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GLOBAL IDENTIFICATION IN NONLINEAR MODELS WITH MOMENT RESTRICTIONS

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This paper derives sufficient conditions for global identification in nonlinear models characterized by a finite number of unconditional moment restrictions. The main contribution of this paper is to provide a set of assumptions that are alternative to those of Gale-Nikaidô-Fisher-Rothenberg, and which when satisfied guarantee that the moment conditions globally identify the parameters of interest.

1. INTRODUCTION

The problem of identification of economic relations has a long-standing history, with systematic discussions given in a collective work of the Cowles Foundation edited by Koopmans (1950).[1] In a nutshell, the identification problem is concerned with the unambiguous definition of the parameters to be estimated. Thus, it precedes the problem of statistical estimation. When identification fails, the properties of conventional statistical procedures are likely to change (see, e.g., Phillips, 1989; Choi and Phillips, 1992). The objective of this paper is to provide primitive conditions under which identification is guaranteed to hold.

Based on the work of Koopmans and Reiersøl (1950), a complete treatment of identification in a parametric context was given in Rothenberg (1971) and Bowden (1973). Using an approach based on information criteria, they provided conditions under which parametric models are locally and globally identified. Unfortunately, such results may only be applied in models in which it is possible to specify the likelihood function of the observed variables.

Situations in which the distribution of the observables is left unspecified require conditions for identification in a nonparametric context. Those have been derived in the work of Brown (1983), Roehrig (1988), Matzkin (1994, 2008), and Benkard and Berry (2006), among others (see, e.g., Matzkin, 2007, for a complete survey). Common to all the studies is an assumption of independence between the observed explanatory variables and latent disturbances to the structural system.

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Moment restriction models, which are the focus of this paper, fall in between the fully parametric and nonparametric models. They arise, for example, when the distribution of the disturbances is only known to satisfy certain moment restrictions. These are typically expressed as conditions for orthogonality between the disturbances and instruments—functions of explanatory variables—and are hence weaker than an assumption of independence.

The present paper examines identification in models defined by unconditional moment restrictions. Thus, its contributions are complementary to the existing literature that considers models with conditional moment restrictions, such as Chesher (2003), Newey and Powell (2003), Chernozhukov and Hansen (2005), Severini and Tripathi (2006), and Chernozhukov, Imbens, and Newey (2007), for example. It is worthwhile distinguishing these two cases, as identification in some unconditional moment models implied by the conditional ones may fail even when the conditional model is identified. Examples of such failures can be found in Dominguez and Lobato (2004).

The basic identification criteria for linear simultaneous equation systems under linear parameter constraints were given in Koopmans (1950). These criteria are the well-known rank conditions that were extended by Fisher (1961, 1965) to nonlinear systems that are still linear in parameters. An important step toward a full treatment of identification in general nonlinear models was made by Fisher (1966) and Rothenberg (1971). Their insight was to treat the identification problem simply as a question of uniqueness of solutions to nonlinear systems of equations. With the exception of Fisher (1966) and Rothenberg (1971), few global identification results apply to models that are nonlinear. Newey and McFadden (1994) remarked that, as a consequence, much of the applied literature has adopted an approach in which identification is simply assumed.

Both Fisher's (1966) and Rothenberg's (1971) results exploit the uniqueness conditions given in Theorem 6w of Gale and Nikaidô (1965, p. 89). The key idea behind the Gale-Nikaidô-Fisher-Rothenberg approach is to look for conditions under which the (nonlinear) equations under consideration correspond to the first order conditions of an optimization problem that involves a strictly convex objective function, called a potential. This approach requires that the derivative matrix of the system of equations be weakly positive quasidefinite, i.e., that its symmetric part be positive semidefinite and that its Jacobian be (strictly) positive everywhere. These conditions ensure that the system derives from a potential that is strictly convex. In many instances, however, this approach produces sufficient conditions for global identification that—in the words of Rothenberg (1971)—are "overly strong."

Indeed, the conditions used in Gale-Nikaidô-Fisher-Rothenberg are sufficient but not necessary. The main contribution of this paper is to provide an alternative set of primitive conditions for global identification. Our uniqueness results exploit the pioneering work by Palais (1959), which does not require the mappings under consideration to derive from a potential. Our key requirement is a *properness* condition introduced by Palais (1959). In addition to properness, we impose two conditions: One concerns the Jacobian of the system, while the other excludes "flats." In particular, we assume that the Jacobian of the system is either everywhere nonnegative or everywhere nonpositive. When the system is continuously differentiable with respect to the structural parameter, this requirement is weaker than the full rank conditions given in Theorem 5.10.2 in Fisher (1966) and Theorem 7 in Rothenberg (1971).[2] In other words, we allow the rank of the derivative matrix to be less than full, provided this only happens over sufficiently small regions in the parameter space. The latter is our second main requirement: that the system does not have any "flats," i.e., does not remain constant over regions in the parameter space that have nonzero dimension. Our results exploit well-established results of nonlinear functional analysis.

The paper is organized as follows: Section 2 sets up the problem. In Section 3 we derive the key mathematical results of the paper. Their implications for identification are discussed in Section 4, which concludes.

2. SETUP

This paper is concerned with models characterized by a finite set of unconditional moment restrictions,

$$\mathrm{E}[r(X,\theta_0)] = 0, \qquad \textbf{(1)}$$

in which $r : \mathbb{D}_X \times \mathbb{R}^k \to \mathbb{R}^k$ is a known mapping and $\mathbb{D}_X \subseteq \mathbb{R}^K$ with $K < \infty$. The variables entering into these equations consist of a set of observed variables $X \in \mathbb{D}_X$, and a finite dimensional parameter $\theta \in \mathbb{R}^k$ ($k < \infty$). Here we shall focus on the case in which the parameter θ is allowed to take any value in $\mathbb{R}^k$, so the parameter space is the entire euclidean space $\mathbb{R}^k$. The object of interest is the true value θ_0 of the structural parameter θ in equation (1).

We call F_X the distribution (measure) of the observables X defined on $\mathbb{D}_X$. The expectations are always taken with respect to F_X, which can itself depend on θ_0. In order to guarantee that the expression in (1) is well defined over the entire parameter space, we assume that $\mathrm{E}[r(X,\theta)]$ exists and is finite for every $\theta \in \mathbb{R}^k$.

The moment function r in equation (1) takes values in $\mathbb{R}^k$; hence, we are in a situation in which there are exactly as many unconditional moment restrictions as there are components of θ to identify. Our main question is then: Under what conditions on r is the true value θ_0 *globally* identified? We shall work with the following definition.

DEFINITION 1. *The true parameter value θ_0 is globally identified if and only if* $\mathrm{E}[r(X,\theta)] = 0$ *has a unique solution $\theta = \theta_0$ on $\mathbb{R}^k$.*

The identification condition in Definition 1 is the well-known generalized method of moment (GMM) identification condition. As pointed out by Newey and McFadden (1994, Sect. 2.2.3, p. 2127), "here conditions for identification are like conditions for unique solutions of nonlinear equations [...], which are known to be difficult." In what follows, define a mapping $g : \mathbb{R}^k \to \mathbb{R}^k$, which to each $\theta \in \mathbb{R}^k$ assigns

$$g(\theta) \equiv \mathrm{E}[r(X,\theta)].$$

As previously, the expectation is taken with respect to F_X. The identification condition in Definition 1 is thus equivalent to the condition that $g(\theta) = 0$ be uniquely solved at $\theta = \theta_0$. Hereafter, we shall maintain the following assumptions on g.

Assumption A. The map g is in $\mathcal{C}^2(\mathbb{R}^k)$.

The mapping g is assumed to be twice continuously differentiable on $\mathbb{R}^k$, and we let $\mathrm{D}g \in L(\mathbb{R}^k, \mathbb{R}^k)$ denote its derivative. The following assumption restricts the behavior of the Jacobian $J_g \equiv \det \mathrm{D}g$ of g on $\mathbb{R}^k$.

Assumption B. For every $\theta \in \mathbb{R}^k$, $J_g(\theta)$ is nonnegative.[3]

The condition on the nonnegativity of the Jacobian J_g is a weakening of the Gale-Nikaidô-Fisher-Rothenberg condition that the latter be positive. Note that unlike Gale-Nikaidô-Fisher-Rothenberg, Assumption B does not require the matrix of derivatives $\mathrm{D}g$ to be quasipositive definite. It is worth pointing out that the sign condition in Assumption B is also a weakening of the condition that the Jacobian be nonvanishing on $\mathbb{R}^k$. Indeed, if g is twice continuously differentiable, then its Jacobian J_g is continuous, so requiring that for every $\theta \in \mathbb{R}^k$, $J_g(\theta) \neq 0$ is equivalent to requiring that J_g be either positive or negative on $\mathbb{R}^k$.

Next, we require that the mapping g be *proper*, i.e., that the inverse image by g of each compact subset of $\mathbb{R}^k$ be a compact in $\mathbb{R}^k$. Since g is continuous, a necessary and sufficient condition is

Assumption C. $\|g(\theta)\| \to \infty$ whenever $\|\theta\| \to \infty$.

Finally, we impose the following.

Assumption D. For every $p \in \mathbb{R}^k$ the equation $g(\theta) = p$ has countably many (possibly zero) solutions in $\mathbb{R}^k$.

Assumption D excludes the situations in which the map g remains "flat" over regions in the parameter space that are of dimension greater than or equal to 1. The requirement that $g(\theta) = p$ have at most countably many solutions is only binding for values of p that are not regular (such values are called critical values). Indeed, if p is a regular value (meaning that the inverse image of $\{p\}$ contains only the parameter values $\theta_r \in \mathbb{R}^k$ for which the Jacobian $J_g(\theta_r)$ is different from 0) then the set of solutions to $g(\theta) = p$ is finite.[4]

We are now ready to derive primitive conditions under which

$$g \text{ is continuous and one-to-one from } \mathbb{R}^k \text{ onto } \mathbb{R}^k \text{ with a continuous inverse.} \quad \textbf{(2)}$$

A mapping g that satisfies the property in (2) is called a *homeomorphism* from $\mathbb{R}^k$ to $\mathbb{R}^k$. According to Definition 1, the homeomorphic property of g is sufficient for θ_0 to be identified. Notice, however, that this property is not strictly necessary, since identification only restricts the behavior of g around $g(\theta) = 0$.

3. HOMEOMORPHISM RESULT

THEOREM 1. *Let $g : \mathbb{R}^k \to \mathbb{R}^k$ satisfy Assumptions A through D. Then, the following results hold:*

(i) if $k = 1$, then g is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$;
(ii) if $k > 2$ and the set of points $\theta_s \in \mathbb{R}^k$ for which $\operatorname{rank} \mathrm{D}g(\theta_s) < k - 1$ *is bounded, then g is a homeomorphism from $\mathbb{R}^k$ to $\mathbb{R}^k$.*

Proof. We start by fixing the notation. Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be a continuous mapping. We denote by $f(A)$ the image by f of any subset $A \subseteq \mathbb{R}^k$, and by $f^{-1}(C)$ the inverse image by f of any subset $C \subseteq f(\mathbb{R}^k)$. The mapping f is said to be *open* if whenever A is open, $f(A)$ is open; f is said to be *light* if for every point $p \in \mathbb{R}^k$, $f^{-1}(p)$ is totally disconnected, i.e., $\dim f^{-1}(p) \leqslant 0$;[5] and f is *proper* if whenever K is compact, $f^{-1}(K)$ is compact. We let B_f denote the set of all points in $\mathbb{R}^k$ at which f fails to be local homeomorphism. The set of points x_s at which $\operatorname{rank} \mathrm{D}f(x_s) \leqslant q$ with $0 \leqslant q \leqslant k$ is denoted by R_f^q; it follows that $B_f \subseteq R_f^{k-1}$. Finally, f is said to be a *covering space map* if one can find a covering of $\mathbb{R}^k$ by open sets U such that for each U, $f^{-1}(U)$ is the disjoint union of open sets, each of which is mapped homeomorphically onto U.

The proof is in five steps.

Step 1. Under Assumption D, the map $g : \mathbb{R}^k \to \mathbb{R}^k$ is light. From Theorem 2 in Titus and Young (1952) we have: *Every $f : \mathbb{R}^k \to \mathbb{R}^k$ of class $\mathcal{C}^1$ that is light and whose Jacobian J_f is nonnegative on $\mathbb{R}^k$ is open*. Thus, Assumptions A, B, and D imply g is open. Now we can use an extension of the inverse function theorem for open maps given in Theorem 1.4 by Church (1963): *If $f : \mathbb{R}^k \to \mathbb{R}^k$ of class $\mathcal{C}^1$ is open, then f is local homeomorphism at $x \in \mathbb{R}^k$ whenever* $\operatorname{rank} J_f(x) \geqslant k - 1$. Hence, $B_g = \emptyset$ when $k = 1$, and $B_g \subseteq R_g^{k-2}$ when $k > 1$.

Step 2. First, we use Corollary 2.3 in Church and Hemmingsen (1960): *If f is light open and* $\dim f(B_f) < k$, *then* $\dim B_f = \dim f(B_f) = \dim f^{-1}(f(B_f))$. From Step 1 we know that g is light open. Moreover, when $k > 1$, we have $g(B_g) \subseteq g(R_g^{k-2})$ so $\dim g(B_g) \leqslant \dim g(R_g^{k-2})$. To show that $\dim g(B_g) < k$ we use Theorem 2 in Sard (1965): *If $f \in \mathcal{C}^n(\mathbb{R}^k)$ with $n \geqslant k - q$, then* $\dim f(R_f^q) \leqslant q$. Under Assumption A, $g \in \mathcal{C}^2(\mathbb{R}^k)$ so letting $q = k - 2$ we get $\dim g(R_g^{k-2}) < k - 1$; hence $\dim g(B_g) < k - 1$. Combining the two results shows that when $k > 1$, $\dim B_g = \dim g(B_g) = \dim g^{-1}(g(B_g)) < k - 1$.

Step 3. We show that $g(\mathbb{R}^k) = \mathbb{R}^k$, that g is proper, and that $\mathbb{R}^k \backslash g^{-1}(g(B_g))$ is connected when $k > 1$.

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Recall from Step 1 that g is open; this implies that $g(\mathbb{R}^k)$ is open in $\mathbb{R}^k$. Now using Assumption C we show that $g(\mathbb{R}^k)$ is closed in $\mathbb{R}^k$: Take a sequence $\{g(\theta_n)\}$ $(n \in \mathbb{N})$ such that $g(\theta_n) \underset{n\to\infty}{\longrightarrow} p$ for some $p \in \mathbb{R}^k$. By Assumption C we then have $\{\theta_n\}$ $(n \in \mathbb{N})$ bounded, so $\theta_n \underset{n\to\infty}{\longrightarrow} \bar{\theta}$ for some $\bar{\theta}$. By continuity of g, $p = g(\bar{\theta})$, so $p \in g(\mathbb{R}^k)$ and $g(\mathbb{R}^k)$ is closed. Since $g(\mathbb{R}^k)$ is both open and closed in $\mathbb{R}^k$, we have $g(\mathbb{R}^k) = \mathbb{R}^k$.

To show that g is proper, take any $K \subset \mathbb{R}^k$ compact. Given that g is continuous, $g^{-1}(K)$ is closed in $\mathbb{R}^k$. It remains to be shown that $g^{-1}(K)$ is bounded. Assume the contrary; then there exists a sequence $\{\theta_n\}$ $(n \in \mathbb{N})$ in $g^{-1}(K)$ such that $\|\theta_n\| \underset{n\to\infty}{\longrightarrow} \infty$. By Assumption C, this implies that the sequence $\{g(\theta_n)\}$ $(n \in \mathbb{N})$ in $g(g^{-1}(K)) = K$ is such that $\|g(\theta_n)\| \underset{n\to\infty}{\longrightarrow} \infty$. So, K is unbounded, which contradicts the fact that K is compact.

Finally, to show that $\mathbb{R}^k \backslash g^{-1}(g(B_g))$ is connected for any $k > 1$, we use Theorem IV.4 in Hurewicz and Wallman (1948): *Any connected k-dimensional set in* $\mathbb{R}^k$ *cannot be disconnected by a subset of dimension* $< k-1$. The desired result follows by using the connectedness of $\mathbb{R}^k$ together with $\dim g^{-1}(g(B_g)) < k-1$ obtained in Step 2.

Step 4. We now show that the restriction of g to $\mathbb{R}^k \backslash g^{-1}(g(B_g))$ is a covering space map.

For this we use Covering Space Theorem 1 in Plastock (1978): *Let A be a connected open set in $\mathbb{R}^k$. Then $f : A \to f(A)$ is a covering space map if (i) f is a local homeomorphism, and (ii) f is proper.* When $k = 1$, Plastock's result applies to $A \equiv \mathbb{R}$ and $f = g$, since from Step 1 we known that g is a local homeomorphism on $\mathbb{R}$, and from Step 3 we know that g is proper. Hence, when $k = 1$, g is a covering map.

Now, consider the case $k > 1$, let $A \equiv \mathbb{R}^k \backslash g^{-1}(g(B_g))$, and let f be a restriction of g to A, which we denote by $f \equiv g|_A$. First, note that $g^{-1}(g(B_g)) \supseteq B_g$ so $A \cap B_g = \emptyset$ and $g|_A : A \to \mathbb{R}^k \backslash g(B_g)$ is a local homeomorphism. Next, we show that $g|_A$ is proper: Let C be a compact subset of $\mathbb{R}^k \backslash g(B_g)$ and note that $g|_A^{-1}(C) = g^{-1}(C)$ since $g|_A^{-1} = g|_{\mathbb{R}^k \backslash g(B_g)}^{-1}$. Then by properness of g we have that $g^{-1}(C)$ is compact in $\mathbb{R}^k$. Since $C \cap g(B_g) = \emptyset$ it follows that $g^{-1}(C) \cap g^{-1}(g(B_g)) = \emptyset$ and so $g^{-1}(C)$ is compact in A.

Finally, we show that A is open. Consider $\theta \in \mathbb{R}^k \backslash B_g$. Then g is a local homeomorphism at θ, i.e., there exists an open neighborhood U of θ such that $g(U)$ is open in $\mathbb{R}^k$ and $g|_U : U \to g(U)$ is a homeomorphism. So $U \cap B_g = \emptyset$ and $U \subset \mathbb{R}^k \backslash B_g$, which shows that $\mathbb{R}^k \backslash B_g$ is open; hence, B_g is closed. It then follows that $g(B_g)$ is closed in $\mathbb{R}^k$, since any continuous proper map g is also closed, i.e., $g(A)$ closed whenever $A \subset \mathbb{R}^k$ closed (see, e.g., corollary in Palais, 1970). Continuity of g then guarantees that $g^{-1}(g(B_g))$ is closed in $\mathbb{R}^k$, thus A is open.

From Step 3 we know that when $k > 1$, the set $\mathbb{R}^k \backslash g^{-1}(g(B_g))$ is connected. We can then apply Plastock's (1978) Covering Space Theorem to show that when $k > 1$ the restriction of g to $\mathbb{R}^k \backslash g^{-1}(g(B_g))$ is a covering map.

Step 5. Finally, we use Theorem 1.3 in Church and Hemmingsen (1960): *Let f be an open map of $\mathbb{R}^k$ onto $\mathbb{R}^k$, $k \neq 2$, such that* $\dim f(B_f) \leqslant k-2$. *If the restriction of f to $\mathbb{R}^k \backslash f^{-1}(f(B_f))$ is a covering map, and if B_f is compact, then f is a homeomorphism.* That g is open follows from Step 1; that g is onto $\mathbb{R}^k$ follows from Step 3. In Step 1 we show that when $k=1$ the set B_g is empty, so $g(B_g)=\emptyset$ and $\dim g(B_g)=-1$. When $k>2$, Step 2 shows that $\dim g(B_g) \leqslant k-2$. That $g|_{\mathbb{R}^k \backslash g^{-1}(g(B_g))}$ is a covering map follows from Step 4. It remains to show that B_g is compact. When $k=1$, the result is trivial. When $k>2$, we know from Step 4 that B_g is closed. From Step 1 we know that $B_g \subseteq R_{k-2}$; so the condition that R_{k-2} is bounded from Theorem 1 implies that B_g is bounded. Thus, g is a homeomorphism from $\mathbb{R}^k$ to $\mathbb{R}^k$. ■

We now comment on the conditions imposed in Theorem 1. First, note that the result of Theorem 1 does not hold if the dimension of the parameter set is $k=2$. Church and Hemmingsen (1960) and Chua and Lam (1972) contain simple examples of mappings that are not one-to-one and yet satisfy all the requirements of Theorem 1 except $k \neq 2$.[6]

Second, when $k>2$, Theorem 1 puts an additional restriction on the set of points $\theta_s \in \mathbb{R}^k$ for which $\operatorname{rank} Dg(\theta_s) \leqslant k-2$, set which we denote by R_g^{k-2}. The restriction is that R_g^{k-2} be bounded. A simple sufficient condition is that the Jacobian J_g does not vanish at infinity. Indeed, if for large enough values of $\|\theta\|$ the Jacobian remains positive (negative), then the set R_g^{k-1} remains bounded. A fortiori, its subset R_g^{k-2} is then bounded as well.

Third, while sufficient, not all the assumptions of Theorem 1 are necessary. It is clear that if g is homeomorphism from $\mathbb{R}^k$ to $\mathbb{R}^k$, then g is light on $\mathbb{R}^k$; hence, Assumption D is necessary. Assuming $g \in \mathcal{C}^k(\mathbb{R}^k)$, Theorem 2.3 in Chua and Lam (1972) further shows that Assumptions C and B are necessary for g to be homeomorphism from $\mathbb{R}^k$ to $\mathbb{R}^k$, and so is the requirement that R_g^{k-1} cannot be a k-dimensional open set. However, the condition of Theorem 1 that R_g^{k-2} be bounded is not necessary. The following theorem replaces the latter with an alternative assumption.

THEOREM 2. *Let $g : \mathbb{R}^k \to \mathbb{R}^k$ satisfy Assumptions A through D. Then, the following results hold:*

(i) if $k=1$, then g is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$;
(ii) if $k>2$ and the set of points $\theta_s \in \mathbb{R}^k$ for which $\operatorname{rank} Dg(\theta_s) < k-1$ *is of dimension less than or equal to $k-3$, then g is a homeomorphism from $\mathbb{R}^k$ to $\mathbb{R}^k$.*

Proof. The proof is identical to that of Theorem 1 except for the proof of the result for $k>2$ in Step 5, which should be modified as follows.

Step 5. We use Lemma 1 in Plastock (1978): *If $f : A \to f(A)$ is a covering space map, A and $f(A)$ pathwise connected, and $f(A)$ simply connected, then f is a global homeomorphism.* Let $A \equiv \mathbb{R}^k \backslash g^{-1}(g(B_g))$ and $f \equiv g|_A$. From Step 2 we know that $\dim g(B_g) = \dim g^{-1}(g(B_g)) \leqslant k-2$. By using the same reasoning as in Step 3, we then have that A and $g|_A(A) = \mathbb{R}^k \backslash g(B_g)$ are connected. Recall in addition from Step 4 that A is open and that $g(B_g)$ is closed, so that $g|_A(A) = \mathbb{R}^k \backslash g(B_g)$ is open in $\mathbb{R}^k$. Hence, A and $g|_A(A)$ are two open subsets of $\mathbb{R}^k$ that are connected; this implies that they are also pathwise connected. To show that $g|_A(A)$ is simply connected, we use Theorem 25 in Basye (1935): *If K is a closed subset of $\mathbb{R}^k$ of dimension $k-3$ or less, then $\mathbb{R}^k \backslash K$ is simply connected.* Letting $K \equiv g(B_g)$, we know that K is closed in $\mathbb{R}^k$. Moreover, from Step 2 we know that $\dim g(B_g) = \dim B_g \leqslant \dim R_g^{k-2}$, which from the condition of Theorem 2 is less or equal than $k-3$; this implies that $g|_A(A)$ is simply connected. Hence, $g|_{\mathbb{R}^k \backslash g^{-1}(g(B_g))}$ is a homeomorphism from $\mathbb{R}^k \backslash g^{-1}(g(B_g))$ onto $\mathbb{R}^k \backslash g(B_g)$.

It remains to show that $g|_{g^{-1}(g(B_g))}$ is a homeomorphism from $g^{-1}(g(B_g))$ onto $g(B_g)$. Then let $\bar{g} \equiv g|_{g^{-1}(g(B_g))}$. By construction, $\bar{g} : g^{-1}(g(B_g)) \to g(B_g)$ is onto. We now show that it is also one-to-one: Let $p \in g(B_g)$ and assume that $g^{-1}(p) \supset \{\theta_1, \theta_2\}$ with $\theta_1 \neq \theta_2$. Since $\mathbb{R}^k$ is separated, there exist two disjoint open sets U_1 and U_2 containing θ_1 and θ_2, respectively. Given that g is open, $V_1 = g(U_1)$ and $V_2 = g(U_2)$ are open, and so $V_1 \cap V_2 \supset \{p\} \neq \emptyset$ is open in $\mathbb{R}^k$; by Theorem IV.3 in Hurewicz and Wallman (1948) then, $\dim V_1 \cap V_2 = k$. In particular, $V_1 \cap V_2$ contains a point $q \in \mathbb{R}^k \backslash g(B_g)$; otherwise, $V_1 \cap V_2 \subseteq g(B_g)$, which would imply $\dim g(B_g) = k$ and is contradictory with $\dim g(B) < k-1$ shown in Step 2. Now, $g|_{\mathbb{R}^k \backslash g^{-1}(g(B_g))}$ being a homeomorphism from $\mathbb{R}^k \backslash g^{-1}(g(B_g))$ onto $\mathbb{R}^k \backslash g(B_g)$ is in contradiction with $U_1 \cap U_2 = \emptyset$. Hence, $\bar{g}$ is one-to-one, onto, continuous, and both open and closed; hence, its inverse is also continuous, and $\bar{g}$ is a homeomorphism from $g^{-1}(g(B_g))$ onto $g(B_g)$. Combining all of the above shows g is a homeomorphism from $\mathbb{R}^k$ to $\mathbb{R}^k$. ■

When $k > 2$, Theorems 1 and 2 give sufficient conditions for g to be a homeomorphism under alternative assumptions on the set R_g^{k-2}. If the latter is bounded, then the result of Theorem 1 applies. If boundedness cannot be established, then Theorem 2 still holds provided the dimension of R_g^{k-2} remains sufficiently small relative to the dimension k of the parameter space.

We now relate the results of Theorems 1 and 2 to the literature.

Important homeomorphism results have been obtained under the assumption that the Jacobian J_g is positive: Corollary 4.3 in Palais (1959) combines $J_g > 0$ with the properness condition on g to show that g is homeomorphism from $\mathbb{R}^k$ to $\mathbb{R}^k$; Theorem 6w in Gale and Nikaidô (1965, p. 89)—also used in Fisher (1966) and Rothenberg (1971)—combines $J_g > 0$ with the restriction that the matrix Dg be weakly positive quasidefinite, i.e., that its symmetric part be everywhere positive semidefinite. These conditions ensure that g derives from a potential that is strictly convex on $\mathbb{R}^k$; hence g is homeomorphism from $\mathbb{R}^k$ to $\mathbb{R}^k$. Both Palais's (1959) and Gale and Nikaidô's (1965) conditions require everywhere positive Jacobian J_g, which is considerably stronger than our Assumption C.

An extension of Palais's (1959) result to the case where J_g is nonnegative can be found in Chua and Lam (1972). Their Theorem 2.2 combines $J_g \geqslant 0$ with the requirement that the set R_g^{k-1} be of dimension less than or equal to 0. Our Theorem 2 imposes Assumption D but relaxes Chua and Lam's dimension requirement on R_g^{k-1} by replacing it with a weaker requirement that dim $R_g^{k-2} \leqslant k-3$. Our Theorem 1 uses an entirely different boundedness condition on R_g^{k-2}. To the best of our knowledge, the use of such boundedness conditions in deriving homeomorphism results is new to the literature. Neither Chua and Lam's nor our results hold in dimension $k=2$.

4. DISCUSSION AND CONCLUSION

We now apply the results of Theorems 1 and 2 to the identification problem. An alternative application of global homeomorphism results is to the problem of indirect inference; see, e.g., Phillips (2012).

Say that the parameter of interest θ in equation (1) is a scalar ($k=1$) whose true value $\theta_0 \in \mathbb{R}$ is known to satisfy $\mathrm{E}[r(X,\theta_0)]=0$. As previously, the expectation is taken with respect to F_X obtained under θ_0, and the map $r : \mathbb{R}^{K+1} \to \mathbb{R}$ is known. The true parameter value θ_0 is globally identified if θ_0 is the unique solution on $\mathbb{R}$ to the equation $g(\theta) \equiv \mathrm{E}[r(X,\theta)]=0$. A simple sufficient condition for global identification of θ_0 is that $g' > 0$ everywhere on $\mathbb{R}$. However, this condition is stronger than necessary. Indeed, Theorems 1 and 2 show that identification obtains under sole Assumptions A through D. A simple sufficient condition for Assumption D is that g' only vanishes over a set of isolated points (see, e.g., Church and Hemmingsen, 1960, Thm. 2.5). This shows that $g' > 0$ is not necessary for identification even in the scalar case.

When $k > 1$, θ and $r(X,\theta)$ are both vectors in $\mathbb{R}^k$, the map g is from $\mathbb{R}^k$ to $\mathbb{R}^k$ and we are brought to consider its Jacobian instead of the above derivative. Unlike in the scalar case, requiring that the Jacobian of g be positive (or negative) on $\mathbb{R}^k$ no longer suffices to show that θ_0 is globally identified. This is because $J_g > 0$ is no longer a sufficient condition for the mapping g to satisfy the property in (2). A standard counterexample is the mapping $c : \mathbb{R}^2 \to \mathbb{R}^2$, which to each $(\theta_1,\theta_2)' \in \mathbb{R}^2$ assigns $c(\theta_1,\theta_2) = (\exp\theta_1 \cos\theta_2, \exp\theta_1 \sin\theta_2)$. It is easy to check that its Jacobian is everywhere positive, yet the inverse image by c of any point in $\mathbb{R}^2\backslash\{0\}$ has an infinite number of distinct elements. Our solution is to first eliminate the mappings such as c by requiring that g be proper (Assumption C), i.e., that the inverse image of any compact set be compact. This condition is clearly violated by c since for any $(p_1,p_2)' \in \mathbb{R}^2\backslash\{0\}$ the inverse image $c^{-1}(\{(p_1,p_2)'\})$ is unbounded (hence not compact) in $\mathbb{R}^2$.

Properness by itself does not guarantee that g is either one-to-one on $\mathbb{R}^k$ or onto $\mathbb{R}^k$. The latter is true if one is willing to assume that in addition its Jacobian J_g never vanishes (see, e.g., Palais, 1959, Cor. 4.3). Still, in models that are nonlinear in θ, everywhere nonvanishing Jacobian might be too strong an assumption. It turns out, however, that when $k \neq 2$, restricting the Jacobian to be either nonnegative on $\mathbb{R}^k$ or nonpositive on $\mathbb{R}^k$ (Assumption B) suffices to make a proper mapping g be one-to-one and onto, provided g is nowhere "flat" (Assumption D). This last assumption is automatically satisfied for any regular value $p = g(\theta)$; however, it needs to be verified whenever p is a critical value.

Working with systems whose Jacobian possibly vanishes requires additional restrictions on the set of points at which rank $\mathrm{D}g \leqslant k-2$. If this set is bounded, then the result of Theorem 1 applies. If boundedness cannot be established, then Theorem 2 still holds provided the dimension of this set remains sufficiently small relative to the dimension k of the parameter space.

Importantly, Theorems 1 and 2 show that Gale-Nikaidô-Fisher-Rothenberg conditions $J_g > 0$ and $\mathrm{D}g$ weakly positive quasidefinite are not necessary for global identification to hold, at least not in the case $k \neq 2$.

NOTES

1. See, e.g., Dufour and Hsiao (2008) for a review of historical and recent developments on identification in economics.

2. It is worth pointing out that we place conditions only on the sign of the Jacobian. Unlike Gale-Nikaidô-Fisher-Rothenberg, we do not make any positive definiteness assumptions on the derivative matrix of the system.

3. Alternatively, Assumption B can be replaced with the requirement that J_g be non-positive.

4. By properness, the inverse image of $\{p\}$ is a compact set in $\mathbb{R}^k$; the inverse function theorem guarantees that this set is discrete, hence it is finite (see, e.g., step 5 in the proof of Theorem by Debreu, 1970).

5. A set A has dimension 0 if every element of A has arbitrary small open neighborhoods with empty boundaries, i.e. A is a totally disconnected set. In particular, every nonempty finite or countable set has dimension 0. The empty set is the only set that has dimension -1 (see, e.g., Hurewicz and Wallman, 1948).

6. A simple counterexample is $b : \mathbb{R}^2 \to \mathbb{R}^2$, $b(\theta_1, \theta_2) = (\theta_1^2 - \theta_2^2, \theta_1\theta_2)$ for any $(\theta_1, \theta_2)' \in \mathbb{R}^2$.

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