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MARKOV CHAIN METHOD FOR COMPUTING THE RELIABILITY OF HAMMOCK NETWORKS

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In this paper, we develop a new method for evaluating the reliability polynomial of a hammock network. The method is based on a homogeneous absorbing Markov chain and provides the exact reliability for networks of width less than 5 and arbitrary length. Moreover, it produces a lower bound for the reliability polynomial for networks of width greater than or equal to 5. To investigate how sharp this lower bound is, we compare our method with other approximation methods and it proves to be the most accurate in terms of absolute as well as relative error. Using the fundamental matrix, we also calculate the average time to absorption, which provides the mean length of a network that is expected to work.

Keywords: graph polynomials, hammock networks, Markov chains, network reliability

1. INTRODUCTION

Providing high-quality services has always been of the utmost importance. The particular cases represented by communications and computations are no exception, and the quality of such services has been a concern since the earliest days—for both providers and users. Unfortunately, the first transatlantic telegraph cables (communications) were very sensitive to say the least, as were the first electronic computers using vacuum tubes (computations). Obviously, such systems were in dire need of practical designs to enhance their reliability. That is why John von Neumann [38], as well as Edward F. Moore and Claude E. Shannon [31,32] , have devised schemes using redundancy for building reliable systems from unreliable components ("arbitrarily good circuits from relays arbitrarily poor in reliability"). And that is how network reliability was established in 1956, around a probabilistic model where the nodes of the network were considered to be perfectly reliable, while the edges could fail independently with a certain probability q. The fundamental problem was that of estimating the probability that two (or more) nodes are connected under these conditions. As all the edges have the same probability to work, $p = 1 - q$, the solution is represented by the well-established reliability polynomial of a network [11,31] . For a detailed presentation of the landscape of network reliability, see the recent survey of Brown *et al.* [6].

Today, networks are ubiquitous starting from roads, railways [4,30] , and power lines [8] to internet [29], telecommunications [27] and wireless sensor networks [7,20,21] , and even models of infectious diseases epidemiology [33,34] , so reliability polynomials can be applied to a plethora of different networks, generating lots of research. However, the seminal work of Moore and Shannon [31] was intended for computation (rather than communications) networks, proposing a very practical redundancy scheme for circuit design. This was built on a thought-provoking type of 2D network called hammock (which can also be drawn as a brick-wall pattern, having alternating vertical connections) of length l and width w.

With the advent of fin field-effect transistors (FinFETs) a decade ago and of various quantum chips lately, the last few years have witness a growing interest on hammocks, not only from a theoretical point of view (see [15,16,17,39]), but also because they are a perfect fit for both FinFETs (in fact any array-based transistors) [13], and 2D arrays of qubits, like Google's Bristlecone [1,5,37] . Besides, extensions of hammocks to 3D have also been suggested and analyzed [12], explaining the high reliability of transport on the axonal cytoskeleton while challenging consecutive systems [3].

Although the brick-wall networks (proposed more than 60 years ago) have a regular, repetitive structure, no general formula for their reliability polynomials has been reported yet. Cowell *et al.* [13] have calculated (using an algorithm based on a recursive depth-first traversal of a binary tree) the exact reliability polynomials for a few particular cases of small size, more precisely for the 29 hammock networks presented by Moore and Shannon in their original paper. Drăgoi *et al.* [17] have compared compositions of series and parallel networks of two devices to size-equivalent hammock networks, showing that compositions of series and parallel networks are not able to surpass hammock networks in terms of reliability. Dăuş and Jianu [16] have calculated the first and second non-zero coefficients of the reliability polynomial and have approximated the reliability polynomial by full Hermite interpolation. Recently, Cowell *et al.* [14] have used the beta distribution to approximate the reliability polynomial. These methods of approximation (based on Hermite interpolation and, respectively, beta distribution) will be compared to our new method based on Markov chains in Section 4.

Markov chains were introduced by the Russian mathematician Andrey Markov in 1907 to describe stochastic processes having the "memoryless" property. We say that a process satisfies this property (named *Markov property*) if the future evolution of the process depends only on the state at the present time and not on its past history.

Markov chains have been proved to be an efficient instrument in a variety of fields, including reliability problems. In their book "Mathematical Theory of Reliability" [2], first published in 1965, Barlow and Proschan dedicated a special chapter (Chapter 5) to Markov chains and their applications. Chao and Fu [9,10] used nonhomogeneous Markov chains for studying the reliability of a large linearly connected system. Consecutive-k-out-of-n:F systems with $(k-1)$-step Markov dependence were analyzed by Fu and Hu in [18,19] . Recently, Zhou *et al.* [40] studied the reliability and importance measures for m-consecutive-k, l-out-of-n systems with nonhomogeneous Markov-dependent components; Markov chains were also used in [22,23] to assess the reliability of a system subject to external shocks; Chakraborty *et al.* [7] developed a Monte-Carlo Markov chain approach to evaluate the coverage-area reliability of mobile wireless sensor networks with multistate nodes. Koutras [28] provides a unified framework for reliability structures which can be described by finite Markov chains.

A discrete-time Markov chain can be described as follows (see [24–26,35]):

Consider a set of states, $S = \{s_1, s_2, \ldots, s_n\}$. The process starts in one of these states and moves in successive steps from one state to another. Assume that the probability of moving from the current state s_i to another state s_j in one step does not depend on the previous states the system was in before the current state s_i (Markov property). For $i, j = 1, 2, \ldots, n$ and $t = 0, 1, \ldots$, we denote by $p_{ij}(t)$ the conditional probability of being in state s_j at the moment $t+1$, knowing that the system was in state s_i at the moment t. If these probabilities (called *transition probabilities*) do not depend on t, $p_{ij}(t) = p_{ij}$ for any $t = 0, 1, \ldots$, we say that the Markov chain is *homogeneous* (or stationary in time). In this paper, homogeneous Markov chains will be used to calculate the reliability polynomials of hammock networks.

We denote by $P = (p_{ij})_{i,j=1,2,\ldots,n}$ the *transition matrix* of the Markov chain. The ijth entry of the matrix P^k, $p_{ij}^{(k)}$, gives the probability that the homogeneous Markov chain, starting in state s_i, will be in state s_j after k steps.

Moreover, if the probability vector $U = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ represents the starting distribution, then the probability that the chain is in state s_i after k steps is the ith entry of the vector UP^k.

A Markov chain is said to be *absorbing* if it has at least one absorbing state (a state impossible to leave from: $p_{ii} = 1$), and if from every state it is possible to go to an absorbing state (not necessarily in one step). In an absorbing Markov chain, a state which is not absorbing is called *transient*. The transition matrix of an absorbing Markov chain is written in the following form:

$$P = \begin{pmatrix} I_r & \mathbf{0}_{r,n-r} \\ R & Q \end{pmatrix}, \quad \textbf{(1.1)}$$

where r is the number of absorbing states and $n-r$ is the number of transient states, Q is the one-step transition matrix between transient states, R is the one-step transition matrix between a transient state and an absorbing state, and I_r is the identity matrix of size r. Consequently, the powers of the transition matrix P are of the form:

$$P^k = \begin{pmatrix} I_r & \mathbf{0}_{r,n-r} \\ R_k & Q^k \end{pmatrix}, \quad \textbf{(1.2)}$$

where $R_k = (I_{n-r} + Q + Q^2 + \cdots + Q^{k-1})R$.

In an absorbing Markov chain, the probability that the process will be absorbed is 1 (i.e., $Q^k \to \mathbf{0}_{n-r}$ as $k \to \infty$). Furthermore, the matrix $I_{n-r} - Q$ is invertible and its inverse,

$$N = (I_{n-r} - Q)^{-1} = I_{n-r} + Q + Q^2 + \cdots \quad \textbf{(1.3)}$$

is called the fundamental matrix of the absorbing chain. The ijth entry of N, n_{ij}, gives the expected number of times that the process is in state s_j given that it starts in state s_i.

If t_i is the expected number of steps before the chain is absorbed, given that the process starts in state s_i, and t is the column vector whose ith entry is t_i, then

$$t = Nc,$$

where $c = (1, 1, \ldots, 1)^T$.

In Section 3, we use an absorbing Markov chain with one absorbing state ("fail"). Using the fundamental matrix N, we calculate the mean time spent in transient states (the average time to absorption), which gives the mean length of a network expected to work.

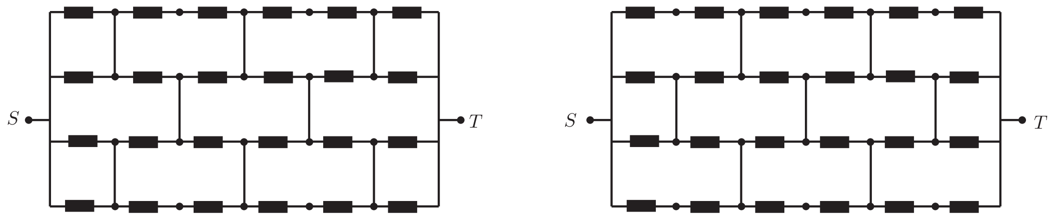

FIGURE 1. Brick-wall networks of dimensions $l = 6$, $w = 4$.

2. THE RELIABILITY POLYNOMIAL OF A HAMMOCK NETWORK

Colbourn [11] defined a network as a probabilistic graph, $H = (V, E)$, where V is the set of nodes and E the set of edges. The edges represent independent relays, and we suppose that each relay operates (is closed) with probability p and fails (is open) with probability $q = 1 - p$. We assume that nodes do not fail; the network failure is always a consequence of edge failures. The two-terminal reliability of the network, denoted by $h(p)$, is the probability that there exists a path (a sequence of adjacent edges) made of operational (closed) edges between two special nodes (called terminals): S (source/input) and T (terminus/output). This probability can be expressed as a polynomial function in p.

If $n = |E|$ is the size of the graph and, for $i = 1, 2, \ldots, n$, N_i is the number of S-T connected random graphs with exactly i edges, then the reliability of the network can be expressed by the following polynomial, called the *reliability polynomial* of the network (see [11,31]):

$$h(p) = \sum_{i=1}^{n} N_i p^i (1 - p)^{n-i}, \tag{2.1}$$

The problem of finding the reliability polynomial of a network belongs to the class of $\#P$-complete problems, a class of computationally equivalent counting problems (introduced by Valiant [36]) that are at least as difficult as the NP-complete problems.

Hammock networks (or brick-wall networks) were introduced by Moore and Shannon [31,32] in order to obtain reliable circuits using less reliable relays. In what follows, we give a short description of these networks (a detailed presentation is given by Dăuş and Jianu [15]).

A *brick-wall network* of length l and width w is formed by $l \times w$ identical relays disposed in w rows, each row consisting of l relays connected in series. Beside the horizontal connections, there also exist vertical connections between consecutive rows, in an alternate way, creating the brick-wall pattern, as can be seen in Figure 1.

All the relays in the leftmost column are connected to a source node, S, and all the relays in the rightmost column are connected to a terminus node, T. We suppose that each relay works with probability p (and fails with probability $q = 1 - p$). The reliability polynomial of the network, $h(p) = h_{l,w}(p)$, is the probability that the source node S and the terminus node T are connected.

For example, the reliability polynomial of a network of width $w = 1$ and length l (l relays connected in series) is

$$h_{l,1}(p) = p^l, \tag{2.2}$$

while the reliability polynomial of a network of width w and length $l = 1$ (w relays connected in parallel) is

$$h_{1,w}(p) = 1 - (1 - p)^w. \tag{2.3}$$

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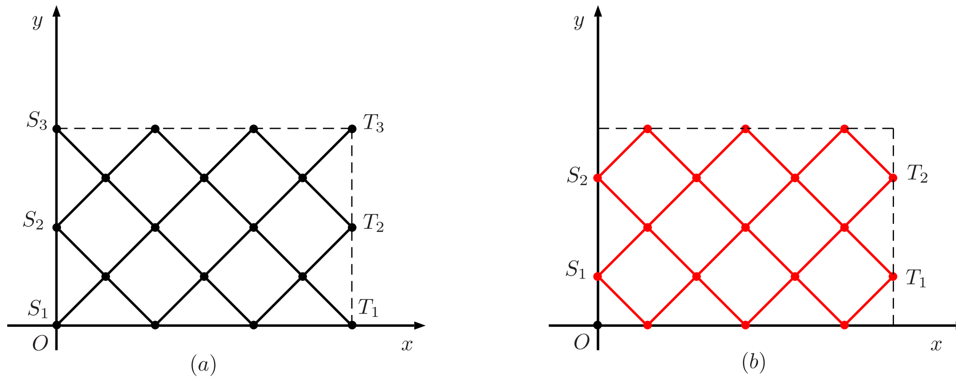

FIGURE **2.** Hammock networks of the first kind (a) and of the second kind (b).

Brick-wall networks were also named *hammock networks* from their appearance when the nodes S and T are pulled apart and every vertical connection collapses into a node, making the rectangular "bricks" to deform into rhombs.

A very convenient way to represent such networks is to consider, in an orthogonal coordinate system, all the points with integer coordinates inside the rectangle $[0, l] \times [0, w]$. These points can be either even ($A(x, y)$ with $x + y =$ even) or odd ($A(x, y)$ with $x + y =$ odd). A hammock network of the first kind is a graph whose vertices are all the even nodes, while the vertices of the second kind network are all the odd nodes inside the rectangle $[0, l] \times [0, w]$. In both cases, the edges (representing the relays) are pairs of nodes $(A(x, y), A'(x', y'))$ such that $|x - x'| = |y - y'| = 1$ and can be represented as line segments of length $\sqrt{2}$ connecting even nodes (odd nodes, respectively). The source node S is replaced by several input (source) nodes, $S_1, S_2, \dots, S_k$ (all the nodes on the vertical line $x = 0$), while the terminus node T is replaced by the output (terminus) nodes $T_1, T_2, \dots, T_{k'}$ (all the nodes on the vertical line $x = l$).

We say that a node *is working* if it is connected to a source node. We consider that all the source nodes are working. Thus, the reliability polynomial of the network is the probability that at least one output node is working.

As one can see, by the symmetry of the network, when at least one dimension is odd, the network of the first kind and the network of the second kind are equivalent, so the reliability polynomials are identical in this case. If both dimensions, l and w, are even, we have two different networks, and we denote by $h^{(1)}_{l,w}(p)$ the reliability polynomial of the first-kind network (which has $\frac{l}{2} + 1$ input nodes and the same number of output nodes) and by $h^{(2)}_{l,w}(p)$ the reliability polynomial of the second-kind network (with $\frac{l}{2} - 1$ input nodes and output nodes, respectively). For example, in Figure 2, the hammock networks of dimensions $w = 4$, $l = 6$ of the first kind (a) and the second kind (b) are presented (corresponding to the brick-wall networks from Figure 1). In addition, all the hammock networks from Moore and Shannon's paper [31] are presented in Figure 3.

3. MARKOV CHAINS FOR HAMMOCK NETWORKS

Consider, firstly, the case of a hammock network of the second kind with $w = 2$ and l even ($l = 2n$). As can be seen in Figure 4(b), this network is formed by n identical rhombs. For any such rhomb (see Figure 4(a)), the probability of working is $1 - (1 - p^2)^2 = 2p^2 - p^4$, so

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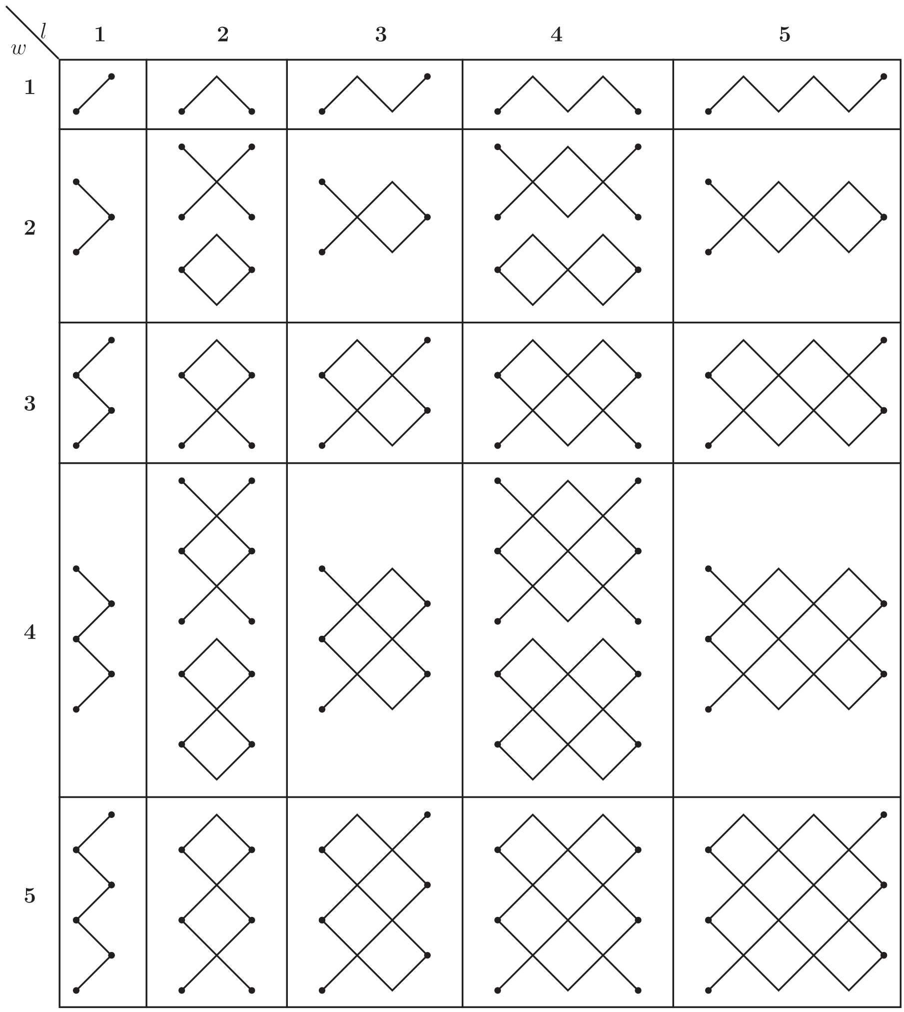

FIGURE 3. Hammock networks of dimensions $l, w = 1, 2, \ldots, 5$.

the reliability polynomial of this network is

$$h(p) = h^{(2)}_{2n,2}(p) = (2p^2 - p^4)^n.$$

We remarked above that if l is odd, then we have only one kind of network. As can be noticed in Figure 4(c), when $l = 2n + 1$ the network is composed by n rhombs and 2 relays connecting the rightmost rhomb to the terminus nodes T_1, T_2, so the reliability polynomial can be written as:

$$h(p) = h_{2n+1,2}(p) = (2p^2 - p^4)^n(2p - p^2).$$

In a similar way (see Figure 4(d)), a first kind network of dimensions $w = 2$ and $l = 2n$ is composed of $n - 1$ rhombs, 2 relays connecting the leftmost rhomb to the source nodes S_1, S_2 and 2 relays connecting the rightmost rhomb to the terminus nodes T_1, T_2, so the reliability polynomial is:

$$h(p) = h^{(1)}_{2n,2}(p) = (2p^2 - p^4)^{n-1}(2p - p^2)^2. \tag{3.1}$$

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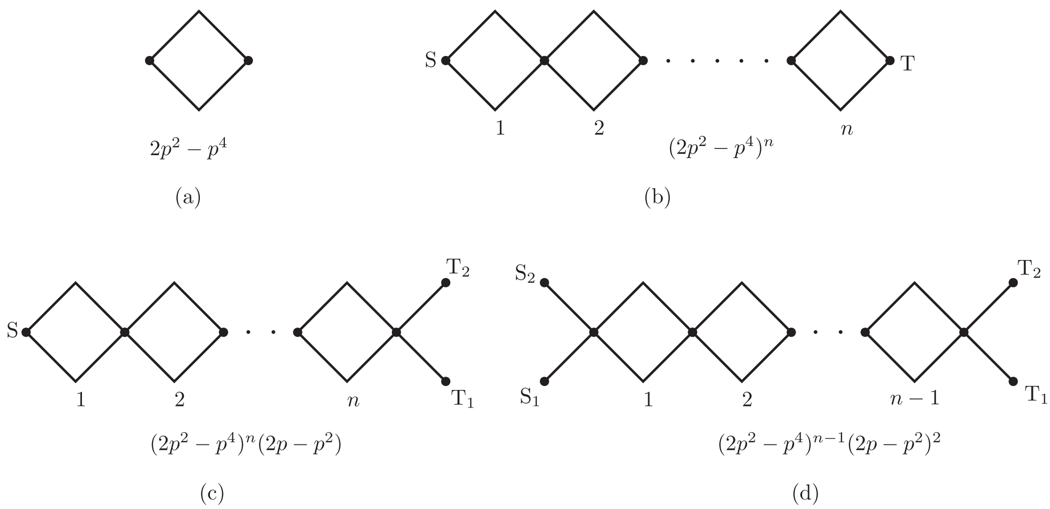

FIGURE 4. (a) An elementary network of the second kind of dimensions $w = 2$, $l = 2$. (b) A hammock network of the second kind having $w = 2$, $l = 2n$. (c) A hammock network of dimensions $w = 2$, $l = 2n + 1$. (d) A hammock network of the first kind of dimensions $w = 2$, $l = 2n$.

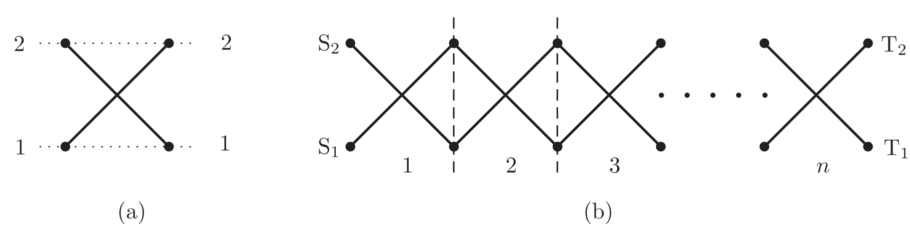

FIGURE 5. (a) An elementary network of the first kind having $w = 2$, $l = 2$. (b) A hammock network of the first kind of dimensions $w = 2$, $l = 2n$.

We shall prove that the same result can be obtained by using Markov chains.

First of all, we notice that a first kind network of dimensions $w = 2$ and $l = 2n$ can be also represented as a chain of n identical elementary structures X-shaped (as can be seen in Figure 5).

We consider the set of states to be $S = \{0, 1, 2, 3\}$, where:

$0 =$ no node is working;
$1 =$ only the node on the first line (corresponding to $S_1(0, 0)$) is working;
$2 =$ only the node on the second line (corresponding to $S_2(0, 2)$) is working;
$3 =$ both nodes are working.

At the beginning we are in state 3 (all the source nodes are working). A *step* means a move to the right by 2 units (if we are on the vertical line $x = 2k$, we move to the line $x = 2k + 2$). The transition probability $Q_{i,j}$ is the probability to be in state j on the line $x = 2k + 2$ knowing that we were in state i on the line $x = 2k$. We need $n = \frac{l}{2}$ steps to reach the output nodes T_1, T_2. Obviously, 0 is an absorbing state and all the other states

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are transient. The transition probabilities are:

$$\begin{aligned} &Q_{1,1} = Q_{1,2} = Q_{2,1} = Q_{2,2} = p^2(1-p), \\ &Q_{1,3} = Q_{2,3} = p^3, \\ &Q_{3,1} = Q_{3,2} = p(1-p)(2p-p^2), \\ &Q_{3,3} = p^2(2p-p^2). \end{aligned}$$

If $Q = (Q_{i,j})_{i,j=1,2,3}$ is the matrix corresponding to the transient states $1,2,3$, then the transition matrix P can be written as follows:

$$P = \begin{pmatrix} 1 & \mathbf{0} \\ R & Q \end{pmatrix}, \tag{3.2}$$

where $\mathbf{0} = (0,0,0)$ and

$$R = \begin{pmatrix} Q_{1,0} \\ Q_{2,0} \\ Q_{3,0} \end{pmatrix} = \begin{pmatrix} 1 - \sum_{i=1}^{3} Q_{1,i} \\ 1 - \sum_{i=1}^{3} Q_{2,i} \\ 1 - \sum_{i=1}^{3} Q_{3,i} \end{pmatrix}.$$

By equation (1.2), we have

$$P^k = \begin{pmatrix} 1 & \mathbf{0} \\ R_k & Q^k \end{pmatrix},$$

where $R_k = (I_3 + Q + Q^2 + \cdots + Q^{k-1})R$. Since the entries of the matrix Q^k, $Q_{i,j}^{(k)}$, $i,j = 1,2,3$, express the probability to be in state j after k steps, starting from state i and the initial state is 3 with probability 1 (both source nodes are working), it follows that the probability to be in state i ($i = 1,2,3$) after k steps is the ith entry of the vector $(0\ 0\ 1)Q^k$ so the probability that at least one node is working after n steps is:

$$h_{2n,2}^{(1)}(p) = (0 \quad 0 \quad 1)Q^n \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = Q_{3,1}^{(n)} + Q_{3,2}^{(n)} + Q_{3,3}^{(n)} \tag{3.3}$$

By mathematical induction, it can be proved that, for any $k = 1,2,\ldots$, the entries of the matrix Q^k are:

$$\begin{aligned} &Q_{1,1}^{(k)} = Q_{1,2}^{(k)} = Q_{2,1}^{(k)} = Q_{2,2}^{(k)} = p^2(1-p)(2p^2-p^4)^{k-1}, \\ &Q_{1,3}^{(k)} = Q_{2,3}^{(k)} = p^3(2p^2-p^4)^{k-1}, \\ &Q_{3,1}^{(k)} = Q_{3,2}^{(k)} = p(1-p)(2p-p^2)(2p^2-p^4)^{k-1}, \\ &Q_{3,3}^{(k)} = p^2(2p-p^2)(2p^2-p^4)^{k-1}. \end{aligned}$$

Using (3.3), we obtain for $h_{2n,2}^{(1)}(p)$ the same expression as in equation (3.1).

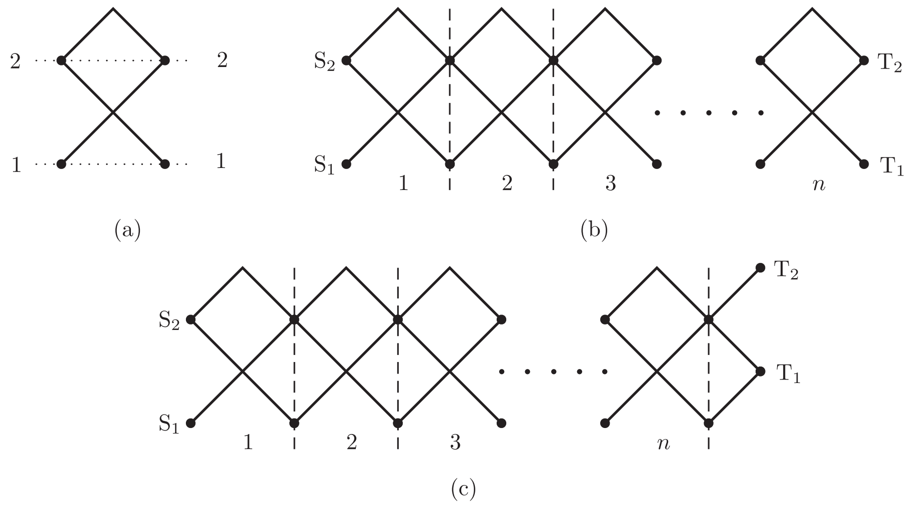

FIGURE 6. (a) An elementary network of dimensions $w = 3$, $l = 2$. (b) A hammock network of dimensions $w = 3$, $l = 2n$. (c) A hammock network of odd length: $l = 2n + 1$.

Consider, now, a hammock network of dimensions $w = 3$ and $l = 2n$. As can be seen in Figure 6(b), in this case we have two input nodes (and two output nodes), so we can use the same set of states as above, but the transition probabilities will be different:

$$\begin{aligned}
Q_{1,1} &= p^2(1 - p - p^3 + p^4)\\
Q_{1,2} &= p^2(1 - p + p^2 - 2p^3 + p^4)\\
Q_{1,3} &= p^3(1 + p^2 - p^3)\\
Q_{2,1} &= p^2(1 - p - p^2 + p^3)\\
Q_{2,2} &= p^2(2 - p - 3p^2 + 2p^3)\\
Q_{2,3} &= p^3(1 + 2p - 2p^2)\\
Q_{3,1} &= p^2(2 - 3p - p^2 + 3p^3 - p^4)\\
Q_{3,2} &= p^2(3 - 3p - 4p^2 + 6p^3 - 2p^4)\\
Q_{3,3} &= p^3(2 + 2p - 5p^2 + 2p^3).
\end{aligned} \tag{3.4}$$

The reliability polynomial is given by the same equation (3.3), but in this case the matrix Q is defined by (3.4):

$$h_{2n,3}(p) = Q_{3,1}^{(n)} + Q_{3,2}^{(n)} + Q_{3,3}^{(n)}.$$

We notice that, for the network of even length, we arrive to the terminus nodes after n steps. This is different when the length of the network is odd, $l = 2n + 1$ (see Figure 6(c)). In this case, we arrive after n steps to the vertical line $x = 2n$, and, since the terminus nodes T_1, T_2 are located on the line $x = 2n + 1$, we need to calculate the conditional probabilities to reach one of them, β_i = the probability to reach an output node from state i, $i = 1, 2, 3$:

$$\begin{aligned}
\beta_1 &= p\\
\beta_2 &= 1 - q^2 = 2p - p^2\\
\beta_3 &= 1 - q^3 = 3p - 3p^2 + p^3.
\end{aligned} \tag{3.5}$$

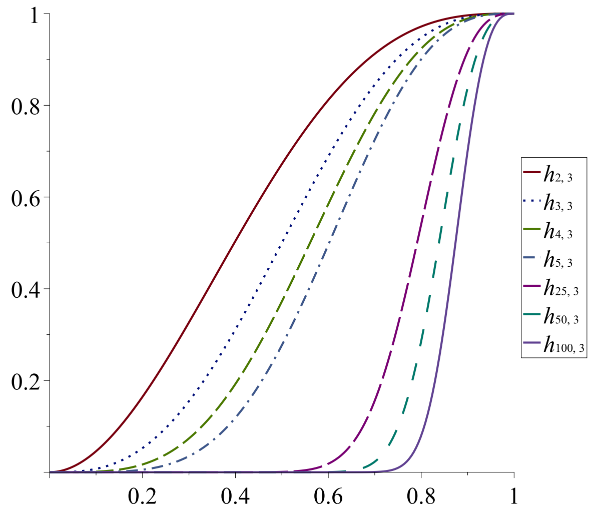

FIGURE **7.** The reliability polynomials of hammock networks of width $w = 3$.

Thus, the reliability polynomial for the network of length $l = 2n + 1$ and $w = 3$ is:

$$h_{2n+1,3}(p) = (0 \quad 0 \quad 1)Q^n \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \beta_1 Q^{(n)}_{3,1} + \beta_2 Q^{(n)}_{3,2} + \beta_3 Q^{(n)}_{3,3}. \tag{3.6}$$

In Figure 7, the reliability polynomials of hammock networks with $w = 3$ and various values of length are represented. The "extreme" cases are $l = 2$ and $l = 100$.

Now, let us study the hammock network of the second kind with $w = 4$ and $l = 2n$. We have two input nodes and two output nodes (see Figure 8(b)), so we have the same set of states as above and the reliability polynomial is given by (3.3), but the matrix Q, in this case, is defined by (3.7):

$$h^{(2)}_{2n,4}(p) = Q^{(n)}_{3,1} + Q^{(n)}_{3,2} + Q^{(n)}_{3,3},$$

$$\begin{aligned}
Q_{1,1} &= p^2(2 - p - 3p^2 + p^3 - p^4 + 4p^5 - 2p^6) \\
Q_{1,2} &= p^2(1 - p - p^3 + 2p^5 - p^6) \\
Q_{1,3} &= p^3(1 + 2p - p^2 + p^3 - 4p^4 + 2p^5) \\
Q_{2,1} &= p^2(1 - p - p^3 + 2p^5 - p^6) \\
Q_{2,2} &= p^2(2 - p - 3p^2 + p^3 - p^4 + 4p^5 - 2p^6) \\
Q_{2,3} &= p^3(1 + 2p - p^2 + p^3 - 4p^4 + 2p^5) \\
Q_{3,1} &= p^2(3 - 3p - 7p^2 + 9p^3 + 2p^4 - 6p^5 + 2p^6) \\
Q_{3,2} &= p^2(3 - 3p - 7p^2 + 9p^3 + 2p^4 - 6p^5 + 2p^6) \\
Q_{3,3} &= p^3(2 + 6p - 10p^2 - 2p^3 + 8p^4 - 3p^5).
\end{aligned} \tag{3.7}$$

For a hammock network of width $w = 4$ and odd length, $l = 2n + 1$, the probability to reach one of the *three* terminus nodes (see Figure 8(c)) is given by equation (3.6), but in

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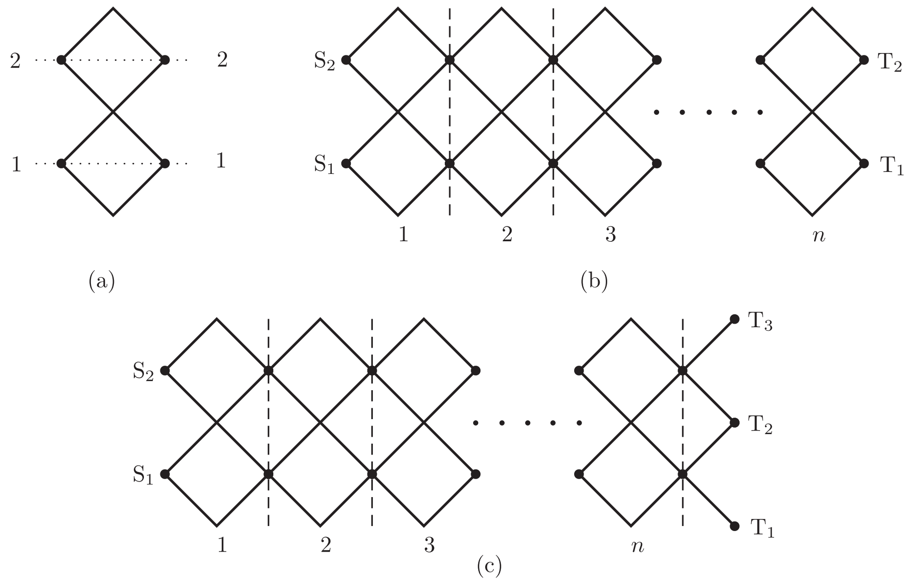

FIGURE 8. (a) An elementary network of the second kind having $w = 4$, $l = 2$. (b) A hammock network of the second kind of dimensions $w = 4$, $l = 2n$. (c) A hammock network of width $w = 4$ and odd length: $l = 2n + 1$.

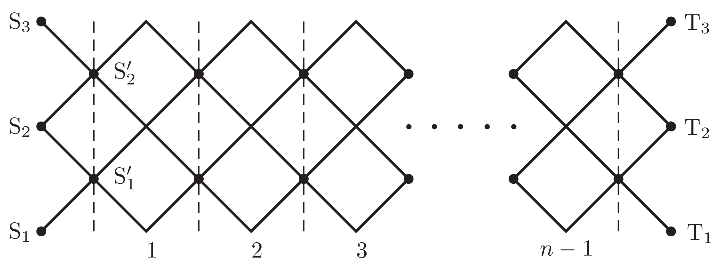

FIGURE 9. A hammock network of the first kind of dimensions $w = 4$, $l = 2n$.

this case the coefficients β_i are defined by:

$$\begin{aligned} \beta_1 = \beta_2 &= 1 - q^2 = 2p - p^2 \\ \beta_3 &= 1 - q^4 = 4p - 6p^2 + 4p^3 - p^4. \end{aligned} \tag{3.8}$$

In the case of a hammock network of the first kind with $w = 4$ and $l = 2n$, since we have *three* input nodes, we should consider a set of states with $8 = 2^3$ elements (the number of subsets of $\{S_1, S_2, S_3\}$). But, as can be seen in Figure 9, there is a way to avoid this if we consider the network of odd length, $l = 2n + 1$, with the input nodes $S'_1(1, 1)$ and $S'_2(1, 3)$ and the starting distribution $(\alpha_1, \alpha_2, \alpha_3)$, where:

$\alpha_1 =$ the probability that only S'_1 is working;
$\alpha_2 =$ the probability that only S'_2 is working;
$\alpha_3 =$ the probability that both input nodes S'_1 and S'_2 are working.

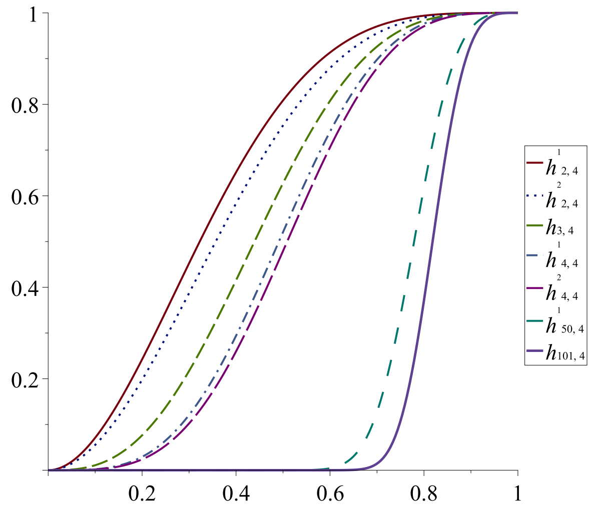

FIGURE **10.** The reliability polynomials of hammock networks of width $w = 4$.

These probabilities are given by:

$$\begin{aligned} \alpha_1 &= (2p - p^2)(1-p)^2 \\ \alpha_2 &= (2p - p^2)(1-p)^2 \\ \alpha_3 &= (2p - p^2)^2. \end{aligned} \tag{3.9}$$

The reliability polynomial for this network is:

$$h^{(1)}_{2n,4}(p) = (\alpha_1 \quad \alpha_2 \quad \alpha_3) Q^{n-1} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \tag{3.10}$$

with α_i defined by (3.9), β_i defined by (3.8), and the matrix Q defined by (3.7).

In Figure 10, the reliability polynomials of hammock networks with $w = 4$ and various values of length are represented. The "extreme" cases are $l = 2$ and $l = 101$.

Time to absorption

Since we built an *absorbing* Markov chain (with a single absorbing state), we can calculate the expected number of steps before the chain is absorbed (knowing that the process starts in state 3 where all the input nodes are working) as follows:

$$\tau(p) = (0 \quad 0 \quad 1) \cdot N(p) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

where $N(p)$ is the fundamental matrix N defined by (1.3).

Recall that a "step" in the Markov chain means an increasing of the length by 2 units. So the average length of a network expected to work would be $\lambda(p) = 2\tau(p)$. Since, for any positive w and n, we have:

$$h_{2n,w}(p) > h_{2n+1,w}(p) > h_{2n+2,w}(p),$$

it suffices to study the average length for networks of even length.

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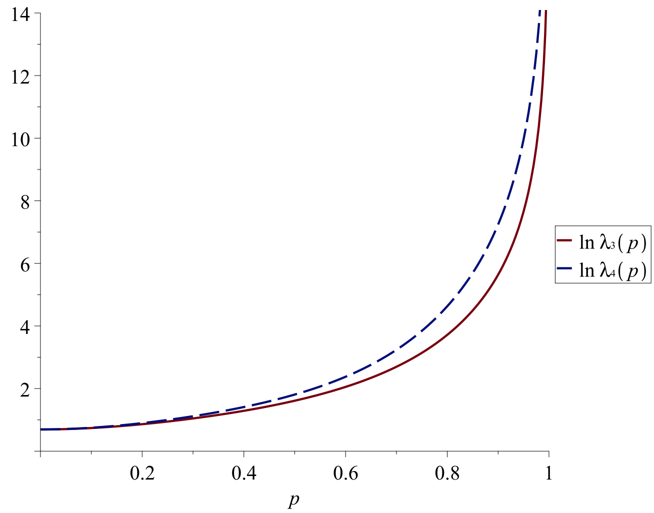

FIGURE 11. Expected length of a functioning network with $w = 3$ ($\lambda_3(p)$) and, respectively, $w = 4$ ($\lambda_4(p)$).

In Figure 11, the graphs of $\lambda(p)$ for networks of width $w = 3$ and $w = 4$, $\lambda_3(p)$ and $\lambda_4(p)$, respectively, are represented (the vertical axis is logarithmic, in base e). Obviously, $\lim_{p\to 1} \lambda(p) = \infty$ in both cases. But, as Kemeny and Snell [26] note, means are fairly unreliable estimates for Markov chains (because of the large variance).

4. DISCUSSION AND COMPARISON WITH OTHER METHODS

Finding the reliability polynomial for networks of high dimensions is a very difficult problem. In spite of the regular form of hammock networks, no efficient method to calculate their reliability polynomial has been developed yet. Several approximation methods have been devised instead, using either interpolation (for example, the Hermite interpolation applied in [16]), or an equivalent statistical distribution (like in the recent approach of Cowell *et al.* [14]).

Our novel method based on Markov chains manages to calculate the *exact* reliability polynomials for hammock networks of width $w \leq 4$ and arbitrary length l. Based on the duality theorem (see [15]), we can also calculate the exact reliability for networks of length $l \leq 4$ and arbitrary width w by using the formula:

$$h_{l,w}^{(i)}(p) = 1 - h_{w,l}^{(2/i)}(1-p). \tag{4.1}$$

When the method is applied for networks of width $w \geq 5$, it provides a lower bound for the reliability polynomial. This happens because some of the paths connecting an input node and an output node "turn back" by more than two units and they will not be considered *connecting paths*; consequently, the reliability calculated by Markov chains will be lower than the exact reliability polynomial. The Markov chain method decomposes the network into elementary networks of width w and length 2, so any path connecting an input node to an output node is decomposed into pieces. If each piece is a continuous path from an input node to an output node of the elementary network such that for any two consecutive elements, the input node of the last one is the output node of the previous one, then (and

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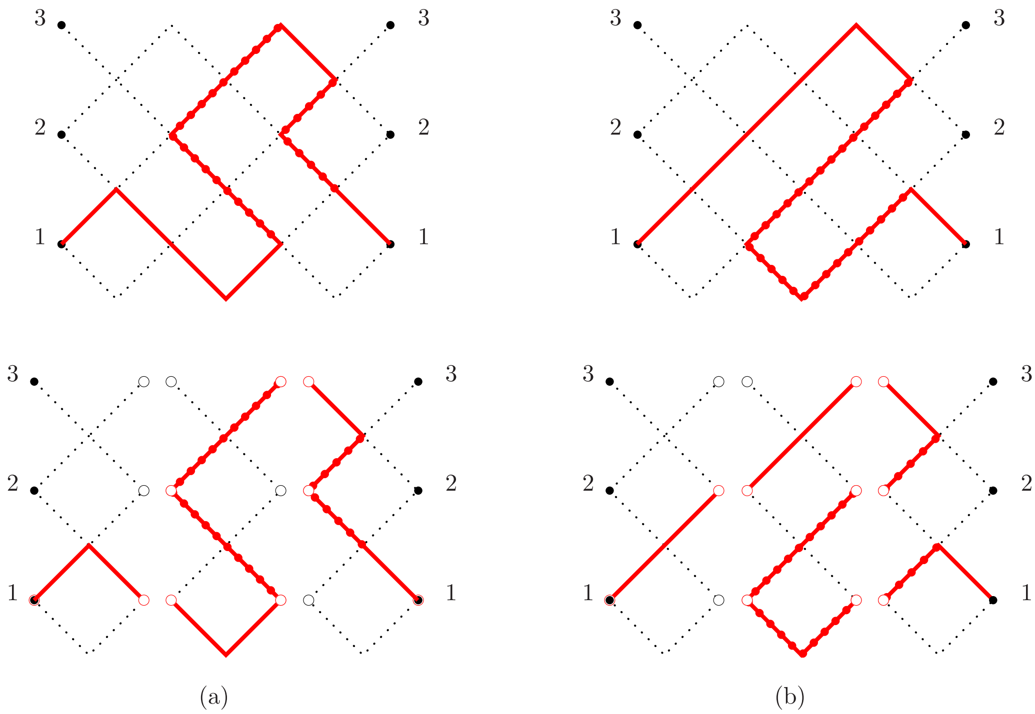

FIGURE **12.** Two connecting paths in a network of width $w = 5$. (a) The path turns back twice: two units and one unit, respectively; it is counted as a connecting path. (b) The path turns back three units; it is not considered a connecting path in the Markov chain method.

only then) the path is counted as a *connecting path* in the Markov chain method. So, if a connecting path has a "turning back" of one unit, it will be considered a connecting path in the Markov chain method; if the "turning back" has two units, the path may still be counted as a connecting path (if both units are in the same elementary network), as can be seen in Figure 12(a). But if the "turning back" has three or more units, then the path cannot be considered a connecting path in the Markov chain method (see Figure 12(b)). In general, in a network of width w, a connecting path may turn back at most $w - 2$ units, hence, for $w \geq 5$, there exist some connecting paths that turn back three units and will not be considered connecting paths in the Markov chain method.

We notice that, for $w = 4$, although it is possible to turn back two units, the two units belong to the same element when the network is decomposed into elementary networks as in Figures 8 and 9, so the Markov chain method calculates the exact reliability polynomial in this case.

In Table 1, we present the reliability polynomials for hammock networks of width $w = 5$ and length $l = 1, 2, \ldots 5$. For $l \leq 3$, the Markov chain method provides the exact reliability polynomials. For $l = 4, 5$ we denoted by $H_{4,5}^{[M]}$ and, respectively, $H_{5,5}^{[M]}$ the polynomials approximated by the Markov chain method.

The results were compared to those obtained by the two methods mentioned above (Hermite interpolation [16] and beta distribution [14], respectively) in the case of a square hammock network of dimension $w = l = 5$. We denote by $H_{5,5}^{[He]}$ the reliability approximated by Hermite interpolation,

$$H_{5,5}^{[He]}(p) = 52p^5 - 46p^6 + 21p^7 - 1983p^8 + 5420p^9 - 7106p^{10} + 5036p^{11} - 1872p^{12} + 288p^{13},$$

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TABLE 1. Reliability polynomials (exact and approximated by the Markov chain method) for networks of width $w = 5$ and length $l = 1, 2, \ldots 5$

$l = 1$	$h_{1,5}(p) = 5p - 10p^2 + 10p^3 - 5p^4 + p^5$
$l = 2$	$h_{2,5}(p) = 9p^2 - 8p^3 - 22p^4 + 40p^5 - 10p^6 - 24p^7 + 23p^8 - 8p^9 + p^{10}$
$l = 3$	$h_{3,5}(p) = 16p^3 - 14p^4 - 13p^5 - 58p^6 + 160p^7 - 31p^8 - 166p^9 + 49p^{10} + 233p^{11} - 300p^{12} + 165p^{13} - 45p^{14} - 5p^{15}$
$l = 4$ (exact)	$h_{4,5}(p) = 29p^4 - 26p^5 - 18p^6 - 80p^7 + 18p^8 + 406p^9 - 128p^{10} - 654p^{11} - 410p^{12} + 2740p^{13} - 2606p^{14} - 242p^{15} + 2267p^{16} - 1960p^{17} + 832p^{18} - 184p^{19} + 17p^{20}$
$l = 4$ approx.)	$H_{4,5}(p) = 29p^4 - 26p^5 - 18p^6 - 80p^7 + 6p^8 + 484p^9 - 316p^{10} - 514p^{11} - 170p^{12} + 2100p^{13} - 2074p^{14} - 230p^{15} + 1867p^{16} - 1590p^{17} + 664p^{18} - 144p^{19} + 13p^{20}$
$l = 5$ (exact)	$h_{5,5}(p) = 52p^5 - 46p^6 - 23p^7 - 204p^8 + 180p^9 + 22p^{10} + 1288p^{11} - 1228p^{12} - 2120p^{13} - 1720p^{14} + 13614p^{15} - 9622p^{16} - 15645p^{17} + 28443p^{18} - 10045p^{19} - 14861p^{20} + 21065p^{21} - 12719p^{22} + 4306p^{23} - 800p^{24} + 64p^{25}$
$l = 5$ (approx.)	$H_{5,5}(p) = 52p^5 - 46p^6 - 23p^7 - 204p^8 + 159p^9 + 127p^{10} + 1160p^{11} - 1371p^{12} - 1966p^{13} - 576p^{14} + 11156p^{15} - 9115p^{16} - 11655p^{17} + 22985p^{18} - 8376p^{19} - 11949p^{20} + 17016p^{21} - 10226p^{22} + 3435p^{23} - 632p^{24} + 50p^{25}$

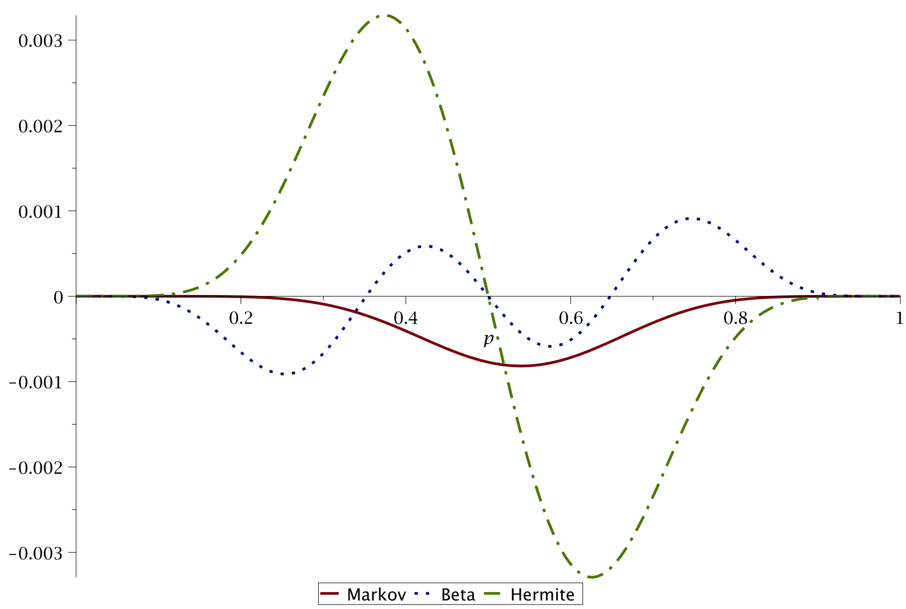

FIGURE 13. The absolute errors obtained by approximating the reliability polynomial using *Markov* chains ($\mathrm{Err}_{5,5}^{[M]}$), *Beta* distribution ($\mathrm{Err}_{5,5}^{[B]}$), and *Hermite* interpolation ($\mathrm{Err}_{5,5}^{[He]}$).

and by $H_{5,5}^{[B]}$ the approximation using beta distribution. The graphs of the error functions,

$$\mathrm{Err}_{5,5}^{[M]} = H_{5,5}^{[M]} - h_{5,5}, \quad \mathrm{Err}_{5,5}^{[B]} = H_{5,5}^{[B]} - h_{5,5}, \quad \mathrm{Err}_{5,5}^{[He]} = H_{5,5}^{[He]} - h_{5,5},$$

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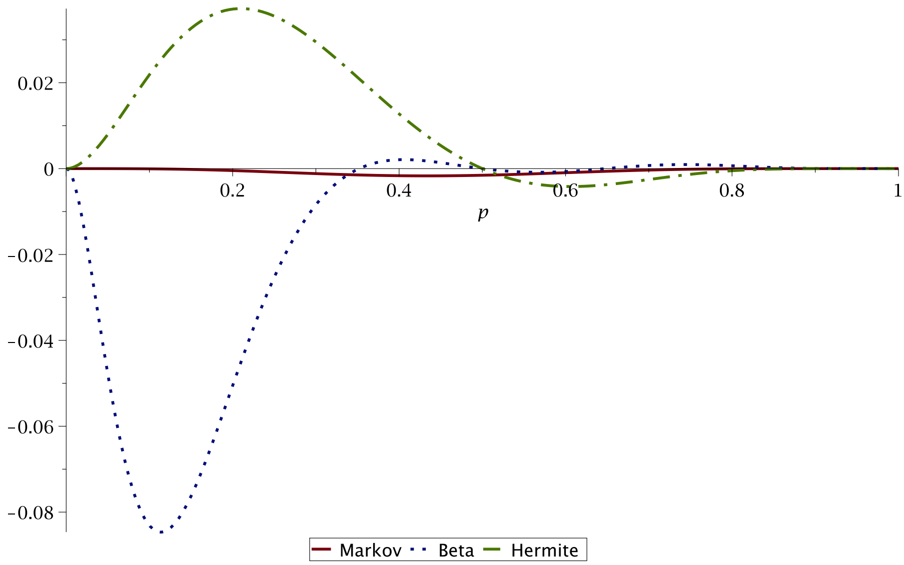

FIGURE 14. The relative errors obtained by approximating the reliability polynomial using *Markov* chains, *Beta* distribution, and *Hermite* interpolation.

are presented in Figure 13. As one can see, the maximum absolute error obtained with the Markov chain method of approximation is smaller than the maximum absolute error obtained in the other two approaches.

In Figure 14, the relative errors are represented. The approximation constructed by the Markov chain method proves to be the most accurate in terms of relative error as well.

5. CONCLUSIONS

In this paper, we introduced a new method to assess the reliability polynomial of a hammock network, a method which is based on the use of Markov chains. As one can easily notice, for $w = 1, 2$ or $l = 1, 2$, the hammock networks are actually made up of relays which are connected in series and/or in parallel, so it is not difficult to find the formula for the reliability polynomial in these cases (see equations (2.2), (2.3), (3.1)). However, for $w, l \geq 3$, there is no such formula and our technique based on Markov chains is the first method to calculate the *exact reliability polynomial* for hammock networks of width $w = 3, 4$ and arbitrary length. By using the dual network (see equation (4.1)), the exact reliability for networks of length $l = 3, 4$ and arbitrary width can also be computed.

On the other hand, for $w \geq 5$, the Markov chain method provides a lower bound of the reliability polynomial which proves to be a very accurate approximation, compared with the approximate polynomials obtained by other methods (based on Hermite interpolation [16] and beta distribution [14], respectively).

The main limitation of the method is related to the dimension of the square matrix Q, which exponentially increases with w (since the set of states has $2^{\lceil \frac{w}{2} \rceil}$ elements, where $\lceil x \rceil$ is the smallest integer greater than or equal to x). This drawback could be overcome by considering the network formed by n hammock networks with a "small" w, connected in parallel. The reliability of this network can be easily computed by equation (2.2) (replacing p with $h(p)$, the reliability of the hammock network of width w), and it obviously constitutes

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a lower bound for the reliability of the hammock network of width $W = n \cdot w$. This could be a direction for future research work.

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