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# On the stabilization of a biped vertical posture in single support using internal torques

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## SUMMARY

We consider the problem of the stabilization in single support of the vertical posture for a two-link, a three-link, and a five-link planar biped without feet. The control torques are applied in the inter-link joints only. Thus, our objects are under-actuated systems. The control laws are designed, using the biped linear models and their associated Jordan forms. The feedback is synthesized to suppress the unstable modes. The limits imposed on the torques are taken into account explicitly. Thus, the feedback laws with saturation are designed. The numerical investigations of the nonlinear models with the designed control laws are presented.

KEYWORDS: Vertical posture; Biped; Limited torques; Single support.

## 1. INTRODUCTION

With the aim to achieve fast walking gaits, some papers have been devoted to the study of walking mechanisms as in the compass and the biped without feet.[1–7] The study and control of walking gait and running gait of these robots is a very difficult problem. This family of robots is statically unstable and under-actuated in the single support phase. The operations of stabilization in the single support for the biped vertical posture, of balancing around this equilibrium posture are also difficult. These problems are interesting, for example, from a biomechanical point of view. The main difficulty is that the vertical posture of the biped without feet is unstable, as an equilibrium of an inverted pendulum. The challenge of the control law is not only to reduce the time of transient oscillations, but also to "suppress" this instability. In reference [2], it is shown that it is possible to track single support stable trajectories with internal stability by a suitable choice of outputs for a two-link robot and a five-link robot. The authors in references [4, 6, 7, 8] realize orbital stabilization for a five-link biped, also in the single support phase. However, usually the limits imposed on the torques are not taken into account explicitly.

In this paper, we propose a strategy of control with limited torques to stabilize the vertical posture of a biped with two, three and five links. A Jordan form is developed to determine and to suppress the unstable modes of the linear model of the biped motion around the vertical posture. This control law is defined such that the torques adjust only the unstable modes of the biped.

The organization of this paper is as follows: Section 2 is devoted to the model of the biped. It contains also the data of the physical parameters of the biped. The linear model of the biped motion around the vertical posture is presented in Section 3. The statement of the problem is defined in Section 4. The control law for the two-link biped is designed in Section 5. The control laws for the three-link and five-link bipeds are developed in Sections 6 and 7. Section 8 presents our conclusion and perspectives.

## 2. MODEL DESCRIPTION OF THE PLANAR BIPED

### 2.1. The dynamic model

We consider an under-actuated planar biped with $n$ degrees of freedom and $n-1$ actuators. The generalized forces (torques) are only due to the actuators in the inter-link joints. The dynamic model of the biped single support motion is given by the following Lagrange matrix equation:

$$D(q)\ddot{q} + C(q,\dot{q}) + F\dot{q} + G(q) = B\Gamma \tag{1}$$

Here, $q$ is the $n \times 1$ configuration vector. Its coordinates are the absolute angle between the trunk and the vertical axis, and the $n-1$ actuated inter-link angles. $D(q)$ is the $n \times n$ inertia positive definite matrix, $C(q,\dot{q})$ is the $n \times 1$ column of Coriolis and centrifugal forces. The matrix $D(q)$ depends on the $n-1$ inter-link angles only. We assume that at each actuated joint there is a viscous friction. Let the friction coefficient $f$ be identical in all actuated joints, $F = (0,\ fI_{n-1})^T$, $I_{n-1}$ is a $(n-1)\times(n-1)$ unit matrix. $G(q)$ is the $n \times 1$ vector of the torques due to gravity. $B$ is a constant $n \times (n-1)$ matrix, $\Gamma$ is the $(n-1)\times 1$ vector of the torques, applied in the knee and hip joints. The diagrams of the two-link biped ($n=2$), the three-link biped ($n=3$), and the five-link biped ($n=5$) are presented in Figure 1. Model (1) is computed considering that the contact between the tip of the stance leg and the ground is an undriven pivot. But in reality there is a unilateral constraint between the ground and the stance leg tip; the ground cannot prevent the stance leg from taking off. We assume there is no take-off and no sliding. Thus, it is necessary to check the ground reaction in the stance leg tip. Its vertical component $R_y$ must

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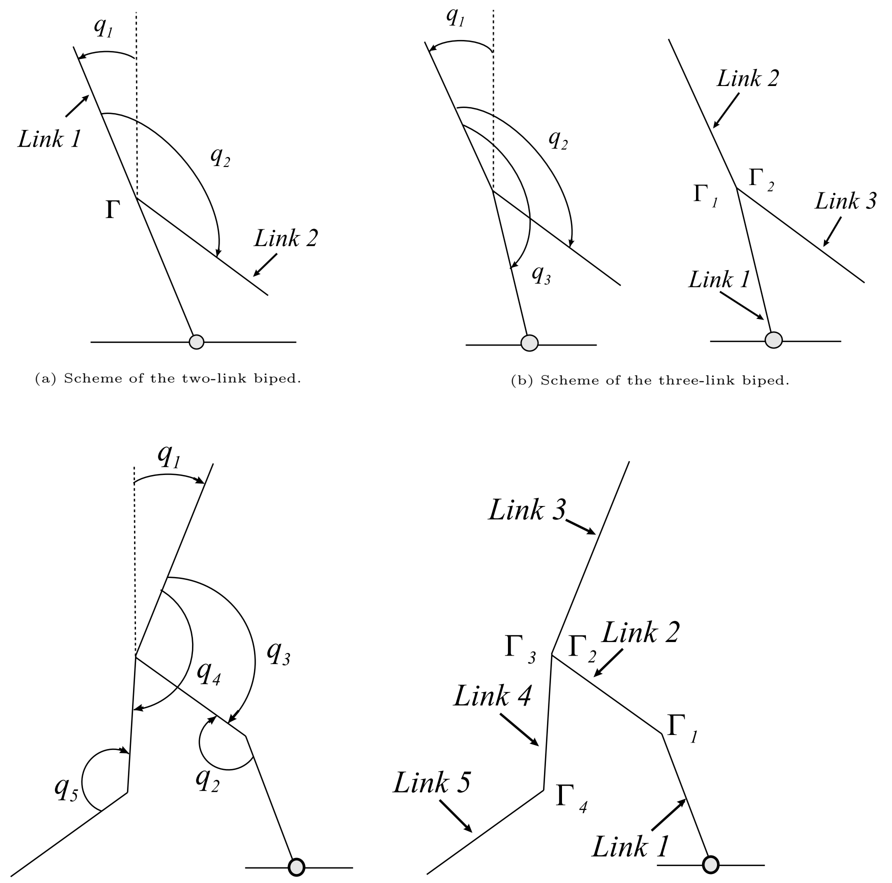


(a) Scheme of the two-link biped.

(b) Scheme of the three-link biped.

(c) Scheme of the five-link biped.

Fig. 1. The three studied bipeds.

be directed upwards. We introduce the following equations applying Newton's second law for the mass center of the biped to determine the ground reaction $R(R_x, R_y)$,

$$\begin{aligned} R_y &= M\ddot{y}_c + Mg \\ R_x &= M\ddot{x}_c \end{aligned} \tag{2}$$

Here, $M$ is the total mass of the biped, $x_c$ and $y_c$ are the coordinates of the mass center of the biped. To check if the ground reaction is located in the friction cone, we have to calculate the ratio $R_x/R_y$.

### 2.2. *Physical parameters of the bipeds*

For the numerical experiments we use the physical parameters of the five-link biped prototype "Rabbit".[7] The masses and the lengths of the five links in Fig. 1 are: $m_1 = m_5 = 3.2\,\text{kg}$, $m_2 = m_4 = 6.8\,\text{kg}$, $m_3 = 20\,\text{kg}$, $l_1 = l_2 = l_4 = l_5 = 0.4\,\text{m}$, $l_3 = 0.625\,\text{m}$.

The distances between the joint and the mass center of each link are the following: $s_1 = s_5 = 0.127\,\text{m}$, $s_2 = s_4 = 0.163\,\text{m}$, $s_3 = 0.2\,\text{m}$.

The inertia moments around the mass center of each link are: $I_1 = I_5 = 0.0484\,\text{kg}\cdot\text{m}^2$, $I_2 = I_4 = 0.0693\,\text{kg}\cdot\text{m}^2$, $I_3 = 1.9412\,\text{kg}\cdot\text{m}^2$. The inertia of the rotor for each DC motor is $I = 3.32\,10^{-4}\,\text{kg}\cdot\text{m}^2$.

The gear ratios equals 50. The maximal value $U$ of the torques equals $150\,\text{N}\cdot\text{m}$.

Using these values we have also calculated the corresponding values for the two-link and three-link bipeds.

## 3. LINEAR MODEL OF THE PLANAR BIPED

In this section, we present the matrix equations (1) linearized around the vertical posture, the state form of this linear model and the Jordan form. This Jordan form will be useful to define the control law in the next section.

Let $q_e$ denote the configuration vector of the biped in the vertical posture. This vertical posture is an equilibrium position. This equilibrium point is $q_e = (0,\ \pi)^T$ the two-link biped, $q_e = (0,\ \pi,\ \pi)^T$ for the three-link biped, and $q_e = (0,\ \pi,\ \pi,\ \pi)^T$ for the five-link biped. The linear model is defined by the variation vector $\nu = q - q_e$ of the configuration vector $q$ around the vertical equilibrium posture $q_e$,

$$D_l\ddot{\nu} + F\dot{\nu} + G_l\nu = B\Gamma \tag{3}$$

Here, $D_l$ is the inertia matrix for the configuration vector $q_e$ : $D_l = D(q_e)$. $G_l$ is the Jacobian of the matrix $G(q)$ computed at the equilibrium point $q_e$. We will consider the following constraint imposed on each torque:

$$|\Gamma_i| \leq U, \ (i = 1, \ldots, n-1), \ U = const \tag{4}$$

We deduce from (3) the state model with $x = (\nu,\ \dot{\nu})^T$:

$$\dot{x} = Ax + b\Gamma \tag{5}$$

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Here,

$$A = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ -D_l^{-1}G_l & -D_l^{-1}F \end{pmatrix}, \quad b = \begin{pmatrix} 0_{n\times n} \\ D_l^{-1}B \end{pmatrix} \tag{6}$$

Introducing a nondegenerate linear transformation $x = Sy$, with a constant matrix $S$, we are able to obtain the well-known Jordan form, (see reference [9]) of equation (5):

$$\dot{y} = \Lambda y + d\Gamma \tag{7}$$

where

$$\Lambda = S^{-1}AS = \begin{pmatrix} \lambda_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & \lambda_{2n} \end{pmatrix}, \quad d = S^{-1}b = [d_i]^T \quad (i = 1, \ldots, n). \tag{8}$$

Here, $\lambda_1, \ldots, \lambda_{2n}$ are the eigenvalues of matrix $A$. Let us assume that the positive eigenvalues have the smaller subscript. For the two-link biped we will obtain $\lambda_1 > 0$, $Re\lambda_i < 0$ $(i = 2, 3, 4)$, for the three-link biped $\lambda_1 > 0$, $\lambda_2 > 0$, $Re\lambda_i < 0$ $(i = 3 - 6)$, and for the five-link biped $\lambda_i > 0$, $(i = 1, 2, 3)$, $Re\lambda_j < 0$ $(j = 4 - 10)$.

## 4. PROBLEM STATEMENT

Let $x = x_e = 0$ (here 0 is a $(2n \times 1)$ zero-column) be the desired equilibrium state of system (5). Let us design the feedback control to stabilize this equilibrium state $x_e = 0$. In other words, we want to design an admissible (satisfying inequality (4)) feedback control to ensure the asymptotic stability of the desired state $x_e = 0$. Let $W$ be the set of vector-functions $\Gamma(t)$ such that their components $\Gamma_i(t)$ $(i = 1, \ldots, n-1)$ are piecewise continuous functions of time, satisfying inequalities (4). Let $Q$ be the set of the initial states $x(0)$ from which the origin $x_e = 0$ can be reached, using an admissible control vector-function. Thus, system (5) can reach the origin $x_e = 0$ with the control $\Gamma(t) \in W$, only when starting from the initial states $x(0) \in Q$. The set $Q$ is called controllability region. If matrix $A$ has eigenvalues with positive real parts and the control torques $\Gamma_i$ $(i = 1, \ldots, n-1)$ are limited, then the controllability region $Q$ is an open subset of the phase space $X$ for system (5) (see reference [10]).

For any admissible feedback control $\Gamma = \Gamma(x)$ $(|\Gamma_i(x)| \leq U, i = 1, \ldots, n-1)$ the corresponding attraction region $V$ belongs to the controllability region: $V \subset Q$. Here as usual, the attraction region is the set of initial states $x(0)$ from which system (5), with the control $\Gamma = \Gamma(x)$ asymptotically tends to the origin point $x_e = 0$ as $t \to \infty$.

Some eigenvalues of matrix $A$ of system (5) are located in the left half of the complex plane. The other eigenvalues of $A$ are in the right half of the complex plane. We will design a control law, which "transfers" these last eigenvalues in the left half of the complex plane.

The structure and the properties of this control law depend on the studied biped and its number of D.O.F. We will detail now these different cases.

## 5. TWO-LINK BIPED ($n = 2$)

Here, we design a control law for the inter-link torque to stabilize the two-link planar biped, with a region of attraction as large as possible.

### 5.1. *Control design for the two-link biped*

If the coefficient of the viscous friction $f$ in the unique joint of the two-link biped is equal to zero, the spectrum of matrix $A$ is symmetric with respect to the imaginary axis because in this case system (5) is conservative. This property is true in the general case, i.e. for a biped with $n$ links. In the case $n = 2$ (under condition $f = 0$), matrix $A$ has two real eigenvalues (positive and negative), and two imaginary conjugate eigenvalues. If $f \neq 0$, then matrix $A$ for $n = 2$ has one real positive eigenvalue, one real negative eigenvalue and two complex conjugate eigenvalues with a negative real part. Let $\lambda_1$ be the real positive eigenvalue, and let us consider the first scalar differential equation of system (7) corresponding to this eigenvalue $\lambda_1$,

$$\dot{y}_1 = \lambda_1 y_1 + d_1\Gamma \tag{9}$$

Assuming that (5) is a Kalman controllable system,[11] then scalar $d_1 \neq 0$. The controllability region $Q$ of equation (9) and consequently of system (7) is described by the following inequality,[10]

$$|y_1| < |d_1|\frac{U}{\lambda_1} \tag{10}$$

We can "suppress" the instability of coordinate $y_1$ by a linear feedback control,

$$\Gamma = \gamma y_1 \tag{11}$$

by the condition,

$$\lambda_1 + d_1\gamma_1 < 0 \tag{12}$$

If we take into account the constraints (4), we obtain from (11) a linear feedback control with saturation,

$$\Gamma = \Gamma(y_1) = \begin{cases} U, & \textit{if} \quad \gamma y_1 \geq U \\ \gamma y_1, & \textit{if} \quad |\gamma y_1| \leq U \\ -U, & \textit{if} \quad \gamma y_1 \leq -U \end{cases} \tag{13}$$

It is possible to see that if $|y_1| < |d_1|U/\lambda_1$ (see inequality (10)), then under condition (12) the right part of equation (9) with control (13) is negative when $y_1 > 0$ and positive when $y_1 < 0$. Consequently, if $|y_1(0)| < |d_1|U/\lambda_1$, then the solution $y_1(t)$ of system (9), (13) tends to 0 as $t \to \infty$. But if $y_1(t) \to 0$, then according to expression (13) $\Gamma(t) \to 0$ as $t \to \infty$. Therefore, the solutions $y_i(t)$ $(i = 2, 3, 4)$ of the second, third and fourth equations of system (7) with any initial conditions $y_i(0)$ $(i = 2, 3, 4)$ converge to zero as $t \to \infty$ because $Re\lambda_i < 0$ for $i = 2, 3, 4$. Thus, under control

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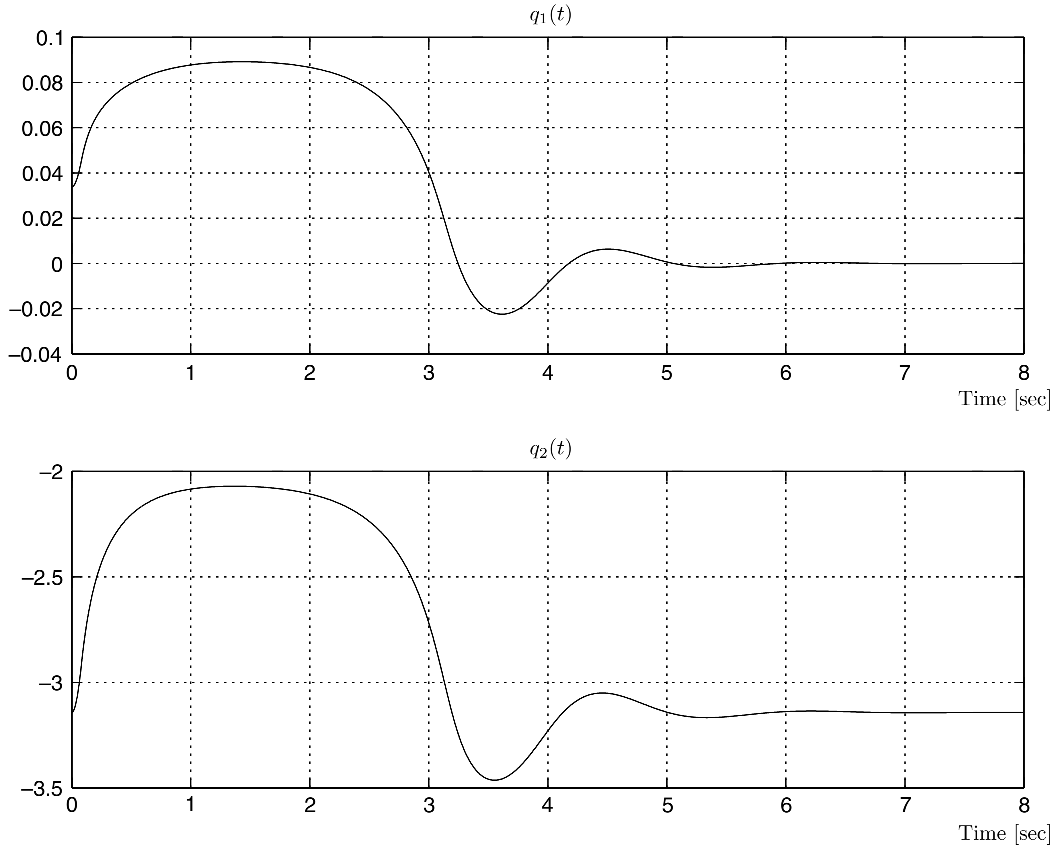


Fig. 2. Two-link biped: Angles $q_1(t)$ and $q_2(t)$.

(13) and with inequality (12), the domain of attraction $V$ coincides with the controllability domain $Q$, [10, 3]: $V = Q$. So, the attraction domain $V$ for system (5), (13) is as large as possible and it is described by inequality (10). Note that the variable $y_1$ depends on the original variables from vector $x$ according to the transformation $x = Sy$ or $y = S^{-1}x$. Due to this, formula (13) defines the control feedback, which depends on vector $x$ of the original variables.

According to Lyapounov's theorem,[12] the equilibrium $q = q_e$ of the nonlinear system (1) is asymptotically stable under control (13) with a given region of attraction.

### *5.2. Numerical results for the two-link biped*

We use the parameters defined in Section 2.2 to compute the parameters of the dynamic model for the two-link biped. In simulation the control law (13) is applied to the nonlinear model (1) for the two-link biped. For the initial configuration $q(0) = [1.94°; 180°]$ the graphs of the variables $q_1(t)$ and $q_2(t)$ as functions of time are shown in Figure 2. At the end the biped is steered to the vertical posture. All the potential of the actuator is applied at the initial time, as shown in Figure 3. The attraction domain for the nonlinear system depends on the feedback gain $\gamma$, which is chosen as $-10\,000$ in our numerical experiment. For the linear case, relation (10) gives the maximum possible deviation of the two-link biped from the vertical axis 7.32°. But for the nonlinear model the maximum possible deviation is close to 1.94°. For each numerical experiment, we check the ground reaction in the stance leg tip to be sure that its vertical component is directed upwards. Figure 4 shows that during the stabilization process of the two-link biped, the vertical component of the ground reaction is always positive and equals the weight of the biped at the end of the process.

We tested our strategy for different values of the actuator power. In fact, we considered different values $U$ for the maximum torque $\Gamma$. Figure 5 shows that when the maximum torque increases, the maximum allowable initial deviation increases too but remains limited.

## 6. THREE-LINK BIPED ($n = 3$)

In this paragraph, we design a control law for the two inter-link torques to stabilize the vertical posture of the three-link planar biped in single support.

### *6.1. Control design for the three-link biped*

Matrix $A$ of the linear model of the biped has two real positive eigenvalues, two real negative eigenvalues and two complex conjugate eigenvalues with negative real part. Let $\lambda_i$, $(i = 1, 2)$ be the real positive eigenvalues, and let us consider the first two lines of matrix equation (7) corresponding to these eigenvalues,

$$\begin{aligned} \dot{y}_1 &= \lambda_1 y_1 + d_{11}\Gamma_1 + d_{12}\Gamma_2 \\ \dot{y}_2 &= \lambda_2 y_2 + d_{21}\Gamma_1 + d_{22}\Gamma_2 \end{aligned} \qquad (14)$$

The torques are chosen such that, the instability of both coordinates $y_i$, $(i = 1, 2)$ is suppressed by choosing the

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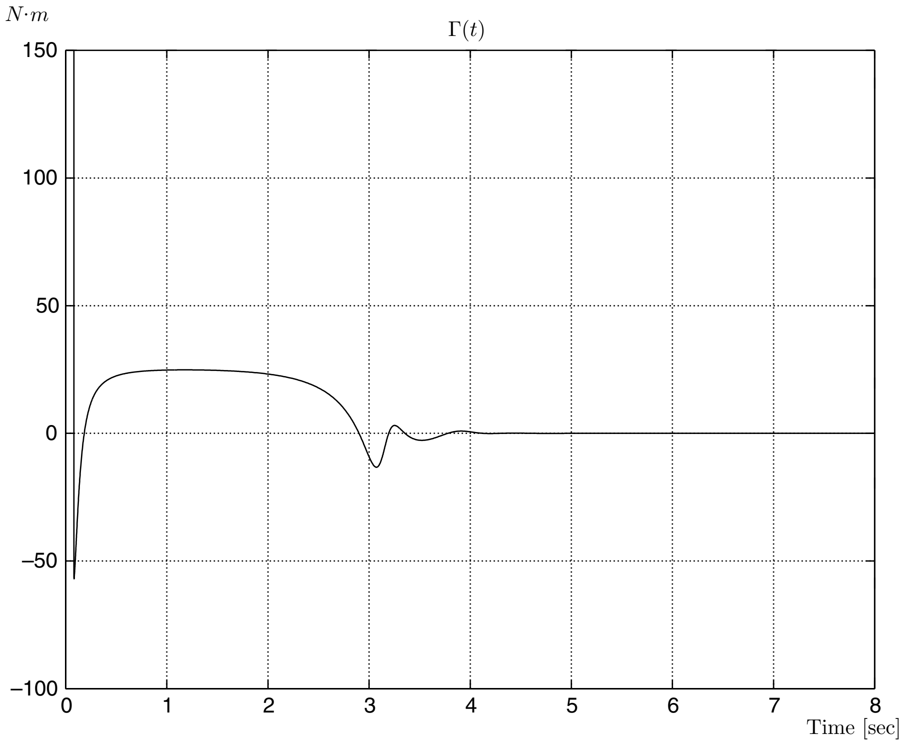


Fig. 3. Two-link biped: Torque in the inter-link joint.

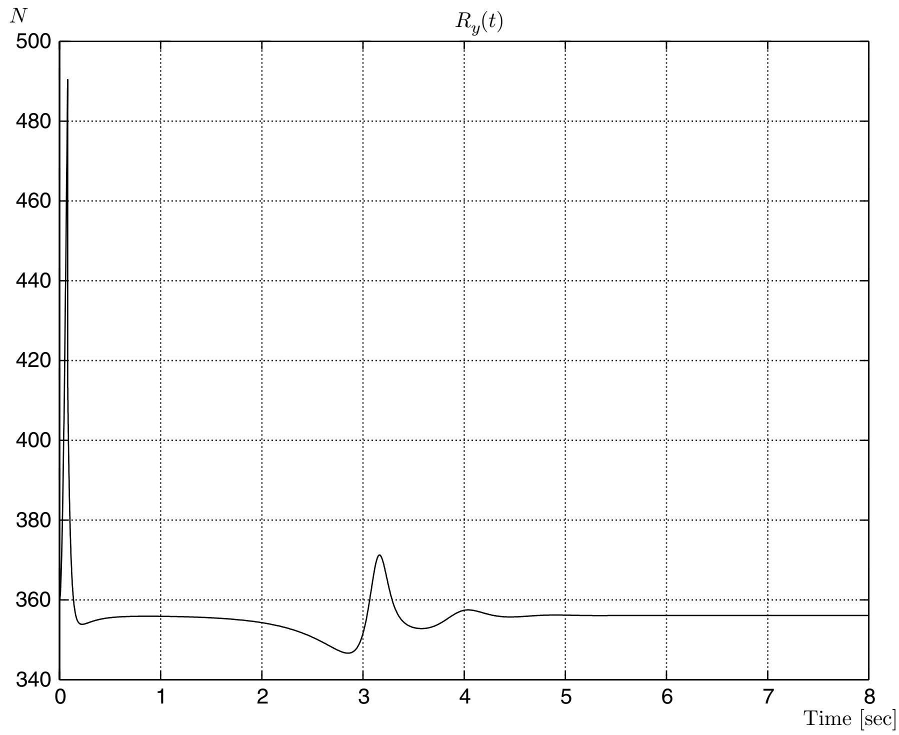


Fig. 4. Two-link biped: Vertical component of the ground reaction in the stance leg tip.

feedback as,

$$\begin{aligned} d_{11}\Gamma_1 + d_{12}\Gamma_2 &= \gamma_1 y_1 \\ d_{21}\Gamma_1 + d_{22}\Gamma_2 &= \gamma_2 y_2 \end{aligned} \tag{15}$$

with both conditions,

$$\lambda_i + \gamma_i < 0 \quad (i = 1, 2) \tag{16}$$

Calculating both torques $\Gamma_1$ and $\Gamma_2$ from algebraic equations (15), we obtain

$$\begin{aligned} \Gamma_1 &= \frac{\gamma_1 d_{22} y_1 - \gamma_2 d_{12} y_2}{d_{11} d_{22} - d_{12} d_{21}} = \Gamma_1^0(y_1, y_2) \\ \Gamma_2 &= \frac{\gamma_2 d_{11} y_2 - \gamma_1 d_{21} y_1}{d_{11} d_{22} - d_{12} d_{21}} = \Gamma_2^0(y_1, y_2) \end{aligned} \tag{17}$$

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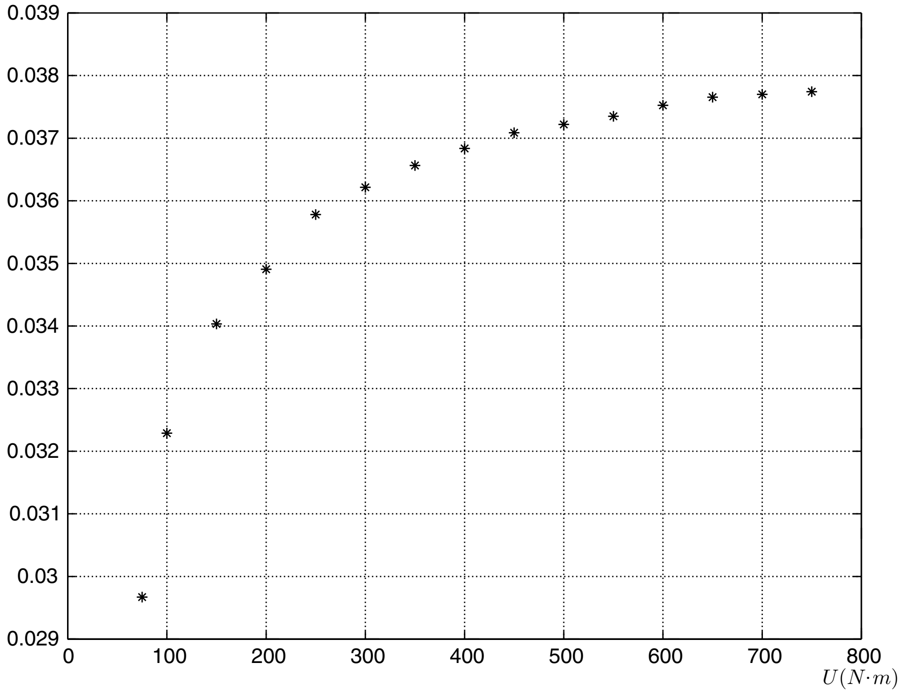


Fig. 5. Two-link biped: Maximum allowable deviation of the biped from the vertical position as a function of the maximum amplitude of the actuator torque.

We assume here that the denominator in expressions (17) is not equal to zero. Taking into account constrains (4), the applied torques $\Gamma_i = \Gamma_i(y_1, y_2)$ are

$$\Gamma_i = \begin{cases} U, & \textit{if} \quad \Gamma_i^0(y_1, y_2) \geq U \\ \Gamma_i^0(y_1, y_2), & \textit{if} \quad \left|\Gamma_i^0(y_1, y_2)\right| \leq U \\ -U, & \textit{if} \quad \Gamma_i^0(y_1, y_2) \leq -U \end{cases} \tag{18}$$

Under the global control law (18), the equilibrium point $x = 0$ is asymptotically stable for system (5) in some attraction region $V \subset Q$. Using Lyapounov's theorem [12], we can prove that the equilibrium point $q = q_e$ of nonlinear system (1), (18) is asymptotically stable as well.

### *6.2. Numerical results for the three-link biped*

We use the parameters defined in Section 2.2 to compute the parameters of the dynamic model for the three-link biped. In the simulation with control law (18) applied to nonlinear model (1) of the three-link biped it is possible to start with an initial configuration $q(0) = [2.9°; 180°; 180°]$. Thus, adding a joint in the haunch between the trunk and the stance leg leads to increasing the attraction domain. The graphs of the variables $q_1(t)$, $q_2(t)$, and $q_3(t)$ as function of time are shown in Figure 6. At the end, the biped is steered to the vertical posture. The maximum torque of the actuator is applied at initial time, as shown in Figure 7. Figure 8 shows that during the stabilization process of the three-link biped the vertical component of the ground reaction is always positive and equals the weight of the biped at the end of the process.

## 7. FIVE-LINK BIPED ($n = 5$)

In this section, we design a control law for four inter-link torques to stabilize the vertical posture of the five-link planar biped without feet in single support.

### *7.1. Control design for the five-link biped*

In this case, matrix $A$ of the linear model has three real positive eigenvalues, three real negative eigenvalues and two pairs of complex conjugate eigenvalues with negative real parts. Let $\lambda_i$, $(i = 1, 2, 3)$ be the real positive eigenvalues. Similarly to the previous cases let us consider the first three lines of system (7) corresponding to these three positive eigenvalues,

$$\begin{aligned} \dot{y}_1 &= \lambda_1 y_1 + d_{11}\Gamma_1 + d_{12}\Gamma_2 + d_{13}\Gamma_3 + d_{14}\Gamma_4 \\ \dot{y}_2 &= \lambda_2 y_2 + d_{21}\Gamma_1 + d_{22}\Gamma_2 + d_{23}\Gamma_3 + d_{24}\Gamma_4 \\ \dot{y}_3 &= \lambda_3 y_3 + d_{31}\Gamma_1 + d_{32}\Gamma_2 + d_{33}\Gamma_3 + d_{34}\Gamma_4 \end{aligned} \tag{19}$$

Now we want to suppress the instability of the three variables $y_i$, $(i = 1, 2, 3)$. This can be achieved by choosing controls $\Gamma_i$ $(i = 1, 2, 3, 4)$ such that

$$\begin{aligned} d_{11}\Gamma_1 + d_{12}\Gamma_2 + d_{13}\Gamma_3 + d_{14}\Gamma_4 &= \gamma_1 y_1 \\ d_{21}\Gamma_1 + d_{22}\Gamma_2 + d_{23}\Gamma_3 + d_{24}\Gamma_4 &= \gamma_2 y_2 \\ d_{31}\Gamma_1 + d_{32}\Gamma_2 + d_{33}\Gamma_3 + d_{34}\Gamma_4 &= \gamma_3 y_3 \end{aligned} \tag{20}$$

with the three conditions,

$$\lambda_i + \gamma_i < 0 \quad (i = 1, 2, 3) \tag{21}$$

System (20) contains three algebraic equations with four unknown variables $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$. Therefore, system (20) has an infinite number of solutions. We find a unique

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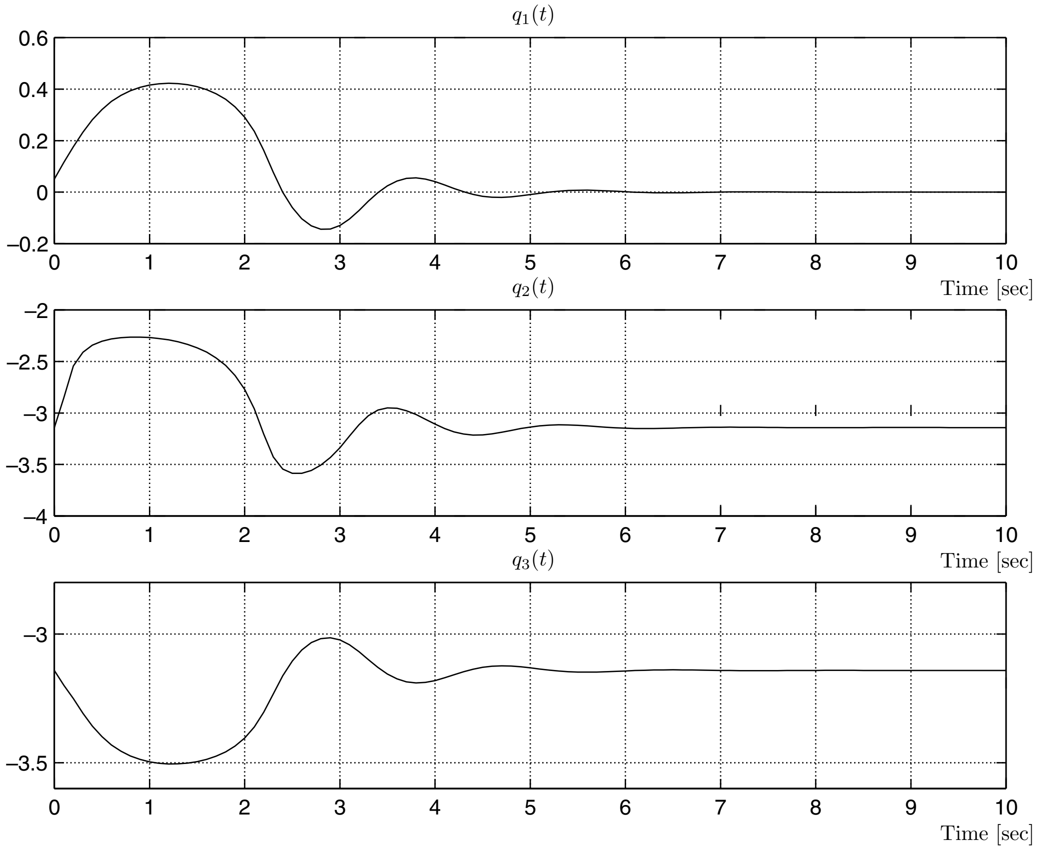


Fig. 6. Three-link biped: Angles $q_1(t)$, $q_2(t)$ and $q_3(t)$.

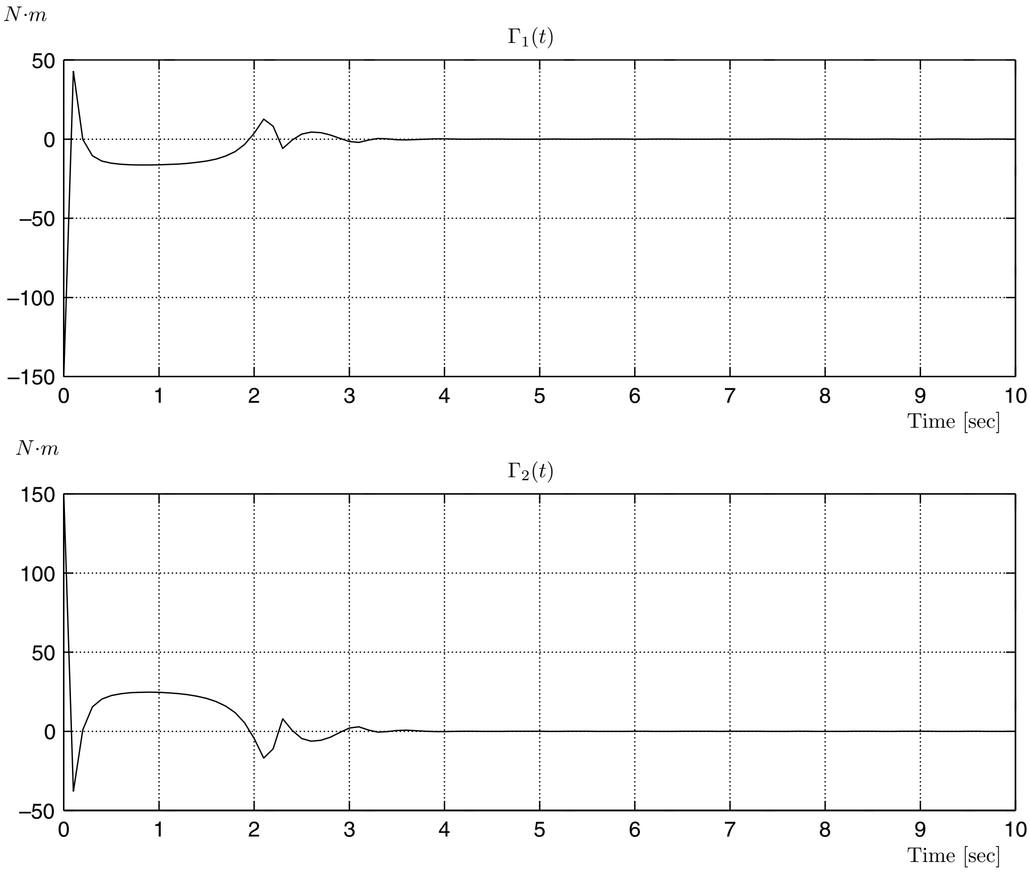


Fig. 7. Three-link biped: Torques in the two inter-link joints.

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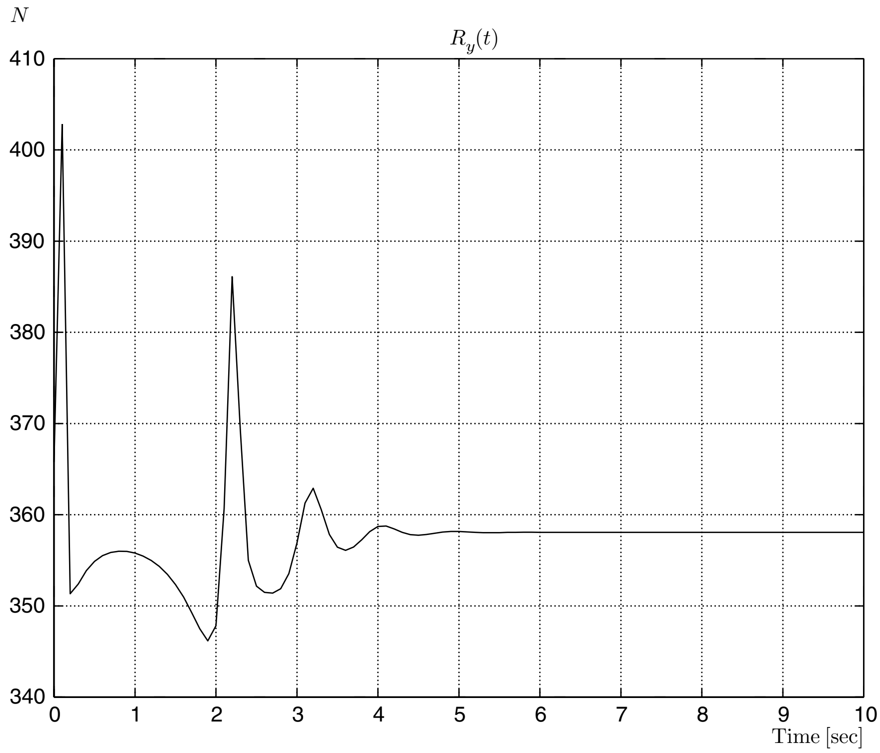


Fig. 8. Three-link biped: Vertical component of the ground reaction in the stance leg tip.

solution of system (20) at each step of the control process by minimizing the following functional

$$J = \max_{i=1,\ldots,4} |\Gamma_i| \tag{22}$$

This yields the torques $\Gamma_i^0(y_1, y_2, y_3)$ $(i = 1, 2, 3, 4)$.

Finally, we apply the torques $\Gamma_i \Gamma_i(y_1, y_2, y_3)$, $(i = 1, 2, 3, 4)$, limited by the same value $U$,

$$\Gamma_i = \Gamma_i(y_1, y_2, y_3) = \begin{cases} U, & \textit{if } \Gamma_i^0(y_1, y_2, y_3) \geq U \\ \Gamma_i^0(y_1, y_2, y_3), & \textit{if } \left|\Gamma_i^0(y_1, y_2, y_3)\right| \leq U \\ -U, & \textit{if } \Gamma_i^0(y_1, y_2, y_3) \leq -U \end{cases} \tag{23}$$

The torques in (23) ensure asymptotical stability of the equilibrium $y_i = 0$ $(i = 1, 2, 3)$ of the first, second and third equations of system (7) (i.e., of system (19)), if the initial state belongs to some attraction region $V$ in the three-dimensional space $y_1$, $y_2$, $y_3$. However, the equilibrium point $y_i = 0$ $(i = 4, \ldots, 10)$ of the fourth – tenth equations of system (7) is also asymptotically stable for all initial conditions $y_i(0)$ $(i = 4, \ldots, 10)$ because $Re\lambda_i < 0$ for $i = 4, \ldots, 10$. Note that $y_i(t) \to 0$ $(i = 1, 2, 3)$ as $t \to \infty$, if $(y_1(0),\ y_2(0),\ y_3(0)) \subset V$. Additionally, according to expressions (20)–(23) $\Gamma_i(t) \to 0$ $(i = 1, 2, 3, 4)$ as $t \to \infty$. Thus, under control (23) and with conditions (21), the origin $x = 0$ is an asymptotically stable equilibrium of system (5) with some attraction region $V \subset Q$.

Variables $y_i$ $(i = 1, 2, 3)$ depend on the original variables from vector $x$ according to the transformation $y = S^{-1}x$. Due to this, formula (23) defines the control feedback, as a function of the vector $x$ of the original variables.

### *7.2. Numerical results for the five-link biped*

In the simulation with the control law (23) applied to nonlinear model (1) of the five-link biped, using the parameters defined in Section 2.2 it is possible to start with an initial configuration $q(0) = [1.7°; 180°; 180°; 180°; 180°]$. Angles $q_1(t)$, $q_2(t)$, $q_3(t)$, $q_4(t)$ and $q_5(t)$ as functions of time are shown in Figure 9. At the end the biped is steered to the vertical posture. All the potential of the actuator is applied at initial time, as shown in Figure 10. Figure 11 shows that during the stabilization process of the five-link biped, the vertical component of the ground reaction is always positive and equals the weight of the biped at the end of the process.

## 8. CONCLUSION

In this paper, we define strategies to stabilize the equilibrium vertical posture of a two-link, a three-link and a five-link biped, explicitly taking into account the limit imposed on the torque amplitude. We use the Jordan form of the linear model of the bipeds to extract the unstable modes that we have to suppress with the feedback control. For the case of the two-link biped, the control law is optimal and the attraction domain for the linear system is as large as possible, *i.e.* it coincides with the controllability domain. For the five-link biped several choices of torques are allowable because we have four torques and three unstable modes. Therefore, we define a criteria to compute these torques. All the numerical results in this paper are realistic. A perspective for the case

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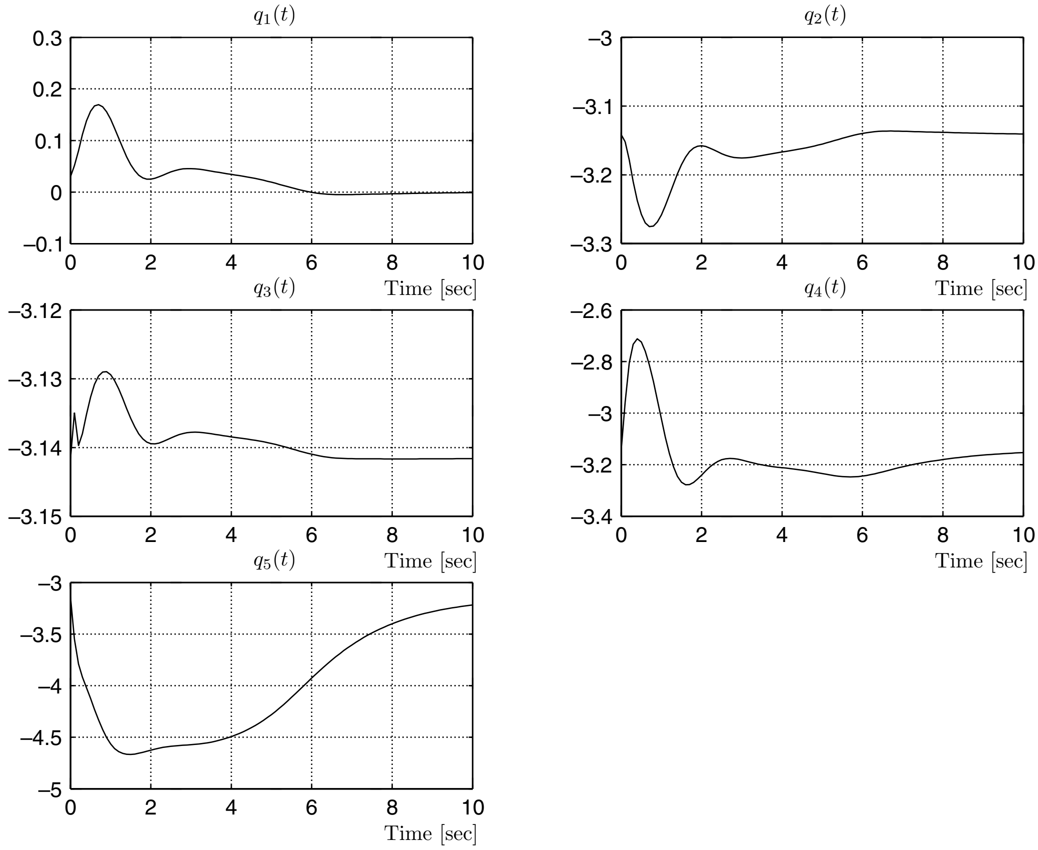


Fig. 9. Five-link biped: Angles $q_1(t)$, $q_2(t)$, $q_3(t)$, $q_4(t)$ and $q_5(t)$.

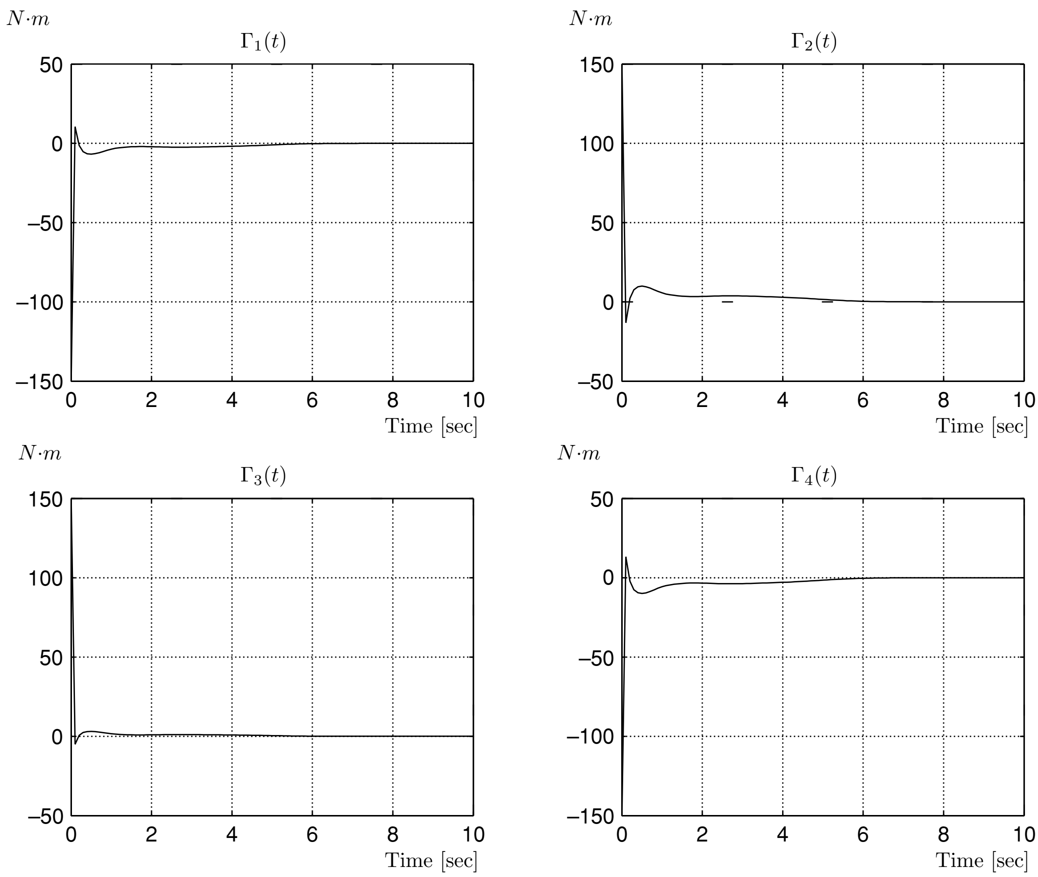


Fig. 10. Five-link biped: Torques in the four inter-link joints.

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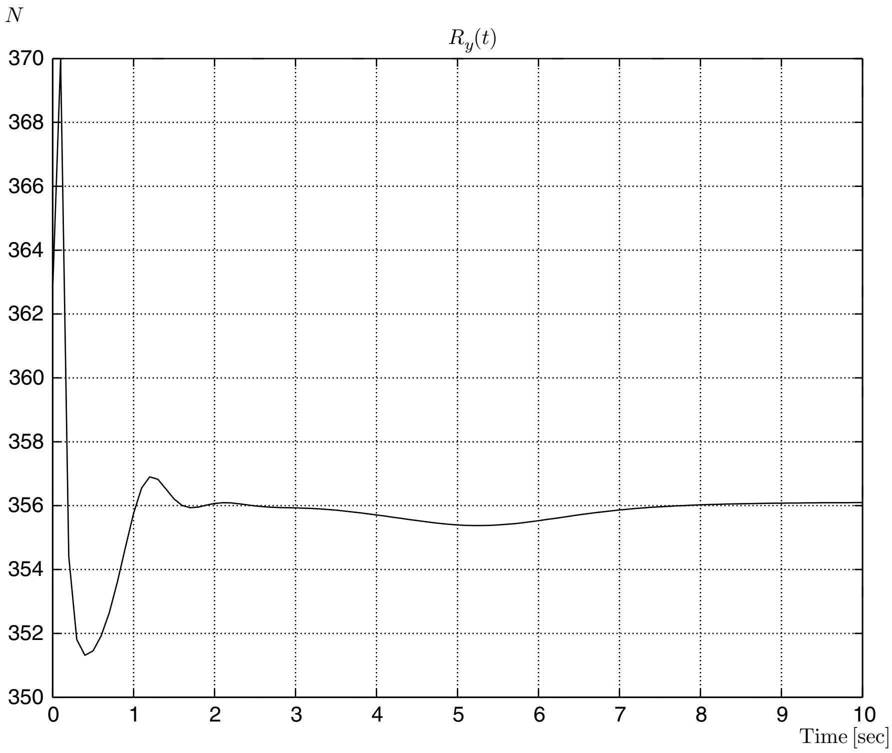


Fig. 11. Five-link biped: Vertical component of the ground reaction in the stance leg tip.

of the three-link biped is to define a control law, for which the attraction domain is as large as allowable.

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