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Infinitely many solutions for a Dirichlet problem involving the p-Laplacian

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The existence of infinitely many solutions for an autonomous elliptic Dirichlet problem involving the p-Laplacian is investigated. The approach is based on variational methods.

1. Introduction

The purpose of this paper is to establish infinitely many solutions for the following autonomous elliptic Dirichlet problem:

$$\left.\begin{aligned} -\Delta_p u + q(x)|u|^{p-2}u &= \lambda f(u) \quad \text{in } \Omega, \\ u &= 0 \qquad\quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{$D_\lambda^{q,f}$}$$

where Ω is a non-empty bounded open subset of the Euclidean space $(\mathbb{R}^N, |\cdot|)$, $N \geqslant 1$, with boundary of class C^1, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $p > N$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $q \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_{x\in\Omega} q(x) \geqslant 0$ and λ is a positive real parameter.

More precisely, under some hypotheses on the behaviour of the potential of the nonlinear term at infinity, the existence of an interval Λ such that, for each $\lambda \in \Lambda$, problem $(D_\lambda^{q,f})$ admits a sequence of pairwise distinct weak solutions is proved (see theorem 3.1).

Moreover, replacing the conditions at infinity of the potential by similar conditions at zero, the same results hold and, in addition, the sequence of pairwise distinct solutions uniformly converges to zero (see theorem 3.4).

As an example we now present a special case of our results.

Theorem 1.1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous non-negative function. Set*

$$F(\xi):=\int_0^\xi f(t)\,\mathrm{d}t$$

for every $\xi\in\mathbb{R}$,

$$\sigma(N,p):=\inf_{\mu\in\,]0,1[}\frac{1-\mu^N}{\mu^N(1-\mu)^p},\qquad \tau:=\sup_{x\in\Omega}\operatorname{dist}(x,\partial\Omega),$$
$$m:=\frac{N^{-1/p}}{\sqrt{\pi}}\left[\Gamma\left(1+\frac{N}{2}\right)\right]^{1/N}\left(\frac{p-1}{p-N}\right)^{1-1/p}|\Omega|^{1/N-1/p},$$

(where Γ denotes the Gamma function) and $\kappa:=\tau^p/m^p|\Omega|\sigma(N,p)$. Assume that

$$\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^p}<\kappa\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^p}. \tag{1.1}$$

Then, for each

$$\lambda\in\left]\frac{\sigma(N,p)}{p\tau^p\limsup_{\xi\to+\infty}F(\xi)/\xi^p},\frac{1}{m^p p|\Omega|\liminf_{\xi\to+\infty}F(\xi)/\xi^p}\right[,$$

the problem

$$\left.\begin{aligned}-\Delta_p u&=\lambda f(u)\quad \textit{in } \Omega,\\ u&=0\qquad\quad \textit{on } \partial\Omega,\end{aligned}\right\} \tag{$D_\lambda^{0,f}$}$$

admits a sequence of pairwise distinct positive weak solutions.

Results of the existence of infinitely many solutions for the problem

$$\left.\begin{aligned}-\Delta_p u&=f(u)\quad \text{in } \Omega,\\ u&=0\qquad \text{on } \partial\Omega,\end{aligned}\right\} \tag{$D_1^{0,f}$}$$

are obtained, for instance, in [2, 4, 5].

As pointed out in remark 3.9, the results contained in [2] are a consequence of ours and, moreover, some examples, where we can apply our theorems while the results in [2] cannot be applied, are given (see remark 4.2 and examples 4.1 and 4.3). In addition, we observe that in [4], Omari and Zanolin assume that

$$\liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^p}=0\quad\text{and}\quad\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^p}=+\infty,$$

which are conditions that imply our key assumption (see (1.1) and condition (ii) of theorem 2.1), when the nonlinear term f is non-negative.

We emphasize that when $q\equiv 0$ our theorems and the results in [5] and [4] are mutually independent (see theorem 1.1, example 4.1 and remark 4.2). The paper is organized as follows. In § 2 we recall some basic definitions and our abstract framework, while § 3 is devoted to infinitely many solutions for the Dirichlet problem ($D_\lambda^{q,f}$). Finally, in § 4, some examples and applications are presented.

2. Preliminaries

Our main tool is the following critical points theorem obtained in [1]. This result is a refinement of the variational principle of Ricceri [7].

THEOREM 2.1 (Bonanno and Molica Bisci [1, theorem 2.1]). *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is strongly continuous, sequentially weakly lower semicontinuous and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, put*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{(\sup_{v \in \Phi^{-1}(]-\infty, r[)} \Psi(v)) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \qquad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then we have the following.

(i) *For every $r > \inf_X \Phi$ and every $\lambda \in]0, 1/\varphi(r)[$, the restriction of the functional $I_\lambda := \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.*

(ii) *If $\gamma < +\infty$, then, for each $\lambda \in]0, 1/\gamma[$, the following alternative holds: either*

   (a) *I_λ possesses a global minimum, or*

   (b) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n \to +\infty} \Phi(u_n) = +\infty$.*

(iii) *If $\delta < +\infty$ then, for each $\lambda \in]0, 1/\delta[$, the following alternative holds: either*

   (a) *there is a global minimum of Φ which is a local minimum of I_λ, or*

   (b) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ which weakly converges to a global minimum of Φ, with $\lim_{n \to +\infty} \Phi(u_n) = \inf_X \Phi$.*

Consider the following autonomous elliptic Dirichlet problem:

$$\left.\begin{aligned} -\Delta_p u + q(x)|u|^{p-2}u = \lambda f(u) \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \end{aligned}\right\} \qquad (D_\lambda^{q,f})$$

where Ω is a non-empty bounded open subset of the Euclidean space $(\mathbb{R}^N, |\cdot|)$, $N \geqslant 1$, with boundary of class C^1, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, with $p > N$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $q \in L^\infty(\Omega)$ satisfies

$$\operatorname*{ess\,inf}_{x \in \Omega} q(x) \geqslant 0 \qquad (2.1)$$

and λ is a positive real parameter.

As usual, $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_* = \left(\int_\Omega |\nabla u(x)|^p \, dx \right)^{1/p}.$$

A function $u : \Omega \to \mathbb{R}$ is said to be a weak solution of $(D_\lambda^{q,f})$ if $u \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\,\mathrm{d}x + \int_\Omega q(x)|u(x)|^{p-2}u(x)v(x)\,\mathrm{d}x = \lambda\int_\Omega f(u(x))v(x)\,\mathrm{d}x$$

for every $v \in W_0^{1,p}(\Omega)$. We are interested in the existence of infinitely many solutions for the problem $(D_\lambda^{q,f})$. The main objective is to use the abstract theorem 2.1.

Set

$$k := \sup\left\{\frac{\max_{x\in\overline{\Omega}}|u(x)|}{\|u\|_*} : u \in W_0^{1,p}(\Omega),\ u \neq 0\right\}.$$

Since $p > N$, one has $k < +\infty$. For our goal it is enough to know an explicit upper bound for the constant k.

In this connection, it is well known [8, formula (6b)] that by setting

$$m := \frac{N^{-1/p}}{\sqrt{\pi}}\left[\Gamma\left(1+\frac{N}{2}\right)\right]^{1/N}\left(\frac{p-1}{p-N}\right)^{1-1/p}|\Omega|^{(1/N)-(1/p)}, \tag{2.2}$$

where Γ denotes the Gamma function and $|\Omega|$ is the Lebesgue measure of Ω, we have $k \leqslant m$ (equality occurs when Ω is a ball). Moreover, we put

$$(u,v) := \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\,\mathrm{d}x + \int_\Omega q(x)|u(x)|^{p-2}u(x)v(x)\,\mathrm{d}x \quad \text{for all } u,v \in W_0^{1,p}(\Omega)$$

and consider in $W_0^{1,p}(\Omega)$ the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p + q(x)|u(x)|^p\,\mathrm{d}x\right)^{1/p},$$

which is equivalent to the usual one.

Since, by hypothesis $p > N$, $W_0^{1,p}(\Omega)$ is compactly embedded in $C^0(\bar{\Omega})$ and, in particular, we have

$$\|u\|_\infty \leqslant m\|u\| \tag{2.3}$$

for every $u \in W_0^{1,p}(\Omega)$.

3. Main results

Define

$$\sigma(N,p) := \inf_{\mu\in\,]0,1\,[}\frac{1-\mu^N}{\mu^N(1-\mu)^p},$$

and consider $\bar{\mu} \in\,]0,1\,[$ such that $\sigma(N,p) = (1-\bar{\mu}^N)/\bar{\mu}^N(1-\bar{\mu})^p$.

Moreover, let

$$\tau := \sup_{x\in\Omega}\operatorname{dist}(x,\partial\Omega). \tag{3.1}$$

Simple calculations show that there is $x_0 \in \Omega$ such that $B(x_0,\tau) \subseteq \Omega$.

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Furthermore, set

$$\kappa := \frac{\tau^p \bar{\mu}^N}{|\Omega| m^p (\bar{\mu}^N \sigma(N,p) + \|q\|_\infty \tau^p g_{\bar{\mu}}(p,N))}, \tag{3.2}$$

where m is given by (2.2),

$$g_{\bar{\mu}}(p,N) := \bar{\mu}^N + \frac{1}{(1-\bar{\mu})^p} N B_{(\bar{\mu},1)}(N,p+1)$$

and $B_{(\bar{\mu},1)}(N,p+1)$ denotes the generalized incomplete beta function defined as follows:

$$B_{(\bar{\mu},1)}(N,p+1) := \int_{\bar{\mu}}^1 t^{N-1}(1-t)^{(p+1)-1}\,\mathrm{d}t.$$

We also denote by

$$\omega_\tau := \tau^N \frac{\pi^{N/2}}{\Gamma(1+N/2)}$$

the measure of the N-dimensional ball of radius τ, where Γ is the Gamma function defined by

$$\Gamma(t) := \int_0^{+\infty} z^{t-1} \mathrm{e}^{-z}\,\mathrm{d}z \quad \text{for all } t > 0.$$

Finally, set

$$A := \liminf_{\xi \to +\infty} \frac{\max_{|t| \leqslant \xi} F(t)}{\xi^p}, \qquad B := \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^p},$$

$$\lambda_1 := \frac{1}{pB}\bigg(\frac{\sigma(N,p)}{\tau^p} + \|q\|_\infty \frac{g_{\bar{\mu}}(p,N)}{\bar{\mu}^N}\bigg), \tag{3.3}$$

$$\lambda_2 := \frac{1}{m^p p A |\Omega|}. \tag{3.4}$$

Our main result is the following result.

THEOREM 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put*

$$F(\xi) := \int_0^\xi f(t)\,\mathrm{d}t$$

for every $\xi \in \mathbb{R}$ and assume that

(i) $\inf_{\xi \geqslant 0} F(\xi) \geqslant 0$,

(ii) $\liminf_{\xi \to +\infty} \dfrac{\max_{|t| \leqslant \xi} F(t)}{\xi^p} < \kappa \limsup_{\xi \to +\infty} \dfrac{F(\xi)}{\xi^p}$, *where κ is given by (3.2).*

Then, for every $\lambda \in]\lambda_1, \lambda_2[$, the problem $(D_\lambda^{q,f})$ admits a sequence of weak solutions which is unbounded in $W_0^{1,p}(\Omega)$.

Proof. Our aim is to apply theorem 2.1(ii). To this end, fix $\lambda \in]\lambda_1, \lambda_2[$ and denote by X the Banach space $W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p + q(x)|u(x)|^p \, \mathrm{d}x \right)^{1/p}.$$

Set $\Phi : X \to \mathbb{R}$ and $\Psi : X \to \mathbb{R}$ defined as follows:

$$\Phi(u) := \frac{\|u\|^p}{p} \quad \text{and} \quad \Psi(u) := \int_\Omega F(u(x)) \, \mathrm{d}x,$$

where

$$F(\xi) := \int_0^\xi f(t) \, \mathrm{d}t$$

for every $\xi \in \mathbb{R}$.

By standard arguments, we have that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = (u, v)$$

for every $v \in X$.

Moreover, Ψ is a Gâteaux differentiable sequentially weakly upper continuous functional whose Gâteaux derivative is given by

$$\Psi'(u)(v) = \int_\Omega f(u(x))v(x) \, \mathrm{d}x$$

for every $v \in X$. Hence, the weak solutions of $(D_\lambda^{q,f})$ are exactly the solutions of the equation

$$\Phi'(u) - \lambda\Psi'(u) = 0.$$

Now, let $\{c_n\}$ be a real sequence such that $\lim_{n\to+\infty} c_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{\max_{|t|\leqslant c_n} F(t)}{c_n^p} = A.$$

Put $r_n = c_n^p/m^p p$ for all $n \in \mathbb{N}$. Taking (2.3) into account, we have $\max_{t\in\Omega} |v(t)| \leqslant c_n$ for all $v \in X$ such that $\|v\|^p < pr_n$. Hence, taking into account that $\|u_0\| = 0$ and

$$\int_\Omega F(u_0(x)) \, \mathrm{d}x = 0,$$

where $u_0(x) = 0$ for all $x \in \Omega$, for all $n \in \mathbb{N}$ we have

$$\begin{aligned}
\varphi(r_n) &= \inf_{\|u\|^p<pr_n} \left(\frac{1}{r_n - (\|u\|^p/p)} \left(\sup_{\|v\|^p<pr_n} \int_\Omega F(v(x)) \, \mathrm{d}x - \int_\Omega F(u(x)) \, \mathrm{d}x \right) \right) \\
&\leqslant \frac{1}{r_n} \sup_{\|v\|^p<pr_n} \int_\Omega F(v(x)) \, \mathrm{d}x \\
&\leqslant |\Omega| \frac{1}{r_n} \max_{|t|\leqslant c_n} F(t) \\
&= |\Omega| m^p p \frac{1}{c_n^p} \max_{|t|\leqslant c_n} F(t).
\end{aligned}$$

Therefore, since from assumption (ii) one has $A < +\infty$, we obtain

$$\gamma \leqslant \liminf_{n\to+\infty} \varphi(r_n) \leqslant |\Omega| m^p p A < +\infty.$$

Setting $I_\lambda := \Phi(u) - \lambda\Psi(u)$ for all $u \in X$, we claim that the functional I_λ is unbounded from below. Let $\{\zeta_n\}$ be a real sequence such that $\lim_{n\to+\infty} \zeta_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{F(\zeta_n)}{\zeta_n^p} = B. \tag{3.5}$$

For each $n \in \mathbb{N}$, define

$$w_n(x) := \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0,\tau), \\ \dfrac{\zeta_n}{\tau(1-\bar{\mu})}(\tau - |x - x_0|) & \text{if } x \in B(x_0,\tau) \setminus B(x_0,\bar{\mu}\tau), \\ \zeta_n & \text{if } x \in B(x_0,\bar{\mu}\tau). \end{cases}$$

We can prove that

$$\begin{aligned} \|w_n\|^p &= \int_\Omega |\nabla w_n(x)|^p \,\mathrm{d}x + \int_\Omega q(x)|w_n(x)|^p \,\mathrm{d}x \\ &\leqslant \zeta_n^p \omega_\tau \biggl[\frac{1-\bar{\mu}^N}{\tau^p(1-\bar{\mu})^p} + \|q\|_\infty g_{\bar{\mu}}(p,N) \biggr]. \end{aligned}$$

Indeed,

$$\begin{aligned} \int_\Omega |\nabla w_n(x)|^p \,\mathrm{d}x &= \int_{B(x_0,\tau)\setminus B(x_0,\bar{\mu}\tau)} \frac{\zeta_n^p}{\tau^p(1-\bar{\mu})^p} \,\mathrm{d}x \\ &= \frac{\zeta_n^p}{(1-\bar{\mu})^p\tau^p}(|B(x_0,\tau)| - |B(x_0,\bar{\mu}\tau)|) \\ &= \frac{\zeta_n^p \omega_\tau}{(1-\bar{\mu})^p\tau^p}(1-\bar{\mu}^N) \end{aligned}$$

and

$$\begin{aligned} \int_\Omega q(x)|w_n(x)|^p \,\mathrm{d}x &= \int_{B(x_0,\bar{\mu}\tau)} q(x)\zeta_n^p \,\mathrm{d}x \\ &\quad + \int_{B(x_0,\tau)\setminus B(x_0,\bar{\mu}\tau)} \frac{q(x)\zeta_n^p}{(1-\bar{\mu})^p\tau^p}(\tau - |x-x_0|)^p \,\mathrm{d}x \\ &\leqslant \|q\|_\infty \zeta_n^p \biggl(\int_{B(x_0,\bar{\mu}\tau)} \mathrm{d}x + \frac{\int_{B(x_0,\tau)\setminus B(x_0,\bar{\mu}\tau)} (\tau - |x-x_0|)^p \,\mathrm{d}x}{(1-\bar{\mu})^p\tau^p} \biggr) \\ &= \|q\|_\infty \omega_\tau \zeta_n^p g_{\bar{\mu}}(p,N). \end{aligned}$$

The last equality holds owing to

$$\begin{aligned} I_p &:= \int_{B(x_0,\tau)\setminus B(x_0,\bar{\mu}\tau)} (\tau - |x-x_0|)^p \,\mathrm{d}x \\ &= N\omega_\tau \tau^p B_{(\bar{\mu},1)}(N, p+1). \end{aligned} \tag{3.6}$$

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The easiest way to compute this integral is to go through a general polar coordinates transformation.

Let

$$\begin{aligned} x_1 &= \rho\cos\theta_1, \\ x_j &= \rho\cos\theta_1\cos\theta_2\cdots\cos\theta_{j-1}\sin\theta_j, \quad j=2,\dots,N-1, \\ x_N &= \rho\cos\theta_1\cos\theta_2\cdots\cos\theta_{N-1}, \end{aligned}$$

for $\rho\in[\bar{\mu}\tau,\tau]$, $\theta_j\in\,]-\pi/2,\pi/2]$, $j=1,\dots,N-2$ and $\theta_{N-1}\in\,]-\pi,\pi]$.

The Jacobian of this transformation is given by

$$\mathrm{d}x_1\cdots\mathrm{d}x_N=\rho^{N-1}\left\{\prod_{j=1}^{N-1}|\cos\theta_j|^{N-j-1}\right\}\mathrm{d}\rho\,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_{N-1}.$$

Hence,

$$I_p=\left(\int_{\bar{\mu}\tau}^{\tau}(\tau-\rho)^p\rho^{N-1}\,\mathrm{d}\rho\right)\left(\int_{-\pi}^{\pi}\mathrm{d}\theta_{N-1}\right)\prod_{j=1}^{N-2}\int_{-\pi/2}^{\pi/2}|\cos\theta_j|^{N-j-1}\,\mathrm{d}\theta_j.$$

Now

$$\prod_{j=1}^{N-2}\int_{-\pi/2}^{\pi/2}|\cos\theta_j|^{N-j-1}\,\mathrm{d}\theta_j=\prod_{j=1}^{N-2}\frac{\Gamma((N-j)/2)\Gamma(\frac{1}{2})}{\Gamma((N-j+1)/2)}$$

and, taking into account the fact that

$$\prod_{j=1}^{N-2}\frac{\Gamma((N-j)/2)\Gamma(\frac{1}{2})}{\Gamma((N-j+1)/2)}=\frac{N\pi^{N/2-1}}{2\Gamma(\frac{1}{2}N+1)},$$

an easy computation gives (3.6).

At this point, bearing in mind theorem 3.1(i), we infer that

$$\int_{\Omega}F(w_n(x))\,\mathrm{d}x\geqslant\int_{B(x_0,\bar{\mu}\tau)}F(\zeta_n)\,\mathrm{d}x=\bar{\mu}^N\omega_\tau F(\zeta_n)\quad\text{for all } n\in\mathbb{N}.$$

Then

$$\Phi(w_n)-\lambda\Psi(w_n)\leqslant\frac{\zeta_n^p\omega_\tau}{p}\left[\frac{1-\bar{\mu}^N}{\tau^p(1-\bar{\mu})^p}+\|q\|_\infty g_{\bar{\mu}}(p,N)\right]-\lambda\bar{\mu}^N\omega_\tau F(\zeta_n),$$

for every $n\in\mathbb{N}$.

If $B<+\infty$, let

$$\varepsilon\in\left]\frac{1}{\lambda B}\left\{\frac{\sigma(N,p)}{p\tau^p}+\|q\|_\infty\frac{g_{\bar{\mu}}(p,N)}{p\bar{\mu}^N}\right\},1\right[.$$

By (3.5), there exists ν_ε such that

$$F(\zeta_n)>\varepsilon B\zeta_n^p\quad\text{for all } n>\nu_\varepsilon.$$

Moreover,

$$\begin{aligned}\Phi(w_n) - \lambda\Psi(w_n) &\leqslant \frac{\zeta_n^p\omega_\tau}{p}\left[\frac{1-\bar{\mu}^N}{\tau^p(1-\bar{\mu})^p} + \|q\|_\infty g_{\bar{\mu}}(p,N)\right] - \lambda\bar{\mu}^N\omega_\tau\varepsilon B\zeta_n^p \\ &= \zeta_n^p\omega_\tau\bar{\mu}^N\left(\frac{\sigma(N,p)}{p\tau^p} + \|q\|_\infty\frac{g_{\bar{\mu}}(p,N)}{p\bar{\mu}^N} - \lambda\varepsilon B\right) \quad \text{for all } n > \nu_\varepsilon.\end{aligned}$$

Taking into account the choice of ε, we have

$$\lim_{n\to+\infty}[\Phi(w_n) - \lambda\Psi(w_n)] = -\infty.$$

If $B = +\infty$, let us consider

$$M > \frac{1}{\lambda}\left\{\frac{\sigma(N,p)}{p\tau^p} + \|q\|_\infty\frac{g_{\bar{\mu}}(p,N)}{p\bar{\mu}^N}\right\}.$$

By (3.5), there exists ν_M such that

$$F(\zeta_n) > M\zeta_n^p \quad \text{for all } n > \nu_M.$$

Moreover,

$$\begin{aligned}\Phi(w_n) - \lambda\Psi(w_n) &\leqslant \frac{\zeta_n^p\omega_\tau}{p}\left[\frac{1-\bar{\mu}^N}{\tau^p(1-\bar{\mu})^p} + \|q\|_\infty g(p,N)\right] - \lambda\bar{\mu}^N\omega_\tau M\zeta_n^p \\ &= \zeta_n^p\omega_\tau\bar{\mu}^N\left(\frac{\sigma(N,p)}{p\tau^p} + \|q\|_\infty\frac{g_{\bar{\mu}}(p,N)}{p\bar{\mu}^N} - \lambda M\right) \quad \text{for all } n > \nu_M.\end{aligned}$$

Taking into account the choice of M, in this case we also have

$$\lim_{n\to+\infty}[\Phi(w_n) - \lambda\Psi(w_n)] = -\infty.$$

Due to theorem 2.1, the functional I_λ admits an unbounded sequence of critical points, and the conclusion is proven. □

REMARK 3.2. In our context, the functionals Φ and Ψ are strongly continuous and Φ is convex, while theorem 2.1 coincides with [7, theorem 2.5], so the existence of infinitely many solutions for the problem $(D_\lambda^{q,f})$ can be obtained by applying [7, theorem 2.5].

Among the consequences of theorem 3.1 we point out the following result.

COROLLARY 3.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that theorem 3.1(i) holds. Furthermore, assume that*

(i)
$$\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^p} > \frac{1}{p}\left(\frac{\sigma(N,p)}{\tau^p} + \|q\|_\infty\frac{g_{\bar{\mu}}(p,N)}{\bar{\mu}^N}\right),$$

(ii)
$$\liminf_{\xi\to+\infty}\frac{\max_{|t|\leqslant\xi}F(t)}{\xi^p} < \frac{1}{m^p p|\Omega|}.$$

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Then the problem

$$\left.\begin{aligned} -\Delta_p u + q(x)|u|^{p-2}u = f(u) \quad & in\ \Omega, \\ u = 0 \quad & on\ \partial\Omega, \end{aligned}\right\} \qquad (D_1^{q,f})$$

possesses a sequence of weak solutions which is unbounded in $W_0^{1,p}(\Omega)$.

Now we present our other main result. First, put

$$A^* := \liminf_{\xi\to 0^+} \frac{\max_{|t|\leqslant\xi} F(t)}{\xi^p},$$

$$B^* := \limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^p},$$

$$\lambda_1^* := \frac{1}{pB^*}\left(\frac{\sigma(N,p)}{\tau^p} + \|q\|_\infty \frac{g_{\bar\mu}(p,N)}{\bar\mu^N}\right), \tag{3.7}$$

$$\lambda_2^* := \frac{1}{m^p p A^* |\Omega|}. \tag{3.8}$$

We have the following theorem.

Theorem 3.4. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a continuous function. Assume that*

(i) $\inf_{\xi\geqslant 0} F(\xi) \geqslant 0$,

(ii) $\liminf_{\xi\to 0^+} \dfrac{\max_{|t|\leqslant\xi} F(t)}{\xi^p} < \kappa \limsup_{\xi\to 0^+} \dfrac{F(\xi)}{\xi^p}$, *where* κ *is given by (3.2).*

Then, for each $\lambda \in\,]\lambda_1^*, \lambda_2^*[$, *problem* $(D_\lambda^{q,f})$ *possesses a sequence of pairwise distinct weak solutions which strongly converges to zero in* $W_0^{1,p}(\Omega)$.

Proof. Arguing in the same way as in the previous result and applying theorem 2.1(iii) instead of theorem 2.1(ii), the conclusion is obtained. □

Remark 3.5. Assumption (ii) in theorem 3.1 and condition (ii) in theorem 3.4 could be replaced, respectively, by the following more general statements.

(i) There exist two sequences $\{a_n\}$ and $\{b_n\}$ such that

$$0 \leqslant a_n < \left(\bar\mu^{N/p} m \left(\frac{\sigma(N,p)}{\tau^p} + \|q\|_\infty \frac{g_{\bar\mu}(p,N)}{\bar\mu^N}\right)^{1/p} \omega_\tau^{1/p}\right)^{-1} b_n \tag{3.9}$$

for every $n\in\mathbb{N}$ and $\lim_{n\to+\infty} b_n = +\infty$ (respectively, $\lim_{n\to+\infty} b_n = 0$) such that

$$A_1 < \kappa|\Omega| \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^p}, \quad \left(\text{respectively, } A_1 < \kappa|\Omega| \limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^p}\right),$$

where κ is given by (3.2) and

$$A_1 := \lim_{n\to+\infty} \frac{|\Omega| \max_{|t|\leqslant b_n} F(t) - \bar\mu^N \omega_\tau F(a_n)}{b_n^p - m^p a_n^p \omega_\tau [\sigma(N,p)/\tau^p + \|q\|_\infty g_{\bar\mu}(p,N)/\bar\mu^N]\bar\mu^N}.$$

We now consider the case involving the conditions on the potential at infinity. If

$$\lambda \in \left]\lambda_1, \frac{1}{m^p p A_1}\right[\quad \text{and} \quad r_n = \frac{b_n^p}{m^p p}$$

for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\varphi(r_n) &= \inf_{\|u\|^p < p r_n} \frac{\sup_{\|v\|^p < p r_n} \int_\Omega F(v(x))\,\mathrm{d}x - \int_\Omega F(u(x))\,\mathrm{d}x}{r_n - \|u\|^p/p} \\ &\leqslant \frac{\sup_{\|v\|^p < p r_n} \int_\Omega F(v(x))\,\mathrm{d}x - \int_\Omega F(w_n(x))\,\mathrm{d}x}{r_n - \|w_n\|^p/p} \\ &\leqslant \frac{m^p p(|\Omega| \max_{|t| \leqslant b_n} F(t) - \bar{\mu}^N \omega_\tau F(a_n))}{b_n^p - m^p a_n^p \omega_\tau [\sigma(N,p)/\tau^p + \|q\|_\infty g_{\bar{\mu}}(p,N)/\bar{\mu}^N]\bar{\mu}^N},\end{aligned}$$

by choosing

$$w_n(x) := \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0,\tau), \\ \dfrac{a_n}{\tau(1-\bar{\mu})}(\tau - |x - x_0|) & \text{if } x \in B(x_0,\tau) \setminus B(x_0,\bar{\mu}\tau), \\ a_n & \text{if } x \in B(x_0,\bar{\mu}\tau) \end{cases}$$

for each $n \in \mathbb{N}$.

Therefore, since, from assumption (i) in remark 3.5, one has $A_1 < +\infty$, we obtain

$$\gamma \leqslant \liminf_{n \to +\infty} \varphi(r_n) \leqslant m^p p A_1 < +\infty.$$

From this point, arguing as in the proof of theorem 3.1, we prove the theorem.

Similar considerations ensure the existence of infinitely many weak solutions for our problem when the hypotheses on the potential at zero are verified.

Remark 3.6. Condition (i) of remark 3.5 is more general than condition (ii) in theorem 3.1 (respectively, condition (ii) of theorem 3.4). In fact, we obtain theorem 3.1 (respectively, theorem 3.4) from (i) of remark 3.5 by choosing $a_n = 0$ for every $n \in \mathbb{N}$.

Remark 3.7. If we assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous non-negative function, then theorem 3.1 and its consequences and variants guarantee infinitely many positive weak solutions. In fact, let u_0 be a weak solution of problem $(D_\lambda^{q,f})$. Arguing by a contradiction, assume that the set $A = \{x \in \Omega : u_0(x) < 0\}$ is non-empty. Put $\bar{v}(x) = \min\{0, u_0(x)\}$ for all $x \in \Omega$. Clearly, $\bar{v} \in W_0^{1,p}(\Omega)$ and we have

$$\int_\Omega |\nabla u_0(x)|^{p-2} \nabla u_0(x) \nabla \bar{v}(x)\,\mathrm{d}x + \int_\Omega q(x)|u_0(x)|^{p-2} u_0(x)\bar{v}(x)\,\mathrm{d}x = \lambda \int_\Omega f(u_0(x))\bar{v}(x)\,\mathrm{d}x,$$

that is,

$$\int_A |\nabla u_0(x)|^p\,\mathrm{d}x + \int_A q(x)|u_0(x)|^p\,\mathrm{d}x = \lambda \int_A f(u_0(x))u_0(x)\,\mathrm{d}x. \tag{3.10}$$

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Therefore, we have

$$0 \leqslant \int_A |\nabla u_0(x)|^p \,\mathrm{d}x + \int_A q(x)|u_0(x)|^p \,\mathrm{d}x \leqslant 0.$$

Hence, $u_0 = 0$ in A, and this is absurd.

Now, taking into account that $-\Delta_p u_0 + q(x)|u_0|^{p-2}u_0 \geqslant 0$ (in the weak sense) and $u_0 \geqslant 0$ in Ω, the strong maximum principle [6, theorem 11.1] ensures that either $u_0 \equiv 0$ or $u_0 > 0$ in Ω.

Otherwise if we assume $f(0) = 0$, then theorem 3.1 and its consequences and variants guarantee infinitely many non-negative weak solutions. In fact, assuming, without loss of generality, that $f(u) = 0$ for all $u \leqslant 0$ and arguing as before, we again obtain (3.10). Therefore,

$$0 \leqslant \int_A |\nabla u_0(x)|^p \,\mathrm{d}x + \int_A q(x)|u_0(x)|^p \,\mathrm{d}x = 0,$$

which, again, is absurd. Hence, $u_0 \geqslant 0$ in Ω.

Remark 3.8. Theorem 1.1 is an immediate consequence of theorem 3.1, taking remark 3.6 into account.

Remark 3.9. We point out that the results contained in [2] are direct consequences of the main theorems. Indeed, as an example, if $\{a_n\}$ and $\{b_n\}$ are two sequences of real numbers satisfying the assumptions in [2, theorem 1.1], we have

$$\frac{b_n}{a_n} > \omega_\tau^{1/p} \bar{\mu}^{N/p} m \frac{\sigma(N,p)^{1/p}}{\tau},$$

definitively. Moreover, an easy computation ensures that

$$\frac{|\Omega| \max_{|t| \leqslant b_n} F(t) - \bar{\mu}^N \omega_\tau F(a_n)}{b_n^p - m^p \bar{\mu}^N \omega_\tau (\sigma(N,p)/\tau^p) a_n^p} \to 0, \quad n \to +\infty.$$

Therefore, substituting condition (ii) with (i) of remark 3.5, theorem 3.1 ensures that problem $(D_1^{0,f})$ admits a sequence of weak solutions which is unbounded in $W_0^{1,p}(\Omega)$.

4. Examples and applications

In this section we present some examples and applications of our results.

Example 4.1. Assume that $p \in \mathbb{N}$ and $1 \leqslant N < p$. Let q_0 be a non-negative real constant and set

$$a_n := \frac{2n!(n+2)! - 1}{4(n+1)!}, \qquad b_n := \frac{2n!(n+2)! + 1}{4(n+1)!}$$

for every $n \in \mathbb{N}$.

Let $\{g_n\}$ be a sequence of non-negative functions such that

(i) $g_n \in C^0([a_n, b_n])$ such that $g(a_n) = g(b_n) = 0$ for every $n \in \mathbb{N}$;

(ii) $\displaystyle\int_{a_n}^{b_n} g_n(t)\,\mathrm{d}t \neq 0$ for every $n \in \mathbb{N}$.

For instance, we can choose the sequence $\{g_n\}$ as follows:

$$g_n(\xi) := \sqrt{\frac{1}{16(n+1)!^2} - \left(\xi - \frac{n!(n+2)}{2}\right)^2} \quad \text{for all } n \in \mathbb{N}.$$

Define the function $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(\xi) := \begin{cases} [(n+1)!^p - n!^p] \dfrac{g_n(\xi)}{\int_{a_n}^{b_n} g_n(t)\,\mathrm{d}t} & \text{if } \xi \in \bigcup_{n=1}^{+\infty} [a_n, b_n], \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\int_{n!}^{(n+1)!} f(t)\,\mathrm{d}t = \int_{a_n}^{b_n} f(t)\,\mathrm{d}t = (n+1)!^p - n!^p$$

and

$$F(a_n) = n!^p - 1, \qquad F(b_n) = (n+1)!^p - 1$$

for every $n \in \mathbb{N}$.

Hence,

$$\lim_{n\to+\infty} \frac{F(b_n)}{b_n^p} = 2^p, \qquad \lim_{n\to+\infty} \frac{F(a_n)}{a_n^p} = 0.$$

Therefore, we can prove that

$$\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^p} = 0 \quad \text{and} \quad \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^p} = 2^p.$$

Then,

$$\begin{aligned} 0 &= \liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^p} \\ &< \frac{2^p \tau^p \bar{\mu}^N}{|\Omega| m^p (\bar{\mu}^N \sigma(N,p) + q_0 \tau^p g_{\bar{\mu}}(p,N))} \\ &= \kappa \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^p}. \end{aligned}$$

By theorem 3.1, for each

$$\lambda > \frac{1}{p2^p}\left(\frac{\sigma(N,p)}{\tau^p} + q_0 \frac{g_{\bar{\mu}}(p,N)}{\bar{\mu}^N}\right),$$

the problem

$$\left.\begin{aligned} -\Delta_p u + q_0|u|^{p-2}u = \lambda f(u) \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \end{aligned}\right\} \qquad (D_\lambda^{q_0,f})$$

possesses a sequence of weak solutions which is unbounded in $W_0^{1,p}(\Omega)$.

Now let $f^* : \mathbb{R} \to \mathbb{R}$ be the positive and continuous function defined as

$$f^*(\xi) = f(\xi) + 1 \quad \text{for all } \xi \in \mathbb{R},$$

where f is given by (4.1).

Clearly,

$$\liminf_{\xi\to+\infty} \frac{F^*(\xi)}{\xi^p} = 0 \quad \text{and} \quad \limsup_{\xi\to+\infty} \frac{F^*(\xi)}{\xi^p} = 2^p.$$

Hence, again due to theorem 3.1 for each

$$\lambda > \frac{1}{2^p p}\left(\frac{\sigma(N,p)}{\tau^p} + q_0 \frac{g_{\bar\mu}(p,N)}{\bar\mu^N}\right),$$

the problem

$$\left.\begin{aligned} -\Delta_p u + q_0|u|^{p-2}u &= \lambda f^*(u) \quad \text{in } \Omega, \\ u &= 0 \qquad\quad\;\; \text{on } \partial\Omega, \end{aligned}\right\} \qquad (D_\lambda^{q_0,f^*})$$

is a sequence of weak solutions which is unbounded in $W_0^{1,p}(\Omega)$.

Remark 4.2. Recently, as pointed out in § 1, in [4] the existence of infinitely many weak solutions of the Dirichlet problem $(D_1^{0,f})$ is studied. More precisely, Omari and Zanolin show their result under the following key assumption:

$$\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^p} = 0$$

and

$$\limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^p} = +\infty. \tag{4.1}$$

Clearly, in this case, the potential F in example 4.1 does not satisfy condition (4.1). Hence, we cannot apply [4, theorem 1.1] to our problem $(D_\lambda^{q_0,f})$ even in the case when $q_0 = 0$. On the other hand, we cannot apply [2, theorem 1.1] to $(D_\lambda^{q_0,f^*})$, since one of the key assumptions is that function f is non-positive in suitable real intervals. Note also that the function f is not positive for all sufficiently large ξ. Hence, in this example, [3, corollary 3.1] fails.

Example 4.3. Let Ω be a non-empty bounded subset of the Euclidean plane $\mathbb{R}^2$. Moreover, letting q_0 be a non-negative real constant, put

$$a_1 := 2, \qquad a_{n+1} := n!(a_n)^{4/3} + 2$$

for every $n \geqslant 1$ and $S := \bigcup_{n\in\mathbb{N}}]a_{n+1} - 1, a_{n+1} + 1[$. Define the continuous function $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(t) := \begin{cases} (a_{n+1})^4 \exp\left\{\dfrac{1}{(t-(a_{n+1}-1))(t-(a_{n+1}+1))} + 1\right\} \\ \qquad\qquad \times \dfrac{2(a_{n+1}-t)}{(t-(a_{n+1}-1))^2(t-(a_{n+1}+1))^2} & \text{if } t \in S, \\ 0 & \text{otherwise,} \end{cases}$$

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for which we have

$$F(\xi) = \int_0^\xi f(t)\,\mathrm{d}t$$
$$= \begin{cases} (a_{n+1})^4 \exp\left\{\dfrac{1}{(\xi-(a_{n+1}-1))(\xi-(a_{n+1}+1))}+1\right\} & \text{if } \xi \in S, \\ 0 & \text{otherwise,} \end{cases}$$

and $F(a_{n+1}) = (a_{n+1})^4$ for every $n \in \mathbb{N}$. Hence,

$$\limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^3} = +\infty.$$

Setting $y_n = a_{n+1} - 1$ for every $n \in \mathbb{N}$, we obtain $\max_{|\xi|\leqslant y_n} F(\xi) = (a_n)^4$ for every $n \in \mathbb{N}$. As

$$\lim_{n\to+\infty} \frac{\max_{|\xi|\leqslant y_n} F(\xi)}{{y_n}^3} = 0,$$

one has

$$\liminf_{\xi\to+\infty} \frac{\max_{|t|\leqslant\xi} F(t)}{\xi^3} = 0.$$

Hence,

$$\liminf_{\xi\to+\infty} \frac{\max_{|t|\leqslant\xi} F(t)}{\xi^3} = 0 < \kappa \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^3} = +\infty.$$

Then, for each $\lambda > 0$, the problem

$$\left.\begin{aligned} -\operatorname{div}(|\nabla u|\nabla u) + q_0 u = \lambda f(u) \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \end{aligned}\right\}$$

possesses a sequence of weak solutions which is unbounded in $W_0^{1,3}(\Omega)$.

Remark 4.4. In [2, theorem 1.1] Cammaroto *et al*. assume that

$$\limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^p} < +\infty.$$

Example 4.2 shows that this condition is not necessary.

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