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A quantitative interpretation of the frequent hypercyclicity criterion

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Abstract. We give a quantitative interpretation of the frequent hypercyclicity criterion. Actually we show that an operator which satisfies the frequent hypercyclicity criterion is necessarily A-frequently hypercyclic, where A refers to some weighted densities sharper than the natural lower density. In that order, we exhibit different scales of weighted densities that are of interest to quantify the 'frequency' measured by the frequent hypercyclicity criterion. Moreover, we construct an example of a unilateral weighted shift which is frequently hypercyclic but not A-frequently hypercyclic on a particular scale.

1. *Introduction*

The notion of frequent hypercyclicity was introduced in the context of linear dynamics by Bayart and Grivaux in 2006 [**1**, **2**]. This latter is now a central notion in that field and is highly connected to combinatorics, number theory and ergodic theory. Let X be a metrizable and complete topological vector space and $L(X)$ be the space of continuous linear operators on X. An operator $T \in L(X)$ is said to be *hypercyclic* if there exists $x \in X$ such that for any non-empty open set $U \subset X$, the return set $\{n \geq 0 : T^n x \in U\}$ is non-empty or equivalently infinite. Such a vector x is called a hypercyclic vector for T. Furthermore, an operator T is called *frequently hypercyclic* if there exists $x \in X$ such that for any non-empty open set $U \subset X$, the set of integers n satisfying $T^n x \in U$ has positive lower density, i.e.

$$\liminf_{N \to +\infty} \frac{\#\{k \leq N : T^k x \in U\}}{N} > 0,$$

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where as usual # denotes the cardinality of the corresponding set. Thus, the notion of frequent hypercyclicity extends the classical hypercyclicity and appraises *how often* the orbit of a hypercyclic vector visits every non-empty open set. In the sequel we denote by $\mathbb{N}$ the set of positive integers and for any $x \in X$ and any subset $U \subset X$ we set $N(x, U) := \{n \in \mathbb{N} : T^n x \in U\}$. Given a subset $E \subset \mathbb{N}$, we define its *lower and upper* densities respectively by

$$\underline{d}(E) = \liminf_{N\to+\infty} \frac{\#\{k \leq N : k \in E\}}{N} \quad \text{and} \quad \overline{d}(E) = \limsup_{N\to+\infty} \frac{\#\{k \leq N : k \in E\}}{N}.$$

In other words, an operator $T \in L(X)$ is hypercyclic (respectively frequently hypercyclic) if there exists $x \in X$ such that for any non-empty open set $U \subset X$, the set $N(x, U)$ is non-empty (respectively has positive lower density). To prove that an operator is hypercyclic, we have at our disposal the so-called hypercyclicity criterion (see [**4**, **9**] and the references therein). In the same spirit, Bayart and Grivaux stated the frequent hypercyclicity criterion, which ensures that an operator is frequently hypercyclic [**2**]. Let us recall it here.

THEOREM 1.1. *Let T be an operator on a separable Fréchet space X. If there are a dense subset X_0 of X and a map $S : X_0 \to X_0$ such that, for any $x \in X_0$:*
(i) $\sum_{n=0}^{+\infty} T^n x$ *converges unconditionally;*
(ii) $\sum_{n=0}^{+\infty} S^n x$ *converges unconditionally;*
(iii) $TSx = x$,
then T is frequently hypercyclic.

We already know that the above result does not characterize frequently hypercyclic operators. Indeed, Bayart and Grivaux have exhibited a frequently hypercyclic weighted shift on c_0 that does not satisfy this criterion [**3**]. A natural question arises: what does the frequent hypercyclicity criterion really quantify? In order to answer this question, Bès *et al* recently generalized the notion of hypercyclic operators by introducing the concept of $\mathcal{A}$-frequent hypercyclicity, where $\mathcal{A}$ refers to a family of subsets of $\mathbb{N}$ satisfying suitable conditions [**6**]. In particular, $\mathcal{A}$ has to satisfy the following separation condition:

$\mathcal{A}$ contains a sequence (A_k) of disjoint sets such that for any $j \in A_k$, any $j' \in A_{k'}$, $j \neq j'$, we have $|j' - j| \geq \max(k, k')$.

In this abstract framework, they also obtained an $\mathcal{A}$-frequent hypercyclicity criterion and proved that the frequent hypercyclicity criterion has very strong consequences in the sense that if T satisfies the frequent hypercyclicity criterion, then T also satisfies the $\mathcal{A}$-frequent hypercyclicity criterion for any suitable family $\mathcal{A}$. Bonilla and Grosse-Erdmann also studied specific notions related to the concept of $\mathcal{A}$-frequent hypercyclicity in the article [**7**].

On the other hand, the notion of frequent hypercyclicity measures the frequency and the length of the intervals when iterates of a hypercyclic vector visit every non-empty open set in a very specific way that is given by the natural density. Actually, there are many types and notions of densities different from the natural one. Our goal is to give quantified consequences to the frequent hypercyclicity criterion in terms of weighted densities. To that purpose, we consider a special kind of lower weighted densities, generalizing the natural one but sharper than that one, by using the formalism of matrix summability methods. For such a matrix A, we use the concepts of A-density and A-frequent hypercyclicity (see Definitions 2.2 and 2.11 below). These general densities were already used in the context of linear dynamics to study frequently universal series [**10**]. In the present paper, we show that the frequent hypercyclicity criterion gives a stronger conclusion than frequent hypercyclicity, which we quantify thanks to explicit weighted densities on different scales. We refer the reader to Proposition 3.5 and Theorem 4.12 below. Therefore, an operator which satisfies the frequent hypercyclicity criterion is necessarily A-frequently hypercyclic, where A refers to some weighted densities sharper than the natural lower density.

Let us return to the formalism of $\mathcal{A}$-frequent hypercyclicity. For instance, $\mathcal{A}$ could be a family of subsets with positive given lower weighted density satisfying the aforementioned separation property. However, from [**6**], there is an underlying question: does there exist a frequently hypercyclic operator not being $\mathcal{A}$-frequently hypercyclic? We give a positive answer by constructing a unilateral weighted shift on c_0 which is frequently hypercyclic but not A-frequently hypercyclic with respect to some A-densities covered by the criterion (see Theorem 5.4 below).

The paper is organized as follows. In §2, we introduce some densities that will be of interest in the sequel and some properties of these densities. Section 3 is devoted to an improvement of the frequent hypercyclicity criterion for a certain scale of weighted densities. In §4, we modify this proof in order to obtain a stronger result to the criterion. Finally, in §5, we exhibit a new example, inspired by [**5**], of an operator which is frequently hypercyclic although it does not satisfy the frequent hypercyclicity criterion. To ensure this latter property, we will show that this operator is not A-frequently hypercyclic for some suitable matrix A.

2. *Densities: preliminary results*

In this section, we state some definitions and results we shall need throughout the paper. Let us first introduce the concept of a summability matrix and its connections with some kind of densities on subsets of $\mathbb{N}$.

Definition 2.1. A *summability matrix* is an infinite matrix $M = (m_{n,k})$ of complex numbers.

Let us recall that, if (x_n) is a sequence and $M = (m_{n,k})$ is a summability matrix, then by Mx we denote the sequence $((Mx)_1, (Mx)_2, \ldots)$, where $(Mx)_n = \sum_{k=1}^{+\infty} m_{n,k}x_k$. The matrix M is called *regular* if the convergence of x to c implies the convergence of Mx to c. By a well-known result of Toeplitz (see for instance [**11**]), M is regular if and only if the following three conditions hold:

$$\begin{cases} \text{(i)} \lim_{n\to+\infty} m_{n,k} = 0 \quad \text{for all } k \in \mathbb{N}; \\ \text{(ii)} \lim_{n\to+\infty} \sum_{k\geq 1} m_{n,k} = 1; \\ \text{(iii)} \sup_n \sum_{k\geq 1} |m_{n,k}| < \infty. \end{cases} \tag{2.1}$$

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Freedman and Sember showed that every regular summability matrix M with non-negative real coefficients defines a density $\underline{d}_M$ on subsets of $\mathbb{N}$, called the lower M-density [**8**].

Definition 2.2. For a regular matrix $M = (m_{n,k})$ with non-negative coefficients and a set $E \subset \mathbb{N}$, the lower M-density of E, denoted $\underline{d}_M(E)$, is defined by

$$\underline{d}_M(E) = \liminf_{n\to+\infty} \sum_{k=1}^{+\infty} m_{n,k} \mathbb{1}_E(k)$$

and the associated upper M-density, denoted $\overline{d}_M(E)$, is defined by

$$\overline{d}_M(E) = 1 - \underline{d}_M(\mathbb{N}\backslash E).$$

Remark 2.3. For a non-negative regular matrix $M = (m_{n,k})$, [**8**, Proposition 3.1] ensures that the upper M-density of any set $E \subset \mathbb{N}$ is given by

$$\overline{d}_M(E) = \limsup_{n\to+\infty} \sum_{k=1}^{+\infty} m_{n,k} \mathbb{1}_E(k).$$

Let $(\alpha_k)_{k\geq 1}$ be a non-negative sequence such that $\sum_{k=1}^{n} \alpha_k \to +\infty$ as n tends to ∞. Then we deal with the special case of A-density, where we write $A = (\alpha_k / \sum_{j=1}^{n} \alpha_j)$, when $A = (\alpha_{n,k})$ with $\alpha_{n,k} = \alpha_k / \sum_{j=1}^{n} \alpha_j$ for $1 \leq k \leq n$ and $\alpha_{n,k} = 0$ for $k > n$. It is easy to check that A is a non-negative regular summability matrix. In summability theory the transformation given by $x = (x_n) \mapsto Ax$ is called the Riesz mean (A, α_n). Here the associated A-density can be viewed as a weighted density with respect to the non-negative weight sequence $(\alpha_k)_{k\geq 1}$.

Definition 2.4. A summability matrix $A = (\alpha_k / \sum_{j=1}^{n} \alpha_j)$ as above will be called an *admissible* matrix. We define its summatory function φ_α as follows: $\varphi_\alpha : \mathbb{N} \to \mathbb{R}_+$, $\varphi_\alpha(n) = \sum_{k\leq n} \alpha_k$.

Example 2.5.

(1) If $\alpha_k = 1$, $k = 1, 2, \ldots,$ then the summability matrix A is the well-known Cesàro matrix and $\underline{d}_A$ is the natural lower density.

(2) If $\alpha_k = 1/k$, $k = 1, 2, \ldots,$ $\underline{d}_A$ is the so-called lower logarithmic density, which is derived from the well-known logarithmic summability method. We have $\varphi_\alpha(k) \sim \log(k)$ as k tends to $+\infty$.

(3) The special case $\alpha_k = k^r$, $r \geq -1$, for $k = 1, 2, \ldots,$ generalizes both the natural density ($r = 0$) and the logarithmic density ($r = -1$). Clearly, we have $\varphi_\alpha(k) \sim k^{r+1}/(r+1)$ as k tends to $+\infty$, when $r > -1$.

(4) If $\alpha_k = e^{k^r}$, $0 < r < 1$, for $k = 1, 2, \ldots,$ an easy calculation gives $\varphi_\alpha(k) \sim (k^{1-r}/r)e^{k^r}$ as k tends to $+\infty$.

(5) If $\alpha_k = e^k$, for $k = 1, 2, \ldots,$ then $\varphi_\alpha(k) \sim (e/(e-1))e^k$ as k tends to $+\infty$.

(6) If $\alpha_1 = 1$ and $\alpha_k = e^{k/\log^r(k)}$, $r > 0$, for $k = 2, 3, \ldots,$ a summation by parts gives $\varphi_\alpha(k) \sim \log^r(k) e^{k/\log^r(k)}$ as k tends to $+\infty$.

(7) Let h_s be the real function defined by $h_s = \log . \log^{(s)}$, with $\log^{(s)} = \log \circ \log \cdots \circ \log$, where log appears s times. If $\alpha_k = e^{k/h_s(k)}$, $l \in \mathbb{N}$, $s \geq 2$, for k large enough, again a summation by parts gives $\varphi_\alpha(k) \sim h_s(k) e^{k/h_s(k)}$ as k tends to $+\infty$.

In the following sections, we shall use the following definitions connected to Example 2.5.

Definition 2.6. We denote by:

(1) C_r the admissible matrix $C_r = (k^r / \sum_{j=1}^{n} j^r)$, $r \geq -1$;

(2) A_r the admissible matrix $A_r = (e^{k^r} / \sum_{j=1}^{n} e^{j^r})$, $r \geq 0$;

(3) B_r the admissible matrix $B_r = (\alpha_k / \sum_{j=1}^{n} \alpha_j)$, with $\alpha_1 = 1$ and $\alpha_k = e^{k/\log^r(k)}$ for $k \geq 2$, $r \geq 0$;

(4) let h_s be the real function defined by $h_s = \log . \log^{(s)}$, with $\log^{(s)} = \log \circ \log \cdots \circ \log$, where log appears s times. We denote by $\widetilde{B}_s$ the admissible matrix $\widetilde{B}_s = (\alpha_k / \sum_{j=1}^{n} \alpha_j)$, with $\alpha_k = e^{k/h_s(k)}$ for k large enough and $s \geq 2$.

For any subset $E \subset \mathbb{N}$, we can write E as a strictly increasing sequence (n_k) of positive integers. It is well known that $\underline{d}(E) = \liminf_{k \to +\infty} (k/n_k)$, which allows us to deduce the following simple fact: $\underline{d}(E) > 0$ if and only if the sequence (n_k/k) is bounded [9]. The following lemma extends this remark to suitable A-densities.

LEMMA 2.7. *Let (α_k) be a non-negative sequence such that $\sum_{k \in \mathbb{N}} \alpha_k = +\infty$. Assume that the sequence $(\alpha_n / \sum_{j=1}^{n} \alpha_j)$ converges to zero as n tends to $+\infty$. Let (n_k) be an increasing sequence of integers forming a subset $E \subset \mathbb{N}$. Then we have*

$$\underline{d}_A(E) = \liminf_{k \to +\infty} \left(\frac{\sum_{j=1}^{k} \alpha_{n_j}}{\sum_{j=1}^{n_k} \alpha_j} \right),$$

where $\underline{d}_A$ is the A-density given by the summability matrix $A = (\alpha_k / \sum_{j=1}^{n} \alpha_j)$.

Proof. Let us consider $n_k \leq N < n_{k+1}$; then

$$\frac{\sum_{j=1}^{k+1} \alpha_{n_j}}{\sum_{j=1}^{n_{k+1}} \alpha_j} - \frac{\alpha_{n_{k+1}}}{\sum_{j=1}^{n_{k+1}} \alpha_j} = \frac{\sum_{j=1}^{k} \alpha_{n_j}}{\sum_{j=1}^{n_{k+1}} \alpha_j} \leq \frac{\sum_{n_j \leq N} \alpha_{n_j}}{\sum_{j=1}^{N} \alpha_j} \leq \frac{\sum_{j=1}^{k} \alpha_{n_j}}{\sum_{j=1}^{n_k} \alpha_j}.$$

Thus, we deduce that

$$\underline{d}_A(E) = \liminf_{N \to +\infty} \left(\frac{\sum_{n_j \leq N} \alpha_{n_j}}{\sum_{j=1}^{N} \alpha_j} \right) = \liminf_{k \to +\infty} \left(\frac{\sum_{j=1}^{k} \alpha_{n_j}}{\sum_{j=1}^{n_k} \alpha_j} \right). \qquad \square$$

In the present paper, we are mainly interested in sharper A-densities than the classical natural density. From this point of view, the following lemma gives some conditions to ensure that the sequence (α_k) leads to a sharper density.

LEMMA 2.8. *Let (α_k) and (β_k) be non-negative sequences such that $\sum_{k \in \mathbb{N}} \alpha_k = \sum_{k \in \mathbb{N}} \beta_k = +\infty$. Assume that the sequence (α_k/β_k) is eventually decreasing to zero. Let $A = (\alpha_k / \sum_{j=1}^{n} \alpha_j)$ and $B = (\beta_k / \sum_{j=1}^{n} \beta_j)$ be the associated admissible matrices. Then, for every subset $E \subset \mathbb{N}$, we have*

$$\underline{d}_B(E) \leq \underline{d}_A(E) \leq \overline{d}_A(E) \leq \overline{d}_B(E).$$

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Proof. Let E be a subset of $\mathbb{N}$. For every $n \geq 1$, let us define $\Lambda_E^\alpha(n) = \sum_{k=1}^n \alpha_k \mathbb{1}_E(k)$ (respectively $\Lambda_E^\beta(n) = \sum_{k=1}^n \beta_k \mathbb{1}_E(k)$). In particular, one may observe that $\Lambda_{\mathbb{N}}^\alpha = \varphi_\alpha$. Now let $N \geq 1$ be an integer such that the sequence $(\alpha_k/\beta_k)_{k\geq N}$ is decreasing. Then, for every $n \geq N+1$, we have

$$\begin{aligned}\sum_{k=N+1}^{n} \alpha_k \mathbb{1}_E(k) &= \sum_{k=N+1}^{n-1} \Lambda_E^\beta(k)\left(\frac{\alpha_k}{\beta_k} - \frac{\alpha_{k+1}}{\beta_{k+1}}\right) + \Lambda_E^\beta(n)\frac{\alpha_n}{\beta_n} - \Lambda_E^\beta(N)\frac{\alpha_{N+1}}{\beta_{N+1}} \\ &= \sum_{k=N+1}^{n-1} \frac{\Lambda_E^\beta(k)}{\varphi_\beta(k)}\varphi_\beta(k)\left(\frac{\alpha_k}{\beta_k} - \frac{\alpha_{k+1}}{\beta_{k+1}}\right) + \frac{\Lambda_E^\beta(n)}{\varphi_\beta(n)}\varphi_\beta(n)\frac{\alpha_n}{\beta_n} \\ &\quad - \Lambda_E^\beta(N)\frac{\alpha_{N+1}}{\beta_{N+1}}.\end{aligned}$$

Moreover, since (α_k/β_k) is a non-negative decreasing sequence and $\sum \alpha_k = +\infty$, we deduce that

$$\begin{aligned}\overline{d}_A(E) &= \limsup_{n\to+\infty}\left[(\varphi_\alpha(n))^{-1}\left(\sum_{k=1}^{N}\alpha_k\mathbb{1}_E(k) + \sum_{k=N+1}^{n}\alpha_k\mathbb{1}_E(k)\right)\right] \\ &= \limsup_{n\to+\infty}\left[(\varphi_\alpha(n))^{-1}\left(\sum_{k=N+1}^{n-1}\frac{\Lambda_E^\beta(k)}{\varphi_\beta(k)}\varphi_\beta(k)\left(\frac{\alpha_k}{\beta_k} - \frac{\alpha_{k+1}}{\beta_{k+1}}\right) + \frac{\Lambda_E^\beta(n)}{\varphi_\beta(n)}\varphi_\beta(n)\frac{\alpha_n}{\beta_n}\right)\right] \\ &\leq \sup_{k>N}\left(\frac{\Lambda_E^\beta(k)}{\varphi_\beta(k)}\right)\limsup_{n\to+\infty}\left[(\varphi_\alpha(n))^{-1}\left(\sum_{k=N+1}^{n-1}\varphi_\beta(k)\left(\frac{\alpha_k}{\beta_k} - \frac{\alpha_{k+1}}{\beta_{k+1}}\right) + \varphi_\beta(n)\frac{\alpha_n}{\beta_n}\right)\right].\end{aligned}$$

Since

$$\sum_{k=N+1}^{n-1}\varphi_\beta(k)\left(\frac{\alpha_k}{\beta_k} - \frac{\alpha_{k+1}}{\beta_{k+1}}\right) + \varphi_\beta(n)\frac{\alpha_n}{\beta_n} = \varphi_\beta(N+1)\frac{\alpha_{N+1}}{\beta_{N+1}} + \sum_{k=N+2}^{n}\alpha_k,$$

we get

$$\overline{d}_A(E) \leq \sup_{k>N}\left(\frac{\Lambda_E^\beta(k)}{\varphi_\beta(k)}\right).$$

Hence, letting $N \to +\infty$, we obtain $\overline{d}_A(E) \leq \overline{d}_B(E)$.

The other inequality is obtained using the relations $\underline{d}_A(E) = 1 - \overline{d}_A(\mathbb{N}\backslash E)$ and $\underline{d}_B(E) = 1 - \overline{d}_B(\mathbb{N}\backslash E)$. □

From now on, we are interested in densities given by special admissible matrices given in Definition 2.6. In this case, Lemma 2.8 leads to the following inequalities.

LEMMA 2.9. *For every subset $E \subset \mathbb{N}$ and for any $0 < r \leq r'$, $0 < s \leq s' < 1$, $1 < t \leq t'$, $2 \leq l \leq l'$, we have*

$$\underline{d}_{A_1}(E) = \underline{d}_{B_0}(E) \leq \underline{d}_{\widetilde{B}_{l'}}(E) \leq \underline{d}_{\widetilde{B}_l}(E) \leq \underline{d}_{B_t}(E) \leq \underline{d}_{B_{t'}}(E)$$

and

$$\underline{d}_{B_{t'}}(E) \leq \underline{d}_{A_{s'}}(E) \leq \underline{d}_{A_s}(E) \leq \underline{d}_{C_{r'}}(E) \leq \underline{d}_{C_r}(E) \leq \underline{d}(E).$$

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Moreover, observe that a subset E of $\mathbb{N}$ possesses a strictly positive natural lower density if and only if it has a strictly positive lower C_r-density for any $r > -1$.

LEMMA 2.10. *Let $r > -1$. Then, for every subset $E \subset \mathbb{N}$, the following assertions are equivalent:*
(i) $\underline{d}_{C_r}(E) > 0$;
(ii) $\underline{d}(E) > 0$.

Proof. Let $(n_k) \subset \mathbb{N}$ be the increasing sequence of elements of E. We divide the proof into two cases.

Case $r \geq 0$: Lemma 2.8 gives $\underline{d}_{C_r}(E) \leq \underline{d}(E)$; hence, (i) $\Rightarrow$ (ii). For the other implication, assume that $\underline{d}(E) > 0$. This means that the sequence (n_j/j) is bounded (see Lemma 2.7). We deduce that there exists an integer $M \geq 1$ such that for every $k \in \mathbb{N}$, $k \leq n_k \leq Mk$ and

$$\sum_{j=1}^{n_k} j^r \leq \sum_{j=1}^{Mk} j^r \leq \frac{(Mk+1)^{r+1}}{r+1} \leq M^{r+1}\frac{(k+1)^{r+1}}{r+1}.$$

Using the fact that $\sum_{j=1}^{k} j^r \sim k^{r+1}/(r+1) \sim (k+1)^{r+1}/(r+1)$ as k tends to $+\infty$ (cf. Example 2.5), we deduce that there exists $C > 0$ such that

$$\sum_{j=1}^{n_k} j^r \leq CM^{r+1}\sum_{j=1}^{k} j^r.$$

Therefore, we have

$$\sum_{j=1}^{n_k} j^r \leq CM^{r+1}\sum_{j=1}^{k} (n_j)^r$$

and

$$\liminf_{k\to+\infty}\left(\frac{\sum_{j=1}^{k}(n_j)^r}{\sum_{j=1}^{n_k} j^r}\right) > 0,$$

which is sufficient to conclude this case thanks to Lemma 2.7.

Case $-1 < r < 0$: As in the previous case, Lemma 2.8 gives the implication (ii) $\Rightarrow$ (i) because $\underline{d}_{C_r}(E) \geq \underline{d}(E)$. For the converse, assume that $\underline{d}_{C_r}(E) > 0$. According to Lemma 2.7, there exists $C > 0$ such that

$$\sum_{j=1}^{n_k} j^r \leq C\sum_{j=1}^{k} (n_j)^r.$$

Using the inequality $j \leq n_j$ and since $r < 0$, we get

$$\frac{(n_k+1)^{r+1}}{r+1} - \frac{1}{r+1} \leq \sum_{j=1}^{n_k} j^r \leq C\sum_{j=1}^{k}(n_j)^r \leq C\sum_{j=1}^{k} j^r \leq C\left(\frac{k^{r+1}}{r+1} - \frac{1}{r+1} + 1\right).$$

The positivity of $r+1$ now ensures that the sequence (n_k/k) is bounded and $\underline{d}(E) > 0$. □

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Finally, let us also extend the definition of frequently hypercyclic operators using the general notion of A-densities introduced before.

Definition 2.11. Let $A = (\alpha_k/\sum_{j=1}^{n} \alpha_j)$ be an admissible matrix. Using the notation of §1, an operator $T \in L(X)$ is said to be *A-frequently hypercyclic* if there exists $x \in X$ such that for any non-empty open set $U \subset X$, the set $N(x, U)$ has positive lower A-density.

3. *Frequent hypercyclicity criterion: classical construction*

According to Lemma 2.10, a frequently hypercyclic operator is necessarily C_r-frequently hypercyclic for any $r > -1$. So, even if this is obvious, the frequent hypercyclicity criterion allows us to obtain the C_r-frequent hypercyclicity too. A careful examination of the well-known proof of this criterion leads to a more precise result. Indeed, in the classical proof of the frequent hypercyclicity criterion the following constructive lemma plays a prominent role [**9**, Lemma 9.5].

LEMMA 3.1. *There exist pairwise disjoint subsets $A(l, \nu)$, $l, \nu \geq 1$, of $\mathbb{N}$ of positive lower density such that, for any $n \in A(l, \nu)$ and $m \in A(k, \mu)$, we have that $n \geq \nu$ and*

$$|n - m| \geq \nu + \mu \quad \textit{if } n \neq m.$$

The proof of this result is based on a specific partition of $\mathbb{N}$ using the dyadic representation $n = \sum_{j=0}^{+\infty} a_j 2^j = (a_0, a_1, \ldots)$ of any positive integer. Actually the authors defined the sets $I(l, \nu)$, $l, \nu \geq 1$, as the sets of all $n \in \mathbb{N}$ whose dyadic representation has the form $n = (0, \ldots, 0, 1, \ldots, 1, 0, *)$ with $l - 1$ leading zeros, exactly followed by ν ones, then one zero and an arbitrary remainder. Let $\delta_k = \nu$ if $k \in I(l, \nu)$ for some $l \geq 1$. Then they constructed the following strictly increasing sequence (n_k) of positive integers by setting

$$n_k = 2\sum_{i=1}^{k-1} \delta_i + \delta_k, \qquad k \geq 1.$$

This construction clearly ensures that for any integers i, j, with $i \neq j$, the separation condition stated in Lemma 3.1 holds, that is,

$$|n_i - n_j| \geq \delta_i + \delta_j.$$

Finally, they defined the sets $A(l, \nu) = \{n_k;\, k \in I(l, \nu)\}$ and they proved that these sets have positive lower density since (n_k) does and the sets $I(l, \nu)$ are arithmetic sequences. Actually we are going to prove that the sequence (n_k) has positive lower B_2-density. To do that, we start by giving an exact formula for this sequence that will allow us to obtain easily its asymptotic behaviour. We obtain the following result, whose proof will be given later (see Lemma 4.8 below).

LEMMA 3.2. *If $k = 2^n + \sum_{i=0}^{n-1} \alpha_i 2^i$, with $\alpha_i \in \{0; 1\}$ for every $0 \leq i \leq n - 1$, then*

$$n_k = 4k - 2\bigg(\sum_{i \in I_k} L_i(i + 1)\bigg) - \delta_k,$$

where I_k stands for the set of integers i such that α_i is the first non-zero integer of a block (of consecutive non-zero coefficients) having length L_i in the dyadic decomposition of k.

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From this lemma, we deduce the following estimate using the same notation.

PROPOSITION 3.3. *The sequence* (n_k) *satisfies the following estimate:* $n_k - 4k = O(\log^2(k))$. *Moreover, this estimate is optimal in the following sense: there exists an increasing subsequence* (λ_n) *of positive integers such that the sequence* $(n_{\lambda_j} - 4\lambda_j)/\log^2(\lambda_j)$ *converges to a non-zero real number.*

Proof. According to Lemma 3.2, we have, for $k = 2^l + \sum_{i=0}^{l-1} \alpha_i 2^i$, with $\alpha_i \in \{0; 1\}$ for every $0 \le i \le l-1$,

$$n_k = 4k - 2\bigg(\sum_{i\in I_k} L_i(i+1)\bigg) - \delta_k,$$

where $I_k = \{i \in \mathbb{N};\ \alpha_i \neq 0 \text{ and } \alpha_{i-1} = 0\}$, with the conventions $\alpha_{-1} = 0$, $\alpha_l = 1$ and, for $i \in I_k$, $L_i = \min\{j;\ \alpha_{i+j} = 0\}$. Obviously, we deduce that

$$n_k \le 4k - 2\log_2(k) - 1.$$

Notice that we have the equality $n_k = 4k - 2(\log_2(k) + 1) - 1$ for $k = 2^l$.

On the other hand, we can write

$$\sum_{i\in I_k} L_i(i+1) = \sum_{i_1<i_2<\cdots<i_{m_k}} L_{i_j}(i_j+1),$$

where $I_k = \{i_1 < i_2 < \cdots < i_{m_k}\}$. Observe that we have

$$i_n + L_{i_n} + 1 \le i_{n+1} \quad \text{for } n = 1, \ldots, m_k - 1 \quad \text{and} \quad L_{i_{m_k}} = l - i_{m_k} + 1.$$

Since $i_{m_k} \le l$, we get

$$\begin{aligned}\sum_{i_1<i_2<\cdots<i_{m_k}} L_{i_j}(i_j+1) &\le \bigg(\sum_{j=1}^{m_k-1} (i_{j+1} - (i_j+1))(i_j+1)\bigg) + (l+1-i_{m_k})(i_{m_k}+1)\\ &\le (l+1)^2.\end{aligned}$$

By construction, we have

$$\delta_k \le \log_2(k) + 1.$$

Since $\log_2(k) \le l \le \log_2(k) + 1$, we conclude that

$$4k - 2(\log_2(k) + 2)^2 - \log_2(k) - 1 \le n_k \le 4k - 2\log_2(k) - 1$$

and the estimate $n_k - 4k = O(\log^2 k)$ holds. Finally, let us consider $\lambda_j = \sum_{l=0}^{j} 2^{2l}$. An easy calculation gives

$$n_{\lambda_j} = 4\lambda_j - 2\sum_{l=0}^{j}(2l+1) - 1 = 4\lambda_j - 2j^2 - 4j - 3.$$

Since $\lambda_j = (4^{j+1} - 1)/3$, the sequence $((n_{\lambda_j} - 4\lambda_j)/\log^2(\lambda_j))$ converges to a non-zero real number. □

We now prove that the sequence (n_k) constructed above not only has positive lower density but has also positive lower B_2-density.

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LEMMA 3.4. *We have* $\underline{d}_{B_2}((n_k)) > 0$.

Proof. Using (6) from Example 2.5, we have

$$\underline{d}_{B_2}(n_k) = \liminf_{k\to+\infty}\bigg(\frac{\sum_{j=2}^{k} e^{n_j/\log^2(n_j)}}{\log^2(n_k)e^{n_k/\log^2(n_k)}}\bigg).$$

According to Proposition 3.3, there exists a constant $C > 0$ such that, for N large enough,

$$\frac{\sum_{j=N}^{k} e^{n_j/\log^2(n_j)}}{\log^2(n_k)e^{n_k/\log^2(n_k)}} \geq \frac{\sum_{j=N}^{k} e^{(4j-C\log^2(j))/\log^2(4j-C\log^2(j))}}{\log^2(4k)e^{4k/\log^2(4k)}}.$$

A summation by parts gives

$$\sum_{j=N}^{k} e^{(4j-C\log^2(j))/\log^2(4j-C\log^2(j))} \sim \frac{\log^2(k)}{4} e^{(4k-C\log^2(k))/\log^2(4k-C\log^2(k))}$$

as $k \to +\infty$.

Finally, a similar computation as those needed for Example 2.5 yields

$$\frac{\sum_{j=N}^{k} e^{(4j-C\log^2(j))/\log^2(4j-C\log^2(j))}}{\log^2(4k)e^{4k/\log^2(4k)}} \sim \frac{e^{-C}}{4} \quad \text{as } k \to +\infty,$$

which finishes the proof. □

Lemma 3.4 allows us to show that the frequent hypercyclicity criterion gives a strengthened result.

PROPOSITION 3.5. *Let T be an operator on a separable Fréchet space X. If there are a dense subset X_0 of X and a map $S : X_0 \to X_0$ such that, for any $x \in X_0$:*
(i) $\sum_{n=0}^{+\infty} T^n x$ *converges unconditionally;*
(ii) $\sum_{n=0}^{+\infty} S^n x$ *converges unconditionally;*
(iii) $TSx = x$,
then T is B_2-frequently hypercyclic.

The proof of this result is the same as the classical proof of the frequent hypercyclicity criterion. Indeed, from Lemma 3.4, we can deduce that the sets $A(l, \nu)$ not only have positive lower density but even have positive lower B_2-density. We will not detail the proof here because we will prove a stronger result in §4.

Thanks to Lemma 2.9, one may actually deduce the following corollary proving that the scale defined by matrices A_r is not fine enough to exhibit the limit in terms of densities of the frequent hypercyclicity criterion.

COROLLARY 3.6. *Under the assumptions of the previous proposition, the operator T is A_r-frequently hypercyclic for every $0 \leq r < 1$.*

The previous result proves that for any $0 \leq r < 1$, the A_r-frequent hypercyclicity phenomenon exists and is even common. On the other hand, one may also notice that the geometric rate of growth (i.e. $r = 1$) is unreachable in terms of dynamics. More precisely, we have the following result.

PROPOSITION 3.7. *There is no A_1-frequently hypercyclic operator.*

Proof. We argue by contradiction. Assume that T is a A_1-frequently hypercyclic operator on a Banach space X and x is a A_1-frequently hypercyclic vector. Let also U be a non-empty open subset in X. Then, by definition and with Example 2.5, we get

$$0 < \liminf_{N\to+\infty} \sum_{k\le N} \frac{e^k}{\sum_{j=1}^N e^j} \mathbb{1}_{N(x,U)}(k) = \liminf_{N\to+\infty} (1-e^{-1}) \sum_{k\le N} e^{k-N} \mathbb{1}_{N(x,U)}(k).$$

Moreover, one may remark that asserting that this limit is non-zero implies that the set $N(x, U)$ has bounded gaps. Indeed, if one supposes that $N(x, U)$ has unbounded gaps, then there exist a sequence (N_i) and a sequence (p_i) tending to $+\infty$ such that for every $i \in \mathbb{N}$, $\{N_i - p_i + 1; N_i - p_i + 2; \ldots; N_i\} \cap N(x, U) = \emptyset$. This gives

$$0 < \liminf_{i\to+\infty}\left((1-e^{-1}) \sum_{k\le N_i} e^{k-N_i} \mathbb{1}_{N(x,U)}(k)\right) \le \liminf_{i\to+\infty}\left((1-e^{-1}) \sum_{k\le N_i-p_i} e^{k-N_i}\right) = 0$$

and this contradiction shows that the set $N(x, U)$ has bounded gaps. Let us denote by M an upper bound of the length of these gaps. It suffices to choose V so far from the origin such that the norm of T forbids $T^k(U)$ from intersecting V for $k \le M$. This means that the orbit of x will never reach the open set V, contradicting the A_1-frequent hypercyclicity of x. □

On the other hand, observe that the following result holds.

LEMMA 3.8. *For every* $0 < r < 1$, $\underline{d}_{B_r}(n_k) = 0$.

Proof. We have to estimate the following limit:

$$\underline{d}_{B_r}(n_k) = \liminf_{k\to+\infty}\left(\frac{\sum_{j=2}^{k} e^{n_j/\log^r(n_j)}}{\sum_{j=2}^{n_{k+1}-1} e^{j/\log^r(j)}}\right).$$

We remark that by definition of δ_k, there exists an increasing sequence of integers $(\lambda_k)_{k\in\mathbb{N}}$ such that $n_{\lambda_k+1} - n_{\lambda_k} - 1 = \delta_{\lambda_k} = k + 1 \sim \log_2(\lambda_k)$ as k tends to ∞ (consider for example $\lambda_k = 2^{k+1} - 1$). Then

$$\underline{d}_{B_r}(n_k) \le \liminf_{k\to+\infty}\left(\frac{\sum_{j=2}^{\lambda_k} e^{n_j/\log^r(n_j)}}{\sum_{j=2}^{n_{\lambda_k+1}-1} e^{j/\log^r(j)}}\right) \le \liminf_{k\to+\infty}\left(\frac{\sum_{j=2}^{n_{\lambda_k}} e^{j/\log^r(j)}}{\sum_{j=2}^{n_{\lambda_k+1}-1} e^{j/\log^r(j)}}\right).$$

Using the estimate (6) from Example 2.5 and the one from Proposition 3.3, we get

$$\begin{aligned}\underline{d}_{B_r}(n_k) &\le \liminf_{k\to+\infty}\left(\frac{\log^r(n_{\lambda_k}) e^{n_{\lambda_k}/\log^r(n_{\lambda_k})}}{\log^r(n_{\lambda_k+1}-1) e^{n_{\lambda_k}/\log^r(n_{\lambda_k})}}\right)\\ &\le \liminf_{k\to+\infty} e^{n_{\lambda_k}/\log^r(n_{\lambda_k}) - (n_{\lambda_k}+\delta_{\lambda_k})/\log^r(n_{\lambda_k}+\delta_{\lambda_k})}.\end{aligned}$$

We begin by studying the term in the exponent

$$\frac{n_{\lambda_k}}{\log^r(n_{\lambda_k})} - \frac{n_{\lambda_k}+\delta_{\lambda_k}}{\log^r(n_{\lambda_k}+\delta_{\lambda_k})} = \frac{n_{\lambda_k}}{\log^r(n_{\lambda_k}+\delta_{\lambda_k})}\left(\left(1 + \frac{\log(1+\delta_{\lambda_k}/n_{\lambda_k})}{\log(n_{\lambda_k})}\right)^r - \left(1 + \frac{\delta_{\lambda_k}}{n_{\lambda_k}}\right)\right),$$

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which reduces to the following thanks to a Taylor expansion:

$$\frac{n_{\lambda_k}}{\log^r(n_{\lambda_k}+\delta_{\lambda_k})}\bigg(\frac{1}{\log(n_{\lambda_k})}-1\bigg)+o\bigg(\frac{\delta_{\lambda_k}}{\log^{1+r}(n_{\lambda_k})}\bigg).$$

Now, combining the estimate $\delta_{\lambda_k}\sim\log_2(\lambda_k)$ as k tends to ∞ with the one given by Proposition 3.3, we deduce that $\delta_{\lambda_k}/\log^{1+r}(n_{\lambda_k})\to 0$ as k tends to ∞. Hence, we get

$$e^{n_{\lambda_k}/\log^r(n_{\lambda_k})-(n_{\lambda_k}+\delta_{\lambda_k})/\log^r(n_{\lambda_k}+\delta_{\lambda_k})}\sim e^{(n_{\lambda_k}/\log^r(n_{\lambda_k}+\delta_{\lambda_k}))(1/\log(n_{\lambda_k})-1)}\underset{k\to+\infty}{\longrightarrow}0.$$

This proves that $\underline{d}_{B_r}(n_k)=0$. □

Notice that Proposition 3.3 combined with Lemma 3.8 do not allow us to conclude the B_r-frequent hypercyclicity or not in the frequent hypercyclicity criterion for $1\le r<2$.

4. *Further results*

In this section, we are going to improve the conclusion of the frequent hypercyclicity criterion given by Proposition 3.5. To do this, we will modify the sequence (n_k) used in the proof of Lemma 3.1 to obtain a new sequence possessing a positive A-density for an admissible matrix A defining a sharper density than the natural density.

Throughout this section, (a_n) will be an increasing sequence of positive integers with $a_1=1$. Using this sequence, we define the function $f:\mathbb{N}\to\mathbb{N}$ by $f(j)=m$ for all $j\in\{a_m,\ldots,a_{m+1}-1\}$. In the spirit of the sequence studied in the previous section, we also define the sequence $(n_k(f))$ by induction:

$$n_1(f)=f(1)=1\quad\text{and}\quad n_k(f)=n_{k-1}(f)+f(\delta_{k-1})+f(\delta_k)\quad\text{for } k\ge 2.$$

Clearly, we obtain the following equality for all $k\ge 2$:

$$n_k(f)=2\sum_{i=1}^{k-1}f(\delta_i)+f(\delta_k). \tag{4.1}$$

Let us notice that, if we set $a_m=m$ for every $m\ge 1$, then the corresponding sequence $(n_k(f))$ is the sequence (n_k) of §3. From now on, we will omit the notation f in $(n_k(f))$ for sake of readability. Our purpose is to compute an exact formula for the new sequence (n_k) to understand its asymptotic behaviour. First of all, we obtain an expression for the subsequence $(n_{2^{a_m}})$.

LEMMA 4.1. *For all $m\in\mathbb{N}$, we have*

$$n_{2^{a_m}}=2^{a_m+1}\sum_{i=1}^{m}\frac{1}{2^{a_i-1}}-2m+1.$$

Proof. Set $\Delta_j^{(m)}=\{1\le l\le 2^{a_m}-1:\delta_l=j\}$. First let us observe that we have, by definition, for every $1\le j\le a_m$,

$$n_{2^{a_m}}=2\sum_{k=1}^{2^{a_m}-1}f(\delta_k)+f(\delta_{2^{a_m}})=2\sum_{j=1}^{a_m}f(j)\#\Delta_j^{(m)}+f(\delta_{2^{a_m}}).$$

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Thus, it suffices to compute the cardinal of the set $\Delta_j^{(m)}$. It easily follows that $\#\Delta_j^{(m)} = 1 + \sum_{i=0}^{a_m-j-1} 2^{a_m-j-i-1}$. Indeed, we separate the case when the first block of ones in the dyadic decomposition of l ends on 2^{a_m-1} and the case when the first block of ones ends before. In the first case, we have no choice, there is only one possibility, but in the second case we have a certain number i of zeros at the beginning, then the first block of ones, which is of length j, then one zero (because the first block of ones has to be of length j) and then we have $2^{a_m-j-i-1}$ possible choices as shown below.

$$(\overbrace{\underbrace{0, 0, \ldots, 0}_{\text{length } i}, \underbrace{1, 1, \ldots, 1, 1, 0}_{\text{length } j+1}, \star, \star, \ldots, \star, \star}^{\text{length } a_m}, 0, 0, \ldots).$$

A quick calculation leads to $\#\Delta_j^{(m)} = 1 + \sum_{i=0}^{a_m-j-1} 2^{a_m-j-i-1} = 2^{a_m-j}$. Therefore, we get

$$n_{2^{a_m}} = 2\sum_{j=1}^{a_m} f(j)2^{a_m-j} + f(\delta_{2^{a_m}}) = 2\sum_{j=1}^{a_m} f(j)2^{a_m-j} + 1.$$

Now we use the link between the values of $f(j)$ and the position of j compared to the sequence (a_m) to compute the sum:

$$\begin{aligned} n_{2^{a_m}} &= 2\sum_{j=1}^{a_m} f(j)2^{a_m-j} + 1 \\ &= 2^{a_m+1}\sum_{j=1}^{a_m-1} f(j)2^{-j} + 2m + 1. \end{aligned}$$

Let us now split the sum according to the values of $f(j)$:

$$\begin{aligned} n_{2^{a_m}} &= 2^{a_m+1}\sum_{i=1}^{m-1}\bigg(\sum_{j=a_i}^{a_{i+1}-1} f(j)2^{-j}\bigg) + 2m + 1 \\ &= 2^{a_m+1}\sum_{i=1}^{m-1} i\bigg(\sum_{j=a_i}^{a_{i+1}-1} 2^{-j}\bigg) + 2m + 1 \\ &= 2^{a_m+1}\sum_{i=1}^{m-1} i\bigg(\frac{1}{2^{a_i-1}} - \frac{1}{2^{a_{i+1}-1}}\bigg) + 2m + 1 \\ &= 2^{a_m+1}\sum_{i=1}^{m}\frac{1}{2^{a_i-1}} - 2m + 1. \end{aligned}$$ □

We strengthen the previous lemma as follows.

LEMMA 4.2. *Let $m \in \mathbb{N}$. For every $q \in \mathbb{N}$ such that $q < a_m - a_{m-1}$, the following equality holds:*

$$n_{2^{a_m-q}} = 2^{a_m-q+1}\sum_{j=1}^{m-1}\frac{1}{2^{a_j-1}} - 2(m-1) + 1.$$

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Proof. This proof works along the same lines as the proof of Lemma 4.1. Thus, we adapt the preceding proof. It yields

$$\begin{aligned} n_{2^{a_m-q}} &= 2\sum_{j=1}^{2^{a_m-q}-1} f(\delta_j) + f(\delta_{2^{a_m-q}}) \\ &= 2\sum_{j=1}^{a_m-q} f(j)2^{a_m-q-j} + 1 \\ &= 2^{a_m-q+1}\bigg(\bigg(\sum_{i=1}^{m-2}\sum_{j=a_i}^{a_{i+1}-1} i2^{-j}\bigg) + \sum_{j=a_{m-1}}^{a_m-q}(m-1)2^{-j}\bigg) + 1 \\ &= 2^{a_m-q+1}\sum_{j=1}^{m-1}\frac{1}{2^{a_j-1}} - 2(m-1) + 1. \end{aligned}$$ □

LEMMA 4.3. *For* $k = 2^n + \sum_{i=0}^{n-2}\alpha_i 2^i$ *with* $\alpha_i \in \{0; 1\}$, $0 \le i \le n-2$, *and* $(\alpha_0, \ldots, \alpha_{n-2}) \neq (0, \ldots, 0)$, *we have*

$$n_{k-2^n} = 2\sum_{i=2^n+1}^{k-1} f(\delta_i) + f(\delta_k).$$

For $k = 2^n + 2^{n-1} + \sum_{i=0}^{n-2}\alpha_i 2^i$ *with* $\alpha_i \in \{0; 1\}$, $0 \le i \le n-2$, *we have*

$$n_{k-2^n} = 2\sum_{i=2^n+1}^{k-1} f(\delta_i) - f(\delta_{k-2^n}) - 2(f(L)-1) + 2f(\delta_k),$$

where L *is the length of the block of ones containing the coefficient of* 2^n *in the dyadic decomposition of* k.

Proof. We begin by proving the first assertion. We have

$$n_{k-2^n} = 2\sum_{i=1}^{k-2^n-1} f(\delta_i) + f(\delta_{k-2^n}).$$

Since $k = 2^n + \sum_{i=0}^{n-2}\alpha_i 2^i$, observe that for all $2^n + 1 \le i \le k-1$ the dyadic decomposition of i contains a one for some 2^l with $0 \le l \le n-2$ and the coefficient of 2^{n-1} is zero. Therefore, the first block of ones in the dyadic decomposition of i does not contain the coefficient of 2^n. Thus, for every such $2^n + 1 \le i \le k-1$, we have $\delta_i = \delta_{i-2^n}$. This proves the first part of the lemma since $\delta_k = \delta_{k-2^n}$.

To prove the second assertion, we begin by observing that

$$2\sum_{i=2^n+1}^{k-1} f(\delta_i) = 2\sum_{i=2^n+1}^{k} f(\delta_i) - 2f(\delta_k).$$

Then, as above if the index i is such that the coefficient of 2^n does not belong to the first block of ones (in the dyadic decomposition of i), then $\delta_i = \delta_{i-2^n}$. On the other hand, if the coefficient of 2^n belongs to the first block of ones and since we have $a_{n-1} = 1$, then i has

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to be of the form $i = \sum_{l=0}^{p} 2^{n-l}$ for $p \in \{1, \ldots, L-1\}$ and $\delta_i = \delta_{i-2^n} + 1$. Now let us pick a particular index i of the form $i = \sum_{l=0}^{p} 2^{n-l}$ with $p \in \{1, \ldots, L-1\}$. We consider two cases.

Case 1: for every $j \in \{2, \ldots, f(L)\}$, we have $p+1 \neq a_j$. Then we have $f(\delta_i) = f(p+1) = f(\delta_i - 1) = f(\delta_{i-2^n})$.

Case 2: there exists an integer $j \in \{2, \ldots, f(L)\}$ with $p+1 = a_j$. Then we get $f(\delta_i) = f(a_j) = j = f(\delta_i - 1) + 1 = f(\delta_{i-2^n}) + 1$.

Finally, we deduce that

$$2 \sum_{i=2^n+1}^{k-1} f(\delta_i) = 2 \sum_{i=2^n+1}^{k} f(\delta_i) - 2f(\delta_k) = 2\left(\sum_{i=1}^{k-2^n} f(\delta_i) + (f(L)-1)\right) - 2f(\delta_k)$$
$$= n_{k-2^n} + f(\delta_{k-2^n}) + 2(f(L)-1) - 2f(\delta_k). \qquad \square$$

From Lemma 4.3, we deduce the following result.

LEMMA 4.4. *Let L be any non-zero integer and q be an integer. If $k = \sum_{j=0}^{L-1} 2^{q+j} + k'$ with $0 \leq k' < 2^{q-1}$, then either $k' \neq 0$ and*

$$n_k = n_{k'} + \sum_{j=0}^{L-1} n_{2^{q+j}} + 2\sum_{j=2}^{L}(f(j)-1) + L$$

or $k' = 0$ and

$$n_k = \sum_{j=0}^{L-1} n_{2^{q+j}} + 2\sum_{j=2}^{L}(f(j)-1) + L - f(L).$$

Proof. We proceed by induction on L. For $L = 1$, set $k = 2^q + k'$. First, observe that if $k' = 0$, the result is clear by Lemma 4.3. So, assume that $0 < k' < 2^{q-1}$. We divide n_k into two sums:

$$n_k = \left(2\sum_{i=1}^{2^q-1} f(\delta_i) + f(\delta_{2^q})\right) + \left(f(\delta_{2^q}) + 2\sum_{i=2^q+1}^{k-1} f(\delta_i) + f(\delta_k)\right).$$

It suffices to apply Lemma 4.3 to obtain

$$n_k = n_{2^q} + f(\delta_{2^q}) + n_{k'} = n_{2^q} + 1 + n_{k'}$$

and we have the desired conclusion. Now choose $L \geq 2$ and suppose that the result holds for every integer l, with $1 \leq l \leq L-1$. By Lemma 4.3, we get

$$n_k = \sum_{i=1}^{k-1} f(\delta_i) + f(\delta_k)$$
$$= \sum_{i=1}^{2^{q+L-1}-1} f(\delta_i) + 2f(\delta_{2^{q+L-1}}) + 2\sum_{i=2^{q+L-1}+1}^{k-1} f(\delta_i) + f(\delta_k)$$
$$= n_{2^{q+L-1}} + f(\delta_{2^{q+L-1}}) + n_{k-2^{q+L-1}} + f(\delta_{k-2^{q+L-1}}) + 2(f(L)-1) - f(\delta_k).$$

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We have $f(\delta_{2^{q+L-1}}) = 1$. Moreover, suppose that $0 < k' < 2^{q-1}$; then the block of ones containing the coefficient of 2^{q+L-1} is not the first one: thus, $\delta_k = \delta_{k'} = \delta_{k-2^{q+L-1}}$. Using the induction hypothesis, we obtain

$$\begin{aligned} n_k &= n_{2^{q+L-1}} + n_{k-2^{q+L-1}} + 2(f(L) - 1) + 1 \\ &= n_{k'} + \sum_{j=0}^{L-2} n_{2^{q+j}} + 2\sum_{j=2}^{L-1}(f(j) - 1) + L - 1 + n_{2^{q+L-1}} + 2(f(L) - 1) + 1 \\ &= n_{k'} + \sum_{j=0}^{L-1} n_{2^{q+j}} + 2\sum_{j=2}^{L}(f(j) - 1) + L. \end{aligned}$$

On the other hand, in the case $k' = 0$, the induction hypothesis gives

$$\begin{aligned} n_k &= n_{2^{q+L-1}} + 1 + n_{k-2^{q+L-1}} + f(L-1) + 2(f(L) - 1) - f(L) \\ &= \sum_{j=0}^{L-2} n_{2^{q+j}} + 2\sum_{j=2}^{L-1}(f(j) - 1) + (L-1) - f(L-1) \\ &\quad + n_{2^{q+L-1}} + 1 + f(L-1) + 2(f(L) - 1) - f(L) \\ &= \sum_{j=0}^{L-1} n_{2^{q+j}} + 2\sum_{j=2}^{L}(f(j) - 1) + L - f(L). \end{aligned}$$

This completes the proof. □

From Lemma 4.4, we immediately get the following result since $f(1) = 1$.

LEMMA 4.5. *Let $L_1, \dots, L_r$ be non-zero integers and $q_1, \dots, q_r$ be integers such that $q_i + L_i < q_{i+1}$ for every $1 \le i \le r-1$. For $k = \sum_{i=1}^{r} \sum_{j=0}^{L_i-1} 2^{q_i+j}$, we have*

$$n_k = \sum_{i=1}^{r} \sum_{j=0}^{L_i-1} n_{2^{q_i+j}} + 2\sum_{i=1}^{r} \sum_{j=1}^{L_i} f(j) - \sum_{i=1}^{r} L_i - f(L_1).$$

We are ready to obtain a general formula for the sequence (n_k). Let us introduce some notation.

Notation 4.6. Let $L_1, \dots, L_r$ be non-zero integers and $q_1, \dots, q_r$ be integers such that $q_i + L_i < q_{i+1}$ for every $1 \le i \le r-1$. We define the integers m_i, t_i, s_i and p_i as follows:

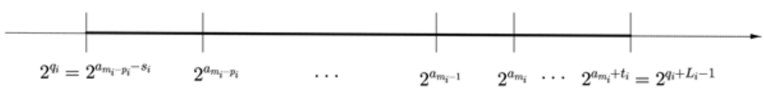
(1) m_i is the greatest integer such that $a_{m_i} \le q_i + L_i - 1$;

(2) $t_i = q_i + L_i - 1 - a_{m_i}$;

(3) $p_i = \#\{l \in \mathbb{N} : l < m_i \text{ and } q_i \le a_l \le q_i + L_i - 1\}$;

(4) $s_i = L_i - 1 - t_i - (a_{m_i} - a_{m_i-p_i})$.

To understand this notation, we give the following representation.



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Now we state an explicit formula for the sequence $(n_k(f))$. This will allow us to obtain a good asymptotic formula for this sequence.

LEMMA 4.7. *Using the notation (4.6), for* $k=\sum_{i=1}^{r}\sum_{j=0}^{L_i-1}2^{q_i+j}$, *we have*

$$\begin{aligned} n_k(f) = 2k\bigg(\sum_{l=1}^{+\infty}\frac{1}{2^{a_l-1}}\bigg) - \sum_{i=1}^{r}\bigg(\sum_{u=0}^{p_i-1}\bigg(\sum_{j=1}^{a_{m_i-u}-a_{m_i-(u+1)}}2^{a_{m_i-u}-j+1}\bigg)\bigg(\sum_{l=m_i-u}^{+\infty}\frac{1}{2^{a_l-1}}\bigg) \\ +\bigg(\sum_{j=0}^{t_i}2^{a_{m_i}+j+1}\bigg)\bigg(\sum_{l=m_i+1}^{+\infty}\frac{1}{2^{a_l-1}}\bigg) + \bigg(\sum_{l=1}^{s_i}2^{a_{m_i-p_i}-l+1}\bigg)\bigg(\sum_{l=m_i-p_i}^{+\infty}\frac{1}{2^{a_l-1}}\bigg)\bigg) \\ -2\sum_{i=1}^{r}\bigg(\sum_{j=a_{m_i-p_i}-s_i}^{a_{m_i}+t_i}f(j) - \sum_{j=1}^{L_i}f(j)\bigg) - f(L_1). \end{aligned}$$

Proof. We only prove the lemma for $t_i < L_i$, the other case being similar but simpler. We use the notation (4.6) to write

$$\sum_{j=0}^{L_i-1}n_{2^{q_i+j}} = \sum_{j=0}^{L_i-1}n_{2^{a_{m_i}+t_i-j}} = \sum_{j=0}^{t_i}n_{2^{a_{m_i}+j}} + \sum_{j=1}^{L_i-1-s_i-t_i}n_{2^{a_{m_i}-j}} + \sum_{j=1}^{s_i}n_{2^{a_{m_i-p_i}-j}}. \quad (4.2)$$

It remains to compute these three sums. We begin with the second one, dropping for the moment the index i for sake of readability. Using Lemma 4.2, we write

$$\begin{aligned} \sum_{j=1}^{L-1-s-t}n_{2^{a_m-j}} &= \sum_{u=0}^{p-1}\sum_{j=1}^{a_{m-u}-a_{m-(u+1)}}n_{2^{a_{m-u}-j}} \\ &= \sum_{u=0}^{p-1}\sum_{j=1}^{a_{m-u}-a_{m-(u+1)}}\bigg(2^{a_{m-u}-j+1}\sum_{l=1}^{m-(u+1)}\frac{1}{2^{a_l-1}} - 2(m-(u+1))+1\bigg). \end{aligned}$$

Thus, we deduce that

$$\begin{aligned} \sum_{j=1}^{L-1-s-t}n_{2^{a_m-j}} = \sum_{u=0}^{p-1}\bigg(\sum_{j=1}^{a_{m-u}-a_{m-(u+1)}}2^{a_{m-u}-j+1}\bigg)\bigg(\sum_{l=1}^{m-(u+1)}\frac{1}{2^{a_l-1}}\bigg) \\ -2\sum_{u=1}^{p}(m-u)(a_{m-(u-1)}-a_{m-u}) + L - s - t. \end{aligned} \quad (4.3)$$

In the same spirit, we compute the first and third sums as follows:

$$\sum_{j=0}^{t}n_{2^{a_m+j}} = \bigg(\sum_{j=0}^{t}2^{a_m+j+1}\bigg)\bigg(\sum_{l=1}^{m}\frac{1}{2^{a_l-1}}\bigg) - 2m(t+1)+t+1 \quad (4.4)$$

and

$$\sum_{j=1}^{s}n_{2^{a_{m-p}-j}} = \bigg(\sum_{l=1}^{s}2^{a_{m-p}-l+1}\bigg)\bigg(\sum_{l=1}^{m-(p+1)}\frac{1}{2^{a_l-1}}\bigg) - 2(m-(p+1))s+s. \quad (4.5)$$

Moreover, since we have by definition $t_i < a_{m_{i+1}} - a_{m_i}$ and $s_i < a_{m_i-p_i} - a_{m_i-p_i-1}$, when we gather equations (4.3)–(4.5), we have to compute the following sum:

$$\begin{aligned}
&\sum_{u=1}^{p_i}(m_i - u)(a_{m_i-(u-1)} - a_{m_i-u}) + m_i(t_i + 1) + (m_i - (p_i + 1))s_i \\
&\quad = \sum_{u=1}^{p_i} \sum_{j=0}^{a_{m_i-(u-1)}-a_{m_i-u}-1} f(a_{m_i-u} + j) + \sum_{j=0}^{t_i} f(a_{m_i} + j) + \sum_{j=1}^{s_i} f(a_{m_i-p_i} - j) \\
&\quad = \sum_{j=a_{m_i-p_i}-s_i}^{a_{m_i}+t_i} f(j).
\end{aligned} \tag{4.6}$$

Thus, thanks to Lemma 4.5 and the equations (4.2)–(4.6), we deduce that

$$\begin{aligned}
n_k &= \sum_{i=1}^{r} \sum_{j=0}^{L_i-1} n_{2^{q_i}+j} + 2\sum_{i=1}^{r}\sum_{j=1}^{L_i} f(j) - \sum_{i=1}^{r} L_i - f(L_1) \\
&= \sum_{i=1}^{r}\Bigg(\sum_{u=0}^{p_i-1}\Bigg(\sum_{j=1}^{a_{m_i-u}-a_{m_i-(u+1)}} 2^{a_{m_i-u}-j+1}\Bigg)\Bigg(\sum_{l=1}^{m_i-(u+1)} \frac{1}{2^{a_l-1}}\Bigg) \\
&\quad + \Bigg(\sum_{j=0}^{t_i} 2^{a_{m_i}+j+1}\Bigg)\Bigg(\sum_{l=1}^{m_i}\frac{1}{2^{a_l-1}}\Bigg) \\
&\quad + \Bigg(\sum_{l=1}^{s_i} 2^{a_{m_i-p_i}-l+1}\Bigg)\Bigg(\sum_{l=1}^{m_i-(p_i+1)}\frac{1}{2^{a_l-1}}\Bigg) - 2\sum_{j=a_{m_i-p_i}-s_i}^{a_{m_i}+t_i} f(j) + L_i\Bigg) \\
&\quad + 2\sum_{i=1}^{r}\sum_{j=1}^{L_i} f(j) - \sum_{i=1}^{r} L_i - f(L_1).
\end{aligned}$$

We remark that a $\sum_{i=1}^{r} L_i$ comes out from the first sum and cancels the term lying at the end of the preceding equality; we also gather the sums over $f(j)$ and we get

$$\begin{aligned}
n_k &= \sum_{i=1}^{r}\Bigg(\sum_{u=0}^{p_i-1}\Bigg(\sum_{j=1}^{a_{m_i-u}-a_{m_i-(u+1)}} 2^{a_{m_i-u}-j+1}\Bigg)\Bigg(\sum_{l=1}^{m_i-(u+1)} \frac{1}{2^{a_l-1}}\Bigg) \\
&\quad + \Bigg(\sum_{j=0}^{t_i} 2^{a_{m_i}+j+1}\Bigg)\Bigg(\sum_{l=1}^{m_i}\frac{1}{2^{a_l-1}}\Bigg) \\
&\quad + \Bigg(\sum_{l=1}^{s_i} 2^{a_{m_i-p_i}-l+1}\Bigg)\Bigg(\sum_{l=1}^{m_i-(p_i+1)}\frac{1}{2^{a_l-1}}\Bigg)\Bigg) \\
&\quad + 2\sum_{i=1}^{r}\Bigg(\sum_{j=1}^{L_i} f(j) - \sum_{j=a_{m_i-p_i}-s_i}^{a_{m_i}+t_i} f(j)\Bigg) - f(L_1).
\end{aligned}$$

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Then we express the partial sums $\sum_{l=1}^{N}(1/2^{a_l-1})$ as the series minus its remainder of order N, which yields

$$\begin{aligned} n_k &= \bigg(\sum_{l=1}^{+\infty}\frac{1}{2^{a_l-1}}\bigg)\sum_{i=1}^{r}\bigg(\sum_{u=0}^{p_i-1}\bigg(\sum_{j=1}^{a_{m_i-u}-a_{m_i-(u+1)}}2^{a_{m_i-u}-j+1}\bigg) \\ &\quad+\sum_{j=0}^{t_i}2^{a_{m_i}+j+1}+\sum_{l=1}^{s_i}2^{a_{m_i-p_i}-l+1}\bigg) \\ &\quad-\sum_{i=1}^{r}\bigg(\sum_{u=0}^{p_i-1}\bigg(\sum_{j=1}^{a_{m_i-u}-a_{m_i-(u+1)}}2^{a_{m_i-u}-j+1}\bigg)\bigg(\sum_{l=m_i-u}^{+\infty}\frac{1}{2^{a_l-1}}\bigg) \\ &\quad+\bigg(\sum_{j=0}^{t_i}2^{a_{m_i}+j+1}\bigg)\bigg(\sum_{l=m_i+1}^{+\infty}\frac{1}{2^{a_l-1}}\bigg) \\ &\quad+\bigg(\sum_{l=1}^{s_i}2^{a_{m_i-p_i}-l+1}\bigg)\bigg(\sum_{l=m_i-p_i}^{+\infty}\frac{1}{2^{a_l-1}}\bigg)\bigg)+2\sum_{i=1}^{r}\bigg(\sum_{j=1}^{L_i}f(j)-\sum_{j=a_{m_i-p_i}-s_i}^{a_{m_i}+t_i}f(j)\bigg) \\ &\quad-f(L_1). \end{aligned}$$

Now it suffices to remark that coming back to notation with the q_i, then k can be expressed in the following way:

$$k=\sum_{i=1}^{r}\bigg(\sum_{u=0}^{p_i-1}\bigg(\sum_{j=1}^{a_{m_i-u}-a_{m_i-(u+1)}}2^{a_{m_i-u}-j}\bigg)+\sum_{j=0}^{t_i}2^{a_{m_i}+j}+\sum_{l=1}^{s_i}2^{a_{m_i-p_i}-l}\bigg).$$

Thus, we obtain

$$\begin{aligned} n_k &= 2k\bigg(\sum_{l=1}^{+\infty}\frac{1}{2^{a_l-1}}\bigg)-\sum_{i=1}^{r}\bigg(\sum_{u=0}^{p_i-1}\bigg(\sum_{j=1}^{a_{m_i-u}-a_{m_i-(u+1)}}2^{a_{m_i-u}-j+1}\bigg)\bigg(\sum_{l=m_i-u}^{+\infty}\frac{1}{2^{a_l-1}}\bigg) \\ &\quad+\bigg(\sum_{j=0}^{t_i}2^{a_{m_i}+j+1}\bigg)\bigg(\sum_{l=m_i+1}^{+\infty}\frac{1}{2^{a_l-1}}\bigg)+\bigg(\sum_{l=1}^{s_i}2^{a_{m_i-p_i}-l+1}\bigg)\bigg(\sum_{l=m_i-p_i}^{+\infty}\frac{1}{2^{a_l-1}}\bigg)\bigg) \\ &\quad+2\sum_{i=1}^{r}\bigg(\sum_{j=1}^{L_i}f(j)-\sum_{j=a_{m_i-p_i}-s_i}^{a_{m_i}+t_i}f(j)\bigg)-f(L_1). \qquad \square \end{aligned}$$

Observe that if we set $(a_m)_m=(m)_m$, then $f(j)=j$ for every integer j and Lemma 4.7 takes the following form.

LEMMA 4.8. *In the aforementioned case, if* $k=2^n+\sum_{i=0}^{n-1}\alpha_i 2^i$ *with* $\alpha_i\in\{0;\,1\}$ *for every* $0\le i\le n-1$*, then*

$$n_k=4k-2\bigg(\sum_{i\in I_k}L_i(i+1)\bigg)-\delta_k,$$

where I_k *stands for the set of integers* i *such that* α_i *is the first non-zero integer of a block (of consecutive non-zero coefficients) having length* L_i *in the dyadic decomposition of* k.

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Proof. Using the notation of Lemma 4.7, we have $t_i = s_i = 0$, $L_i = p_i + 1$, $m_i = i + L_i - 1$. Therefore, we deduce that

$$\begin{aligned} n_k &= 4k - 4\sum_{i=1}^{r}(p_i+1) - 2\sum_{i=1}^{r}\bigg(\sum_{j=i}^{i+L_i-1} j - \frac{L_i(L_i+1)}{2}\bigg) - L_1 \\ &= 4k - 4\sum_{i=1}^{r} L_i - \sum_{i=1}^{r}(2i+L_i-1)L_i + \sum_{i=1}^{r} L_i(L_i+1) - L_1 \\ &= 4k - 2\sum_{i=1}^{r} L_i(i+1) - L_1. \end{aligned}$$

This last equality gives the result since $L_1 = \delta_k$. □

Lemma 4.8 is exactly Lemma 3.4 announced in the previous section.

Let us return to the general situation, using the notation $(n_k(f))$ again. From Lemma 4.7, we deduce the following estimate on the sequence $(n_k(f))$ for specific choices of functions f.

LEMMA 4.9. *Using the previous notation, assume that* $a_m = 2^{2^{\cdot^{\cdot^{2^m}}}}$, *where* 2 *appears* s *times* ($s \geq 1$). *Then the associated function* $f_s : \mathbb{N} \to \mathbb{N}$ *is given by* $f_s(j) = m$, *for* $j \in \{a_m, \ldots, a_{m+1} - 1\}$, *and the following estimate holds:*

$$\begin{aligned} 2k\bigg(\sum_{l=1}^{+\infty}\frac{1}{2^{a_l-1}}\bigg) - 2\log_2(k) f_s(\lfloor\log_2(k)\rfloor) - 14\log_2(k) - 8 f_s(\lfloor\log_2(k)\rfloor) \leq n_k^{(s)} \\ \leq 2k\bigg(\sum_{l=1}^{+\infty}\frac{1}{2^{a_l-1}}\bigg), \end{aligned}$$

with $n_k(f_s) = n_k^{(s)}$.

Proof. We need Lemma 4.7 and its notation. The proof of the upper bound is obvious. For the lower bound, observe first that the subadditivity of f_s implies that for $k = \sum_{i=1}^{r}\sum_{j=0}^{L_i-1} 2^{q_i+j}$,

$$\begin{aligned} \sum_{j=1}^{L_i} f_s(j) - \sum_{j=a_{m_i-p_i}-s_i}^{a_{m_i}+t_i} f_s(j) &= \sum_{j=1}^{L_i}(f_s(j) - f_s(a_{m_i-p_i} - s_i - 1 + j)) \\ &\geq -L_i f_s(a_{m_i-p_i} - s_i). \end{aligned}$$

In addition, since for every $u \geq 1$ we have $a_u + 2 < a_{u+2}$ and $\sum_{l=q}^{+\infty} 2^{-j} = 2^{1-q}$, we obtain

$$\begin{aligned} \bigg(\sum_{j=0}^{t_i} 2^{a_{m_i}+j+1}\bigg)\bigg(\sum_{l=m_i+1}^{+\infty}\frac{1}{2^{a_l-1}}\bigg) &= 4\bigg(\sum_{j=0}^{t_i} 2^{a_{m_i}+j}\bigg)\bigg(\sum_{l=m_i+1}^{+\infty}\frac{1}{2^{a_l}}\bigg) \\ &\leq 4\bigg(\sum_{j=0}^{t_i} 2^{a_{m_i}+j}\bigg)\bigg(\frac{1}{2^{a_{m_i+1}}} + \frac{1}{2^{a_{m_i+1}+1}} + \frac{1}{2^{a_{m_i+1}+2}}\bigg) \end{aligned}$$

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$$
\begin{aligned}
&\leq \frac{7}{2^{a_{m_i+1}-a_{m_i}}}\left(2^{t_i+1}-1\right)\\
&\leq \frac{7}{2^{a_{m_i+1}-(a_{m_i}+t_i+1)}}\\
&\leq 7.
\end{aligned}
$$

In the same spirit, we also have

$$
\begin{aligned}
&\bigg(\sum_{l=1}^{s_i} 2^{a_{m_i-p_i}-l+1}\bigg)\bigg(\sum_{l=m_i-p_i}^{+\infty}\frac{1}{2^{a_l-1}}\bigg)\\
&\quad = 4\bigg(\sum_{l=1}^{s_i} 2^{a_{m_i-p_i}-l}\bigg)\bigg(\sum_{l=m_i-p_i}^{+\infty}\frac{1}{2^{a_l}}\bigg)\\
&\quad \leq 4\bigg(\sum_{l=1}^{s_i} 2^{a_{m_i-p_i}-l}\bigg)\bigg(\frac{1}{2^{a_{m_i-p_i}}}+\frac{1}{2^{a_{m_i-p_i}+1}}+\frac{1}{2^{a_{m_i-p_i}+2}}\bigg)\\
&\quad \leq 7\bigg(\sum_{l=1}^{s_i} 2^{-l}\bigg)\\
&\quad \leq 7.
\end{aligned}
$$

The same method gives again

$$
\sum_{u=0}^{p_i-1}\bigg(\sum_{j=1}^{a_{m_i-u}-a_{m_i-(u+1)}} 2^{a_{m_i-u}-j+1}\bigg)\bigg(\sum_{l=m_i-u}^{+\infty}\frac{1}{2^{a_l-1}}\bigg) \leq 7\sum_{u=0}^{p_i-1}\bigg(\sum_{j=1}^{a_{m_i-u}-a_{m_i-(u+1)}} 2^{-j}\bigg) \leq 7p_i.
$$

Finally, we gather these estimates and we obtain

$$
\begin{aligned}
n_k^{(s)} &\geq 2k\bigg(\sum_{l=1}^{+\infty}\frac{1}{2^{a_l-1}}\bigg) - 7\sum_{i=1}^{r}(2+p_i) - 2\sum_{i=1}^{r} L_i f_s(a_{m_i-p_i}-s_i) - f_s(L_1)\\
&\geq 2k\bigg(\sum_{l=1}^{+\infty}\frac{1}{2^{a_l-1}}\bigg) - 7(2r+m_r) - 2\bigg(\sum_{i=1}^{r} L_i\bigg) f_s(a_{m_r}+t_r) - f_s(L_1).
\end{aligned}
$$

Using the fact that

$$
a_{m_r}+t_i = q_r+L_r-1 \leq \log_2(k) < q_r+L_r \quad \text{and} \quad k=\sum_{i=1}^{r}\sum_{j=0}^{L_i-1} 2^{q_i+j},
$$

we get

$$
\begin{aligned}
n_k^{(s)} &\geq 2k\bigg(\sum_{l=1}^{+\infty}\frac{1}{2^{a_l-1}}\bigg) - 7(2\log_2(k)+f_s(\lfloor\log_2(k)\rfloor)) - 2\log_2(k) f_s(\lfloor\log_2(k)\rfloor)\\
&\quad - f_s(\lfloor\log_2(k)\rfloor)\\
&\geq 2k\bigg(\sum_{l=1}^{+\infty}\frac{1}{2^{a_l-1}}\bigg) - 2\log_2(k) f_s(\lfloor\log_2(k)\rfloor) - 14\log_2(k) - 8 f_s(\lfloor\log_2(k)\rfloor).
\end{aligned}
$$

This finishes the proof. □

We now prove that the sequence $(n_k^{(s)})$ constructed above not only has positive lower density but has also positive lower $\widetilde{B}_s$-density for every $s \geq 2$.

LEMMA 4.10. *We have* $\underline{d}_{\widetilde{B}_s}((n_k^{(s)})) > 0$.

Proof. According to (7) from Example 2.5, we write

$$\underline{d}_{B_s}(n_k^{(s)}) = \liminf_{k\to+\infty}\bigg(\frac{\sum_{j=1}^{k} e^{n_j^{(s)}/h_s(n_j^{(s)})}}{h_s(n_k^{(s)})e^{n_k^{(s)}/h_s(n_k^{(s)})}}\bigg).$$

Observe that Lemma 4.9 ensures the existence of two constants $C_1, C_2 > 1$ such that for N large enough,

$$\frac{\sum_{j=N}^{k} e^{n_j^{(s)}/h_s(n_j^{(s)})}}{h_s(n_k^{(s)})e^{n_k^{(s)}/h_s(n_k^{(s)})}} \geq \frac{\sum_{j=N}^{k} e^{(C_1 j - C_2 h_s(j))/h_s(C_1 j - C_2 h_s(j))}}{h_s(C_1 k)e^{C_1 k/h_s(C_1 k)}}.$$

A summation by parts gives

$$\sum_{j=N}^{k} e^{(C_1 j - C_2 h_s(j))/h_s(C_1 j - C_2 h_s(j))} \sim \frac{h_s(k)}{C_1} e^{(C_1 k - C_2 h_s(k))/h_s(C_1 k - C_2 h_s(k))} \quad \text{as } k \to +\infty.$$

Then a similar computation as those needed for (7) from Example 2.5 leads to the following estimate:

$$\frac{\sum_{j=N}^{k} e^{(C_1 j - C_2 h_s(j))/h_s(C_1 j - C_2 h_s(j))}}{h_s(C_1 k)e^{C_1 k/h_s(C_1 k)}} \sim \frac{e^{-C_2}}{C_1} \quad \text{as } k \to +\infty,$$

which gives the desired conclusion. □

This allows us to prove the following combinatorial lemma, which extends Lemma 3.1.

LEMMA 4.11. *There exist pairwise disjoint subsets* $B^{(s)}(l, \nu)$, $l, \nu \geq 1$, *of* $\mathbb{N}$ *having positive* $\widetilde{B}_s$*-density such that, for any* $n \in B^{(s)}(l, \nu)$ *and* $m \in B^{(s)}(k, \mu)$, *we have that* $n \geq f_s(\nu)$ *and*

$$|n - m| \geq f_s(\nu) + f_s(\mu) \quad \text{if } n \neq m.$$

Proof. We consider the sequence $(n_k^{(s)})$ constructed above and also sets $I(l, \nu)$ constructed in [**9**] that we recalled just after Lemma 3.1. We also define $B^{(s)}(l, \nu) := \{n_k^{(s)}; k \in I(l, \nu)\}$. These sets are clearly pairwise disjoint since the sets $I(l, \nu)$ are, and the sequence $(n_k^{(s)})$ is increasing. Moreover, by definition of the sets $I(l, \nu)$, that are arithmetic sequences, and Lemma 4.9, the conclusion of Lemma 4.10 remains true, i.e. the sets $B^{(s)}(l, \nu)$ have positive lower $\widetilde{B}_s$-density. Then by definition of $n_k^{(s)}$ from (4.1), we get $n_k^{(s)} \geq f_s(\delta_k) = f_s(\nu)$. Finally, if $n_j^{(s)} \in B^{(s)}(l, \nu)$ and $n_m^{(s)} \in B^{(s)}(k, \mu)$, with $j > m$, then

$$n_j^{(s)} - n_m^{(s)} = f_s(\delta_m) + 2\sum_{i=m+1}^{j-1} f_s(\delta_i) + f_s(\delta_j) \geq f_s(\mu) + f_s(\nu). \qquad \square$$

This strengthened version of Lemma 3.1 allows us to give a stronger conclusion to the so-called frequent hypercyclicity criterion, whose proof will be only sketched since it is an adaptation of the classical proof given in [**9**, Theorem 9.9].

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THEOREM 4.12. *Let T be an operator on a separable Fréchet space X. If there are a dense subset X_0 of X and a map $S : X_0 \to X_0$ such that, for any $x \in X_0$:*
(i) $\sum_{n=0}^{\infty} T^n x$ *converges unconditionally;*
(ii) $\sum_{n=0}^{\infty} S^n x$ *converges unconditionally;*
(iii) $TSx = x$,
then T is $\widetilde{B}_s$-frequently hypercyclic for every $s \geq 2$.

Proof. Let (y_n) be a dense sequence from X_0 that is dense in X. Let $\|.\|$ denote an F-norm that defines the topology of X. The unconditional convergence of the series (ii) and (iii) allows us to find, for every $l \in \mathbb{N}$, an integer $N_l \geq 1$ such that for every $j \leq l$ and every finite set $F \subset \{N_l; N_l + 1; \ldots\}$,

$$\left\| \sum_{n \in F} T^n y_l \right\| \leq \frac{1}{l2^l} \quad \text{and} \quad \left\| \sum_{n \in F} S^n y_l \right\| \leq \frac{1}{l2^l}.$$

Now let (M_l) be an increasing sequence such that $f_s(M_l) \geq N_l$ and $(f_s(M_l))$ is increasing. We also define

$$B^{(s)} := \bigcup_{l=1}^{+\infty} B^{(s)}(l, M_l)$$

and

$$z_n = y_l \quad \text{if } n \in B^{(s)}(l, M_l).$$

Finally, we claim that

$$x = \sum_{n \in \mathbb{N}} S^n(z_n)$$

defines a $\widetilde{B}_s$-frequently hypercyclic vector for T. From this point, the proof is just an adaptation of the proof of the frequent hypercyclicity criterion from [**9**] replacing Lemma 3.1 by Lemma 4.11 stated above. □

We may also deduce the following corollary using Lemma 2.9.

COROLLARY 4.13. *Under the assumptions of the previous proposition, the operator T is B_r-frequently hypercyclic for every $r > 1$.*

5. *A frequently hypercyclic operator which is not A_r-frequently hypercyclic*

In this final section, we are going to show that there exist frequently hypercyclic operators that do not belong to the class of A_r-frequently hypercyclic operators for any $0 < r \leq 1$. According to Proposition 3.5 or Theorem 4.12, such an operator cannot satisfy the frequent hypercyclicity criterion. To build it, we are going to use several ideas of the work [**5**], where the authors provided some counterexamples to questions regarding frequent hypercyclicity.

In a recent paper, Bayart and Ruzsa gave a characterization of frequently hypercyclic weighted shifts on the sequence spaces ℓ^p and c_0. We recall here their result on $c_0(\mathbb{N})$ that will be useful in the following [**5**, Theorem 13].

THEOREM 5.1. *Let $w = (\omega_n)_{n \in \mathbb{N}}$ be a bounded sequence of positive integers. Then B_w is frequently hypercyclic on $c_0(\mathbb{N})$ if and only if there exist a sequence $(M(p))$ of positive real numbers tending to $+\infty$ and a sequence (E_p) of subsets of $\mathbb{N}$ such that:*

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(a) *for any* $p \geq 1$, $\underline{d}(E_p) > 0$;
(b) *for any* $p, q \geq 1$, $p \neq q$, $(E_p + [0, p]) \cap (E_q + [0, q]) = \emptyset$;
(c) $\lim_{n\to\infty,\ n\in E_p+[0,p]} \omega_1 \cdots \omega_n = +\infty$;
(d) *for any* $p, q \geq 1$, *for any* $n \in E_p$ *and any* $m \in E_q$ *with* $m > n$, *for any* $t \in \{0, \ldots, q\}$,

$$\omega_1 \cdots \omega_{m-n+t} \geq M(p)M(q).$$

In the same paper, the authors also provided examples of a $\mathcal{U}$-frequently hypercyclic weighted shift which is not frequently hypercyclic and of a frequently hypercyclic weighted shift which is not distributionally chaotic. In what follows, we modify these constructions in order to provide a frequently hypercyclic weighted shift on $c_0(\mathbb{N})$ which is not A_r-frequently hypercyclic for any $0 < r < 1$. To that purpose, we will need the following lemma [**5**, Lemma 1].

LEMMA 5.2. *There exist* $a > 1$ *and* $\varepsilon > 0$ *such that* $\overline{d}(\bigcup_{u\geq 1} I_u^{a,4\varepsilon}) < 1$ *and, for any integer* $u > v \geq 1$,

$$I_u^{a,2\varepsilon} \cap I_v^{a,2\varepsilon} = \emptyset, \quad I_u^{a,2\varepsilon} - I_v^{a,2\varepsilon} \subset I_u^{a,4\varepsilon},$$

where $I_u^{a,\varepsilon} = [(1-\varepsilon)a^u, (1+\varepsilon)a^u]$.

The philosophy of the previous lemma is that it suffices to choose a very large and at the same time ε very small to obtain the result stated. This allows us to strengthen this lemma demanding also that the following condition holds:

$$\frac{1-\varepsilon}{1+\varepsilon} a > 1. \tag{5.1}$$

From now on, we suppose that a and ε are given by the previous lemma with the additional condition (5.1).

Let also (b_p) be an increasing sequence of integers such that

$$\sum_{q\geq 1} \frac{(4q+1)(2q+1)}{b_q} e^{2q} < \infty \quad \text{and} \quad b_p \geq 8p. \tag{5.2}$$

Finally, let (A_p) be any syndetic partition of $\mathbb{N}$ and

$$E_p = \bigcup_{u\in A_p} (I_u^{a,\varepsilon} \cap (b_p\mathbb{N} + [0, p])).$$

Bayart and Ruzsa constructed such sets and they proved that these sets have positive lower density [**5**, Lemma 2]. Then, for the same reasons, we have $\underline{d}(E_p) > 0$. Further, the following lemma is almost the same as [**5**, Lemma 3] once again and it still holds in our context.

LEMMA 5.3. *Let* $p, q \geq 1$, $n \in E_p$, $m \in E_q$ *with* $n \neq m$. *Then* $|n - m| > \max(p, q)$.

In particular, $(E_p + [0, p]) \cap (E_q + [0, q]) = \emptyset$ if $p \neq q$.

Thus, the sequence of sets (E_p) satisfies conditions (a) and (b) from Theorem 5.1.

We now turn to the construction of the weights of the weighted shift we are looking for. For this construction, we also draw our inspiration from constructions made in [**5**]. We set

$$w_0^p \cdots w_{k-1}^p = \begin{cases} 1 & \text{if } k \notin b_p\mathbb{N} + [-4p, 4p], \\ 2^p & \text{if } k \in b_p\mathbb{N} + [-2p, 2p], \end{cases}$$

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and, for every $k \in \mathbb{N}$, $1/2 \le w_k^p \le 2$. Then, for $p, q \ge 1$, $u \in A_p$ and $v \in A_q$ with $u > v$, we define

$$w_0^{u,v} \cdots w_{k-1}^{u,v} = \begin{cases} 1 & \text{if } k \notin I_u^{a,4\varepsilon}, \\ \max(2^p, 2^q) & \text{if } k \in I_u^{a,\varepsilon} - I_v^{a,\varepsilon} + [0, p], \end{cases}$$

and, for every $k \in \mathbb{N}$, $1/2 \le w_k^{u,v} \le 2$.

We are now able to give the definition of the weight w. This one is constructed in order to satisfy the following equality:

$$w_0 \cdots w_{n-1} = \max_{p,u,v}(w_0^p \cdots w_{n-1}^p, w_0^{u,v} \cdots w_{n-1}^{u,v}).$$

It is clear by construction that for every $n \in \mathbb{N}$, $1/2 \le w_n \le 2$, so the weighted backward shift B_w is bounded and invertible. Moreover, this construction satisfies condition (c) in Theorem 5.1.

Since we want to prove that B_w is frequently hypercyclic, the only condition left to prove is condition (d) from Theorem 5.1. Thus, let $p, q \ge 1$, $n \in E_p$ and $m \in E_q$ with $m > n$ and $t \in [0, q]$. Then we have two cases.

- If $p = q$, then $m - n + t \in b_q\mathbb{N} + [-q, 2q]$ and the definition of w ensures that $w_0 \cdots w_{m-n+t} \ge 2^q$.
- If $p \ne q$, then there exists $u > v$ such that $n \in I_v^{a,\varepsilon}$ and $m \in I_u^{a,\varepsilon}$. Thus, by definition of w,
$$w_0 \cdots w_{m-n+t} \ge \max(2^p; 2^q) \ge 2^{(p+q)/2} \ge \lfloor 2^{p/2} \rfloor \cdot \lfloor 2^{q/2} \rfloor.$$

Now one may define $M(p) := \lfloor 2^{p/2} \rfloor$ and each case above satisfies condition (d) from Theorem 5.1. Thus, we have proved that the weighted shift B_w is frequently hypercyclic.

We now turn to the A_r-frequent hypercyclicity of B_w. We are going to prove by contradiction that B_w is not A_r-frequently hypercyclic for every $0 < r < 1$.

Let us suppose that B_w is A_r-frequently hypercyclic and that $E \subset \mathbb{N}$ is such that $\underline{d}_{A_r}(E) > 0$ and $\lim_{n\to\infty, n\in E} w_1 \cdots w_n = +\infty$. Such a set exists since B_w is A_r-frequently hypercyclic. Indeed, it suffices to consider $E = \{n \in \mathbb{N} : \|B_w^n(x) - e_0\| \le 1/2\}$, where x is a A_r-frequently hypercyclic vector.

For every $p \ge 1$, we consider the set

$$F_p = \{n \in E : w_1 \cdots w_n > 2^p\}.$$

This set is a cofinite subset of E, so it has the same lower A_r-density. We also consider an increasing enumeration (n_k) of A_p.

Then

$$\underline{d}_{A_r}(F_p) \le \liminf_{k\to\infty}\left(\frac{\sum_{n\le(1+\varepsilon)a^{n_k},\, n\in F_p} e^{n^r} + \sum_{(1+\varepsilon)a^{n_k}<n\le(1-\varepsilon)a^{n_{k+1}},\, n\in\bigcup_{q>p}(b_q\mathbb{N}+[-2q,2q])} e^{n^r}}{\sum_{n\le(1-\varepsilon)a^{n_{k+1}}} e^{n^r}}\right).$$

Moreover, since we have $\sum_{n\le N} e^{n^r} \sim (1/r)N^{1-r}e^{N^r}$, as N tends to ∞, we get

$$\underline{d}_{A_r}(F_p) \le \liminf_{k\to\infty}\left(\frac{(1/r)((1+\varepsilon)a^{n_k})^{1-r}e^{((1+\varepsilon)a^{n_k})^r} + \sum_{(1+\varepsilon)a^{n_k}<n\le(1-\varepsilon)a^{n_{k+1}},\, n\in\bigcup_{q>p}(b_q\mathbb{N}+[-2q,2q])} e^{n^r}}{(1/r)((1-\varepsilon)a^{n_{k+1}})^{1-r}e^{((1-\varepsilon)a^{n_{k+1}})^r}}\right).$$

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A straightforward computation using inequality (5.1) proves that the first term on the right-hand side tends to 0. We now focus on the second term:

$$\underline{d}_{A_r}(F_p) \leq \liminf_{k\to\infty}\left(\frac{\sum_{\substack{(1+\varepsilon)a^{n_k}<n\leq(1-\varepsilon)a^{n_{k+1}},\\ n\in\bigcup_{q>p}(b_q\mathbb{N}+[-2q,2q])}} e^{n^r}}{(1/r)((1-\varepsilon)a^{n_{k+1}})^{1-r}e^{((1-\varepsilon)a^{n_{k+1}})^r}}\right)$$

$$\leq \liminf_{k\to\infty}\frac{\sum_{q>p}(4q+1)\sum_{j=1}^{(1-\varepsilon)a^{n_{k+1}}/(b_q+2q)} e^{(jb_q+2q)^r}}{(1/r)((1-\varepsilon)a^{n_{k+1}})^{1-r}e^{((1-\varepsilon)a^{n_{k+1}})^r}}.$$

A classical calculation ensures that we have

$$\sum_{j=1}^{(1-\varepsilon)a^{n_{k+1}}/(b_q+2q)} e^{(jb_q+2q)^r} \sim \frac{(b_q(1-\varepsilon)a^{n_{k+1}}/(b_q+2q)+2q)^{1-r}}{rb_q} \times e^{(b_q(1-\varepsilon)a^{n_{k+1}}/(b_q+2q)+2q)^r}$$

as k tends to ∞. Thus, we obtain

$$\begin{aligned}
&\underline{d}_{A_r}(F_p)\\
&\leq \liminf_{k\to\infty}\frac{\sum_{q>p}(4q+1)(b_q(1-\varepsilon)a^{n_{k+1}}/(b_q+2q)+2q)^{1-r}e^{(b_q(1-\varepsilon)a^{n_{k+1}}/(b_q+2q)+2q)^r}}{b_q((1-\varepsilon)a^{n_{k+1}})^{1-r}e^{((1-\varepsilon)a^{n_{k+1}})^r}}\\
&\leq \liminf_{k\to\infty}\sum_{q>p}\frac{4q+1}{b_q}\left(\frac{b_q}{b_q+2q}+\frac{2q}{(1-\varepsilon)a^{n_{k+1}}}\right)^{1-r} e^{((1-\varepsilon)a^{n_{k+1}}+2q)^r-((1-\varepsilon)a^{n_{k+1}})^r}\\
&\leq \sum_{q>p}\frac{4q+1}{b_q}(1+2q)^{1-r}e^{(2q)^r}\\
&\leq \sum_{q>p}\frac{4q+1}{b_q}(1+2q)e^{2q}.
\end{aligned}$$

Recall that this does not require any property on p, so we can let p tend to infinity, which, thanks to (5.2), implies that $\underline{d}_{A_r}(E)=\lim_{p\to\infty}\underline{d}_{A_r}(F_p)=0$; hence, we obtain a contradiction. Thus, the weighted shift B_w is not A_r-frequently hypercyclic. From this construction together with Corollary 3.6, we deduce the following result.

THEOREM 5.4. *There exists a frequently hypercyclic operator, being not A_r-frequently hypercyclic for any $0<r\leq 1$, and hence which does not satisfy the frequent hypercyclicity criterion.*

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