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HARDY SPACES ESTIMATES FOR MULTILINEAR OPERATORS WITH HOMOGENEOUS KERNELS

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Abstract. In this paper the authors prove that a class of multilinear operators formed by the singular integral or fractional integral operators with homogeneous kernels are bounded operators from the product spaces $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_K}(\mathbb{R}^n)$ to the Hardy spaces $H^q(\mathbb{R}^n)$ and the weak Hardy space $H^{q,\infty}(\mathbb{R}^n)$, where the kernel functions Ω_{ij} satisfy only the L^s-Dini conditions. As an application of this result, we obtain the (L^p, L^q) boundedness for a class of commutator of the fractional integral with homogeneous kernels and BMO function.

§1. Introduction and statements of results

It is known that the Jacobin determinant $J(f, g)$ of the functions pair (f, g) is defined by $J(f,g) = \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1}$. Lions and Meyer proved that J is bounded from the product space $L_1^2 \times L_1^2$ of the Sobolev spaces to the Hardy space $H^1(\mathbb{R}^n)$. Because of its importance in the harmonic analysis and partial differential equations, it has been studied by some authors in recent years. In 1992, Coifman and Grafakos extended the above result to give the mapping properties on the Hardy space for a class of the multilinear operator formed by the Calderón-Zygmund singular integrals [CG], [G]. Recently, Miyachi [Mi] studied the similar mapping properties for a class of the multilinear operator formed by the Calderón-Zygmund singular integrals and the Riesz potential operators. This is an extension of Coifman-Grafakos's results. In [DL4], we also obtained the same conclusion as in [G] but under a weaker condition.

In this paper, we will use the L^p boundedness of the singular integral with rough kernel and the (L^p, L^q) boundedness of rough fractional integral operator to prove that the multilinear operator formed by the products

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of the singular integral or fractional integral operators with homogeneous kernels are bounded from the product space $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_K}(\mathbb{R}^n)$ into the Hardy spaces $H^q(\mathbb{R}^n)$ and the weak Hardy space $H^{q,\infty}(\mathbb{R}^n)$. In our results, we need only that the kernel function Ω to satisfy L^s-Dini condition, which is weaker than the smoothness condition assumed in [CG], [G] and [Mi], respectively.

Now let us give some definitions. Suppose that S^{n-1} is the unit sphere of $\mathbb{R}^n$ ($n \geq 2$) equipped with normalized Lebesgue measure $d\sigma(x')$ and $\Omega \in L^1(S^{n-1})$ is homogeneous of degree zero on $\mathbb{R}^n$. Then the singular integral and fractional integral operators with homogeneous kernel are respectively defined by

$$T_\Omega f(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy \tag{1.1}$$

and

$$T_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)\, dy, \tag{1.2}$$

where $0 < \alpha < n$. Obviously, when $\Omega \equiv 1$ the operator $T_{\Omega,\alpha}$ is just the Riesz potential operator. See [CWW], [MW], [DL1]–[DL3] and [D] for the boundedness of $T_{\Omega,\alpha}$ on the various spaces (or weighted spaces).

We say that Ω satisfies the L^s-Dini condition ($s \geq 1$) if Ω is homogeneous of degree zero on $\mathbb{R}^n$ with $\Omega \in L^s(S^{n-1})$ and

$$\int_0^1 \omega_s(\delta) \frac{d\delta}{\delta} < \infty,$$

where $\omega_s(\delta)$ denotes the integral modulus of continuity of order s of Ω defined by

$$\omega_s(\delta) = \sup_{\|\rho\|<\delta} \left(\int_{S^{n-1}} |\Omega(\rho x') - \Omega(x')|^s \, d\sigma(x') \right)^{1/s}$$

and ρ is a rotation on S^{n-1} with $\|\rho\| = \sup_{x' \in S^{n-1}} |\rho x' - x'|$.

Throughout this article, N and K will denote fixed integers satisfying $K, N \geq 2$. It is said that r is the harmonic mean of $p_1, p_2, \ldots, p_K > 1$ if $1/r = 1/p_1 + 1/p_2 + \cdots + 1/p_K$. Moreover, for $i = 1, 2, \ldots, N$; $j = 1, 2, \ldots, K$,

(i) $0 \leq \alpha_{ij} < n/p_j$ and $\alpha := \sum_{j=1}^K \alpha_{ij} \geq 0$ ($i = 1, 2, \ldots, N$);

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(ii) Ω_{ij} satisfies $L^{s_{ij}}$-Dini condition with $s_{ij} \geq 1$.
(iii) If $\alpha_{ij} = 0$, then

$$\int_{S^{n-1}} \Omega_{ij}(x')\, d\sigma(x') = 0. \tag{1.3}$$

Now we define the K-linear operator by

$$L_{\Omega,\alpha}(\vec{f})(x) = \sum_{i=1}^{N} [T_{\Omega_{i1},\alpha_{i1}} f_1(x)][T_{\Omega_{i2},\alpha_{i2}} f_2(x)] \cdots [T_{\Omega_{iK},\alpha_{iK}} f_K(x)],$$

where $\vec{f} = (f_1, f_2, \ldots, f_K)$ and $T_{\Omega_{ij},\alpha_{ij}}$ defined in (1.2). When $\alpha_{ij} = 0$, $T_{\Omega_{ij},\alpha_{ij}}$ becomes indeed into the singular integral operator $T_{\Omega_{ij}}$ defined in (1.1). In this case, we need Ω_{ij} to satisfy (1.3).

Let us now formulate our main result as follows. In Theorem 1, the multilinear operator $L_{\Omega,\alpha}$ is formed by the fractional integral operators $T_{\Omega_{ij},\alpha_{ij}}$ or the singular integral operators $T_{\Omega_{ij}}$.

THEOREM 1. *Suppose that r is the harmonic mean of $p_1, p_2, \ldots, p_K > 1$. For $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, K$, $\alpha_{ij} \geq 0$ satisfy* (i) *and Ω_{ij} satisfies* (ii) *with $s_{ij} > p'_j$ and* (iii), *respectively. Moreover, $0 \leq \alpha < n - 1$ and $1/q = 1/r - \alpha/n$. If the harmonic mean of any proper subset of the set $\{p_1, p_2, \ldots, p_K\}$ is greater than one and for all $\vec{f} \in (C_0^\infty)^K$,*

$$\int x^\beta L_{\Omega,\alpha}(\vec{f})(x)\, dx = 0 \quad \text{for } |\beta| \leq m, \tag{1.4}$$

where m satisfies $0 \leq m < n - 1 - \alpha$. Then we have the following conclusions:

(a) *When $\frac{n}{n+m+1+\alpha} < r \leq \frac{n}{n+\alpha}$ (equivalently $\frac{n}{n+m+1} < q \leq 1$), $L_{\Omega,\alpha}$ can be extended into a bounded mapping from $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_K}$ to H^q.*
(b) *When $r = \frac{n}{n+m+1+\alpha}$ (equivalently $q = \frac{n}{n+m+1}$), $L_{\Omega,\alpha}$ can be extended into a bounded mapping from $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_K}$ to $H^{q,\infty}$.*

Remark 1. If $\alpha = m = 0$, then the Theorem 1 is just the main conclusion obtained in [DL4]. Thus Theorem 1 generalized the main theorem in [DL4] in two ways, i.e., $\alpha \geq 0$ and $m \geq 0$.

As a direct application of Theorem 1, we prove that the commutator $T_{\Omega,\alpha,b}$ formed by the homogeneous fractional integral operator $T_{\Omega,\alpha}$ Gand a function $b(x)$ in BMO is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. This is an extension of Chanillo's famous result on commutator of the Riesz potential operator [Cha].

THEOREM 2. *Suppose that* $0 < \alpha < n-1$, $1 < p < n/\alpha$ *and* $1/q = 1/p - \alpha/n$. *If* $\Omega \in L^s(S^{n-1})$ $(s > \max\{p', q\})$ *satisfying* L^s*-Dini condition, then there is a constant* $C > 0$, *independent of* f, *such that*

$$\|T_{\Omega,\alpha,b}(f)\|_q \le C\|b\|_{BMO}\|f\|_p,$$

where the commutator $T_{\Omega,\alpha,b}$ *is defined by* $T_{\Omega,\alpha,b}(f)(x) = b(x)T_{\Omega,\alpha}f(x) - T_{\Omega,\alpha}(bf)(x)$.

§2. Some elementary results and lemmas

In this section let us recall some known results and give some lemmas which will be used in the proofs of our theorems.

THEOREM A. ([DL3]) *Suppose that* $0 < \alpha < n$, $1 < p < n/\alpha$ *and* $1/q = 1/p - \alpha/n$. *If* $\Omega \in L^s(S^{n-1})$ $(s > n/(n-\alpha))$ *to be homogeneous of degree zero on* $\mathbb{R}^n$, *then* $T_{\Omega,\alpha}$ *is bounded operator from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$.

THEOREM B. ([CZ], [Che]) *Let* $\Omega \in L^s(S^{n-1})$ $(s > 1)$ *to be homogeneous of degree zero on* $\mathbb{R}^n$ *and satisfy* (1.3). *Then for* $1 < p < \infty$ *there exists a* $C > 0$, *independent of* f, *such that*

$$\|T_\Omega f\|_p \le C\|f\|_p \quad \textit{and} \quad \|T_\Omega^* f\|_p \le C\|f\|_p,$$

where T_Ω^* *denotes the maximal operator of* T_Ω *defined by*

$$T_\Omega^* f(x) = \sup_{\varepsilon>0}\left|\int_{|x-y|\ge\varepsilon} \frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy\right|.$$

LEMMA 1. ([KW]) *Suppose that* Ω *satisfies the* L^s*-Dini condition* $(s > 1)$. *If there is a constant* a_0 *with* $0 < a_0 < 1/2$ *such that* $|x| < a_0 R$, *then*

$$\begin{aligned}\left(\int_{R<|y|<2R}\left|\frac{\Omega(y-x)}{|y-x|^n} - \frac{\Omega(y)}{|y|^n}\right|^s dy\right)^{1/s} \\ \le CR^{n/s-n}\left\{\frac{|x|}{R} + \int_{|x|/2R<\delta<|x|/R} \omega_s(\delta)\frac{d\delta}{\delta}\right\},\end{aligned}$$

where the constant C *is independent of* R *and* x.

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LEMMA 2. *Suppose that Ω satisfies the L^s-Dini condition for $s > 1$. Then there is a $C > 0$, i ndependent of f, such that for any $x_0 \in \mathbb{R}^n$, $t > 0$ and any x with $|x - x_0| \le t$,*

$$\left|\int_{|y-x_0|\ge 2t}\left(\frac{\Omega(x-y)}{|x-y|^n}-\frac{\Omega(x_0-y)}{|x_0-y|^n}\right)f(y)\,dy\right| \le C\big[M(|f|^{s'})(x_0)\big]^{1/s'},$$

where M denotes the Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{r>0}\frac{1}{r^n}\int_{|x-y|<r}|f(y)|\,dy.$$

In fact, by Hölder's inequality and Lemma 1 we have

$$\begin{aligned}
&\left|\int_{|y-x_0|\ge 2t}\left(\frac{\Omega(x-y)}{|x-y|^n}-\frac{\Omega(x_0-y)}{|x_0-y|^n}\right)f(y)\,dy\right| \\
&\le \sum_{j=1}^{\infty}\left(\int_{2^j t\le|y-x_0|<2^{j+1}t}\left|\frac{\Omega(x-y)}{|x-y|^n}-\frac{\Omega(x_0-y)}{|x_0-y|^n}\right|^s dy\right)^{1/s} \\
&\qquad\qquad\qquad\qquad \times\left(\int_{|y-x_0|<2^{j+1}t}|f(y)|^{s'}dy\right)^{1/s'} \\
&\le C\sum_{j=1}^{\infty}\left(\frac{1}{2^j}+\int_{|x-x_0|/2^{j+1}t}^{|x-x_0|/2^j t}\omega_s(\delta)\frac{d\delta}{\delta}\right)\left(\frac{1}{(2^{j+1}t)^n}\int_{|y-x_0|<2^{j+1}t}|f(y)|^{s'}dy\right)^{1/s'} \\
&\le C\big[M(|f|^{s'})(x_0)\big]^{1/s'}\left(1+\int_0^1\omega_s(\delta)\frac{d\delta}{\delta}\right) \le C\big[M(|f|^{s'})(x_0)\big]^{1/s'}.
\end{aligned}$$

LEMMA 3. ([DL3]) *Suppose that $0 < \alpha < n$ and Ω satisfies the L^s-Dini condition with $s > 1$. If there is a constant a_0 with $0 < a_0 < 1/2$ such that $|x| < a_0 R$, then*

$$\begin{aligned}
&\left(\int_{R<|y|<2R}\left|\frac{\Omega(y-x)}{|y-x|^{n-\alpha}}-\frac{\Omega(y)}{|y|^{n-\alpha}}\right|^s dy\right)^{1/s} \\
&\qquad \le CR^{n/s-(n-\alpha)}\left\{\frac{|x|}{R}+\int_{|x|/2R<\delta<|x|/R}\frac{\omega_r(\delta)}{\delta}\,d\delta\right\},
\end{aligned}$$

where the constant $C > 0$ is independent of R and x.

Lemma 4. *Suppose that $0 < \alpha < n$ and Ω satisfies the L^s-Dini condition with $s > 1$. Then there is a $C > 0$, independent of f, such that for any $x_0 \in \mathbb{R}^n$, $t > 0$ and any x with $|x - x_0| \le t$,*

$$\left|\int_{|y-x_0|\ge 2t}\left(\frac{\Omega(x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(x_0-y)}{|x_0-y|^{n-\alpha}}\right) f(y)\,dy\right| \le C\big[M_{\alpha s'}(|f|^{s'})(x_0)\big]^{1/s'},$$

where M_λ denotes the fractional maximal operator defined by

$$M_\lambda f(x) = \sup_{r>0}\frac{1}{r^{n-\lambda}}\int_{|x-y|<r}|f(y)|\,dy \quad \textit{for } 0 < \lambda < n.$$

By Lemma 3 and using the same method in proving Lemma 2, we may get Lemma 4.

Lemma 5. *Suppose that $0 < \alpha < n$, $1 < p < n/\alpha$ and $1/r = 1/p - \alpha/n$. If $1 \le s' < p$, then*

$$\big\|\big[M_{\alpha s'}(|f|^{s'})(\,\cdot\,)\big]^{1/s'}\big\|_r \le C\|f\|_p.$$

In fact, this is a direct result of the (L^p, L^r) boundedness the fractional maximal operator M_λ (see [T], for example).

§3. Proof of Theorem 1

First let us consider the case of $\alpha_{ij} > 0$ for all i, j. In the proof of Theorem 1 we use some idea from [G]. Take $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\text{supp}(\phi) \subset \{x \in \mathbb{R}^n : |x| \le 1\}$. For $x, x_0 \in \mathbb{R}^n$ we define $\phi_{t,x_0}(x) = \frac{1}{t^n}\phi\big(\frac{x-x_0}{t}\big)$. By the maximal function characterization of the Hardy spaces and weak Hardy spaces (see [S] and [FS], or [Lu]), we need to verify that

(3.1)
$$\sup_{t>0}\left|\int \phi_{t,(\,\cdot\,)}(x) L_{\Omega,\alpha}(\vec{f})(x)\,dx\right| \in \begin{cases} L^q(\mathbb{R}^n), & \text{for } n/(n+m+1) < q \le 1\\ L^{q,\infty}(\mathbb{R}^n), & \text{for } q = n/(n+m+1).\end{cases}$$

Fix a smooth cut-off $\eta(x)$ such that $\eta(x) \ge 0$ and $\eta(x) \equiv 1$ on $|x| < 2$, $\eta(x) = 0$ on $|x| \ge 4$. Let $\eta_0(x) = \eta\big(\frac{x-x_0}{t}\big)$ and $\eta_1(x) = 1 - \eta_0(x)$. By expanding the following equality

$$L_{\Omega,\alpha}(\eta_0 f_1, \dots, \eta_0 f_K) = L_{\Omega,\alpha}(f_1 - \eta_1 f_1, \dots, f_K - \eta_1 f_K),$$

and solving out $L_{\Omega,\alpha}(\vec{f})$, we may get $L_{\Omega,\alpha}(\vec{f})(x) = L_0 + L_1 + \cdots + L_K$, where

$$
\begin{aligned}
L_0 &= L_{\Omega,\alpha}(\eta_0 f_1, \eta_0 f_2, \ldots, \eta_0 f_K)\\
L_1 &= \sum_{j=1}^{K} L_{\Omega,\alpha}(f_1, \ldots, \eta_1 f_j, \ldots, f_K)\\
L_2 &= -\sum_{1\le j_1<j_2\le K} L_{\Omega,\alpha}(f_1, \ldots, \eta_1 f_{j_1}, \ldots, \eta_1 f_{j_2}, \ldots, f_K)\\
&\quad \cdots\cdots\cdots\\
L_l &= (-1)^{l+1} \sum_{1\le j_1<j_2<\cdots<j_l\le K} L_{\Omega,\alpha}(f_1, \ldots, \eta_1 f_{j_1}, \ldots, \eta_1 f_{j_2}, \ldots, \eta_1 f_{j_l}, \ldots, f_K)\\
&\quad \cdots\cdots\cdots\\
L_K &= (-1)^{K+1} L_{\Omega,\alpha}(\eta_1 f_1, \eta_1 f_2, \ldots, \eta_1 f_K).
\end{aligned}
$$

To prove (3.1), it suffices to show for $0 \le l \le K$

$$
\sup_{t>0}\left|\int \phi_{t,(\,\cdot\,)}(x) L_l(x)\,dx\right| \in L^q(\mathbb{R}^n), \quad \text{if } n/(n+m+1) < q \le 1, \tag{3.2}
$$

and

$$
\left|\left\{x_0 \in \mathbb{R}^n : \sup_{t>0}\left|\int \phi_{t,x_0}(x) L_l(x)\,dx\right| > \lambda\right\}\right| \le \frac{C}{\lambda^q} \prod_{j=1}^{K} \|f_j\|_{p_j}^q, \tag{3.3}
$$
$$
\text{if } q = n/(n+m+1),
$$

where $C > 0$ is independent of λ and $\vec{f}$.

Let us begin by giving the estimate for the term L_1. Note that

$$
\begin{aligned}
|L_1(x)| \le \sum_{j=1}^{K}\Bigg(\sum_{i=1}^{N} &|T_{\Omega_{i1},\alpha_{i1}} f_1(x)| \cdots \big[|T_{\Omega_{ij},\alpha_{ij}}(\eta_1 f_j)(x) - T_{\Omega_{ij},\alpha_{ij}}(\eta_1 f_j)(x_0)|\\
&+ |T_{\Omega_{ij},\alpha_{ij}}(\eta_1 f_j)(x_0)|\big] \cdots |T_{\Omega_{iK},\alpha_{iK}} f_K(x)|\Bigg),
\end{aligned}
$$

and by Lemmas 3 and 4 if $|x - x_0| \le t$, $\Omega \in L^{s'}(S^{n-1})$ then

(3.4)

$$
\begin{aligned}
&\sup_{t>0} |T_{\Omega,\alpha}(\eta_1 f)(x_0)| \le \sup_{t>0} \int_{|x_0-y|\ge 2t} \frac{|\Omega(x_0-y)|}{|x_0-y|^{n-\alpha}} |f(y)|\,dy \le T_{|\Omega|,\alpha}(|f|)(x_0),\\
&\sup_{t>0} |T_{\Omega,\alpha}(\eta_1 f)(x) - T_{\Omega,\alpha}(\eta_1 f)(x_0)| \le C[M_{\alpha s'}(|f|^{s'})(x_0)]^{1/s'}.
\end{aligned}
$$

Thus we get

(3.5)

$$
\begin{aligned}
&\sup_{t>0}\left|\int \phi_{t,x_0}(x)L_1(x)\,dx\right| \\
&\quad\le \sum_{l=1}^{K}\sum_{i=1}^{N}\sup_{t>0}\int |\phi_{t,x_0}(x)| \prod_{\substack{1\le j\le K\\ j\ne l}} |T_{\Omega_{ij},\alpha_{ij}}f_j(x)| \\
&\qquad\times \Big(|T_{\Omega_{il},\alpha_{il}}(\eta_1 f_l)(x)-T_{\Omega_{il},\alpha_{il}}(\eta_1 f_l)(x_0)|+|T_{\Omega_{il},\alpha_{il}}(\eta_1 f_l)(x_0)|\Big)\,dx \\
&\quad\le C\sum_{l=1}^{K}\sum_{i=1}^{N}\Big(\prod_{\substack{1\le j\le K\\ j\ne l}}|T_{\Omega_{ij},\alpha_{ij}}f_j|\Big)^{*}(x_0) \\
&\qquad\times \Big([M_{\alpha_{il}s'_{il}}(|f_l|^{s'_{il}})(x_0)]^{1/s'_{il}}+T_{|\Omega_{il}|,\alpha_{il}}(f_l)(x_0)\Big),
\end{aligned}
$$

where we denote the Hardy-Littlewood maximal function of g at x_0 by $g^*(x_0)$. Let $1/r_{ij}=1/p_j-\alpha_{ij}/n$ for $1\le j\le K$. If taking $1/\sigma_{il}=\sum_{\substack{1\le j\le K\\ j\ne l}}1/r_{ij}$, then we have $1/q=1/\sigma_{il}+1/r_{ij}$. Since the harmonic mean of any proper subset of the p_j's is greater than one, we have

$$1/\sigma_{il}=\sum_{\substack{1\le j\le K\\ j\ne l}}(1/p_j-\alpha_{ij}/n)<\sum_{\substack{1\le j\le K\\ j\ne l}}1/p_j<1.$$

Thus by Hölder's inequality, Theorem A and the L^p-boundedness of the Hardy-Littlewood maximal function and Lemma 5 (note that $s'_{il}<p_l$), the L^q norm in x_0 of the last term in (3.5) is bounded by

$$
\begin{aligned}
&C\sum_{l=1}^{K}\sum_{i=1}^{N}\Big\|[M_{\alpha_{il}s'_{il}}(|f_l|^{s'_{il}})(\,\cdot\,)]^{1/s'_{il}}+T_{|\Omega_{il}|,\alpha_{il}}(|f_l|)(\,\cdot\,)\Big\|_{r_{il}} \\
&\qquad\qquad\times\Big\|\Big(\prod_{\substack{1\le j\le K\\ j\ne l}}|T_{\Omega_{ij},\alpha_{il}}f_j|\Big)^{*}\Big\|_{\sigma_{il}} \\
&\quad\le C\sum_{l=1}^{K}\sum_{i=1}^{N}\Big(\|[M_{\alpha_{il}s'_{il}}(|f_l|^{s'_{il}})^{1/s'_{il}}\|_{r_{il}}+\|T_{|\Omega_{il}|,\alpha_{il}}(|f_l|)\|_{r_{il}}\Big) \\
&\qquad\qquad\times\Big\|\prod_{\substack{1\le j\le K\\ j\ne l}}T_{\Omega_{ij},\alpha_{il}}(f_j)\Big\|_{\sigma_{il}}
\end{aligned}
$$

$$\leq C\sum_{l=1}^{K}\sum_{i=1}^{N}\|f_l\|_{p_l}\prod_{\substack{1\leq j\leq K\\ j\neq l}}\|T_{\Omega_{ij},\alpha_{il}}(f_j)\|_{r_{il}}$$

$$\leq C\sum_{l=1}^{K}\|f_l\|_{p_l}\prod_{\substack{1\leq j\leq K\\ j\neq l}}\|f_j\|_{p_j}\leq C\prod_{j=1}^{K}\|f_j\|_{p_j}.$$

Hence

$$\left\|\sup_{t>0}\left|\int\phi_{t,(\,\cdot\,)}(x)L_1(x)\,dx\right|\right\|_q\leq C\prod_{j=1}^{K}\|f_j\|_{p_j}. \tag{3.6}$$

From the proof (3.6) it is easy to see that (3.6) holds indeed for $n/(n+m+1)\leq q\leq 1$. Thus for $q=n/(n+m+1)$ we can get the following inequality by (3.6)

$$\left|\left\{x_0\in\mathbb{R}^n:\sup_{t>0}\left|\int\phi_{t,x_0}(x)L_1(x)\,dx\right|>\lambda\right\}\right|\leq\frac{C}{\lambda^q}\prod_{j=1}^{K}\|f_j\|_{p_j}^q, \tag{3.7}$$

where C is independent of λ and $\vec{f}$.

Term L_2 is treated similarly. We write $L_2=L_{21}+L_{22}+L_{23}+L_{24}$, where

$$\begin{aligned}L_{21}=-\sum_{1\leq j_1<j_2\leq K}\Bigg(\sum_{i=1}^{N}&[T_{\Omega_{i1},\alpha_{i1}}f_1(x)]\cdots\\ &\cdots[T_{\Omega_{ij_1},\alpha_{ij_1}}(\eta_1 f_{j_1})(x)-T_{\Omega_{ij_1},\alpha_{ij_1}}(\eta_1 f_{j_1})(x_0)]\cdots\\ &\cdots[T_{\Omega_{ij_2},\alpha_{ij_2}}(\eta_1 f_{j_2})(x)-T_{\Omega_{ij_2},\alpha_{ij_2}}(\eta_1 f_{j_2})(x_0)]\cdots\\ &\cdots[T_{\Omega_{iK},\alpha_{iK}}f_K(x)]\Bigg)\end{aligned}$$

$$\begin{aligned}L_{22}=\sum_{1\leq j_1<j_2\leq K}\Bigg(\sum_{i=1}^{N}&[T_{\Omega_{i1},\alpha_{i1}}f_1(x)]\cdots[T_{\Omega_{ij_1},\alpha_{ij_1}}(\eta_1 f_{j_1})(x_0)]\cdots\\ &\cdots[T_{\Omega_{ij_2},\alpha_{ij_2}}(\eta_1 f_{j_2})(x)-T_{\Omega_{ij_2},\alpha_{ij_2}}(\eta_1 f_{j_2})(x_0)]\cdots\\ &\cdots[T_{\Omega_{iK},\alpha_{iK}}f_K(x)]\Bigg)\end{aligned}$$

$$L_{23}=\sum_{1\leq j_1<j_2\leq K}\Bigg(\sum_{i=1}^{N}[T_{\Omega_{i1},\alpha_{i1}}f_1(x)]\cdots$$

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$$\cdots[T_{\Omega_{ij_1},\alpha_{ij_1}}(\eta_1 f_{j_1})(x) - T_{\Omega_{ij_1},\alpha_{ij_1}}(\eta_1 f_{j_1})(x_0)]\cdots$$
$$\cdots[T_{\Omega_{ij_2},\alpha_{ij_2}}(\eta_1 f_{j_2})(x_0)]\cdots[T_{\Omega_{iK},\alpha_{iK}} f_K(x)]\Big)$$

$$L_{24} = -\sum_{1\le j_1<j_2\le K}\Big(\sum_{i=1}^{N}[T_{\Omega_{i1},\alpha_{i1}} f_1(x)]\cdots[T_{\Omega_{ij_1},\alpha_{ij_1}}(\eta_1 f_{j_1})(x_0)]\cdots$$
$$\cdots[T_{\Omega_{ij_2},\alpha_{ij_2}}(\eta_1 f_{j_2})(x_0)]\cdots[T_{\Omega_{iK},\alpha_{iK}} f_K(x)]\Big).$$

By Lemma 4 and (3.4) we may show that any term L_{2u} $(u = 1, 2, 3, 4)$ satisfy the following estimate

$$\sup_{t>0}\left|\int \phi_{t,x_0}(x) L_{2u}(x)\,dx\right| \le C\sum_{1\le j_1<j_2\le K}\sum_{i=1}^{N}\Big(\prod_{\substack{1\le l\le K\\ l\ne j_1, j_2}}|T_{\Omega_{il},\alpha_{il}} f_l|\Big)^*(x_0)$$
$$\times C_{ij_1}(f_{j_1})(x_0) C_{ij_2}(f_{j_2})(x_0).$$

By (3.4), we know that each $C_{ij_m}(f_{j_m})$ $(m = 1, 2)$ is either

$$[M_{\alpha_{ij_m} s'_{ij_m}}(|f_{j_m}|^{s'_{ij_m}})]^{1/s'_{ij_m}} \quad \text{or} \quad T_{|\Omega_{ij_m}|,\alpha_{ij_m}}(|f_{j_m}|).$$

By Theorem A and Lemma 5 (note that $s'_{ij_m} < p_{j_m}$), we have $\|C_{ij_m}(f_{j_m})\|_{r_{ij_m}} \le C\|f_{j_m}\|_{p_{j_m}}$. Now define σ_{iv} by $1/q = 1/r_{ij_1} + 1/r_{ij_2} + 1/\sigma_{iv}$. From the conditions of Theorem 1 we know that $\sigma_{iv} > 1$. Using the same method treated L_1, for $u = 1, 2, 3, 4$, we get

(3.8)
$$\left\|\sup_{t>0}\left|\int \phi_{t,(\,\cdot\,)}(x) L_{2u}(x)\,dx\right|\right\|_q$$
$$\le C\sum_{1\le j_1<j_2\le K}\sum_{i=1}^{N}\left\|\Big(\prod_{\substack{1\le l\le K\\ l\ne j_1, j_2}}|T_{\Omega_{il},\alpha_{il}} f_l|\Big)^*\right\|_{\sigma_{iv}}\|C_{ij_1}(f_{j_1})\|_{r_{ij_1}}\|C_{ij_2}(f_{j_2})\|_{r_{ij_2}}$$
$$\le C\sum_{1\le j_1<j_2\le K}\sum_{i=1}^{N}\left\|\prod_{\substack{1\le l\le K\\ l\ne j_1, j_2}} T_{\Omega_{il},\alpha_{il}} f_l\right\|_{\sigma_{iv}}\|f_{j_1}\|_{p_{j_1}}\|f_{j_2}\|_{p_{j_2}}$$
$$\le C\sum_{1\le j_1<j_2\le K}\sum_{i=1}^{N}\Big(\prod_{\substack{1\le l\le K\\ l\ne j_1, j_2}}\|f_l\|_{p_l}\Big)\|f_{j_1}\|_{p_{j_1}}\|f_{j_2}\|_{p_{j_2}} \le C\prod_{1\le l\le K}\|f_l\|_{p_l}.$$

Since (3.8) holds for $n/(n+m+1) \leq q \leq 1$, we have

$$\left|\left\{x_0 \in \mathbb{R}^n : \sup_{t>0}\left|\int \phi_{t,x_0}(x)L_2(x)\,dx\right| > \lambda\right\}\right| \leq \frac{C}{\lambda^q}\prod_{j=1}^{K}\|f_j\|_{p_j}^q. \tag{3.9}$$

where $q = n/(n+m+1)$ and C is independent of λ and $\vec{f}$.

Applying the method treated L_1 and L_2, we may prove that (3.2) and (3.3) hold also for the terms $L_3, L_4, \ldots, L_K$. Here we omit the details. Thus, to complete the proof of Theorem 1, it remains to verify that (3.2) and (3.3) hold still for L_0. In order to do this, we need the following lemma.

LEMMA 6. *Under the conditions of Theorem* 1, *we have*

(i) *if* $n/(n+m+1+\alpha) < r \leq n/(n+\alpha)$, *then there exists* $1 < d_j < p_j$, $(1 \leq j \leq K)$ *such that*

$$\sup_{t>0}\left|\int \phi_{t,x_0}(x)L_0(x)\,dx\right| \leq C\prod_{j=1}^{K}\left[M_{\alpha_{ij}d_j}(|f_j|^{d_j})(x_0)\right]^{1/d_j}; \tag{3.10}$$

(ii) *if* $r = n/(n+m+1+\alpha)$, *then*

$$\sup_{t>0}\left|\int \phi_{t,x_0}(x)L_0(x)\,dx\right| \leq C\prod_{j=1}^{K}\left[M_{\alpha_{ij}p_j}(|f_j|^{p_j})(x_0)\right]^{1/p_j}. \tag{3.11}$$

Proof. First we give the proof of (3.11). By the definition of $L_0(x)$ and the moment condition (1.4), we have

$$\begin{aligned}
&\left|\int \phi_{t,x_0}(x)L_0(x)\,dx\right| \\
&\quad= \Bigg|\sum_{i=1}^{N}\int\left[\phi_{t,x_0}(x) - \sum_{|\beta|\leq m}\frac{1}{\beta!}\phi_{t,x_0}^{(\beta)}(y_1)(x-y_1)^{\beta}\right] \\
&\qquad\qquad \times \int \frac{\Omega_{i1}(x-y_1)}{|x-y_1|^{n-\alpha_{i1}}}(\eta_0 f_1)(y_1)\,dy_1 \prod_{j=2}^{K} T_{\Omega_{ij},\alpha_{ij}}(\eta_0 f_j)(x)\,dx\Bigg| \\
&\quad\leq \sum_{i=1}^{N}\iint\left|\phi_{t,x_0}(x) - \sum_{|\beta|\leq m}\frac{1}{\beta!}\phi_{t,x_0}^{(\beta)}(y_1)(x-y_1)^{\beta}\right| \\
&\qquad\qquad \times \frac{|\Omega_{i1}(x-y_1)|}{|x-y_1|^{n-\alpha_{i1}}}|(\eta_0 f_1)(y_1)|\,dy_1 \prod_{j=2}^{K}|T_{\Omega_{ij},\alpha_{ij}}(\eta_0 f_j)(x)|\,dx.
\end{aligned} \tag{3.12}$$

Since

$$\left|\phi_{t,x_0}(x) - \sum_{|\beta|\le m} \frac{1}{\beta!}\phi_{t,x_0}^{(\beta)}(y_1)(x-y_1)^\beta\right| \le Ct^{-n-m-1}|x-y_1|^{m+1},$$

we have

(3.13)

$$\begin{aligned}
&\int\left|\phi_{t,x_0}(x) - \sum_{|\beta|\le m} \frac{1}{\beta!}\phi_{t,x_0}^{(\beta)}(y_1)(x-y_1)^\beta\right|\frac{|\Omega_{i1}(x-y_1)|}{|x-y_1|^{n-\alpha_{i1}}}|(\eta_0 f_1)(y_1)|\,dy_1 \\
&\le C\int t^{-n-m-1}|x-y_1|^{m+1}\frac{|\Omega_{i1}(x-y_1)|}{|x-y_1|^{n-\alpha_{i1}}}|(\eta_0 f_1)(y_1)|\,dy_1 \\
&= Ct^{-n-m-1}T_{|\Omega_{i1}|,m+1+\alpha_{i1}}(|\eta_0 f_1|)(x).
\end{aligned}$$

Since $\sum_{j=1}^K 1/p_j = (n+m+1+\alpha)/n$ and $\sum_{j=2}^K 1/p_j < 1$, we have $1/p_1 > (m+1+\alpha_{i1})/n$. Denote $1/\sigma_{i1} = 1/p_1 - (m+1+\alpha_{i1})/n$ and $1/r_{ij} = 1/p_j - \alpha_{ij}/n$ for $j = 2,\dots,K$, then by (3.12), (3.13) and applying Hölder's inequality and Theorem A we have

$$\begin{aligned}
&\sup_{t>0}\left|\int \phi_{t,x_0}(x)L_0(x)\,dx\right| \\
&\le C\sup_{t>0}\sum_{i=1}^N t^{-n-m-1}\|T_{|\Omega_{i1}|,m+1+\alpha_{i1}}(|\eta_0 f_1|)\|_{\sigma_{i1}}\prod_{j=2}^K\|T_{|\Omega_{ij}|,\alpha_{ij}}(|\eta_0 f_j|)\|_{r_{ij}} \\
&\le C\sup_{t>0} t^{-n-m-1}\|\eta_0 f_1\|_{p_1}\prod_{j=2}^K\|\eta_0 f_j\|_{p_j} \\
&\le C\prod_{j=1}^K\left[M_{\alpha_{ij}p_j}(|f_j|^{p_j})(x_0)\right]^{1/p_j},
\end{aligned}$$

where we use the assumption $\alpha = \sum_{j=1}^K \alpha_{ij}$ for all $i = 1,2,\dots,N$. This is just (3.11). Let us now consider (3.10). First we show that by

$$1/p_1 + \cdots + 1/p_K = 1/r < (n+m+1+\alpha)/n,$$

we can choose some d_j $(1 \le j \le K)$ such that

(a) $1 < d_j < p_j$ for $1 \le j \le K$;
(b) $1/d_2 + \cdots + 1/d_K < 1$;

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(c) $1/d_1 + \cdots + 1/d_K = (n + m + 1 + \alpha)/n$.

In fact, under the conditions of Theorem 1, it is easy to see that

$$\max\{1/p_2 + \cdots + 1/p_K, (m + 1 + \alpha)/n\} < \min\{1, 1/p_1' + (m + 1 + \alpha)/n\}.$$

Thus, we can choose δ such that

$$\max\{1/p_2+\cdots+1/p_K, (m+1+\alpha)/n\} < \delta < \min\{1, 1/p_1'+(m+1+\alpha)/n\}.$$

Taking $\varepsilon_2, \dots, \varepsilon_K > 0$, such that

$$(1 + \varepsilon_2)/p_2 + \cdots + (1 + \varepsilon_K)/p_K = \delta. \tag{3.14}$$

Obviously, for $2 \le j \le K$ we have $1 + \varepsilon_j < p_j$. Let

$$\varepsilon = (n + m + 1 + \alpha)/n - (1/p_1 + \cdots + 1/p_K) > 0.$$

Then by (3.14) we know that $\varepsilon > \varepsilon_2/p_2 + \cdots + \varepsilon_K/p_K$. Denote

$$\varepsilon_1 = [\varepsilon - (\varepsilon_2/p_2 + \cdots + \varepsilon_K/p_K)]p_1 > 0,$$

then

$$(n+m+1+\alpha)/n = \varepsilon+1/p_1+\cdots+1/p_K = (1+\varepsilon_1)/p_1+\cdots+(1+\varepsilon_K)/p_K$$

and $1 + \varepsilon_1 < p_1$. In fact, by $(m + 1 + \alpha)/n < \delta$, we have

$$\begin{aligned}
1 + \varepsilon_1 &= [\varepsilon - (\varepsilon_2/p_2 + \cdots + \varepsilon_K/p_K)]p_1 + 1 \\
&= \left[\frac{n + m + 1 + \alpha}{n} - \frac{1}{p_1} - \left(\frac{1 + \varepsilon_2}{p_2} + \cdots + \frac{1 + \varepsilon_K}{p_K}\right)\right]p_1 + 1 \\
&= \left[1/p_1' + (1 + \alpha)/n - \delta\right]p_1 + 1 < p_1/p_1' + 1 = p_1.
\end{aligned}$$

Now set $d_j = p_j/(1 + \varepsilon_j)$, $1 \le j \le K$, then $d_j{}'s$ satisfy (a), (b) and (c). Thus by the conclusion of (3.11), we obtain (3.10). Thus we complete the proof of Lemma 6. ∎

Below we use Lemma 6 to prove that (3.2) and (3.3) hold for L_0. First let us consider (3.2). By (3.10) and $1/q = 1/r_{i1} + \cdots + 1/r_{iK}$, we have

$$\begin{aligned}
\left\|\sup_{t>0}\left|\int \phi_{t,(\cdot)}(x)L_0(x)\,dx\right|\right\|_{L^q} &\le C\left\|\prod_{j=1}^{K}\left[M_{\alpha_{ij}d_j}(|f_j|^{d_j})(\cdot)\right]^{1/d_j}\right\|_{L^q} \\
&\le C\prod_{j=1}^{K}\left\|\left[M_{\alpha_{ij}d_j}(|f_j|^{d_j})(\cdot)\right]^{1/d_j}\right\|_{L^{r_{ij}}}.
\end{aligned}$$

Since $d_j < p_j$ and $1/r_{ij} = 1/p_j - \alpha_{ij}/n$, by the above inequality and Lemma 5 we have

$$\left\|\sup_{t>0}\left|\int \phi_{t,(\,\cdot\,)}(x)L_0(x)\,dx\right|\right\|_{L^q} \le C\prod_{j=1}^{K}\|f_j\|_{L^{p_j}}.$$

Finally let us give the weak estimate for L_0. For any $\lambda > 0$, let $\theta_0 = \lambda$, $\theta_K = 1$, and $\theta_1, \theta_2, \dots, \theta_{K-1} > 0$ be arbitrary which be chosen later. Then by (3.11) we get

$$\begin{aligned}(3.15)\qquad &\left\{x_0 : \sup_{t>0}\left|\int \phi_{t,x_0}L_0\,dx\right| > \lambda\right\}\\ &\subset \bigcup_{j=1}^{K}\left\{x_0 : \left[M_{\alpha_{ij}p_j}(|f_j|^{p_j})(x_0)\right]^{1/p_j} > \theta_{j-1}/\theta_j\right\}.\end{aligned}$$

We now take $\theta_1, \theta_2, \dots, \theta_{K-1} > 0$ such that

$$(3.16)\qquad \left(\frac{\theta_j}{\theta_{j-1}}\right)^{r_{ij}} = \frac{\prod_{j=1}^{K}\|f_j\|_{p_j}^{q}}{\lambda^q\|f_j\|_{p_j}^{r_{ij}}}, \quad j = 1, 2, \dots, K.$$

By $1/q = 1/r_{i1} + 1/r_{i2} + \cdots + 1/r_{iK}$, we have

$$\prod_{j=1}^{K}\left(\frac{\theta_j}{\theta_{j-1}}\right) = \prod_{j=1}^{K}\frac{\left(\prod_{j=1}^{K}\|f_j\|_{p_j}\right)^{q/r_{ij}}}{\lambda^{q/r_{ij}}\|f_j\|_{p_j}} = \frac{1}{\lambda}.$$

Combining (3.15), (3.16) with the weak boundedness of the fractional maximal operator M_α (see [T]) and noting that $1/(r_{ij}/p_j) = 1 - (\alpha_{ij}p_j)/n$, we have

$$\begin{aligned}(3.17)\qquad &\left|\left\{x_0 \in \mathbb{R}^n : \sup_{t>0}\left|\int \phi_{t,x_0}L_0\,dx\right| > \lambda\right\}\right|\\ &\le \sum_{j=1}^{K}\left|\left\{x_0 \in \mathbb{R}^n : M_{\alpha_{ij}p_j}(|f_j|^{p_j})(x_0) > \left(\frac{\theta_{j-1}}{\theta_j}\right)^{p_j}\right\}\right|\\ &\le C\sum_{j=1}^{K}\left(\left(\frac{\theta_j}{\theta_{j-1}}\right)^{p_j}\int_{\mathbb{R}^n}|f_j(x_0)|^{p_j}\,dx_0\right)^{r_{ij}/p_j}\\ &= C\sum_{j=1}^{K}\left(\frac{\theta_j}{\theta_{j-1}}\|f_j\|_{p_j}\right)^{r_{ij}} \le \frac{C}{\lambda^q}\prod_{j=1}^{K}\|f_j\|_{p_j}^{q}.\end{aligned}$$

In the last inequality we use (3.16). Thus (3.3) holds for L_0 and we prove Theorem 1 for the case of all $\alpha_{ij} > 0$.

As for the case of $\alpha_{ij} = 0$ with some i, j, the proof is almost the same as before. In fact, when $\alpha_{ij} = 0$, we have $r_{ij} = p_j$. By the definition of T_Ω^* and Lemmas 1 and 2 we get

$$\sup_{t>0} |T_\Omega(\eta_1 f)(x_0)| \le T_\Omega^* f(x_0) + C[M(|f|^{s'})(x_0)]^{1/s'}. \tag{3.18}$$

Moreover, for any x of satisfying $|x - x_0| \le t$ and $\Omega \in L^{s'}(S^{n-1})$

$$|T_\Omega(\eta_1 f)(x) - T_\Omega(\eta_1 f)(x_0)| \le C[M(|f|^{s'})(x_0)]^{1/s'}. \tag{3.19}$$

By (3.18), (3.19) and Theorem B and using the same method above, we may obtain the conclusion of Theorem1 for this case. Here we omit the details.

§4. Proof for Theorem 2

Now we apply the conclusion of Theorem 1 (taking $K = N = 2$) to give the proof of Theorem 2. In fact, it is easy to verify that

$$\begin{aligned} \|T_{\Omega,\alpha,b}(f)\|_q &= \sup_g \left| \int_{\mathbb{R}^n} g(x)\big[b(x)T_{\Omega,\alpha}f(x) - T_{\Omega,\alpha}(bf)(x)\big]\,dx \right| \\ &= \sup_g \left| \int_{\mathbb{R}^n} b(x)\big[g(x)T_{\Omega,\alpha}f(x) - f(x)T'_{\Omega,\alpha}g(x)\big]\,dx \right|, \end{aligned} \tag{4.1}$$

where the supremun is taken over all functions $g(x) \in L^{q'}$ with $\|g\|_{q'} \le 1$. Moreover, $T'_{\Omega,\alpha}$ denotes the adjoint operator of $T_{\Omega,\alpha}$.

On the other hand, if replacing the singular integral operator by the identical operator in the K-linear operator $L_{\Omega,\alpha}(\vec{f})$, then the conclusion of Theorem 2 still holds. Note that

$$\int_{\mathbb{R}^n} \Big(g(x)T_{\Omega,\alpha}f(x) - f(x)T'_{\Omega,\alpha}g(x)\Big)\,dx = 0. \tag{4.2}$$

Hence, by the conditions of Theorem 2 and (4.2), and using the result of Theorem 1, we have

$$\|gT_{\Omega,\alpha}f - fT'_{\Omega,\alpha}g\|_{H^1} \le C\|f\|_p\|g\|_{q'}. \tag{4.3}$$

Since $b(x) \in BMO$ and $(H^1)^* = BMO$, by (4.1), (4.3) and the choice of g, we get

$$\|T_{\Omega,\alpha,b}(f)\|_q \leq \sup_g \|b\|_{BMO}\|gT_{\Omega,\alpha}f - fT'_{\Omega,\alpha}g\|_{H^1} \leq C\|b\|_{BMO}\|f\|_p.$$

Thus, we obtain the conclusion of Theorem 2.

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