s+\frac{2}{\rho_1}}_{r,1}} \leqslant C\left(\|u_0\|_{\dot{B}^s_{r,1}} + \|f\|_{\tilde{L}^\rho_T \dot{B}^{s-2+\frac{2}{\rho}}_{r,1}}\right). \tag{2.1}$$

Lemma 2.5. *[1] Let $T > 0$, $s > 0$ and $1 \leqslant r, \rho \leqslant \infty$. Assume that $F \in W^{[s]+3,\infty}_{\text{loc}}(\mathbb{R})$ with $F(0) = 0$. Then we have*

$$\|F(f)\|_{\tilde{L}^\rho_T \dot{B}^s_{r,1}} \leqslant C(1 + \|f\|_{L^\infty_T L^\infty})^{[s]+2} \|f\|_{\tilde{L}^\rho_T \dot{B}^s_{r,1}}. \tag{2.2}$$

The Kato–Ponce estimate can be given by

Lemma 2.6. *[19] Let $s > 0$, $1 \leqslant p, r \leqslant \infty$, then*

$$\|fg\|_{\dot{B}^s_{p,r}(\mathbb{R}^3)} \leqslant C\left\{\|f\|_{L^{p_1}(\mathbb{R}^3)}\|g\|_{\dot{B}^s_{p_2,r}(\mathbb{R}^3)} + \|g\|_{L^{r_1}(\mathbb{R}^3)}\|g\|_{\dot{B}^s_{r_2,r}(\mathbb{R}^3)}\right\}, \tag{2.3}$$

where $1 \leqslant p_1, r_1 \leqslant \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2}$.

For the readers' convenience, we refer to [1] for more details.

Thanks to the above Bony's decomposition and Bernstein inequalities, we can obtain the following product estimates.

Lemma 2.7. *Let*

$$1 \leqslant s, s_1, s_2, s_{i1}, s_{i2} \leqslant \infty, \quad 3 < r < \infty, \quad 3 < q < 6, \quad |\alpha| \geqslant 1,$$
$$\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s_{i1}} + \frac{1}{s_{i2}}, \quad i = 1, 2,$$
$$\frac{3}{p_0} + \frac{3}{r} > 1, \quad p_0 > 6.$$

Then the following estimates hold:

(a)

$$\|fg\|_{\tilde{L}^s_T \dot{B}^{\frac{3}{r}}_{r,1}} \leqslant C\left(\|g\|_{L^{s_{11}}_T L^\infty} \|f\|_{\tilde{L}^{s_{12}}_T \dot{B}^{\frac{3}{r}}_{r,1}} + \|\partial^\alpha g\|_{L^{s_{21}}_T L^\infty} \|f\|_{\tilde{L}^{s_2}_T \dot{B}^{\frac{3}{r}-|\alpha|}_{r,1}}\right); \tag{2.4}$$

(b)

$$\|fg\|_{\tilde{L}^s_T \dot{B}^{\frac{3}{r}-1}_{r,1}} \leqslant C\left(\|f\|_{L^{s_{11}}_T L^\infty} \|g\|_{\tilde{L}^{s_{12}}_T \dot{B}^{\frac{3}{r}-1}_{r,1}} + \|f\|_{L^{s_{21}}_T \dot{B}^{\frac{3}{p_0}+2}_{p_0,1}} \|g\|_{\tilde{L}^{s_2}_T \dot{B}^{\frac{3}{r}-3}_{r,1}}\right); \tag{2.5}$$

(c)

$$\|fg\|_{\tilde{L}^s_T \dot{B}^{\frac{3}{r}-1}_{r,1}} \leqslant C\|f\|_{\tilde{L}^{s_1}_T \dot{B}^{\frac{3}{p_0}}_{p_0,1}} \|g\|_{\tilde{L}^{s_2}_T \dot{B}^{\frac{3}{r}-1}_{r,1}}. \tag{2.6}$$

In particular, we have

$$\|fg\|_{\tilde{L}^s_T \dot{B}^{\frac{3}{q}-1}_{q,1}} \leqslant C\|f\|_{\tilde{L}^{s_1}_T \dot{B}^{\frac{3}{q}}_{q,1}} \|g\|_{\tilde{L}^{s_2}_T \dot{B}^{\frac{3}{q}-1}_{q,1}}. \tag{2.7}$$

Proof. (a) Using Bony decomposition and Young's inequality for series, we have

$$\begin{aligned}
\|\Delta_j T_g f\|_{L^r} + \|\Delta_j R(f, g)\|_{L^r} &\leqslant C \sum_{|k-j|\leqslant 4} \|S_{k-1} g\|_{L^\infty} \|\Delta_k f\|_{L^r} \\
&\quad + C \sum_{k \geqslant j-3} \|\Delta_k g\|_{L^\infty} \|\Delta_k f\|_{L^r} \\
&\leqslant C\|g\|_{L^\infty} \sum_{k \geqslant j-4} \|\Delta_k f\|_{L^r} \\
&\leqslant C 2^{-j\frac{3}{r}} \|g\|_{L^\infty} \sum_{k \geqslant j-4} 2^{(j-k)\frac{3}{r}} 2^{k\frac{3}{r}} \|\Delta_k f\|_{L^r} \\
&\leqslant C c_j 2^{-j\frac{3}{r}} \|g\|_{L^\infty} \|f\|_{\dot{B}^{\frac{3}{r}}_{r,1}}.
\end{aligned}$$

And by Bernstein's inequality, one can get

$$\|\Delta_j T_f g\|_{L^r} \leqslant C \sum_{|k-j|\leqslant 4} \|S_{k-1} f\|_{L^r} \|\Delta_k g\|_{L^\infty}$$

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$$
\begin{aligned}
&\leqslant C\sum_{|k-j|\leqslant 4}2^{-k|\alpha|}\|\partial^\alpha\Delta_k g\|_{L^\infty}\|S_{k-1}f\|_{L^r}\\
&\leqslant C\|\partial^\alpha g\|_{L^\infty}\sum_{|k-j|\leqslant 4}2^{-k|\alpha|}\|S_{k-1}f\|_{L^r}\\
&\leqslant C2^{-j\frac{3}{r}}\|\partial^\alpha g\|_{L^\infty}\sum_{|k-j|\leqslant 4}2^{(j-k)\frac{3}{r}}2^{k(\frac{3}{r}-|\alpha|)}\|S_{k-1}f\|_{L^r}\\
&\leqslant Cc_j2^{-j\frac{3}{r}}\|\partial^\alpha g\|_{L^\infty}\|f\|_{\dot B^{\frac{3}{r}-|\alpha|}_{r,1}},
\end{aligned}
$$

where we have used

$$
|\alpha|\geqslant 1,\quad r>3
$$

and

$$
\|2^{js}\|S_jf\|_{L^p}\|_{l^r(\mathbb{Z})}\approx\|f\|_{\dot B^s_{p,r}},\quad s<0,
$$

Adding the above estimates, then multiplying $2^{j\frac{3}{r}}$ and summing over $j\in\mathbb{Z}$, applying Hölder's inequality to resulting estimate can yield the desired result (2.4).

(b) Similarly, we have

$$
\begin{aligned}
\|\Delta_jT_fg\|_{L^r}+\|\Delta_jT_gf\|_{L^r}&\leqslant C\sum_{|k-j|\leqslant 4}\|S_{k-1}f\|_{L^\infty}\|\Delta_kg\|_{L^r}+\sum_{|k-j|\leqslant 4}\|S_{k-1}g\|_{L^r}\|\Delta_kf\|_{L^\infty}\\
&\leqslant C2^{-j(\frac{3}{r}-1)}\|f\|_{L^\infty}\Bigg(\sum_{|k-j|\leqslant 4}2^{(j-k)(\frac{3}{r}-1)}2^{k(\frac{3}{r}-1)}\|\Delta_kg\|_{L^r}\\
&\quad+\sum_{|k-j|\leqslant 4}2^{(j-k)(\frac{3}{r}-1)}2^{k(\frac{3}{r}-1)}\|S_{k-1}g\|_{L^r}\Bigg)\\
&\leqslant Cc_j2^{-j(\frac{3}{r}-1)}\|f\|_{L^\infty}\|g\|_{\dot B^{\frac{3}{r}-1}_{r,1}}.
\end{aligned}\tag{2.8}
$$

Let $\frac{3}{r_1}=\frac{3}{p_0}+\frac{3}{r}$, then $\frac{3}{r_1}>1$. By Bernstein's inequality,

$$
\begin{aligned}
\|\Delta_jR(f,g)\|_{L^r}&\leqslant C2^{j(\frac{3}{r_1}-\frac{3}{r})}\sum_{k\geqslant j-3}\|\Delta_kf\tilde\Delta_kg\|_{L^{r_1}}\\
&\leqslant C2^{j(\frac{3}{r_1}-\frac{3}{r})}\sum_{k\geqslant j-3}\|\Delta_kf\|_{L^{p_0}}\|\tilde\Delta_kg\|_{L^r}\\
&\leqslant C2^{j(1-\frac{3}{r})}\sum_{k\geqslant j-3}2^{(j-k)(\frac{3}{r_1}-1)}2^{k(\frac{3}{r_1}-1)}\|\Delta_kf\|_{L^{p_0}}\|\tilde\Delta_kg\|_{L^r}\\
&\leqslant C2^{j(1-\frac{3}{r})}\|f\|_{\dot B^{\frac{3}{p_0}+2}_{p_0,1}}\|g\|_{\dot B^{\frac{3}{r}-3}_{r,1}}\sum_{k\geqslant j-3}2^{(j-k)(\frac{3}{r_1}-1)}c_k\\
&\leqslant Cc_j2^{j(1-\frac{3}{r})}\|f\|_{\dot B^{\frac{3}{p_0}+2}_{p_0,1}}\|g\|_{\dot B^{\frac{3}{r}-3}_{r,1}}.
\end{aligned}
$$

Following the same lines as the upper estimate of (2.4) leads the estimate (2.5).

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(c) (2.6) can lead to (2.7) if we choose $r = q$ and use $\dot{B}^{\frac{3}{q}}_{q,1} \hookrightarrow \dot{B}^{\frac{3}{p_0}}_{p_0,1}$. For the proof of (2.6), indeed, By (2.8) and

$$\|\Delta_j R(f,g)\|_{L^r} \leqslant C c_j 2^{j(1-\frac{3}{r})} \|f\|_{\dot{B}^{\frac{3}{p_0}}_{p_0,1}} \|g\|_{\dot{B}^{\frac{3}{r}-1}_{r,1}},$$

together with $\dot{B}^{\frac{3}{p_0}}_{p_0,1} \hookrightarrow L^\infty$ can imply the desired result. □

To meet the requirement in our proof, we also need some estimates of the following generalized transport equation, the proofs of which will be given in the Appendix.

Lemma 2.8. *Let $T > 0$, $p_0 > 6$ and $q \in (3, 6)$. If $\mathfrak{B}$ is the solution of the generalized transport equation*

$$\begin{cases} \partial_t \mathfrak{B} + u \cdot \nabla \mathfrak{B} - \mathfrak{B}\,\mathrm{div}\, u = f, \\ \mathfrak{B}(0,x) = \mathfrak{B}_0(x), \end{cases}$$

then we have

(a)

$$\|\mathfrak{B}\|_{L^\infty_T L^\infty} \leqslant C(\|\mathfrak{B}_0\|_{L^\infty} + \|f\|_{L^1_T L^\infty}) \exp\{C\|\nabla u\|_{L^1_T L^\infty}\}; \tag{2.9}$$

(b)

$$\|\mathfrak{B}\|_{\tilde{L}^\infty_T \dot{B}^{\frac{3}{q}}_{q,1}} \leqslant C\left(\|\mathfrak{B}_0\|_{\dot{B}^{\frac{3}{q}}_{q,1}} + \|\mathfrak{B}\|_{L^\infty_T L^\infty} \|u\|_{L^1_T \dot{B}^{\frac{3}{q}+1}_{q,1}} + \|f\|_{L^1_T \dot{B}^{\frac{3}{q}}_{q,1}}\right) \exp\left\{C\|\nabla u\|_{L^1_T \dot{B}^{\frac{3}{p_0}}_{p_0,1}}\right\}; \tag{2.10}$$

(c)

$$\|\mathfrak{B}\|_{\tilde{L}^\infty_T \dot{B}^{\frac{3}{p_0}}_{p_0,1}} \leqslant C\left(\|\mathfrak{B}_0\|_{\dot{B}^{\frac{3}{p_0}}_{p_0,1}} + \|f\|_{L^1_T \dot{B}^{\frac{3}{p_0}}_{p_0,1}}\right) \exp\left\{C\|\nabla u\|_{L^1_T \dot{B}^{\frac{3}{p_0}}_{p_0,1}}\right\}. \tag{2.11}$$

Remark 2.9. The limitation of the parameters like q and p_0 in Lemmas 2.7 and 2.8 can be easily relaxed. To make the following proof clear, we only focus on some special scope.

3. Proof of Theorem 1.1

Firstly, we need to get a new form of system (1.1). Let

$$\mathcal{A} = \bar{\mu}\Delta + (\bar{\lambda} + \bar{\mu})\nabla\mathrm{div}, \quad K(a) = \frac{P'(\bar{\rho}(1+a))}{1+a}, \quad L(a) = \frac{a}{1+a},$$

and $\bar{\mu} = \frac{\mu}{\bar{\rho}}$, $\bar{\lambda} = \frac{\lambda}{\bar{\rho}}$, $\bar{\nu} = \bar{\lambda} + 2\bar{\mu}$. then we can rewrite (1.1) as follows:

$$\begin{cases} \partial_t a + u \cdot \nabla a + \mathrm{div}\, u(1+a) = 0, \\ \partial_t h - \bar{\nu}\Delta h = -|D|^{-1}\,\mathrm{div}(u \cdot \nabla u + L(a)\mathcal{A}u + K(a)\nabla a), \\ \partial_t \Omega - \bar{\mu}\Delta\Omega = -|D|^{-1}\,\mathrm{curl}(u \cdot \nabla u + L(a)\mathcal{A}u), \\ (a(0,x), h(0,x), \Omega(0,x)) = (a_0(x), h_0(x), \Omega_0(x)), \end{cases} \tag{3.1}$$

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by introducing the new unknowns

$$a = \frac{\rho}{\bar{\rho}} - 1, \quad h = |D|^{-1}\operatorname{div} u, \quad \Omega = |D|^{-1}\operatorname{curl} u$$

leading to

$$u = -|D|^{-1}\nabla h + |D|^{-1}\operatorname{curl}\Omega.$$

The above can also be seen in [7]. Let $b = L(a)$, then we can rewrite (3.1) as follows:

$$\begin{cases} \partial_t b + u\cdot\nabla b + (1-b)\operatorname{div} u = 0, \\ \partial_t h - \bar{\nu}\Delta h = -|D|^{-1}\operatorname{div}(u\cdot\nabla u + b\mathcal{A}u + \tilde{K}(b)\nabla b), \\ \partial_t\Omega - \bar{\mu}\Delta\Omega = -|D|^{-1}\operatorname{curl}(u\cdot\nabla u + b\mathcal{A}u), \\ (b(0,x), h(0,x), \Omega(0,x)) = (b_0(x), h_0(x), \Omega_0(x)), \end{cases} \tag{3.2}$$

where $b_0 = L(a_0)$, $\tilde{K}(b) = \frac{P'(\frac{\bar{\rho}}{1-b})}{1-b}$. By Duhamel principle, we have

$$\begin{aligned} h(t,x) &= e^{\bar{\nu}\Delta t}h_0 - \int_0^t e^{\bar{\nu}(t-\tau)\Delta}|D|^{-1}\operatorname{div}(u\cdot\nabla u + b\mathcal{A}u + \tilde{K}(b)\nabla b)\,d\tau, \\ \Omega(t,x) &= e^{\bar{\mu}\Delta t}\Omega_0 - \int_0^t e^{\bar{\mu}(t-\tau)\Delta}|D|^{-1}\operatorname{curl}(u\cdot\nabla u + b\mathcal{A}u)\,d\tau. \end{aligned}$$

Denote

$$\begin{aligned} U_0(t) &= -|D|^{-2}\nabla\operatorname{div}(e^{\bar{\nu}\Delta t}u_0) + |D|^{-2}\operatorname{curlcurl}(e^{\bar{\mu}\Delta t}u_0), \\ U_1(t) &= -|D|^{-1}\nabla h_1 + |D|^{-1}\operatorname{curl}\Omega_1, \\ U_2(t) &= -|D|^{-1}\nabla h_2 + |D|^{-1}\operatorname{curl}\Omega_2, \end{aligned}$$

where

$$\begin{aligned} h_1 &= -|D|^{-1}\operatorname{div}\int_0^t e^{\bar{\nu}(t-\tau)\Delta}(b_0\mathcal{A}U_0)\,d\tau, \\ \Omega_1 &= -|D|^{-1}\operatorname{curl}\int_0^t e^{\bar{\mu}(t-\tau)\Delta}(b_0\mathcal{A}U_0)\,d\tau, \\ h_2 &= -|D|^{-1}\operatorname{div}\int_0^t e^{\bar{\nu}(t-\tau)\Delta}(F_1 + u\cdot\nabla u + \tilde{K}(b)\nabla b)\,d\tau, \\ \Omega_2 &= -|D|^{-1}\operatorname{curl}\int_0^t e^{\bar{\mu}(t-\tau)\Delta}(F_1 + u\cdot\nabla u)\,d\tau, \end{aligned}$$

with

$$F_1 = b_0\mathcal{A}(U_1+U_2) + b_1\mathcal{A}u.$$

Now, we can decompose u and b as

$$u(t,x) = U_0(t,x) + U_1(t,x) + U_2(t,x)$$

and

$$b(t,x) = b_0(x) + b_1(t,x)$$

where $b_1(t,x)$ satisfies the generalized transport equation

$$\begin{cases} \partial_t b_1 + u\cdot\nabla(b_0+b_1) + (1-b_0-b_1)\operatorname{div} u = 0, \\ b_1(0,x) = 0. \end{cases} \tag{3.3}$$

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3.1. The choice of initial data

Let $C(N)=2^{\frac{N}{2}(\frac{1}{2}-\frac{3}{p}-2\epsilon_1)}$ for some $0<\epsilon_1<\frac{1}{2}(\frac{1}{2}-\frac{3}{p})$ and $p>6$, where $N>0$ determined later is a sufficiently large constant leading to $\frac{N}{C(N)}\ll 1$. We construct the initial data (b_0,u_0) as follows

$$\widehat{b_0}(\xi)=\frac{1}{C(N)}\sum_{k=100}^{N}2^{-k\frac{3}{p}}(f(\xi-2^k e_1)+f(\xi+2^k e_1)),$$

$$\widehat{u_0}(\xi)=\frac{1}{C(N)}\sum_{k=100}^{N}2^{\frac{k}{2}}(f(\xi-2^k e_1)+f(\xi+2^k e_1),0,0),$$

where $e_1=(1,0,0)$ and f is a smooth, radial and nonnegative function in $\mathbb{R}^3$ satisfying

$$f=\begin{cases}1 & \text{for } |\xi|\leqslant 1,\\ 0 & \text{for } |\xi|\geqslant 2.\end{cases}$$

It is clear that b_0 and u_0 are real valued function and real vector-valued function, respectively. One can also check the following estimates hold, i.e.,

$$\|b_0\|_{L^\infty}\leqslant\frac{C}{C(N)},\quad \|b_0\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\leqslant\frac{CN}{C(N)},\quad \|u_0\|_{\dot{B}^{-\frac{1}{2}}_{6,1}}\leqslant\frac{CN}{C(N)} \tag{3.4}$$

and

$$\|b_0\|_{\dot{B}^{s}_{r,1}}\leqslant\frac{C2^{N(s-\frac{3}{p})}}{C(N)},\quad \|u_0\|_{\dot{B}^{s_1}_{r_1,1}}\leqslant\frac{C2^{N(s_1+\frac{1}{2})}}{C(N)}, \tag{3.5}$$

with $(r,r_1)\in[1,\infty]^2$ and $s>\frac{3}{p}$, $s_1>-\frac{1}{2}$. In fact, one has

$$\|b_0\|_\infty\leqslant\|\widehat{b_0}\|_{L^1}\leqslant\frac{C\|f\|_{L^1}}{C(N)}\leqslant\frac{C}{C(N)}$$

and

$$\|b_0\|_{\dot{B}^{\frac{3}{p}}_{p,1}}\leqslant\sum_{j=99}^{N+1}2^{j\frac{3}{p}}\|\Delta_j b_0\|_{L^p}\leqslant C\sum_{j=99}^{N+1}\frac{\|f\|_{L^p}}{C(N)}\leqslant\frac{CN}{C(N)},$$

and other terms can be bounded similarly.

Next, a lemma is given.

Lemma 3.1. *Let $p>6$. Then there exist some positive constants q, p_0, ϵ, ϵ_1 satisfying*

$$\begin{cases}1<\dfrac{3}{q}+\dfrac{3}{p_0}<\dfrac{5}{4}-\dfrac{3}{2p}+3\epsilon-3\epsilon_1,\\[2mm] \dfrac{3}{p_0}<\dfrac{1}{4}+\dfrac{3}{2p}+2\epsilon-\epsilon_1,\ p_0\in(6,p),\\[2mm] \dfrac{3}{q}<\dfrac{3}{4}-\dfrac{3}{2p}+2\epsilon-\epsilon_1,\ q\in(3,6),\\[2mm] 0<2\epsilon<2\epsilon_1<\min\left\{\dfrac{1}{2}-\dfrac{3}{p},\dfrac{1}{3}\left(\dfrac{1}{2}-\dfrac{3}{p}\right)+2\epsilon,4\epsilon\right\}.\end{cases} \tag{3.6}$$

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Remark 3.2. We give the following explanations of the limitations in (3.6). Let $T = 2^{-2(1+\epsilon)N}$, then we have

$$\begin{cases} \dfrac{T2^{N(\frac{3}{p_0}-\frac{3}{p}+2)}}{C(N)} \ll 1 \Longleftrightarrow \dfrac{3}{p_0} < \dfrac{1}{4} + \dfrac{3}{2p} + 2\epsilon - \epsilon_1, \\ \dfrac{T2^{N(\frac{3}{q}+\frac{3}{2})}}{C(N)} \ll 1 \Longleftrightarrow \dfrac{3}{q} < \dfrac{3}{4} - \dfrac{3}{2p} + 2\epsilon - \epsilon_1, \\ \dfrac{T2^{N(\frac{3}{p_0}+\frac{3}{2})}}{C(N)} \ll 1 \Longleftarrow p_0 \in (6, p), \\ \dfrac{T^{\frac{3}{2}}2^{N(\frac{3}{q}+\frac{3}{p_0}-\frac{3}{p}+\frac{5}{2})}}{C(N)^3} \ll 1 \Longleftrightarrow \dfrac{3}{q} + \dfrac{3}{p_0} < \dfrac{5}{4} - \dfrac{3}{2p} + 3\epsilon - 3\epsilon_1. \end{cases} \tag{3.7}$$

As a matter of fact, we assume the conditions on the right hand side of (3.7) to ensure the conditions on the left hand side which are required in our proof. Furthermore, we use $\frac{3}{q} + \frac{3}{p_0} > 1$ (with (3.7) leads $2\epsilon_1 < 4\epsilon$ and $2\epsilon_1 < \frac{1}{3}(\frac{1}{2} - \frac{3}{p}) + 2\epsilon$) to ensure some product estimates like (2.6). The choice of $C(N)$ needs $2\epsilon_1 < \frac{1}{2} - \frac{3}{p}$, while $\epsilon < \epsilon_1$ ensures the norm inflation of U_1 (see the end of § 3.2).

Lemma 3.1 can be proved easily, here we use the following example in this article:

$$\epsilon = \frac{1}{4}\left(\frac{1}{2} - \frac{3}{p}\right), \quad \epsilon_1 = \frac{1}{3}\left(\frac{1}{2} - \frac{3}{p}\right)$$

and

$$\frac{3}{p_0} = \frac{1}{4} + \frac{3}{2p}, \quad \forall \frac{3}{q} \in \left(\frac{3}{4} - \frac{3}{2p}, \frac{5}{6} - \frac{2}{p}\right).$$

This ensures that

$$2(\epsilon_1 - \epsilon) = \frac{1}{2}\left(\frac{1}{2} - \frac{3}{p} - 2\epsilon_1\right). \tag{3.8}$$

3.2. The large lower bound of U_1

Let G^j be the j-th component of the vector G. Thanks to the choice of the initial velocity $u_0 = (u_0^1, 0, 0)$, we have

$$\begin{aligned} \mathcal{A}U_0 &= (\bar{\mu}\Delta + (\bar{\lambda} + \bar{\mu})\nabla \operatorname{div})(-|D|^{-2}\nabla \operatorname{div}(e^{\bar{\nu}\Delta t}u_0) + |D|^{-2}\operatorname{curlcurl}(e^{\bar{\mu}\Delta t}u_0)) \\ &= (\bar{\lambda} + 2\bar{\mu})\nabla \operatorname{div}(e^{\bar{\nu}\Delta t}u_0) + \bar{\mu}(\Delta - \nabla \operatorname{div})(e^{\bar{\mu}\Delta t}u_0), \end{aligned} \tag{3.9}$$

where

$$\begin{cases} (\mathcal{A}U_0)^1 = (\bar{\lambda} + 2\bar{\mu})\partial_1^2(e^{\bar{\nu}\Delta t}u_0^1) + \bar{\mu}(\partial_2^2 + \partial_3^2)(e^{\bar{\mu}\Delta t}u_0^1), \\ (\mathcal{A}U_0)^2 = (\bar{\lambda} + 2\bar{\mu})\partial_1\partial_2(e^{\bar{\nu}\Delta t}u_0^1) - \bar{\mu}\partial_1\partial_2(e^{\bar{\mu}\Delta t}u_0^1), \\ (\mathcal{A}U_0)^3 = (\bar{\lambda} + 2\bar{\mu})\partial_1\partial_3(e^{\bar{\nu}\Delta t}u_0^1) - \bar{\mu}\partial_1\partial_3(e^{\bar{\mu}\Delta t}u_0^1). \end{cases} \tag{3.10}$$

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One can also get a new form of U_1 given by

$$U_1 = -\int_0^t e^{\bar{\mu}(t-\tau)\Delta} b_0 \mathcal{A} U_0 \, d\tau + |D|^{-2} \nabla \operatorname{div} \int_0^t (e^{\bar{\nu}(t-\tau)\Delta} - e^{\bar{\mu}(t-\tau)\Delta})(b_0 \mathcal{A} U_0) \, d\tau.$$

Due to $\dot{B}^{-\frac{1}{2}}_{6,1} \hookrightarrow \dot{B}^{-1}_{\infty,\infty}$, we have

$$\begin{aligned} \|U_1(t)\|_{\dot{B}^{-\frac{1}{2}}_{6,1}} &\geqslant c\|U_1(t)\|_{\dot{B}^{-1}_{\infty,\infty}} \geqslant c\|U_1^1(t)\|_{\dot{B}^{-1}_{\infty,\infty}} \\ &\geqslant c\left|\int \varphi(2^9\xi)\widehat{U_1^1}(\xi)\, d\xi\right| \geqslant |B_1| - |B_2|, \end{aligned}$$

where

$$B_i = \int \varphi(2^9\xi)\widehat{U_{1i}^1}(\xi)\, d\xi, \quad i = 1, 2$$

with

$$U_{11}^1 = \int_0^t e^{\bar{\mu}(t-\tau)\Delta} b_0 (\mathcal{A} U_0)^1 \, d\tau$$

and

$$U_{12}^1 = |D|^{-2} \partial_1 \operatorname{div} \int_0^t (e^{\bar{\nu}(t-\tau)\Delta} - e^{\bar{\mu}(t-\tau)\Delta})(b_0 \mathcal{A} U_0)\, d\tau.$$

Now, we give the estimates of B_i.

- *The estimate of* B_2.
  Let $\xi = (\xi_1, \xi_2, \xi_3)$. Using $\operatorname{div}(\mathcal{A}U_0) = \partial_1(\mathcal{A}U_0)^1 + \sum_{i=2,3} \partial_i(\mathcal{A}U_0)^i$, one can split B_2 into

two parts, that is,

$$\begin{aligned} B_2 &= -\int \varphi(2^9\xi)\frac{\xi_1^2}{|\xi|^2}\int_0^t (e^{-\bar{\nu}(t-\tau)|\xi|^2} - e^{-\bar{\mu}(t-\tau)|\xi|^2})\mathcal{F}(b_0(\mathcal{A}U_0)^1)\, d\tau\, d\xi \\ &\quad + \sum_{i=2,3}\int \varphi(2^9\xi)\frac{\xi_1\xi_i}{|\xi|^2}\int_0^t (e^{-\bar{\nu}(t-\tau)|\xi|^2} - e^{-\bar{\mu}(t-\tau)|\xi|^2})\mathcal{F}(b_0(\mathcal{A}U_0)^i)\, d\tau\, d\xi \\ &= B_{21} + B_{22}. \end{aligned}$$

Applying (3.10), one gets

$$B_{21} = B_{211} + B_{212},$$

where

$$B_{211} = -(\bar{\lambda} + 2\bar{\mu})\int \varphi(2^9\xi)\frac{\xi_1^2}{|\xi|^2}\int_0^t (e^{-\bar{\nu}(t-\tau)|\xi|^2} - e^{-\bar{\mu}(t-\tau)|\xi|^2})\mathcal{F}\{b_0\partial_1^2(e^{\bar{\nu}\Delta\tau}u_0^1)\}\, d\tau\, d\xi$$

and

$$B_{212} = -\bar{\mu}\int \varphi(2^9\xi)\frac{\xi_1^2}{|\xi|^2}\int_0^t (e^{-\bar{\nu}(t-\tau)|\xi|^2} - e^{-\bar{\mu}(t-\tau)|\xi|^2})\mathcal{F}\{b_0(\partial_2^2 + \partial_3^2)(e^{\bar{\mu}\Delta\tau}u_0^1)\}\, d\tau\, d\xi.$$

Using the constrcture of b_0 and u_0, we can obtain

$$B_{211} = \frac{C}{C(N)^2}\sum_{k=100}^{N} 2^{k(\frac{1}{2}-\frac{3}{p})}\int\int \varphi(2^9\xi)\frac{\xi_1^2\eta_1^2}{|\xi|^2}\mathfrak{A}(\xi,\eta,k)A(t,\xi,\eta)\, d\xi\, d\eta,$$

where

$$\mathfrak{A}(\xi,\eta,k)=f(\xi-\eta-2^k e_1)f(\eta+2^k e_1)+f(\xi-\eta+2^k e_1)f(\eta-2^k e_1)$$

and

$$A(t,\xi,\eta)=\int_0^t (e^{-\bar{\nu}(t-\tau)|\xi|^2}-e^{-\bar{\mu}(t-\tau)|\xi|^2})e^{-\bar{\nu}|\eta|^2\tau}\,d\tau.$$

For the estimate of $A(t,\xi,\eta)$, by Taylor's expansion $e^x=\sum_{r\geqslant 0}\frac{x^r}{r!}$, using $|\eta|\approx 2^k$ and $|\xi|<1$, one obtain

$$A(t,\xi,\eta)=\frac{e^{-\bar{\nu}t|\eta|^2}-e^{-\bar{\nu}t|\xi|^2}}{\bar{\nu}(|\xi|^2-|\eta|^2)}-\frac{e^{-\bar{\nu}t|\eta|^2}-e^{-\bar{\mu}t|\xi|^2}}{\bar{\mu}|\xi|^2-\bar{\nu}|\eta|^2}=\mathcal{O}(t^2|\eta|^2),$$

when $t2^{2N}<1$. This implies that

$$B_{211}=\frac{C}{C(N)^2}\sum_{k=100}^{N}2^{k(\frac{1}{2}-\frac{3}{p})}\int\int \mathcal{O}(t^2|\eta|^2)\varphi(2^9\xi)\frac{\xi_1^2\eta_1^2}{|\xi|^2}\mathfrak{A}(\xi,\eta,k)\,d\xi\,d\eta,$$

which leads

$$|B_{211}|\leqslant \frac{Ct^2}{C(N)^2}\sum_{k=100}^{N}2^{k(\frac{9}{2}-\frac{3}{p})}=\frac{Ct^2 2^{N(\frac{9}{2}-\frac{3}{p})}}{C(N)^2}. \tag{3.11}$$

Similarly,

$$B_{212}=\frac{C}{C(N)^2}\sum_{k=100}^{N}2^{k(\frac{1}{2}-\frac{3}{p})}\int\int \mathcal{O}(t^2|\eta|^2)\varphi(2^9\xi)\frac{\xi_1^2(\eta_2^2+\eta_3^2)}{|\xi|^2}\mathfrak{A}(\xi,\eta,k)\,d\xi\,d\eta$$

following

$$|B_{212}|\leqslant \frac{Ct^2 2^{N(\frac{5}{2}-\frac{3}{p})}}{C(N)^2}. \tag{3.12}$$

Combining with (3.11) and (3.12), we have

$$|B_{21}|\leqslant \frac{Ct^2 2^{N(\frac{9}{2}-\frac{3}{p})}}{C(N)^2}. \tag{3.13}$$

Using (3.10) again, we have

$$B_{22}=B_{221}+B_{222},$$

where

$$B_{221}=(\bar{\lambda}+2\bar{\mu})\sum_{i=2,3}\int \varphi(2^9\xi)\frac{\xi_1\xi_i}{|\xi|^2}\int_0^t (e^{-\bar{\nu}(t-\tau)|\xi|^2}-e^{-\bar{\mu}(t-\tau)|\xi|^2})\mathcal{F}(b_0\partial_1\partial_i e^{\bar{\nu}\Delta\tau}u_0^1)\,d\tau\,d\xi$$

and

$$B_{222}=-\bar{\mu}\sum_{i=2,3}\int \varphi(2^9\xi)\frac{\xi_1\xi_i}{|\xi|^2}\int_0^t (e^{-\bar{\nu}(t-\tau)|\xi|^2}-e^{-\bar{\mu}(t-\tau)|\xi|^2})\mathcal{F}(b_0\partial_1\partial_i e^{\bar{\mu}\Delta\tau}u_0^1)\,d\tau\,d\xi.$$

The upper bound of B_{22i} is smaller than B_{211} since nonlinear term $\mathcal{F}(b_0\partial_1\partial_2 e^{\bar{\nu}\Delta\tau}u_0^1)$ and $\mathcal{F}(b_0\partial_1\partial_2 e^{\bar{\mu}\Delta\tau}u_0^1)$ contain one helpful derivative ∂_1 less than $\mathcal{F}(b_0\partial_1^2 e^{\bar{\nu}\Delta\tau}u_0^1)$.

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In fact, we have

$$|B_{22i}| \leqslant \frac{Ct}{C(N)^2} \sum_{k=100}^{N} 2^{k(\frac{1}{2}-\frac{3}{p})} \int\int \varphi(2^9\xi)|\eta_1|\mathfrak{A}(\xi,\eta,k)\,d\xi\,d\eta$$
$$\leqslant \frac{Ct}{C(N)^2} \sum_{k=100}^{N} 2^{k(\frac{3}{2}-\frac{3}{p})} \leqslant \frac{Ct2^{N(\frac{3}{2}-\frac{3}{p})}}{C(N)^2}$$

following

$$|B_{22}| \leqslant \frac{Ct2^{N(\frac{3}{2}-\frac{3}{p})}}{C(N)^2}.$$

Together with (3.13) leads

$$|B_2| \leqslant \frac{Ct^2 2^{N(\frac{9}{2}-\frac{3}{p})}}{C(N)^2} + \frac{Ct2^{N(\frac{3}{2}-\frac{3}{p})}}{C(N)^2}. \tag{3.14}$$

- *The estimate of* B_1.
  Applying (3.10), then

$$B_1 = \int \varphi(2^9\xi) \int_0^t e^{-\bar{\mu}(t-\tau)|\xi|^2} \mathcal{F}(b_0(\mathcal{A}U_0)^1)\,d\tau\,d\xi = B_{11} + B_{12},$$

where

$$B_{11} = (\bar{\lambda}+2\bar{\mu}) \int \varphi(2^9\xi) \int_0^t e^{-\bar{\mu}(t-\tau)|\xi|^2} \mathcal{F}\{b_0\partial_1^2(e^{\bar{\nu}\Delta t}u_0^1)\}\,d\tau\,d\xi,$$
$$B_{12} = \bar{\mu} \int \varphi(2^9\xi) \int_0^t e^{-\bar{\mu}(t-\tau)|\xi|^2} \mathcal{F}\{b_0(\partial_2^2+\partial_3^2)(e^{\bar{\nu}\Delta t}u_0^1)\}\,d\tau\,d\xi.$$

Thanks to the choice of the initial data, we have

$$B_{11} = -c \int\int \varphi(2^9\xi)\widehat{b_0}(\xi-\eta)\eta_1^2\widehat{u_0^1}(\eta)A_1(\xi,\eta,t)\,d\xi\,d\eta,$$

where

$$A_1(\xi,\eta,t) = \int_0^t e^{-\bar{\mu}(t-\tau)|\xi|^2-\bar{\nu}|\eta|^2\tau}\,d\tau.$$

By simple computation and using Taylor's formula yields that

$$A_1(\xi,\eta,t) = e^{-\bar{\mu}t|\xi|^2} \int_0^t e^{(\bar{\mu}|\xi|^2-\bar{\nu}|\eta|^2)\tau}\,d\tau$$
$$= \frac{e^{-\bar{\nu}t|\eta|^2} - e^{-\bar{\mu}t|\xi|^2}}{\bar{\mu}|\xi|^2-\bar{\nu}|\eta|^2}$$
$$= t + \mathcal{O}(t^2|\eta|^2),$$

when $t2^{2N} < 1$. So we can get

$$|B_{11}| \geqslant \frac{ct}{C(N)^2} \sum_{k=100}^{N} 2^{k(\frac{1}{2}-\frac{3}{p})} \int\int \varphi(2^9\xi)\eta_1^2\mathfrak{A}(\xi,\eta,k)\,d\xi\,d\eta$$

$$
\begin{aligned}
&- \frac{Ct^2}{C(N)^2}\sum_{k=100}^{N} 2^{k(\frac{1}{2}-\frac{3}{p})}\int\int \varphi(2^9\xi)\eta_1^2|\eta|^2\mathfrak{A}(\xi,\eta,k)\,d\xi\,d\eta\\
&\geqslant \frac{ct}{C(N)^2}\sum_{k=100}^{N} 2^{k(\frac{5}{2}-\frac{3}{p})} - \frac{Ct^2}{C(N)^2}\sum_{k=100}^{N} 2^{k(\frac{9}{2}-\frac{3}{p})}\\
&\geqslant \frac{ct2^{N(\frac{5}{2}-\frac{3}{p})}}{C(N)^2} - \frac{Ct^2 2^{N(\frac{9}{2}-\frac{3}{p})}}{C(N)^2}. \qquad (3.15)
\end{aligned}
$$

Since there is no helpful derivative in the nonlinear term $\mathcal{F}\{b_0(\partial_2^2+\partial_3^2)(e^{\bar{\nu}\Delta t}u_0^1)\}$ of B_{12}, it is easy to bound B_{12} as follows

$$
\begin{aligned}
|B_{12}| &\leqslant \frac{Ct}{C(N)^2}\sum_{k=100}^{N} 2^{k(\frac{1}{2}-\frac{3}{p})}\int\int \varphi(2^9\xi)\mathfrak{A}(\xi,\eta,k)\,d\xi\,d\eta\\
&\leqslant \frac{Ct2^{N(\frac{1}{2}-\frac{3}{p})}}{C(N)^2},
\end{aligned}
$$

together with (3.15) and choosing N such that $C2^{-2N} < \frac{c}{3}$, yields that

$$
|B_1| \geqslant |B_{11}| - |B_{12}| \geqslant \frac{2ct2^{N(\frac{5}{2}-\frac{3}{p})}}{3C(N)^2} - \frac{Ct^2 2^{N(\frac{9}{2}-\frac{3}{p})}}{C(N)^2}. \qquad (3.16)
$$

Therefore, by setting

$$
t = T_0 := 2^{-2(1+\epsilon)N} \qquad (3.17)
$$

with $0<\epsilon<\epsilon_1$, we have $t2^{2N} < 1$, and combining with (3.14) and (3.16) follows

$$
\begin{aligned}
\|U_1(t)\|_{\dot{B}^{-\frac{1}{2}}_{6,1}} &\geqslant \frac{2ct2^{N(\frac{5}{2}-\frac{3}{p})}}{3C(N)^2} - \frac{Ct^2 2^{N(\frac{9}{2}-\frac{3}{p})}}{C(N)^2} - \frac{Ct2^{N(\frac{3}{2}-\frac{3}{p})}}{C(N)^2}\\
&\geqslant \frac{ct2^{N(\frac{5}{2}-\frac{3}{p})}}{2C(N)^2} = \frac{c}{2}2^{2N(\epsilon_1-\epsilon)}. \qquad (3.18)
\end{aligned}
$$

3.3. The analysis of U_2

In this Subsection, we split the analysis of U_2 into several steps. Let $0 \leqslant T \leqslant T_0$. In step 1, we give firstly some estimates of U_1; In step 2, the estimate of $\|u\cdot\nabla u\|_{L^1_T\dot{B}^{\frac{3}{q}-1}_{q,1}}$ is given; Step 3 devotes to the estimate of $\|(b_0\mathcal{A}(U_1+U_2),\ b_1\mathcal{A}u)\|_{L^1_T\dot{B}^{\frac{3}{q}-1}_{q,1}}$; In step 4, we show the estimate of $\|\tilde{K}(b)\nabla b\|_{L^1_T\dot{B}^{\frac{3}{q}-1}_{q,1}}$. In step 5, we conclude the analysis of U_2.

Step 1: Some estimates of U_1. Let (p, p_0, q) be given as in (3.6). Firstly, by using 2.1, (3.4) and (3.5), we give some estimates on U_0 as follows:

$$
\|\widehat{\mathcal{A}U_0}\|_{L^\infty_T L^1} \leqslant C2^{2N}\|\widehat{u_0}\|_{L^1} \leqslant \frac{C2^{\frac{5}{2}N}}{C(N)},
$$

$$
\|U_0\|_{L^1_T\dot{B}^s_{q,1}} \leqslant CT\|u_0\|_{\dot{B}^s_{q,1}} \leqslant \frac{CT2^{N(s+\frac{1}{2})}}{C(N)}, \quad s\geqslant 0,
$$

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$$\|\nabla^2 U_0\|_{L^1_T L^\infty} \leqslant CT2^{2N}\|u_0\|_{L^\infty} \leqslant \frac{CT2^{\frac{5}{2}N}}{C(N)}$$

and

$$\|U_0\|_{\tilde{L}^2_T \dot{B}^{\frac{3}{q}}_{q,1}} \leqslant T^{\frac{1}{2}}\|U_0\|_{\tilde{L}^\infty_T \dot{B}^{\frac{3}{q}}_{q,1}} \leqslant CT^{\frac{1}{2}}\|u_0\|_{\dot{B}^{\frac{3}{q}}_{q,1}} \leqslant \frac{CT^{\frac{1}{2}}2^{N(\frac{3}{q}+\frac{1}{2})}}{C(N)}.$$

Now, we give the estimates of U_1. By $\|g\|_{L^\infty} \leqslant C\|\widehat{g}\|_{L^1}$ and Young's inequality, we have $\forall\, r \in [1,\infty]$,

$$\begin{aligned}\|U_1\|_{L^r_T L^\infty} &\leqslant T^{\frac{1}{r}}\|\widehat{U_1}\|_{L^\infty_T L^1} \leqslant T^{\frac{1}{r}}\int_0^T \|\widehat{b_0\mathcal{A}U_0}\|_{L^1}\,d\tau\\ &\leqslant T^{1+\frac{1}{r}}\|b_0\|_{L^1}\|\widehat{\mathcal{A}U_0}\|_{L^\infty_T L^1} \leqslant \frac{CT^{1+\frac{1}{r}}2^{\frac{5}{2}N}}{C(N)^2}.\end{aligned}$$

By (2.1), the Kato–Ponce estimate (2.3), (3.4) and (3.5), we have

$$\begin{aligned}&\|U_1\|_{L^1_T \dot{B}^{\frac{3}{q}+1}_{q,1}} + \|U_1\|_{\tilde{L}^2_T \dot{B}^{\frac{3}{q}}_{q,1}}\\ &\leqslant T^{\frac{1}{2}}\left(\|U_1\|_{\tilde{L}^2_T \dot{B}^{\frac{3}{q}+1}_{q,1}} + \|U_1\|_{\tilde{L}^\infty_T \dot{B}^{\frac{3}{q}}_{q,1}}\right) \leqslant CT^{\frac{1}{2}}\|b_0\mathcal{A}U_0\|_{L^1_T \dot{B}^{\frac{3}{q}}_{q,1}}\\ &\leqslant CT^{\frac{1}{2}}\left(\|b_0\|_{L^\infty}\|U_0\|_{L^1_T \dot{B}^{\frac{3}{q}+2}_{q,1}} + \|\nabla^2 U_0\|_{L^1_T L^\infty}\|b_0\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\right)\\ &\leqslant \frac{CT^{\frac{3}{2}}2^{N(\frac{3}{q}+\frac{5}{2})}}{C(N)^2}.\end{aligned}$$

Similarly, we also have

$$\|U_1\|_{L^1_T \dot{B}^{\frac{3}{p_0}+1}_{p_0,1}} + \|U_1\|_{\tilde{L}^2_T \dot{B}^{\frac{3}{p_0}}_{p_0,1}} \leqslant \frac{CT^{\frac{3}{2}}2^{N(\frac{3}{p_0}+\frac{5}{2})}}{C(N)^2}.$$

Step 2: The estimate of $\|u\cdot\nabla u\|_{L^1_T \dot{B}^{\frac{3}{q}-1}_{q,1}}$. This estimate can be split into nine terms. By the product estimate (2.6), we have

$$\|U_0\cdot\nabla U_0\|_{L^1_T \dot{B}^{\frac{3}{q}-1}_{q,1}} \leqslant C\|U_0\|_{\tilde{L}^2_T \dot{B}^{\frac{3}{p_0}}_{p_0,1}}\|U_0\|_{\tilde{L}^2_T \dot{B}^{\frac{3}{q}}_{q,1}} \leqslant \frac{CT2^{N(\frac{3}{q}+\frac{3}{p_0}+1)}}{C(N)^2},$$

$$\|U_0\cdot\nabla U_1\|_{L^1_T \dot{B}^{\frac{3}{q}-1}_{q,1}} \leqslant C\|U_0\|_{\tilde{L}^2_T \dot{B}^{\frac{3}{p_0}}_{p_0,1}}\|U_1\|_{\tilde{L}^2_T \dot{B}^{\frac{3}{q}}_{q,1}} \leqslant \frac{CT^2 2^{N(\frac{3}{q}+\frac{3}{p_0}+3)}}{C(N)^3},$$

$$\|U_0\cdot\nabla U_2\|_{L^1_T \dot{B}^{\frac{3}{q}-1}_{q,1}} \leqslant C\|U_0\|_{\tilde{L}^2_T \dot{B}^{\frac{3}{p_0}}_{p_0,1}}\|U_2\|_{\tilde{L}^2_T \dot{B}^{\frac{3}{q}}_{q,1}} \leqslant \frac{CT^{\frac{1}{2}}2^{N(\frac{3}{p_0}+\frac{1}{2})}}{C(N)}\|U_2\|_{\tilde{L}^2_T \dot{B}^{\frac{3}{q}}_{q,1}},$$

$$\|U_1\cdot\nabla U_0\|_{L^1_T \dot{B}^{\frac{3}{q}-1}_{q,1}} \leqslant C\|U_1\|_{\tilde{L}^2_T \dot{B}^{\frac{3}{p_0}}_{p_0,1}}\|U_0\|_{\tilde{L}^2_T \dot{B}^{\frac{3}{q}}_{q,1}} \leqslant \frac{CT^2 2^{N(\frac{3}{q}+\frac{3}{p_0}+3)}}{C(N)^3},$$

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$$\|U_1\cdot\nabla U_1\|_{L^1_T\dot B^{\frac{3}{q}-1}_{q,1}}\leqslant C\|U_1\|_{\tilde L^2_T\dot B^{\frac{3}{p_0}}_{p_0,1}}\|U_1\|_{\tilde L^2_T\dot B^{\frac{3}{q}}_{q,1}}\leqslant\frac{CT^3 2^{N(\frac{3}{q}+\frac{3}{p_0}+5)}}{C(N)^4},$$

$$\|U_1\cdot\nabla U_2\|_{L^1_T\dot B^{\frac{3}{q}-1}_{q,1}}\leqslant C\|U_1\|_{\tilde L^2_T\dot B^{\frac{3}{p_0}}_{p_0,1}}\|U_2\|_{\tilde L^2_T\dot B^{\frac{3}{q}}_{q,1}}\leqslant\frac{CT^{\frac{3}{2}}2^{N(\frac{3}{p_0}+\frac{5}{2})}}{C(N)^2}\|U_2\|_{\tilde L^2_T\dot B^{\frac{3}{q}}_{q,1}},$$

$$\|U_2\cdot\nabla U_0\|_{L^1_T\dot B^{\frac{3}{q}-1}_{q,1}}\leqslant C\|\nabla U_0\|_{L^1_T\dot B^{\frac{3}{p_0}}_{p_0,1}}\|U_2\|_{\tilde L^\infty_T\dot B^{\frac{3}{q}-1}_{q,1}}\leqslant\frac{CT2^{N(\frac{3}{p_0}+\frac{3}{2})}}{C(N)}\|U_2\|_{\tilde L^\infty_T\dot B^{\frac{3}{q}-1}_{q,1}},$$

$$\|U_2\cdot\nabla U_1\|_{L^1_T\dot B^{\frac{3}{q}-1}_{q,1}}\leqslant C\|\nabla U_1\|_{L^1_T\dot B^{\frac{3}{p_0}}_{p_0,1}}\|U_2\|_{\tilde L^\infty_T\dot B^{\frac{3}{q}-1}_{q,1}}\leqslant\frac{CT^{\frac{3}{2}}2^{N(\frac{3}{p_0}+\frac{5}{2})}}{C(N)^2}\|U_2\|_{\tilde L^\infty_T\dot B^{\frac{3}{q}-1}_{q,1}}.$$

By (2.7),

$$\|U_2\cdot\nabla U_2\|_{L^1_T\dot B^{\frac{3}{q}-1}_{q,1}}\leqslant C\|U_2\|^2_{\tilde L^2_T\dot B^{\frac{3}{q}}_{q,1}}.$$

Adding the above estimates, we have

$$\|u\cdot\nabla u\|_{L^1_T\dot B^{\frac{3}{q}-1}_{q,1}}\leqslant\frac{CT2^{N(\frac{3}{q}+\frac{3}{p_0}+1)}}{C(N)^2}+C\|U_2\|^2_{\tilde L^2_T\dot B^{\frac{3}{q}}_{q,1}}$$
$$+\frac{CT^{\frac{1}{2}}2^{N(\frac{3}{p_0}+\frac{1}{2})}}{C(N)}\left(\|U_2\|_{\tilde L^\infty_T\dot B^{\frac{3}{q}-1}_{q,1}}+\|U_2\|_{L^1_T\dot B^{\frac{3}{q}+1}_{q,1}}\right).\tag{3.19}$$

Step 3: The estimate of $\|(b_0\mathcal{A}(U_1+U_2),b_1\mathcal{A}u)\|_{L^1_T\dot B^{\frac{3}{q}-1}_{q,1}}$. Thanks to the product estimate (2.6), one can get

$$\|b_0\mathcal{A}U_1\|_{L^1_T\dot B^{\frac{3}{q}-1}_{q,1}}\leqslant C\|b_0\|_{\tilde L^\infty_T\dot B^{\frac{3}{p_0}}_{p_0,1}}\|U_1\|_{L^1_T\dot B^{\frac{3}{q}+1}_{q,1}}\leqslant\frac{CT^{\frac{3}{2}}2^{N(\frac{3}{q}+\frac{3}{p_0}-\frac{3}{p}+\frac{5}{2})}}{C(N)^3},$$

$$\|b_1\mathcal{A}U_0\|_{L^1_T\dot B^{\frac{3}{q}-1}_{q,1}}\leqslant C\|b_1\|_{\tilde L^\infty_T\dot B^{\frac{3}{p_0}}_{p_0,1}}\|U_0\|_{L^1_T\dot B^{\frac{3}{q}+1}_{q,1}}\leqslant\frac{CT2^{N(\frac{3}{q}+\frac{3}{2})}}{C(N)}\|b_1\|_{\tilde L^\infty_T\dot B^{\frac{3}{p_0}}_{p_0,1}},$$

$$\|b_1\mathcal{A}U_1\|_{L^1_T\dot B^{\frac{3}{q}-1}_{q,1}}\leqslant C\|b_1\|_{\tilde L^\infty_T\dot B^{\frac{3}{p_0}}_{p_0,1}}\|U_1\|_{L^1_T\dot B^{\frac{3}{q}+1}_{q,1}}\leqslant\frac{CT^{\frac{3}{2}}2^{N(\frac{3}{q}+\frac{5}{2})}}{C(N)^2}\|b_1\|_{\tilde L^\infty_T\dot B^{\frac{3}{p_0}}_{p_0,1}},$$

$$\|b_1\mathcal{A}U_2\|_{L^1_T\dot B^{\frac{3}{q}-1}_{q,1}}\leqslant C\|b_1\|_{\tilde L^\infty_T\dot B^{\frac{3}{p_0}}_{p_0,1}}\|U_2\|_{L^1_T\dot B^{\frac{3}{q}+1}_{q,1}}.$$

By (2.5),

$$\|b_0\mathcal{A}U_2\|_{L^1_T\dot B^{\frac{3}{q}-1}_{q,1}}\leqslant C\left(\|b_0\|_{L^\infty}\|U_2\|_{L^1_T\dot B^{\frac{3}{q}+1}_{q,1}}+T\|b_0\|_{\dot B^{\frac{3}{p_0}+2}_{p_0,1}}\|U_2\|_{\tilde L^\infty_T\dot B^{\frac{3}{q}-1}_{q,1}}\right)$$
$$\leqslant C\frac{T2^{N(\frac{3}{p_0}-\frac{3}{p}+2)}+1}{C(N)}\left(\|U_2\|_{\tilde L^\infty_T\dot B^{\frac{3}{q}-1}_{q,1}}+\|U_2\|_{L^1_T\dot B^{\frac{3}{q}+1}_{q,1}}\right).$$

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Combining with these estimates leads

$$\begin{aligned}
&\|(b_0\mathcal{A}(U_1+U_2), b_1\mathcal{A}u)\|_{L^1_T\dot{B}^{\frac{3}{q}-1}_{q,1}}\\
&\leqslant \frac{CT^{\frac{3}{2}}2^{N(\frac{3}{q}+\frac{3}{p_0}-\frac{3}{p}+\frac{5}{2})}}{C(N)^3}+C\frac{T2^{N(\frac{3}{p_0}-\frac{3}{p}+2)}+1}{C(N)}\left(\|U_2\|_{L^1_T\dot{B}^{\frac{3}{q}+1}_{q,1}}+\|U_2\|_{\tilde{L}^\infty_T\dot{B}^{\frac{3}{q}-1}_{q,1}}\right)\\
&\quad+\frac{CT2^{N(\frac{3}{q}+\frac{3}{2})}}{C(N)}\|b_1\|_{\tilde{L}^\infty_T\dot{B}^{\frac{3}{p_0}}_{p_0,1}}+\|U_2\|_{L^1_T\dot{B}^{\frac{3}{q}+1}_{q,1}}\|b_1\|_{\tilde{L}^\infty_T\dot{B}^{\frac{3}{p_0}}_{p_0,1}}. \qquad (3.20)
\end{aligned}$$

Step 4: The estimate of $\|\tilde{K}(b)\nabla b\|_{L^1_T\dot{B}^{\frac{3}{q}-1}_{q,1}}$. Denote

$$\mathfrak{V}(t):=(1+\|b\|_{L^\infty_T L^\infty})^2.$$

Using (2.6), and applying (2.2) yields

$$\begin{aligned}
\|(\tilde{K}(b)-1)\nabla b\|_{L^1_T\dot{B}^{\frac{3}{q}-1}_{q,1}} &\leqslant C\|\tilde{K}(b)-1\|_{L^1_T\dot{B}^{\frac{3}{p_0}}_{p_0,1}}\|b\|_{\tilde{L}^\infty_T\dot{B}^{\frac{3}{q}}_{q,1}}\\
&\leqslant C\mathfrak{V}(t)\|b\|_{L^1_T\dot{B}^{\frac{3}{p_0}}_{p_0,1}}\|b\|_{\tilde{L}^\infty_T\dot{B}^{\frac{3}{q}}_{q,1}}\\
&\leqslant CT\mathfrak{V}(t)\|b\|_{\tilde{L}^\infty_T\dot{B}^{\frac{3}{p_0}}_{p_0,1}}\|b\|_{\tilde{L}^\infty_T\dot{B}^{\frac{3}{q}}_{q,1}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\tilde{K}(b)\nabla b\|_{L^1_T\dot{B}^{\frac{3}{q}-1}_{q,1}} &\leqslant \|(\tilde{K}(b)-1)\nabla b\|_{L^1_T\dot{B}^{\frac{3}{q}-1}_{q,1}}+T\|b\|_{\tilde{L}^\infty_T\dot{B}^{\frac{3}{q}}_{q,1}}\\
&\leqslant CT\mathfrak{V}(t)\left(1+\|b\|_{L^\infty_T\dot{B}^{\frac{3}{p_0}}_{p_0,1}}\right)\|b\|_{\tilde{L}^\infty_T\dot{B}^{\frac{3}{q}}_{q,1}}\\
&\leqslant CT\mathfrak{V}(t)\left(1+\|b_0\|_{\dot{B}^{\frac{3}{p_0}}_{p_0,1}}+\|b_1\|_{L^\infty_T\dot{B}^{\frac{3}{p_0}}_{p_0,1}}\right)\left(\|b_0\|_{\dot{B}^{\frac{3}{q}}_{q,1}}+\|b_1\|_{\tilde{L}^\infty_T\dot{B}^{\frac{3}{q}}_{q,1}}\right)\\
&\leqslant CT\mathfrak{V}(t)\left(1+\frac{2^{N(\frac{3}{p_0}-\frac{3}{p})}}{C(N)}\right)\left(\frac{2^{N(\frac{3}{q}-\frac{3}{p})}}{C(N)}+\|b_1\|_{\tilde{L}^\infty_T\dot{B}^{\frac{3}{q}}_{q,1}}\right)\\
&\leqslant CT^{\frac{3}{4}}\mathfrak{V}(t)\left(\frac{2^{N(\frac{3}{q}-\frac{3}{p})}}{C(N)}+\|b_1\|_{\tilde{L}^\infty_T\dot{B}^{\frac{3}{q}}_{q,1}}\right). \qquad (3.21)
\end{aligned}$$

To achieve the goal of this step, we need to bound $\|b_1\|_{\tilde{L}^\infty_T\dot{B}^{\frac{3}{q}}_{q,1}}$, which is proved in the following lemma.

Lemma 3.3. *Let us consider (3.3), then the following estimates of b_1 hold:*
(1)

$$\|b_1\|_{L^\infty_T L^\infty}\leqslant C\left(\frac{T2^{\frac{3N}{2}}}{C(N)^2}+\|\nabla U_2\|_{L^1_T L^\infty}\right)\exp\{C\|\nabla u\|_{L^1_T L^\infty}\}. \qquad (3.22)$$

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(2)

$$\begin{aligned}\|b_1\|_{\tilde{L}^\infty_T \dot{B}^{\frac{3}{q}}_{q,1}} \leqslant &\left\{ \frac{CT2^{N(\frac{3}{q}+\frac{3}{2})}}{C(N)} + C\left(\frac{T2^{\frac{3N}{2}}}{C(N)^2} + \frac{T2^{N(\frac{3}{q}+\frac{3}{2})}}{C(N)}\right)\|U_2\|_{L^1_T \dot{B}^{\frac{3}{q}+1}_{q,1}} \right.\\ &\left. + \ \|U_2\|_{\tilde{L}^\infty_T \dot{B}^{\frac{3}{q}-1}_{q,1}} + \|U_2\|_{L^1_T \dot{B}^{\frac{3}{q}+1}_{q,1}} + \|U_2\|^2_{L^1_T \dot{B}^{\frac{3}{q}+1}_{q,1}} \right\} \exp\left\{C\|\nabla u\|_{L^1_T \dot{B}^{\frac{3}{p_0}}_{p_0,1}}\right\}. \quad (3.23)\end{aligned}$$

Proof of Lemma 3.3. (1) From (2.9) in Lemma 2.8, we have

$$\|b_1\|_{L^\infty_T L^\infty} \leqslant \int_0^T \|(u\cdot\nabla b_0, (b_0-1)\mathrm{div} u)\|_{L^\infty}\, dt \exp(C\|\nabla u\|_{L^1_T L^\infty}).$$

Simple computation yields

$$\begin{aligned}\|u\cdot\nabla b_0\|_{L^1_T L^\infty} &\leqslant \|U_0\cdot\nabla b_0\|_{L^1_T L^\infty} + \|U_1\cdot\nabla b_0\|_{L^1_T L^\infty} + \|U_2\cdot\nabla b_0\|_{L^1_T L^\infty}\\ &\leqslant \frac{CT2^{N(\frac{3}{2}-\frac{3}{p})}}{C(N)^2} + \frac{CT^2 2^{N(\frac{7}{2}-\frac{3}{p})}}{C(N)^3} + T^{\frac{1}{2}}\|\nabla b_0\|_{L^\infty}\|U_2\|_{L^2_T L^\infty}\\ &\leqslant \frac{CT2^{N(\frac{3}{2}-\frac{3}{p})}}{C(N)^2} + \frac{CT^{\frac{1}{2}}2^{N(1-\frac{3}{p})}}{C(N)}\|U_2\|_{L^2_T L^\infty}\end{aligned}$$

and

$$\begin{aligned}\|(b_0-1)\mathrm{div}\, u\|_{L^1_T L^\infty} &\leqslant (1+\|b_0\|_{L^\infty})\|\nabla u\|_{L^1_T L^\infty}\\ &\leqslant C\left(1+\frac{1}{C(N)}\right)\left(\frac{T2^{\frac{3N}{2}}}{C(N)^2} + \|\nabla U_2\|_{L^1_T L^\infty}\right)\\ &\leqslant \frac{CT2^{\frac{3N}{2}}}{C(N)^2} + C\|\nabla U_2\|_{L^1_T L^\infty}.\end{aligned}$$

Combining with the above estimates yields (3.22).

(2) From (2.10) in Lemma 2.8, we can get

$$\begin{aligned}\|b_1\|_{\tilde{L}^\infty_T \dot{B}^{\frac{3}{q}}_{q,1}} \leqslant &\ C\left(\|b_1\|_{L^\infty_T L^\infty}\|u\|_{L^1_T \dot{B}^{\frac{3}{q}+1}_{q,1}} + \|(u\cdot\nabla b_0, (b_0-1)\mathrm{div}\, u)\|_{L^1_T \dot{B}^{\frac{3}{q}}_{q,1}}\right)\\ &\times \exp\left(C\|\nabla u\|_{L^1_T \dot{B}^{\frac{3}{p_0}}_{p_0,1}}\right). \quad (3.24)\end{aligned}$$

By the Kato–Ponce estimate (2.3) and (2.4), we can obtain

$$\|U_0\cdot\nabla b_0\|_{L^1_T \dot{B}^{\frac{3}{q}}_{q,1}} \leqslant C\left(\|U_0\|_{L^1_T \dot{B}^{\frac{3}{q}}_{q,1}}\|\nabla b_0\|_{L^\infty} + \|U_0\|_{L^1_T L^\infty}\|\nabla b_0\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\right) \leqslant \frac{CT2^{N(\frac{3}{q}-\frac{3}{p}+\frac{3}{2})}}{C(N)^2},$$

$$\|U_1\cdot\nabla b_0\|_{L^1_T \dot{B}^{\frac{3}{q}}_{q,1}} \leqslant C\left(\|U_1\|_{L^1_T \dot{B}^{\frac{3}{q}}_{q,1}}\|\nabla b_0\|_{L^\infty} + \|U_1\|_{L^1_T L^\infty}\|\nabla b_0\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\right) \leqslant \frac{CT^2 2^{N(\frac{3}{q}-\frac{3}{p}+\frac{7}{2})}}{C(N)^3},$$

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and

$$\|U_2\cdot\nabla b_0\|_{L^1_T\dot B^{\frac{3}{q}}_{q,1}} \leqslant C\Big(T^{\frac12}\|\nabla b_0\|_{L^\infty}\|U_2\|_{\tilde L^2_T\dot B^{\frac{3}{q}}_{q,1}} + T\|\nabla^2 b_0\|_{L^\infty}\|U_2\|_{\tilde L^\infty_T\dot B^{\frac{3}{q}-1}_{q,1}}\Big)$$
$$\leqslant \frac{CT^{\frac12}2^{N(1-\frac{3}{p})}}{C(N)}\Big(\|U_2\|_{\tilde L^\infty_T\dot B^{\frac{3}{q}-1}_{q,1}} + \|U_2\|_{\tilde L^2_T\dot B^{\frac{3}{q}}_{q,1}}\Big).$$

Inserting the above estimates into (3.24) implies

$$\|u\cdot\nabla b_0\|_{L^1_T\dot B^{\frac{3}{q}}_{q,1}} \leqslant \frac{CT2^{N(\frac{3}{q}-\frac{3}{p}+\frac32)}}{C(N)^2} + \frac{CT^{\frac12}2^{N(1-\frac{3}{p})}}{C(N)}\Big(\|U_2\|_{\tilde L^\infty_T\dot B^{\frac{3}{q}-1}_{q,1}} + \|U_2\|_{\tilde L^2_T\dot B^{\frac{3}{q}}_{q,1}}\Big). \quad (3.25)$$

Using the product estimate (2.4) again,

$$\|(b_0-1)\operatorname{div} u\|_{L^1_T\dot B^{\frac{3}{q}}_{q,1}} \leqslant C\|\operatorname{div} u\|_{L^1_T\dot B^{\frac{3}{q}}_{q,1}} + \|b_0\operatorname{div} u\|_{L^1_T\dot B^{\frac{3}{q}}_{q,1}}$$
$$\leqslant C\Big(\|u\|_{L^1_T\dot B^{\frac{3}{q}+1}_{q,1}} + \|b_0\|_{L^\infty}\|u\|_{L^1_T\dot B^{\frac{3}{q}+1}_{q,1}} + T^{\frac12}\|\nabla b_0\|_{L^\infty}\|u\|_{\tilde L^2_T\dot B^{\frac{3}{q}}_{q,1}}\Big)$$
$$\leqslant C\|u\|_{L^1_T\dot B^{\frac{3}{q}+1}_{q,1}} + \frac{CT^{\frac12}2^{N(1-\frac{3}{p})}}{C(N)}\|u\|_{\tilde L^2_T\dot B^{\frac{3}{q}}_{q,1}}$$
$$\leqslant \frac{CT2^{N(\frac{3}{q}+\frac32)}}{C(N)} + C\Big(\|U_2\|_{L^1_T\dot B^{\frac{3}{q}+1}_{q,1}} + \|U_2\|_{\tilde L^2_T\dot B^{\frac{3}{q}}_{q,1}}\Big). \quad (3.26)$$

Combining the above estimates with (3.22) in (3.24) follows the desired result (3.23). □

Now, we continue (3.21). Thanks to (3.23), we have

$$\|\tilde K(b)\nabla b\|_{L^1_T\dot B^{\frac{3}{q}-1}_{q,1}} \leqslant C\mathfrak{V}(t)\Bigg\{\frac{T^{\frac34}2^{N(\frac{3}{q}-\frac{3}{p})}}{C(N)} + \Bigg(\frac{T^{\frac74}2^{\frac{3N}{2}}}{C(N)^2} + \frac{T^{\frac74}2^{N(\frac{3}{q}+\frac32)}}{C(N)}\Bigg)\|U_2\|_{L^1_T\dot B^{\frac{3}{q}+1}_{q,1}}$$
$$+T^{\frac34}\Big(\|U_2\|_{\tilde L^\infty_T\dot B^{\frac{3}{q}-1}_{q,1}} + \|U_2\|_{L^1_T\dot B^{\frac{3}{q}+1}_{q,1}}\Big)$$
$$+\ T^{\frac34}\|U_2\|^2_{L^1_T\dot B^{\frac{3}{q}+1}_{q,1}}\Bigg\}\exp\Big(C\|U_2\|_{L^1_T\dot B^{\frac{3}{q}+1}_{q,1}}\Big). \quad (3.27)$$

Step 5: The estimate of $\|U_2\|_{L^1_T\dot B^{\frac{3}{q}-1}_{q,1}}$ *and* $\|b_1\|_{\tilde L^\infty_T\dot B^{\frac{3}{p_0}}_{p_0,1}}$. Adding the above estimate (3.19), (3.20) and (3.27), one gets

$$\|U_2\|_{L^1_T\dot B^{\frac{3}{q}-1}_{q,1}} \leqslant C\|u\cdot\nabla u\|_{L^1_T\dot B^{\frac{3}{q}-1}_{q,1}} + C\|b_0\mathcal{A}(U_1+U_2)\|_{L^1_T\dot B^{\frac{3}{q}-1}_{q,1}}$$
$$+C\|b_1\mathcal{A}u\|_{L^1_T\dot B^{\frac{3}{q}-1}_{q,1}} + C\|\tilde K(b)\nabla b\|_{L^1_T\dot B^{\frac{3}{q}-1}_{q,1}}$$

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$$
\begin{aligned}
&\leqslant \mathfrak{V}(t)\Bigg\{\frac{CT^{\frac{3}{2}}2^{N(\frac{3}{q}+\frac{3}{p_0}-\frac{3}{p}+\frac{5}{2})}}{C(N)^3}+C\frac{T2^{N(\frac{3}{p_0}-\frac{3}{p}+2)}+1}{C(N)}\\
&\quad\times\Big(\|U_2\|_{L^1_T\dot B^{\frac{3}{q}+1}_{q,1}}+\|U_2\|_{\tilde L^\infty_T\dot B^{\frac{3}{q}-1}_{q,1}}\Big)+\frac{CT2^{N(\frac{3}{q}+\frac{3}{2})}}{C(N)}\|b_1\|_{\tilde L^\infty_T\dot B^{\frac{3}{p_0}}_{p_0,1}}\\
&\quad+\|U_2\|_{L^1_T\dot B^{\frac{3}{q}+1}_{q,1}}\|b_1\|_{\tilde L^\infty_T\dot B^{\frac{3}{p_0}}_{p_0,1}}+\|U_2\|^2_{\tilde L^2_T\dot B^{\frac{3}{q}}_{q,1}}+\|U_2\|^2_{L^1_T\dot B^{\frac{3}{q}+1}_{q,1}}\Bigg\}\\
&\quad\times\exp\Big(C\|U_2\|_{L^1_T\dot B^{\frac{3}{q}+1}_{q,1}}\Big).
\end{aligned}\tag{3.28}
$$

For the estimate of $\|b_1\|_{\tilde L^\infty_T\dot B^{\frac{3}{p_0}}_{p_0,1}}$, from (2.9) in Lemma 2.8, we have

$$
\|b_1\|_{\tilde L^\infty_T\dot B^{\frac{3}{p_0}}_{p_0,1}}\leqslant C\|\big(u\cdot\nabla b_0,(b_0-1)\operatorname{div}u\|_{L^1_T\dot B^{\frac{3}{p_0}}_{p_0,1}}\big)\exp\Big(C\|\nabla u\|_{L^1_T\dot B^{\frac{3}{p_0}}_{p_0,1}}\Big).
$$

Like (3.25) and (3.26), we have

$$
\|u\cdot\nabla b_0\|_{L^1_T\dot B^{\frac{3}{p_0}}_{p_0,1}}\leqslant\frac{CT2^{N(\frac{3}{p_0}-\frac{3}{p}+\frac{3}{2})}}{C(N)^2}+\frac{CT^{\frac{1}{2}}2^{N(1-\frac{3}{p})}}{C(N)}\Big(\|U_2\|_{\tilde L^\infty_T\dot B^{\frac{3}{p_0}-1}_{p_0,1}}+\|U_2\|_{\tilde L^2_T\dot B^{\frac{3}{p_0}}_{p_0,1}}\Big)
$$

and

$$
\|(b_0-1)\operatorname{div}u\|_{L^1_T\dot B^{\frac{3}{p_0}}_{p_0,1}}\leqslant\frac{CT2^{N(\frac{3}{p_0}+\frac{3}{2})}}{C(N)}+C\Big(\|U_2\|_{L^1_T\dot B^{\frac{3}{p_0}+1}_{p_0,1}}+\|U_2\|_{\tilde L^2_T\dot B^{\frac{3}{p_0}}_{p_0,1}}\Big),
$$

respectively. Hence, we get

$$
\|b_1\|_{\tilde L^\infty_T\dot B^{\frac{3}{p_0}}_{p_0,1}}\leqslant C\left(\frac{T2^{N(\frac{3}{p_0}+\frac{3}{2})}}{C(N)}+\|U_2\|_{\tilde L^\infty_T\dot B^{\frac{3}{q}-1}_{q,1}}+\|U_2\|_{L^1_T\dot B^{\frac{3}{q}+1}_{q,1}}\right)\exp\Big(C\|U_2\|_{L^1_T\dot B^{\frac{3}{p_0}+1}_{p_0,1}}\Big).\tag{3.29}
$$

3.4. Proof of Theorem 1.1

Let us keep the definition of T_0 (see the § 3.2) in mind. Denote

$$
X_T:=\|b_1\|_{\tilde L^\infty_T\dot B^{\frac{3}{p_0}}_{p_0,1}},\qquad Y_T:=\|U_2\|_{\tilde L^\infty_T\dot B^{\frac{3}{q}-1}_{q,1}}+\|U_2\|_{L^1_T\dot B^{\frac{3}{q}+1}_{q,1}}
$$

and

$$
T_1=:\sup\left\{t\in(0,T_0):X_T+Y_T\leqslant M\left(\frac{T2^{N(\frac{3}{p_0}+\frac{3}{2})}}{C(N)}+\frac{T^{\frac{3}{2}}2^{N(\frac{3}{q}+\frac{3}{p_0}-\frac{3}{p}+\frac{5}{2})}}{C(N)^3}\right)\right\},
$$

where M is a large enough positive constant, which will be determined later on. The definition of T_0 can be seen in (3.17). Assume that $T_1<T_0$. For all $T\leqslant T_1$, we have

$$
\mathfrak{V}(t)\leqslant(1+\|b_0\|_{L^\infty}+\|b_1\|_{L^\infty_T L^\infty})^2\leqslant 2,
$$

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provided that X_T is small enough (In fact, we assume that the below (3.32) holds, so X_T is small enough, and then $\mathfrak{V}(t) \leqslant 2$, which is absorbed by the constant C in the following text.) Thanks to the above estimates, we have $\forall T \in (0, T_1)$,

$$X_T \leqslant C\left(\frac{T2^{N(\frac{3}{p_0}+\frac{3}{2})}}{C(N)} + Y_T\right), \tag{3.30}$$

and

$$\begin{aligned} Y_T &\leqslant \frac{CT^{\frac{3}{2}}2^{N(\frac{3}{q}+\frac{3}{p_0}-\frac{3}{p}+\frac{5}{2})}}{C(N)^3} + C\left(\frac{T2^{N(\frac{3}{p_0}-\frac{3}{p}+2)}+1}{C(N)} + X_T + Y_T\right)Y_T + \frac{CT2^{N(\frac{3}{q}+\frac{3}{2})}}{C(N)}X_T \\ &\leqslant \frac{1}{2}Y_T + 2C\frac{T^{\frac{3}{2}}2^{N(\frac{3}{q}+\frac{3}{p_0}-\frac{3}{p}+\frac{5}{2})}}{C(N)^3} + \frac{CT2^{N(\frac{3}{q}+\frac{3}{2})}}{C(N)}X_T, \end{aligned} \tag{3.31}$$

where we have set N such that

$$\frac{T2^{N(\frac{3}{p_0}-\frac{3}{p}+2)}+1}{C(N)} + M\left(\frac{T2^{N(\frac{3}{p_0}+\frac{3}{2})}}{C(N)} + \frac{T^{\frac{3}{2}}2^{N(\frac{3}{q}+\frac{3}{p_0}-\frac{3}{p}+\frac{5}{2})}}{C(N)^3}\right) \ll 1. \tag{3.32}$$

Equation (3.31) yields

$$Y_T \leqslant 4C\frac{T^{\frac{3}{2}}2^{N(\frac{3}{q}+\frac{3}{p_0}-\frac{3}{p}+\frac{5}{2})}}{C(N)^3} + \frac{2CT2^{N(\frac{3}{q}+\frac{3}{2})}}{C(N)}X_T. \tag{3.33}$$

Multiplying (3.33) by $2C$, and adding to (3.30), choosing N such that

$$\frac{4C^2T2^{N(\frac{3}{q}+\frac{3}{2})}}{C(N)} < \frac{1}{2}, \tag{3.34}$$

we get

$$X_T + Y_T \leqslant \frac{1}{2}X_T + 8C^2\left(\frac{T2^{N(\frac{3}{p_0}+\frac{3}{2})}}{C(N)} + \frac{T^{\frac{3}{2}}2^{N(\frac{3}{q}+\frac{3}{p_0}-\frac{3}{p}+\frac{5}{2})}}{C(N)^3}\right).$$

This implies that

$$X_T + Y_T \leqslant 16C^2\left(\frac{T2^{N(\frac{3}{p_0}+\frac{3}{2})}}{C(N)} + \frac{T^{\frac{3}{2}}2^{N(\frac{3}{q}+\frac{3}{p_0}-\frac{3}{p}+\frac{5}{2})}}{C(N)^3}\right).$$

One can see from Lemma 3.1 and Remark 3.2 that the above requirement of N (3.32) and (3.34) can be met. If we choose $M = 32C^2$, and then a contradiction is obtained. So we have $T_1 = T_0$ via using the continuation arguments. That is $\forall T \leqslant T_0$, we have

$$X_T + Y_T \leqslant 32C^2\left(\frac{T2^{N(\frac{3}{p_0}+\frac{3}{2})}}{C(N)} + \frac{T^{\frac{3}{2}}2^{N(\frac{3}{q}+\frac{3}{p_0}-\frac{3}{p}+\frac{5}{2})}}{C(N)^3}\right).$$

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Now, we can get

$$\begin{aligned}\|u(T_0)\|_{\dot{B}^{-\frac{1}{2}}_{6,1}} &\geqslant \|U_1\|_{\dot{B}^{-\frac{1}{2}}_{6,1}} - \left(\|U_0(T_0)\|_{\dot{B}^{-\frac{1}{2}}_{6,1}} + \|U_2(T_0)\|_{\dot{B}^{-\frac{1}{2}}_{6,1}}\right)\\ &\geqslant \frac{c}{2}2^{2N(\epsilon_1-\epsilon)} - \frac{C}{C(N)} - 32C^2\left(\frac{T2^{N(\frac{3}{p_0}+\frac{3}{2})}}{C(N)} + \frac{T^{\frac{3}{2}}2^{N(\frac{3}{q}+\frac{3}{p_0}-\frac{3}{p}+\frac{5}{2})}}{C(N)^3}\right)\\ &\geqslant \frac{c}{8}2^{2N(\epsilon_1-\epsilon)}.\end{aligned}$$

Thanks to the choice of ϵ and ϵ_1, see (3.8) in § 3.1, we can conclude the proof of Theorem 1.1.

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Appendix

In this Section, we prove three estimates in Lemma 2.8.

Proof of Lemma 2.8(a). Let $p > 2$, integrating by parts, we have

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|\mathfrak{B}\|^p_{L^p} &\leqslant -\frac{1}{p}\int u\cdot\nabla|\mathfrak{B}|^p + \int \operatorname{div} u|\mathfrak{B}|^p + \int f\mathfrak{B}|\mathfrak{B}|^{p-2}\\ &\leqslant \frac{p+1}{p}\|\nabla u\|_{L^\infty}\|\mathfrak{B}\|^p_{L^p} + \|f\|_{L^\infty}\|\mathfrak{B}\|^{p-1}_{L^p}\end{aligned}$$

Dividing $\|\mathfrak{B}\|^{p-1}_{L^p}$ on the both sides of the above inequality, then let $p \to \infty$, we get

$$\frac{d}{dt}\|\mathfrak{B}\|_{L^\infty} \leqslant \|\nabla u\|_{L^\infty}\|\mathfrak{B}\|_{L^\infty} + \|f\|_{L^\infty}.$$

With Gronwall's lemma leads the desired result (2.9). □

Proof of Lemma 2.8(b). Denote

$$(\mathfrak{B}_\lambda(t), f_\lambda(t)) := (\mathfrak{B}, f)\exp\left\{-\lambda\int_0^t \|\nabla u\|_{\dot{B}^{\frac{3}{p_0}}_{p_0,1}}\,d\tau\right\},$$

where λ is a large enough constant, which will be set later. So we have

$$\partial_t\mathfrak{B}_\lambda + \lambda\|\nabla u\|_{\dot{B}^{\frac{3}{p_0}}_{p_0,1}}\mathfrak{B}_\lambda + u\cdot\nabla\mathfrak{B}_\lambda - \mathfrak{B}_\lambda\operatorname{div} u = f_\lambda.$$

By a standard process, we have

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\|\Delta_j\mathfrak{B}_\lambda\|^q_{L^q} + \lambda\|\nabla u\|_{\dot{B}^{\frac{3}{p_0}}_{p_0,1}}\|\Delta_j\mathfrak{B}_\lambda\|^q_{L^q} \leqslant{}& C(\|\nabla u\|_{L^\infty}\|\Delta_j\mathfrak{B}_\lambda\|_{L^q} + \|[\Delta_j, u\cdot\nabla]\mathfrak{B}_\lambda\|_{L^q}\\ &+ \|\Delta_j(\mathfrak{B}_\lambda\operatorname{div} u)\|_{L^q} + \|\Delta_j f_\lambda\|_{L^q})\|\Delta_j\mathfrak{B}_\lambda\|^{q-1}_{L^q}.\end{aligned}$$

Dividing $\|\Delta_j \mathfrak{B}_\lambda\|_{L^q}^{q-1}$ on the both sides, integrating in time $(0,t)$, we obtain

$$\begin{aligned}
&\|\Delta_j \mathfrak{B}_\lambda\|_{L^\infty_t L^q} + \lambda \int_0^t \|\nabla u\|_{\dot B^{\frac{3}{p_0}}_{p_0,1}} \|\Delta_j \mathfrak{B}_\lambda\|_{L^q}\, d\tau \\
&\leqslant C \Bigg(\int_0^t \|\nabla u\|_{\dot B^{\frac{3}{p_0}}_{p_0,1}} \|\Delta_j \mathfrak{B}_\lambda\|_{L^q}\, d\tau + \|[\Delta_j, u\cdot\nabla]\mathfrak{B}_\lambda\|_{L^1_t L^q} \\
&\quad + \|\Delta_j(\mathfrak{B}_\lambda \operatorname{div} u)\|_{L^1_t L^q} + \|\Delta_j f_\lambda\|_{L^1_t L^q} \Bigg).
\end{aligned}$$

Then multiplying $2^{j\frac{3}{q}}$ and summing over $j \in \mathbb{Z}$, using Kato–Ponce estimate (2.3) and the commutator estimate (see, [1, Lemma 2.100])

$$\sum_{j\in\mathbb{Z}} 2^{j\frac{3}{q}} \|[\Delta_j, u\cdot\nabla]\mathfrak{B}\|_{L^q} \leqslant C\|\nabla u\|_{\dot B^{\frac{3}{p_0}}_{p_0,1}} \|\mathfrak{B}\|_{\dot B^{\frac{3}{q}}_{q,1}}$$

yields

$$\begin{aligned}
&\|\mathfrak{B}_\lambda\|_{\tilde L^\infty_t \dot B^{\frac{3}{q}}_{q,1}} + \lambda \int_0^t \|\nabla u\|_{\dot B^{\frac{3}{p_0}}_{p_0,1}} \|\mathfrak{B}_\lambda\|_{\dot B^{\frac{3}{q}}_{q,1}}\, d\tau \\
&\leqslant C \left(\int_0^t \|\nabla u\|_{\dot B^{\frac{3}{p_0}}_{p_0,1}} \|\mathfrak{B}_\lambda\|_{\dot B^{\frac{3}{q}}_{q,1}}\, d\tau + \|\mathfrak{B}_\lambda\|_{L^\infty_t L^\infty} \|u\|_{L^1_t \dot B^{\frac{3}{q}+1}_{q,1}} + \|f_\lambda\|_{L^1_t \dot B^{\frac{3}{q}}_{q,1}} \right).
\end{aligned}$$

Choosing $\lambda = 4C$, so we can absorb the first term on the right hand side of the above inequality and then obtain the desired inequality (2.10). □

Proof of Lemma 2.8(c). Using Kato–Ponce estimate and $\dot B^{\frac{3}{p_0}}_{p_0,1} \hookrightarrow L^\infty$, we have

$$\|\mathfrak{B} \operatorname{div} u\|_{\dot B^{\frac{3}{p_0}}_{p_0,1}} \leqslant C\|\mathfrak{B}\|_{\dot B^{\frac{3}{p_0}}_{p_0,1}} \|\nabla u\|_{\dot B^{\frac{3}{p_0}}_{p_0,1}}.$$

Following the same idea as the upper estimate of Lemma 2.8(b), one can get the desired result (2.11). □

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