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ON THE UNIFORM APPROXIMATION OF SMOOTH FUNCTIONS BY JACOBI POLYNOMIALS

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1. Introduction. Let $\omega_n(x)$ denote the Jacobi polynomials with the weight function

$$p(x) = (1 - x)^\alpha(1 + x)^\beta, \alpha > -1 \text{ and } \beta > -1.$$

If we denote the corresponding normalized Jacobi polynomials by $\bar{\omega}_n(x)$ we have

$$(1.1) \qquad \bar{\omega}_n(x) = \left[\frac{(2n + \alpha + \beta + 1)\Gamma(n + 1)\Gamma(n + \alpha + \beta + 1)}{2^{\alpha+\beta+1}\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)}\right]^{\frac{1}{2}} \omega_n(x).$$

Now let

$$S_n(x) = \sum_{k=0}^{n} b_k \bar{\omega}_k(x)$$

be the nth partial sum of the Fourier series of Jacobi polynomials of a function $f(x)$. In the second of three volumes on *Constructive function theory* Natanson proved the following:

THEOREM 1 [**1**]. *Let* $\sigma = \max(\alpha, \beta) \geqq -\frac{1}{2}$ *and let* p *be a positive integer which is not less than* $2\sigma + 2$. *Then on the interval* $[-1, 1]$ *every function* $f(x)$ *with a continuous* pth *derivative can be expanded in a uniformly convergent Fourier series of Jacobi polynomials* $\bar{\omega}_n(x)$.

As far as we know this is the latest result on this topic. In our note we improve Natanson's result by proving the following:

THEOREM 2. *If* $f(x)$ *has* p *continuous derivatives on* $[-1, 1]$ *and* $f^{(p)}(x) \in \text{Lip}\, \mu$ $(0 < \mu < 1)$, *then for* $p + \mu \geqq \sigma + \frac{1}{2}$ *and* $-1 \leqq x \leqq 1$,

$$(1.2) \qquad |f(x) - S_n(x)| \leqq c_1^* \ln n/n^{p+\mu-\sigma-\frac{1}{2}};$$

for $p + \mu \geqq \frac{1}{2}$,

$$(1.3) \qquad (1 - x)^{\frac{1}{4}(2\alpha+1)}(1 + x)^{\frac{1}{4}(2\beta+1)}|f(x) - S_n(x)| \leqq c_2^* \ln n/n^{p+\mu};$$

and for $p + \mu \geqq \sigma + 2r + \frac{1}{2}$ *and* $r \geqq 1$,

$$(1.4) \qquad |f^{(r)}(x) - S_n^{(r)}(x)| \leqq c_3^* \ln n/n^{p+\mu-\sigma-2r-\frac{1}{2}},$$

where $\sigma = \max(\alpha, \beta), \alpha \geqq 0, \beta \geqq 0$.

It is worthwhile to point out that the recent results of Suetin [**4**] and that of Saxena [**3**] are particular cases of Theorem 2.

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2. Jacobi polynomials. We state in this section some well-known results which will be required later.

From [**1**] we have for $\gamma > -1$ and $\lambda > -1$,

$$\frac{\Gamma(n+\gamma+\lambda+1)}{\Gamma(n+\gamma+1)} < d_1 n^{\lambda}, \tag{2.1}$$

where d_1 is a positive constant. Hence we obtain that

$$\frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \leqq d_2 n. \tag{2.2}$$

Also from [**5**] we have for $-1 \leqq x \leqq 1$,

$$|\omega_n(x)| \leqq d_3 n^{\sigma}, \tag{2.3}$$

where $\sigma = \max(\alpha, \beta) \geqq -\frac{1}{2}$ and d_3 is a constant depending on α and β and

$$(1-x)^{\frac{1}{4}(2\alpha+1)}(1+x)^{\frac{1}{4}(2\beta+1)}|\omega_n(x)| \leqq d_4 n^{-\frac{1}{2}} \tag{2.4}$$

for $\alpha \geqq -\frac{1}{2}, \beta \geqq -\frac{1}{2}$. Then from (1.1), (2.2), (2.3) and (2.4) it follows that for $-1 \leqq x \leqq 1$,

$$|\bar{\omega}_n(x)| \leqq d_5 n^{\sigma+\frac{1}{2}} \tag{2.5}$$

and

$$(1-x)^{\frac{1}{4}(2\alpha+1)}(1+x)^{\frac{1}{4}(2\beta+1)}|\bar{\omega}_n(x)| \leqq d_6 \tag{2.6}$$

for $\alpha \geqq -\frac{1}{2}, \beta \geqq -\frac{1}{2}$. Further upon applying Markov's inequality [**1**] to (2.3) and (2.5) we obtain

$$|\omega_n^{(r)}(x)| \leqq d_3^* n^{\sigma+2r} \tag{2.7}$$

and

$$|\bar{\omega}_n^{(r)}(x)| \leqq d_5^* n^{\sigma+2r+\frac{1}{2}}. \tag{2.8}$$

3. Some lemmas. In order to prove Theorem 2 we need the following lemmas.

LEMMA 1. *If* $-1 \leqq x \leqq 1$ *and* $\alpha \geqq -\frac{1}{2}, \beta \geqq -\frac{1}{2}$ *then*

$$\int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta}\left|\sum_{k=0}^{n} \bar{\omega}_k(x)\bar{\omega}_k(t)\right| dt \leqq c_5^* n^{\sigma+1}, \tag{3.1}$$

$$(1-x)^{\frac{1}{4}(2\alpha+1)}(1+x)^{\frac{1}{4}(2\beta+1)} \times \int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta}\left|\sum_{k=0}^{n} \bar{\omega}_k(x)\bar{\omega}_k(t)\right| dt \leqq c_6^* n^{\frac{1}{2}} \tag{3.2}$$

and

$$\int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta}\left|\sum_{k=r}^{n} \bar{\omega}_k^{(r)}(x)\bar{\omega}_k(t)\right| dt \leqq c_7^* n^{\sigma+2r+1}, \tag{3.3}$$

where $\sigma = \max(\alpha, \beta)$.

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Proof. First of all, we evaluate the integral

$$\int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta}dt.$$

The substitution $t = 2u - 1$, yields

$$\text{(3.4)} \qquad \int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta}dt = 2^{\alpha+\beta+1}\int_{0}^{1} (1-u)^{\alpha}u^{\beta}du$$
$$= 2^{\alpha+\beta+1}B(\alpha+1, \beta+1)$$
$$= \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}.$$

Now, if $\sigma = \max(\alpha, \beta) \geqq -\frac{1}{2}$, we get by making use of (2.5),

$$\text{(3.5)} \qquad \int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta}\left[\sum_{k=0}^{n} \bar{\omega}_k(t)\bar{\omega}_k(x)\right]^2 dt = \sum_{k=0}^{n} |\bar{\omega}_k(x)|^2$$
$$\leqq c_1\sum_{k=0}^{n} k^{2\sigma+1}$$
$$\leqq c_2 n^{2\sigma+2}.$$

Finally, with the help of Cauchy's inequality and (3.4), (3.5) we obtain

$$\int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta}\left|\sum_{k=0}^{n} \bar{\omega}_k(t)\bar{\omega}_k(x)\right|dt$$
$$\leqq \left[\int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta}\left\{\sum_{k=0}^{n} \bar{\omega}_k(t)\bar{\omega}_k(x)\right\}^2 dt\right]^{\frac{1}{2}}\left[\int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta}dt\right]^{\frac{1}{2}}$$
$$\leqq c_3 n^{\sigma+1},$$

from which (3.1) follows. By similar arguments (2.6) will yield (3.2), while (2.8) will yield (3.3).

LEMMA 2. *If* $-1 \leqq x \leqq 1, \alpha \geqq 0, \beta \geqq 0$ *and* $p + \mu \geqq \frac{1}{2}$ *then*

$$\text{(3.6)} \quad \int_{-1}^{1} (1-t^2)^{\frac{1}{2}(p+\mu)}(1-t)^{\alpha}(1+t)^{\beta}\left|\sum_{k=0}^{n} \bar{\omega}_k(x)\bar{\omega}_k(t)\right|dt \leqq c_8{}^* n^{\sigma+\frac{1}{2}}\ln n,$$

$$\text{(3.7)} \quad (1-x)^{\frac{1}{4}(2\alpha+1)}(1+x)^{\frac{1}{4}(2\beta+1)}$$
$$\times \int_{-1}^{1} (1-t^2)^{\frac{1}{2}(p+\mu)}(1-t)^{\alpha}(1+t)^{\beta}\left|\sum_{k=0}^{n} \bar{\omega}_k(x)\bar{\omega}_k(t)\right| dt \leqq c_9{}^*\ln n,$$

and

$$\text{(3.8)} \quad \int_{-1}^{1} (1-t^2)^{\frac{1}{2}(p+\mu)}(1-t)^{\alpha}(1+t)^{\beta}$$
$$\times \left|\sum_{k=r}^{n} \bar{\omega}_k{}^{(r)}(x)\bar{\omega}_k(t)\right| dt \leqq c_{10}{}^* n^{\sigma+2r+\frac{1}{2}} \ln n.$$

Proof. We denote by $\Delta_n(x)$ the part of $[-1, 1]$ on which $|x - t| \leqq 1/n$ and by $\delta_n(x)$ the rest of the interval. Consider now

$$(3.9) \qquad \int_{-1}^{1} (1 - t^2)^{\frac{1}{2}(p+\mu)}(1 - t)^\alpha(1 + t)^\beta \left| \sum_{k=0}^{n} \bar{\omega}_k(x)\bar{\omega}_k(t) \right| dt = \int_{\Delta_n(x)} + \int_{\delta_n(x)} = J_1 + J_2.$$

Since

$$J_1 = \int_{\Delta_n(x)} (1 - t^2)^{\frac{1}{2}(p+\mu)}(1 - t)^\alpha(1 + t)^\beta \left| \sum_{k=0}^{n} \bar{\omega}_k(x)\bar{\omega}_k(t) \right| dt$$
$$\leqq c_4 \int_{\Delta_n(x)} (1 - t^2)^{\frac{1}{4}(2p+2\mu-1)} \sum_{k=0}^{n} [(1 - t)^{\frac{1}{4}(2\alpha+1)}(1 + t)^{\frac{1}{4}(2\beta+1)}|\bar{\omega}_k(t)|]|\bar{\omega}_k(x)|dt,$$

making use of (2.5) and (2.6) we obtain

$$J_1 \leqq c_5 \sum_{k=0}^{n} k^{\sigma+\frac{1}{2}} \int_{\Delta_n(x)} dt, \text{ for } p + \mu \geqq 1/2$$
$$\leqq c_6 n^{-1} \sum_{k=0}^{n} k^{\sigma+\frac{1}{2}}$$
$$(3.10) \qquad \leqq c_7 n^{\sigma+\frac{1}{2}}.$$

To find an estimate for the integral over $\delta_n(x)$ we make use of the Christoffel formula [**5**]:

$$(3.11) \qquad \sum_{k=0}^{n} \bar{\omega}_k(x)\bar{\omega}_k(t) = \frac{\Gamma(n + 2)\Gamma(n + \alpha + \beta + 2)2^{-(\alpha+\beta)}}{\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)(2n + \alpha + \beta + 2)} \times \left[\frac{\omega_{n+1}(x)\omega_n(t) - \omega_n(x)\omega_{n+1}(t)}{x - t}\right].$$

Since $|x - t| > 1/n$ for $t \in \delta_n(x)$, we have therefore, making use of (2.1), (2.3), (2.4) and (3.11),

$$J_2 = \int_{\delta_n(x)} (1 - t^2)^{\frac{1}{2}(p+\mu)}(1 - t)^\alpha(1 + t)^\beta \left| \sum_{k=0}^{n} \bar{\omega}_k(t)\bar{\omega}_k(x) \right| dt$$
$$\leqq c_8 n \int_{\delta_n(x)} (1 - t^2)^{\frac{1}{2}(p+\mu)}(1 - t)^\alpha(1 + t)^\beta \times \left[\frac{|\omega_{n+1}(x)||\omega_n(t)| + |\omega_n(x)||\omega_{n+1}(t)|}{|x - t|}\right] dt$$
$$\leqq c_9 n^{\sigma+\frac{1}{2}} \int_{\delta_n(x)} (1 - t^2)^{\frac{1}{2}(p+\mu)-\frac{1}{4}}(1 - t)^{\alpha/2}(1 + t)^{\beta/2} \frac{dt}{|x - t|}$$
$$\leqq c_{10} n^{\sigma+\frac{1}{2}} \int_{\delta_n(x)} \frac{dt}{|x - t|}, \text{ for } p + \mu \geqq \tfrac{1}{2}$$
$$(3.12) \qquad \leqq c_{11} n^{\sigma+\frac{1}{2}} \ln n.$$

From (3.9), (3.10) and (3.12) we obtain (3.6). The proof of (3.7) can be given in same manner, using (2.1), (2.4), (2.6) and (3.11).

The proof of (3.8) is as follows:

$$(3.13)\qquad \int_{-1}^{1} (1-t^2)^{\frac{1}{2}(p+\mu)}(1-t)^{\alpha}(1+t)^{\beta}\left|\sum_{k=0}^{n} \bar{\omega}_k(t)\bar{\omega}_k^{(r)}(x)\right|dt = \int_{\Delta_n(x)} + \int_{\delta_n(x)} = J_1^* + J_2^*.$$

Making use of (2.6) and (2.8) we obtain

$$(3.14)\quad J_1^* \leqq \int_{\Delta_n(x)} (1-t^2)^{\frac{1}{2}(p+\mu)-\frac{1}{4}}(1-t)^{\alpha/2}(1+t)^{\beta/2} \times \sum_{k=r}^{n} [(1-t)^{\frac{1}{4}(2\alpha+1)}(1+t)^{\frac{1}{4}(2\beta+1)}|\bar{\omega}_k(t)|]|\bar{\omega}_k^{(r)}(x)|dt$$
$$\leqq c_{12}\sum_{k=0}^{n} k^{\sigma+2r+\frac{1}{2}}\int_{\Delta_n(x)} dt, \text{ for } p+\mu \geqq \tfrac{1}{2}$$
$$\leqq c_{13}n^{\sigma+2r+\frac{1}{2}}.$$

To estimate the integral over $\delta_n(x)$ we differentiate both sides of (3.11) r times to obtain

$$\sum_{k=r}^{n} \bar{\omega}_k(t)\bar{\omega}_k^{(r)}(x) = \theta_n\left[\frac{\omega_n(t)\omega_{n+1}^{(r)}(x) - \omega_{n+1}(t)\omega_n^{(r)}(x)}{x-t}\right] + \theta_n\sum_{\nu=0}^{r-1}\frac{(-1)^{r-\nu}r![\omega_n(t)\omega_{n+1}^{(\nu)}(x) - \omega_{n+1}(t)\omega_n^{(\nu)}(x)]}{\nu!(x-t)^{r-\nu+1}},$$

where

$$\theta_n = \frac{\Gamma(n+2)\Gamma(n+\alpha+\beta+2)2^{-\alpha-\beta}}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)(2n+\alpha+\beta+2)}.$$

Hence we have

(3.15)

$$J_2^* \leqq c_{14}n\int_{\delta_n(x)} (1-t^2)^{\frac{1}{4}(2p+2\mu-1)}(1-t)^{\alpha/2}(1+t)^{\beta/2}[(1-t)^{\frac{1}{4}(2\alpha+1)}(1+t)^{\frac{1}{4}(2\beta+1)} \times |\omega_n(t)||\omega_{n+1}^{(r)}(x)| + (1-t)^{\frac{1}{4}(2\alpha+1)}(1+t)^{\frac{1}{4}(2\beta+1)}|\omega_{n+1}(t)||\omega_n^{(r)}(x)|]\frac{dt}{|x-t|}$$
$$+ c_{14}n\int_{\delta_n(x)} (1-t^2)^{\frac{1}{4}(2p+2\mu-1)}(1-t)^{\alpha/2}(1+t)^{\beta/2} \times\sum_{\nu=0}^{r-1} r![(1-t)^{\frac{1}{4}(2\alpha+1)}(1+t)^{\frac{1}{4}(2\beta+1)}|\omega_n(t)||\omega_{n+1}^{(\nu)}(x)|$$
$$+ (1-t)^{\frac{1}{4}(2\alpha+1)}(1+t)^{\frac{1}{4}(2\beta+1)}|\omega_{n+1}(t)||\omega_n^{(\nu)}(x)|]\frac{dt}{\nu!(|x-t|)^{r-\nu+1}}$$
$$= u_1 + u_2.$$

With the help of (2.4) and (2.7) and bearing in mind that for $t \in \delta_n(x)$, $|x - t| > 1/n$, we get

$$u_1 \leqq c_{15} n^{\sigma+2r+\frac{1}{2}} \int_{\delta_n(x)} \frac{dt}{|x-t|}, \text{ for } p + \mu \geqq \tfrac{1}{2} \qquad (3.16)$$

$$\leqq c_{16} n^{\sigma+2r+\frac{1}{2}} \ln n.$$

Again using (2.4) and (2.7) we have for u_2

$$u_2 \leqq c_{17} n^{\sigma+\frac{1}{2}} \sum_{\nu=0}^{r-1} \frac{n^{2\nu}}{\nu!} \int_{\delta_n(x)} \frac{dt}{(|x-t|)^{r-\nu+1}}, \text{ for } p + \mu \geqq \tfrac{1}{2} \qquad (3.17)$$

$$\leqq c_{18} n^{\sigma+\frac{1}{2}} \sum_{\nu=0}^{r-1} \frac{n^{r+\nu}}{\nu!}$$

$$\leqq c_{18} n^{r+\sigma+\frac{1}{2}} \sum_{\nu=0}^{r-1} n^{\nu}$$

$$\leqq c_{19} n^{2r+\sigma+\frac{1}{2}}.$$

Thus from (3.13), (3.14), (3.15), (3.16) and (3.17) we obtain (3.8).

LEMMA 3 [**2**]. *Let $f^{(q)}(x) \in$ Lip μ $(0 < \mu < 1)$, in $[-1, 1]$; then there is a polynomial $Q_n(x)$ of degree at most n possessing the following properties:*

$$|f(x) - Q_n(x)| \leqq \frac{c_{11}^*}{n^{q+\mu}} \left[(1 - x^2)^{\frac{1}{2}(q+\mu)} + \frac{1}{n^{q+\mu}} \right] \qquad (3.18)$$

and

$$|f^{(r)}(x) - Q_n^{(r)}(x)| \leqq \frac{c_{12}^*}{n^{q+\mu-r}} \left[(1 - x^2)^{\frac{1}{2}(q+\mu-r)} + \frac{1}{n^{q+\mu-r}} \right] \qquad (3.19)$$

uniformly in $[-1, 1]$ and $r = 1, 2, \ldots, q$.

4. The proof of Theorem 2. We shall confine ourselves to proving (1.2). The proof of (1.3) and (1.4) can be given along the same lines.

Since $f^{(p)}(x) \in$ Lip μ $(0 < \mu < 1)$, and hence there exists a polynomial $\pi_n(x)$ due to Lemma 3, we write

$$|f(x) - S_n(x)| \leqq |f(x) - \pi_n(x)| + |\pi_n(x) - S_n(x)| \qquad (4.1)$$

$$= I_1 + I_2.$$

With the help of (3.18) we obtain

$$I_1 \leqq \frac{c_{20}}{n^{p+\mu}} \left[(1 - x^2)^{\frac{1}{2}(p+\mu)} + \frac{1}{n^{p+\mu}} \right] \leqq \frac{c_{21}}{n^{p+\mu}}. \qquad (4.2)$$

Now consider

$$I_2 \leqq \int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta} |\pi_n(t) - f(t)| \left| \sum_{k=0}^{n} \bar{\omega}_k(t) \bar{\omega}_k(x) \right| dt.$$

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From (3.18) it follows that

$$I_2 \leqq \frac{c_{22}}{n^{p+\mu}} \int_{-1}^{1} (1-t)^\alpha (1+t)^\beta (1-t^2)^{\frac{1}{2}(p+\mu)} \left| \sum_{k=0}^{n} \bar{\omega}_k(t)\bar{\omega}_k(x) \right| dt$$
$$+ \frac{c_{22}}{n^{2(p+\mu)}} \int_{-1}^{1} (1-t)^\alpha (1+t)^\beta \left| \sum_{k=0}^{n} \bar{\omega}_k(t)\bar{\omega}_k(x) \right| dt.$$

Now using (3.1) and (3.6) we obtain

(4.3) $$I_2 \leqq c_{23} n^{\sigma-p-\mu+\frac{1}{2}} \ln n + c_{24} n^{\sigma-2p-2\mu+1},$$

consequently from (4.1), (4.2) and (4.3) it follows that

(4.4) $$|f(x) - S_n(x)| \leqq c_{21} n^{-p-\mu} + c_{23} n^{\sigma-p-\mu+\frac{1}{2}} \ln n + c_{24} n^{\sigma-2p-2\mu+1}$$
$$\leqq c_{25} n^{\sigma-p-\mu+\frac{1}{2}} \ln n, \; p+\mu \geqq \sigma + \tfrac{1}{2}.$$

This completes the proof of (1.2). The proof of (1.4) requires both parts of Lemma 3.

Remark 1. If $E_n(f)$ is the best approximation of the function $f(x)$ by polynomials from H_n, where H_n is the set of all polynomials of degree less than or equal to n, then one can easily see from (4.4) that

$$E_n(f) \leqq c^* \ln n / n^{p+\mu-\sigma-\frac{1}{2}}, \text{ for } p+\mu \geqq \sigma + \tfrac{1}{2}.$$

Remark 2. If $E_n^{(r)}$ is the best approximation of $f^{(r)}(x)$ by polynomials of degree $\leqq n$ then it is easy to verify from (1.4) that

$$E_n^{(r)} \leqq c_3^* \ln n / n^{p+\mu-\sigma-2r-\frac{1}{2}}, \text{ for } p+\mu \geqq \sigma + 2r + \tfrac{1}{2}.$$

Following word for word the proof of the above lemmas and Theorem 2 and making some minor changes there we also easily establish the following:

THEOREM 3. *If $f(x)$ has p continuous derivatives on $[-1, 1]$ and $f^{(p)}(x) \in \text{Lip}\,\mu$, $0 < \mu < 1$, then for $-1 \leqq x \leqq 1$ and $p + \mu \geqq \max(\sigma + \frac{1}{2}, \frac{1}{2} - \tau)$,*

(4.5) $$|f(x) - S_n(x)| \leqq c_{26} \ln n / n^{p+\mu-\sigma-\frac{1}{2}},$$

and for $p + \mu \geqq \max(\frac{1}{2}, \frac{1}{2} - \tau)$,

(4.6) $$(1-x)^{\frac{1}{4}(2\alpha+1)}(1+x)^{\frac{1}{4}(2\beta+1)}|f(x) - S_n(x)| \leqq c_{27} \ln n / n^{p+\mu},$$

where $\sigma = \max(\alpha, \beta)$, $\tau = \min(\alpha, \beta)$; $\alpha \geqq -\frac{1}{2}, \beta \geqq -\frac{1}{2}$.

In view of the above theorem one can now also find an estimate for the best approximation as in Remark 1 under the condition

$$p + \mu \geqq \max(\sigma + \tfrac{1}{2}, \tfrac{1}{2} - \tau).$$

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