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DERIVED NON-ARCHIMEDEAN ANALYTIC HILBERT SPACE

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Abstract In this short paper, we combine the representability theorem introduced in [Porta and Yu, Representability theorem in derived analytic geometry, preprint, 2017, arXiv:1704.01683; Porta and Yu, Derived Hom spaces in rigid analytic geometry, preprint, 2018, arXiv:1801.07730] with the theory of derived formal models introduced in [António, p-adic derived formal geometry and derived Raynaud localization theorem, preprint, 2018, arXiv:1805.03302] to prove the existence representability of the derived Hilbert space $\mathbf{R}\mathrm{Hilb}(X)$ for a separated k-analytic space X. Such representability results rely on a localization theorem stating that if $\mathfrak{X}$ is a quasi-compact and quasi-separated formal scheme, then the ∞-category $\mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}})$ of almost perfect complexes over the generic fiber can be realized as a Verdier quotient of the ∞-category $\mathrm{Coh}^-(\mathfrak{X})$. Along the way, we prove several results concerning the ∞-categories of formal models for almost perfect modules on derived k-analytic spaces.

Keywords: derived geometry; rigid geometry; Hilbert scheme; formal models; derived generic fiber

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Contents

1. Introduction

Let k be a non-archimedean field equipped with a non-trivial valuation of rank 1. We let k° denote its ring of integers and $\mathfrak{m}$ an ideal of definition. Given a separated k-analytic space X, we are concerned with the existence of the *derived* moduli space $\mathbf{R}\mathrm{Hilb}(X)$, which parametrizes flat families of closed subschemes of X. The truncation of $\mathbf{R}\mathrm{Hilb}(X)$ coincides with the classical Hilbert scheme functor, $\mathrm{Hilb}(X)$, which has been shown to be representable by a k-analytic space in [7]. On the other hand, in algebraic geometry, the representability of the derived Hilbert scheme is an easy consequence of the Artin–Lurie representability theorem. In this paper, we combine the analytic version of Lurie's representability obtained by T. Y. Yu and the second author in [16] together with a theory of derived formal models developed by the first author in [2]. The only missing step is to establish the existence of the cotangent complex.

Indeed, the techniques introduced in [17] allows us to prove the existence of the cotangent complex at points $x\colon S \to \mathbf{R}\mathrm{Hilb}(X)$ corresponding to families of closed subschemes $j\colon Z \hookrightarrow S \times X$, which are of finite presentation in the derived sense. However, not every point of $\mathbf{R}\mathrm{Hilb}(X)$ satisfies this condition: typically, we are concerned with families which are *almost* of finite presentation. The difference between the two situations is governed by the relative analytic cotangent complex $\mathbb{L}^{\mathrm{an}}_{Z/S\times X}$: Z is (almost) of finite presentation if $\mathbb{L}^{\mathrm{an}}_{Z/S\times X}$ is (almost) perfect. We can explain the main difficulty as follows: if $p\colon Z \to S$ denotes the projection to S, then the cotangent complex of $\mathbf{R}\mathrm{Hilb}(X)$ at $x\colon S \to \mathbf{R}\mathrm{Hilb}(X)$ is computed by $p_+(\mathbb{L}^{\mathrm{an}}_{Z/S\times X})$. Here, p_+ is a (partial) left adjoint for the functor p^*, which has been introduced in the k-analytic setting in [17]. However, in loc. cit. the functor p_+ has only been defined on perfect complexes, rather than on almost perfect complexes. From this point of view, the main contribution of this paper is to provide an extension of the construction p_+ to almost perfect complexes. Our construction relies heavily on the existence results for formal models of derived k-analytic spaces obtained by the first author in [2]. Along the way, we establish three results that we deem to be of independent interest, and which we briefly summarize below.

Let $\mathfrak{X}$ be a derived formal k°-scheme topologically almost of finite presentation. One of the main constructions of [1–3] is the generic fiber $\mathfrak{X}^{\mathrm{rig}}$, which is a derived k-analytic space. The formalism introduced in loc. cit. provides as well an exact functor

$$(-)^{\mathrm{rig}}\colon \mathrm{Coh}^-(\mathfrak{X}) \longrightarrow \mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}}), \tag{1.1}$$

where Coh^- denotes the stable ∞-category of *almost perfect complexes on* $\mathfrak{X}$ and *on* $\mathfrak{X}^{\mathrm{rig}}$, respectively. When $\mathfrak{X}$ is underived, this functor has been considered at length in [8], where in particular it has been shown to be essentially surjective, thereby extending the classical theory of formal models for coherent sheaves on k-analytic spaces. In this paper, we extend this result to the case where $\mathfrak{X}$ is derived, which is a key technical step in our construction of the plus pushforward. In order to do so, we will establish the following descent statement, which is an extension of [8, Theorem 7.3].

Theorem 1. *The functor* $\mathrm{Coh}^-_{\mathrm{loc}}\colon \mathrm{dAn}_k \to \mathrm{Cat}^{\mathrm{st}}_\infty$*, which associates with every derived formal derived scheme*

$$\mathfrak{X} \in \mathrm{dfDM} \mapsto \mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}}) \in \mathrm{Cat}^{\mathrm{st}}_\infty,$$

satisfies Zariski hyperdescent.

We refer the reader to Theorem 3.6 for the precise statement. We obtain as a consequence of Theorem 1 above the following statement, concerning the properties of ∞-categories of formal models for almost perfect complexes on $X \in \mathrm{dAn}_k$.

Theorem 2 (Theorem 3.21). *Let $X \in \mathrm{dAn}_k$ be a derived k-analytic space and let $\mathcal{F} \in \mathrm{Coh}^-(X)$ be a bounded below almost perfect complex on X. For any derived formal model $\mathfrak{X}$ of X, there exist $\mathcal{G} \in \mathrm{Coh}^-(\mathfrak{X})$ and an equivalence $\mathcal{G}^{\mathrm{rig}} \simeq \mathcal{F}$. Furthermore, the full subcategory of $\mathrm{Coh}^-(\mathfrak{X}) \times_{\mathrm{Coh}^-(X)} \mathrm{Coh}^-(X)_{/\mathcal{F}}$ spanned by formal models of $\mathcal{F}$ is filtered.*

Theorem 2 is another key technical ingredient in the proof of the existence of a plus pushforward construction. The third auxiliary result we need is a refinement of the existence theorem for formal models for morphisms of derived analytic spaces proven in [2]. It can be stated as follows.

Theorem 3 (Theorem 4.1). *Let $f : X \to Y$ be a flat map between derived k-analytic spaces. Then there are formal models $\mathfrak{X}$ and $\mathfrak{Y}$ for X and Y, respectively, and a flat map $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ whose generic fiber is equivalent to f.*

The classical analogue of Theorem 3 was proven by Bosch and Lutkëbohmert in [6]. The proof of this theorem is not entirely obvious: indeed the algorithm provided in [2] proceeds by induction on the Postnikov tower of both X and Y, and at each step uses [8, Theorem 7.3] to choose appropriately formal models for $\pi_i(\mathcal{O}_X^{\mathrm{alg}})$ and $\pi_i(\mathcal{O}_Y^{\mathrm{alg}})$. In the current situation, however, the flatness requirement on $\mathfrak{f}$ makes it impossible to freely choose a formal model for $\pi_i(\mathcal{O}_X^{\mathrm{alg}})$. We circumvent the problem by proving a certain lifting property for morphisms of almost perfect complexes.

Theorem 4 (Corollary 3.22). *Let $X \in \mathrm{dAn}_k$ be a derived k-analytic space and let $f : \mathcal{F} \to \mathcal{G}$ be a morphism in $\mathrm{Coh}^-(X)$. Let $\mathfrak{X}$ denote a given formal model for X. Suppose, furthermore, that we are given formal models $\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}} \in \mathrm{Coh}^-(X)$ for $\mathcal{F}$ and $\mathcal{G}$, respectively. Then, there exists a non-zero element $t \in \mathfrak{m}$ such that the map $t^n f$ admits a lift $\widetilde{f} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$, in the ∞-category $\mathrm{Coh}^-(\mathfrak{X})$.*

Finally, the techniques of the current text allow us to prove the following generalization of [17, Theorem 8.6].

Theorem 5 (Theorem 6.4). *Let S be a rigid k-analytic space. Let X, Y be rigid k-analytic spaces over S. Assume that X is proper and flat over S and that Y is separated over S. Then the ∞-functor $\mathbf{Map}_S(X, Y)$ is representable by a derived k-analytic space separated over S.*

Notation and conventions. In this paper, we freely use the language of ∞-categories. Although the discussion is often independent of the chosen model for ∞-categories, whenever needed we identify them with quasi-categories and refer to [10] for the necessary foundational material.

The notations $\mathcal{S}$ and Cat_∞ are reserved to denote the ∞-categories of spaces and of ∞-categories, respectively. If $\mathcal{C} \in \mathrm{Cat}_\infty$, we denote by $\mathcal{C}^{\simeq}$ the maximal ∞-groupoid contained in $\mathcal{C}$. We let $\mathrm{Cat}_\infty^{\mathrm{st}}$ denote the ∞-category of stable ∞-categories with exact functors between them. We also let $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ denote the ∞-category of presentable ∞-categories with left adjoints between them. Similarly, we let $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}$ denote the ∞-categories of stably presentable ∞-categories with left adjoints between them. Finally, we set

$$\mathrm{Cat}_{\infty}^{\mathrm{st},\otimes} := \mathrm{CAlg}(\mathrm{Cat}_{\infty}^{\mathrm{st}}), \quad \mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}_{\mathrm{st}} := \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{st}}).$$

Given an ∞-category $\mathcal{C}$, we denote by $\mathrm{PSh}(\mathcal{C})$ the ∞-category of $\mathcal{S}$-valued presheaves. We follow the conventions introduced in [15, §2.4] for ∞-categories of sheaves on an ∞-site.

For a field k, we reserve the notation CAlg_k for the ∞-category of simplicial commutative rings over k. We often refer to objects in CAlg_k simply as *derived commutative rings*. We denote its opposite by dAff_k, and we refer to it as the ∞-category of *derived affine schemes*. We say that a derived ring $A \in \mathrm{CAlg}_k$ is *almost of finite presentation* if $\pi_0(A)$ is of finite presentation over k and $\pi_i(A)$ is a finitely presented $\pi_0(A)$-module.[1] We denote by $\mathrm{dAff}_k^{\mathrm{afp}}$ the full subcategory of dAff_k spanned by derived affine schemes $\mathrm{Spec}(A)$ such that A is almost of finite presentation. When k is either a non-archimedean field equipped with a non-trivial valuation or is the field of complex numbers, we let An_k denote the category of analytic spaces over k. We denote by $\mathrm{Sp}(k)$ the analytic space associated with k.

2. Preliminaries on derived formal and derived non-archimedean geometries

Let k denote a non-archimedean field equipped with a rank 1 valuation. We let $k^\circ = \{x \in k : |x| \leqslant 1\}$ denote its ring of integers. We denote by $\mathfrak{m}$ an ideal of definition generated by a specified pseudo-uniformizer $t \in \mathfrak{m}$.

Notation 2.1.

(1) Let R be a discrete commutative ring. Let $\mathcal{T}_{\mathrm{disc}}(R)$ denote the full subcategory of R-schemes spanned by affine spaces $\mathbb{A}^n_R$. We say that a morphism in $\mathcal{T}_{\mathrm{disc}}(R)$ is *admissible* if it is an isomorphism. We endow $\mathcal{T}_{\mathrm{disc}}(R)$ with the trivial Grothendieck topology.

(2) Let $\mathcal{T}_{\mathrm{adic}}(k^\circ)$ denote the full subcategory of k°-schemes spanned by formal schemes that are formally smooth and topologically finitely generated over k°. A morphism in $\mathcal{T}_{\mathrm{adic}}(k^\circ)$ is said to be *admissible* if it is formally étale. We equip the category $\mathcal{T}_{\mathrm{adic}}(k^\circ)$ with the formally étale topology, $\tau\text{ét}$.

(3) Denote by $\mathcal{T}_{\mathrm{an}}(k)$ the category of smooth k-analytic spaces. A morphism in $\mathcal{T}_{\mathrm{an}}(k)$ is said to be *admissible* if it is étale. We endow $\mathcal{T}_{\mathrm{an}}(k)$ with the étale topology, $\tau\text{ét}$.

In what follows, we will let $\mathcal{T}$ denote either one of the categories introduced above. We let τ denote the corresponding Grothendieck topology.

Definition 2.2. Let $\mathcal{X}$ be an ∞-topos. A *$\mathcal{T}$-structure* on $\mathcal{X}$ is a functor $\mathcal{O}\colon \mathcal{T} \to \mathcal{X}$, which commutes with finite products, pullbacks along admissible morphisms and takes

[1] Equivalently, A is almost of finite presentation if $\pi_0(A)$ is of finite presentation and the cotangent complex $\mathbb{L}_{A/k}$ is an almost perfect complex over A.

τ-coverings in effective epimorphisms. We denote by $\mathrm{Str}_{\mathcal{T}}(\mathcal{X})$ the full subcategory of $\mathrm{Fun}_{\mathcal{T}}(\mathcal{T}, \mathcal{X})$ spanned by $\mathcal{T}$-structures. A *$\mathcal{T}$-structured ∞-topos* is a pair $(\mathcal{X}, \mathcal{O})$, where $\mathcal{X}$ is an ∞-topos and $\mathcal{O} \in \mathrm{Str}_{\mathcal{T}}(\mathcal{X})$.

We can assemble $\mathcal{T}$-structured ∞-topoi into an ∞-category denoted by ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T})$. We refer to [12, Definition 1.4.8] for the precise construction.

Definition 2.3. Let $\mathcal{X}$ be an ∞-topos. A morphism of $\mathcal{T}$-structures $\alpha: \mathcal{O} \to \mathcal{O}'$ is said to be *local* if for every admissible morphism $f: U \to V$ in $\mathcal{T}$ the diagram

$$\begin{array}{ccc} \mathcal{O}(U) & \xrightarrow{\mathcal{O}(f)} & \mathcal{O}(V) \\ \downarrow{\alpha_U} & & \downarrow{\alpha_V} \\ \mathcal{O}'(U) & \xrightarrow{\mathcal{O}'(f)} & \mathcal{O}'(V) \end{array}$$

is a pullback square in $\mathcal{X}$. We denote by $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}}(\mathcal{X})$ the (non-full) subcategory of $\mathrm{Str}_{\mathcal{T}}(\mathcal{X})$ spanned by local structures and local morphisms between these.

Example 2.4.

(1) Let R be a discrete commutative ring. A $\mathcal{T}_{\mathrm{disc}}(R)$-structure on an ∞-topos $\mathcal{X}$ is simply a product preserving functor $\mathcal{O}: \mathcal{T}_{\mathrm{disc}}(R) \to \mathcal{X}$. When $\mathcal{X} = \mathcal{S}$ is the ∞-topos of spaces, we can therefore use [10, Proposition 5.5.9.2] to identify the ∞-category $\mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(R)}(\mathcal{X})$ with the underlying ∞-category CAlg_R of the model category of simplicial commutative R-algebras. It follows that $\mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(R)}(\mathcal{X})$ is canonically identified with the ∞-category of sheaves on $\mathcal{X}$ with values in CAlg_R. For this reason, we write $\mathrm{CAlg}_R(\mathcal{X})$ rather than $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{disc}}(R)}(\mathcal{X})$.

(2) Let $\mathfrak{X}$ denote a formal scheme over k° complete along $t \in k^\circ$. Denote by $\mathfrak{X}_{\mathrm{f\acute{e}t}}$ the small formal étale site on $\mathfrak{X}$ and denote by $\mathcal{X} := \mathrm{Shv}(\mathfrak{X}_{\mathrm{f\acute{e}t}}, \tau\mathrm{\acute{e}t})^\wedge$ the hypercompletion of the ∞-topos of formally étale sheaves on $\mathfrak{X}$. We define a $\mathcal{T}_{\mathrm{adic}}(k^\circ)$-structure on $\mathcal{X}$ as the functor that sends $U \in \mathfrak{X}_{\mathrm{f\acute{e}t}}$ to the sheaf $\mathcal{O}(U) \in \mathcal{X}$ defined by the association

$$V \in \mathfrak{X}_{\mathrm{f\acute{e}t}} \mapsto \mathrm{Hom}_{\mathrm{fSch}_{k^\circ}}(V, U) \in \mathcal{S}.$$

In this case, $\mathcal{O}(\mathbf{A}^1_{k^\circ})$ corresponds to the sheaf of functions on $\mathfrak{X}$ whose support is contained in the (t)-locus of $\mathfrak{X}$. To simplify the notation, we write $\mathrm{fCAlg}_{k^\circ}(\mathcal{X})$ rather than $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{adic}}(k^\circ)}(\mathcal{X})$.

(3) Let X be a k-analytic space and denote by $X_{\mathrm{\acute{e}t}}$ the associated small étale site on X. Let $\mathcal{X} := \mathrm{Shv}(X_{\mathrm{\acute{e}t}}, \tau_{\mathrm{\acute{e}t}})^\wedge$ denote the hypercompletion of the ∞-topos of étale sheaves on X. We can attach to X a $\mathcal{T}_{\mathrm{an}}(k)$-structure on $\mathcal{X}$ as follows: given $U \in \mathcal{T}_{\mathrm{an}}(k)$, we define the sheaf $\mathcal{O}(U) \in \mathcal{X}$ by

$$X_{\mathrm{\acute{e}t}} \ni V \mapsto \mathrm{Hom}_{\mathrm{An}_k}(V, U) \in \mathcal{S}.$$

As in the previous case, we can canonically identify $\mathcal{O}(\mathbf{A}^1_k)$ with the usual sheaf of analytic functions on X. We write $\mathrm{AnRing}_k(\mathcal{X})$ rather than $\mathrm{Str}^{\mathrm{loc}}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X})$.

Construction 2.5. Let $\mathcal{X}$ be an ∞-topos. We can relate the ∞-categories $\mathrm{CAlg}_{k^\circ}(\mathcal{X})$, $\mathrm{CAlg}_k(\mathcal{X})$, $\mathrm{fCAlg}_{k^\circ}(\mathcal{X})$ and $\mathrm{AnRing}_k(\mathcal{X})$ as follows. Consider the following functors:

(1) The functor

$$- \otimes_{k^\circ} k \colon \mathcal{T}_{\mathrm{disc}}(k^\circ) \longrightarrow \mathcal{T}_{\mathrm{disc}}(k)$$

induced by base change along the map $k^\circ \to k$.

(2) The functor

$$(-)^\wedge_t \colon \mathcal{T}_{\mathrm{disc}}(k^\circ) \longrightarrow \mathcal{T}_{\mathrm{adic}}(k^\circ)$$

induced by the (t)-completion.

(3) The functor

$$(-)^{\mathrm{an}} \colon \mathcal{T}_{\mathrm{disc}}(k) \longrightarrow \mathcal{T}_{\mathrm{an}}(k)$$

induced by the analytification.

(4) The functor

$$(-)^{\mathrm{rig}} \colon \mathcal{T}_{\mathrm{adic}}(k^\circ) \longrightarrow \mathcal{T}_{\mathrm{an}}(k)$$

induced by Raynaud's generic fiber construction (cf. [4, Theorem 8.4.3]).

These functors respect the classes of admissible morphisms and are continuous morphisms of sites. It follows that precomposition with them induce well-defined functors

$$\mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k)}(\mathcal{X}) \longrightarrow \mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k^\circ)}(\mathcal{X}), \quad (-)^{\mathrm{alg}} \colon \mathrm{Str}_{\mathcal{T}_{\mathrm{adic}}(k^\circ)}(\mathcal{X}) \longrightarrow \mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k^\circ)}(\mathcal{X})$$
$$(-)^{+} \colon \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X}) \longrightarrow \mathrm{Str}_{\mathcal{T}_{\mathrm{adic}}(k^\circ)}(\mathcal{X}), \quad (-)^{\mathrm{alg}} \colon \mathrm{Str}_{\mathcal{T}_{\mathrm{an}}(k)}(\mathcal{X}) \longrightarrow \mathrm{Str}_{\mathcal{T}_{\mathrm{disc}}(k)}(\mathcal{X}).$$

The first functor simply forgets the k-algebra structure to a k°-algebra one via the natural map $k^\circ \to k$. We refer to the second and fourth functors as the *underlying algebra functors*. The third functor is an analogue of taking the subring of power-bounded elements in rigid geometry.

Using the underlying algebra functors introduced in the above construction, we can at last introduce the definitions of derived formal scheme and derived k-analytic space. They are analogous to each other.

Definition 2.6. A $\mathcal{T}_{\mathrm{adic}}(k^\circ)$-structured ∞-topos $\mathfrak{X} := (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ is said to be a *derived formal Deligne–Mumford k°-stack* if there exists a collection of objects $\{U_i\}_{i \in I}$ in $\mathcal{X}$ such that $\coprod_{i \in I} U_i \to \mathbf{1}_{\mathcal{X}}$ is an effective epimorphism and the following conditions are met:

(1) For every $i \in I$, the $\mathcal{T}_{\mathrm{adic}}(k^\circ)$-structured ∞-topos $(\mathcal{X}_{/U_i}, \pi_0(\mathcal{O}_{\mathfrak{X}}|_{U_i}))$ is equivalent to the $\mathcal{T}_{\mathrm{adic}}(k^\circ)$-structured ∞-topos arising from an affine formal k°-scheme via the construction given in Example 2.4.

(2) For each $i \in I$ and each integer $n \geqslant 0$, the sheaf $\pi_n(\mathcal{O}_{\mathfrak{X}}^{\mathrm{alg}}|_{U_i})$ is a quasi-coherent sheaf over $(\mathcal{X}_{/U_i}, \pi_0(\mathcal{O}_{\mathfrak{X}}|_{U_i}))$.

We say that $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ is a *formal derived k°-scheme* if it is a derived formal Deligne–Mumford stack and furthermore its 0-*truncation* $\mathrm{t}_0(\mathfrak{X}) := (\mathcal{X}, \pi_0(\mathcal{O}_{\mathfrak{X}}))$ is equivalent to the $\mathcal{T}_{\mathrm{adic}}(k^\circ)$-structured ∞-topos associated with a formal scheme via Example 2.4.

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Definition 2.7. A $\mathcal{T}_{\mathrm{an}}(k)$-structured ∞-topos $X := (\mathcal{X}, \mathcal{O}_X)$ is said to be a *derived k-analytic space* if $\mathcal{X}$ is hypercomplete and there exists a collection of objects $\{U_i\}_{i\in I}$ in $\mathcal{X}$ such that $\coprod_{i\in I} U_i \to \mathbf{1}_{\mathcal{X}}$ is an effective epimorphism and the following conditions are met:

(1) For each $i \in I$, the $\mathcal{T}_{\mathrm{an}}(k)$-structured ∞-topos $(\mathcal{X}_{/U_i}, \pi_0(\mathcal{O}_X|_{U_i}))$ is equivalent to the $\mathcal{T}_{\mathrm{an}}(k)$-structured ∞-topos arising from an ordinary k-analytic space via the construction given in Example 2.4.

(2) For each $i \in I$ and each integer $n \geqslant 0$, the sheaf $\pi_n(\mathcal{O}_X^{\mathrm{alg}}|_{U_i})$ is a coherent sheaf on $(\mathcal{X}_{/U_i}, \mathcal{O}_X|_{U_i})$.

We shall denote by $\mathrm{t}_0(X) := (\mathcal{X}, \pi_0(\mathcal{O}_X))$ the 0*-truncation of* X. By construction, the latter is always isomorphic to an ordinary k-analytic space.

Theorem 2.8 (cf. [2, 11, 14]). *Derived formal Deligne–Mumford k°-stacks and derived k-analytic spaces assemble into ∞-categories, denoted respectively by* dfDM_{k° *and* dAn_k*, which enjoy the following properties:*

(1) *Fiber products exist in both* dfDM_{k° *and* dAn_k*.*

(2) *The constructions given in Example 2.4 induce full faithful embeddings from the categories of ordinary formal Deligne–Mumford k°-stacks* fDM_{k° *and of ordinary k-analytic spaces* An_k *in* dfDM_{k° *and* dAn_k*, respectively.*

Following [9, §8.1], we let $\mathrm{CAlg}^{\mathrm{ad}}_{k^\circ}$ denote the ∞-category of simplicial commutative rings equipped with an adic topology on their 0th truncation. Morphisms are morphisms of simplicial commutative rings that are furthermore continuous for the adic topologies on their 0th truncations. We set

$$\mathrm{CAlg}^{\mathrm{ad}}_{k^\circ} := \mathrm{CAlg}^{\mathrm{ad}}_{k^\circ/},$$

where we regard k° equipped with its $\mathfrak{m}$-adic topology. Thanks to [2, Remark 3.1.4], the underlying algebra functor $(-)^{\mathrm{alg}} \colon \mathrm{fCAlg}_{k^\circ}(\mathcal{X}) \to \mathrm{CAlg}_{k^\circ}(\mathcal{X})$ factors through $\mathrm{CAlg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X})$. We denote by $(-)^{\mathrm{ad}}$ the resulting functor:

$$(-)^{\mathrm{ad}} \colon \mathrm{fCAlg}_{k^\circ}(\mathcal{X}) \longrightarrow \mathrm{CAlg}^{\mathrm{ad}}_{k^\circ}(\mathcal{X}).$$

Definition 2.9. Let $A \in \mathrm{fCAlg}_{k^\circ}(\mathcal{X})$. We say that A is *topologically almost of finite presentation over* k° if the underlying sheaf of k°-adic algebras A^{ad} is (t)-complete, $\pi_0(A^{\mathrm{alg}})$ is a sheaf of topologically finitely presented k°-adic algebras and for each $i > 0$, $\pi_i(A)$ is finitely generated as $\pi_0(A)$-module.

We say that a derived formal Deligne–Mumford stack $\mathfrak{X} := (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ is *topologically almost of finite presentation over* k° if its underlying ∞-topos is coherent (cf. [13, §3]) and $\mathcal{O}_{\mathfrak{X}} \in \mathrm{fCAlg}_{k^\circ}(\mathcal{X})$ is topologically almost of finite presentation over k°. We denote by $\mathbf{dfDM}^{\mathrm{tafp}}$ (respectively, $\mathbf{dfSch}^{\mathrm{tafp}}$) the full subcategory of $\mathbf{dfDM}_{k^\circ}$ spanned by those derived formal Deligne–Mumford stacks $\mathfrak{X}$ that are topologically almost of finite presentation over k° (respectively and whose truncation $\mathrm{t}_0(\mathfrak{X})$ is equivalent to a formal k°-scheme).

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The transformation of pregeometries

$$(-)^{\mathrm{rig}}\colon \mathcal{T}_{\mathrm{adic}}(k^\circ) \longrightarrow \mathcal{T}_{\mathrm{an}}(k)$$

induced by Raynaud's generic fiber functor induces ${}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{adic}}(k^\circ))$. [12, Theorem 2.1.1] provides a right adjoint to this last functor, which we still denote as

$$(-)^{\mathrm{rig}}\colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{adic}}(k^\circ)) \longrightarrow {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k)).$$

We refer to this functor as the *derived generic fiber functor* or as the *derived rigidification functor*.

Theorem 2.10 [2, Corollary 4.1.4, Proposition 4.1.6]. *The functor* $(-)^{\mathrm{rig}}\colon {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{adic}}(k^\circ)) \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}(\mathcal{T}_{\mathrm{an}}(k))$ *enjoys the following properties:*

(1) *It restricts to a functor*

$$(-)^{\mathrm{rig}}\colon \mathrm{dfDM}^{\mathrm{tafp}} \longrightarrow \mathrm{dAn}_k.$$

(2) *The restriction of* $(-)^{\mathrm{rig}}\colon \mathrm{dfDM}^{\mathrm{tafp}} \to \mathrm{dAn}_k$ *to the full subcategory* $\mathrm{fSch}^{\mathrm{tafp}}_{k^\circ}$ *is canonically equivalent to Raynaud's generic fiber functor.*

(3) *Every derived analytic space* $X \in \mathrm{dAn}_k$ *whose truncation is an ordinary* k*-analytic space*[2] *lies in the essential image of the functor* $(-)^{\mathrm{rig}}$.

Fix a derived formal Deligne–Mumford stack $\mathfrak{X} := (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ and a derived k-analytic space $Y := (\mathcal{Y}, \mathcal{O}_Y)$. We set

$$\mathcal{O}_{\mathfrak{X}}\text{-Mod} := \mathcal{O}_{\mathfrak{X}}^{\mathrm{alg}}\text{-Mod}, \quad \mathcal{O}_Y\text{-Mod} := \mathcal{O}_Y^{\mathrm{alg}}\text{-Mod}.$$

We refer to $\mathcal{O}_{\mathfrak{X}}$-Mod as the *stable* ∞*-category of* $\mathcal{O}_{\mathfrak{X}}$*-modules*. Similarly, we refer to $\mathcal{O}_Y$-Mod as the *stable* ∞*-category of* $\mathcal{O}_Y$*-modules*. The derived generic fiber functor induces a functor

$$(-)^{\mathrm{rig}}\colon \mathcal{O}_{\mathfrak{X}}\text{-Mod} \longrightarrow \mathcal{O}_{\mathfrak{X}^{\mathrm{rig}}}\text{-Mod}.$$

Definition 2.11. Let $\mathfrak{X} \in \mathrm{dfDM}_{k^\circ}$ be a derived k°-adic Deligne–Mumford stack and let $X \in \mathrm{dAn}_k$ be a derived k-analytic space. The ∞-category $\mathrm{Coh}^-(\mathfrak{X})$ (respectively, $\mathrm{Coh}^-(X)$) of almost perfect complexes on $\mathfrak{X}$ (respectively, on X) is the full subcategory of $\mathcal{O}_{\mathfrak{X}}$-Mod (respectively, of $\mathcal{O}_X$-Mod) spanned by those $\mathcal{O}_{\mathfrak{X}}$-modules (respectively, $\mathcal{O}_X$-modules) $\mathcal{F}$ such that $\pi_i(\mathcal{F})$ is a coherent sheaf on $\mathrm{t}_0(\mathfrak{X})$ (respectively, on $\mathrm{t}_0(X)$) for every $i \in \mathbb{Z}$ and $\pi_i(\mathcal{F}) \simeq 0$ for $i \gg 0$.

For later use, let us record the following result.

Proposition 2.12 ([9] & [17, Theorem 3.4]). *Let* $\mathfrak{X}$ *be a derived affine* k°*-adic scheme. Let* $A := \Gamma(\mathfrak{X}; \mathcal{O}_{\mathfrak{X}}^{\mathrm{alg}})$. *Then the functor* $\Gamma(\mathfrak{X}; -)$ *restricts to*

$$\mathrm{Coh}^-(\mathfrak{X}) \longrightarrow \mathrm{Coh}^-(A),$$

[2]The ∞-category dAn_k also contains k-analytic Deligne–Mumford stacks.

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and furthermore this is an equivalence. Similarly, if X is a derived k-affinoid space,[3] *and* $B := \Gamma(X; \mathcal{O}_X^{\mathrm{alg}})$*, then* $\Gamma(X; -)$ *restricts to*

$$\mathrm{Coh}^-(X) \longrightarrow \mathrm{Coh}^-(B),$$

and furthermore this is an equivalence.

Remark 2.13. Let $\mathfrak{X}$ denote a derived affine k°-adic scheme topologically almost of finite presentation. Let $X := \mathfrak{X}^{\mathrm{rig}}$ and write

$$A := \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{alg}}), \quad B := \Gamma(X, \mathcal{O}_X^{\mathrm{alg}}).$$

There is a natural map $A \to B$, and Proposition 2.12 implies that the diagram

$$\begin{array}{ccc} \mathrm{Coh}^-(\mathfrak{X}) & \xrightarrow{\Gamma} & \mathrm{Coh}^-(A) \\ \downarrow{\scriptstyle (-)^{\mathrm{rig}}} & & \downarrow{\scriptstyle -\otimes_A B} \\ \mathrm{Coh}^-(X) & \xrightarrow{\Gamma} & \mathrm{Coh}^-(B) \end{array}$$

commutes. On the other hand, [2, Proposition 3.1.12] implies that the natural map $A \otimes_{k^\circ} k \to B$ is an equivalence. In particular, $A \to B$ is a Zariski open immersion and hence [8, Theorem 2.12] implies that the functor $(-)^{\mathrm{rig}}$ above is essentially surjective.

To complete this short review, we briefly discuss the notion of the k°-adic and k-analytic cotangent complexes. The two theories are parallel, and for the sake of brevity, we limit ourselves to the first one. We refer to the introduction of [16] for a more thorough review of the k-analytic theory.

In [2, §3.4], it was constructed a functor

$$\Omega^\infty_{\mathrm{ad}} \colon \mathcal{O}_{\mathfrak{X}}\text{-Mod} \longrightarrow \mathrm{fCAlg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}_{\mathfrak{X}}},$$

which we refer to as the *k°-adic split square-zero extension functor*. Given $\mathcal{F} \in \mathcal{O}_{\mathfrak{X}}$-Mod, we often write $\mathcal{O}_{\mathfrak{X}} \oplus \mathcal{F}$ instead of $\Omega^\infty_{\mathrm{ad}}(\mathcal{F})$.

Remark 2.14. Although the ∞-category $\mathcal{O}_{\mathfrak{X}}$-Mod is *not* sensitive to the $\mathcal{T}_{\mathrm{adic}}(k^\circ)$-structure on $\mathcal{O}_{\mathfrak{X}}$, the functor $\Omega^\infty_{\mathrm{ad}}$ depends on it in an essential way.

Definition 2.15. The functor of *k°-adic derivations* is the functor

$$\mathrm{Der}^{\mathrm{ad}}_{k^\circ}(\mathfrak{X}; -) \colon \mathcal{O}_{\mathfrak{X}}\text{-Mod} \longrightarrow \mathcal{S}$$

defined by

$$\mathrm{Der}^{\mathrm{ad}}_{k^\circ}(\mathfrak{X}; \mathcal{F}) := \mathrm{Map}_{\mathrm{fCAlg}_{k^\circ}(\mathcal{X})_{/\mathcal{O}_{\mathfrak{X}}}}(\mathcal{O}_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}} \oplus \mathcal{F}).$$

For formal reasons, the functor $\mathrm{Der}^{\mathrm{ad}}_{k^\circ}(\mathfrak{X}; -)$ is corepresentable by an object $\mathbb{L}^{\mathrm{ad}}_{\mathfrak{X}} \in \mathcal{O}_{\mathfrak{X}}$-Mod. We refer to it as the *k°-adic cotangent complex of* $\mathfrak{X}$. The following theorem summarizes its main properties.

[3] By definition, X is a derived k-affinoid space if $\mathrm{t}_0(X)$ is a k-affinoid space.

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Theorem 2.16 [2, Proposition 3.4.4, Corollary 4.3.5, Proposition 3.5.8]. *Let* $\mathfrak{X} := (\mathcal{X}, \mathcal{O}_{\mathfrak{X}})$ *be a derived* k°*-adic Deligne–Mumford stack. Let* $\mathrm{t}_{\leqslant n}\mathfrak{X} := (\mathcal{X}, \tau_{\leqslant n}\mathcal{O}_{\mathfrak{X}})$ *be the* n*th truncation of* $\mathfrak{X}$*. Then*

(1) *the* k°*-adic cotangent complex* $\mathbb{L}^{\mathrm{ad}}_{\mathfrak{X}}$ *belongs to* $\mathrm{Coh}^-(\mathfrak{X})$*;*

(2) *in* $\mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}})$*, there is a canonical equivalence*

$$(\mathbb{L}^{\mathrm{ad}}_{\mathfrak{X}})^{\mathrm{rig}} \simeq \mathbb{L}^{\mathrm{an}}_{\mathfrak{X}^{\mathrm{rig}}},$$

where $\mathbb{L}^{\mathrm{an}}_{\mathfrak{X}^{\mathrm{rig}}}$ *denotes the analytic cotangent complex of the derived* k*-analytic space* $\mathfrak{X}^{\mathrm{rig}}$*;*

(3) *the algebraic derivation classifying canonical map* $(\mathfrak{X}, \tau_{\leqslant n+1}\mathcal{O}_{\mathfrak{X}}) \to (\mathfrak{X}, \tau_{\leqslant n}\mathcal{O}_{\mathfrak{X}})$ *can be canonically lifted to a* k°*-adic derivation*

$$\mathbb{L}^{\mathrm{ad}}_{\mathrm{t}_{\leqslant n}\mathfrak{X}} \longrightarrow \pi_{n+1}(\mathcal{O}_{\mathfrak{X}})[n+2].$$

3. Formal models for almost perfect complexes

3.1. Formal descent statements

We fix a pseudo-uniformizer t for $\mathfrak{m}$. We start by recalling the notion of $\mathfrak{m}$-nilpotent almost perfect complexes.

Definition 3.1. Let $\mathfrak{X}$ be a derived k°-adic Deligne–Mumford stack topologically almost of finite presentation. We let $\mathrm{Coh}^-_{\mathrm{nil}}(\mathfrak{X})$ denote the fiber of the generic fiber functor (1.1):

$$\mathrm{Coh}^-_{\mathrm{nil}}(\mathfrak{X}) := \mathrm{fib}\Big(\mathrm{Coh}^-(\mathfrak{X}) \xrightarrow{(-)^{\mathrm{rig}}} \mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}})\Big).$$

We refer to $\mathrm{Coh}^-_{\mathrm{nil}}(\mathfrak{X})$ as the full subcategory of $\mathfrak{m}$-nilpotent almost perfect complexes on X.

A morphism $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ in $\mathrm{dfDM}^{\mathrm{tafp}}_{k^\circ}$ induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Coh}^-(\mathfrak{Y}) & \xrightarrow{\mathfrak{f}^*} & \mathrm{Coh}^-(\mathfrak{X}) \\ \downarrow{\scriptstyle (-)^{\mathrm{rig}}} & & \downarrow{\scriptstyle (-)^{\mathrm{rig}}} \\ \mathrm{Coh}^-(\mathfrak{Y}^{\mathrm{rig}}) & \xrightarrow{(\mathfrak{f}^{\mathrm{rig}})^*} & \mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}}). \end{array} \tag{3.1}$$

In particular, we see that $\mathfrak{f}^*$ preserves the subcategory of $\mathfrak{m}$-nilpotent almost perfect complexes on X. Moreover, as both $\mathrm{Coh}^-(\mathfrak{X})$ and $\mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}})$ satisfy étale descent, we conclude that $\mathrm{Coh}^-_{\mathrm{nil}}(\mathfrak{X})$ satisfies étale descent as well.

Lemma 3.2. *Let* $\mathfrak{X}$ *be a derived* k°*-adic Deligne–Mumford stack topologically almost of finite presentation. Then an almost perfect sheaf* $\mathcal{F} \in \mathrm{Coh}^-(X)$ *is* $\mathfrak{m}$*-nilpotent if and only if for every* $i \in \mathbb{Z}$*, the coherent sheaf* $\pi_i(\mathcal{F})$ *is annihilated by some power of the ideal* $\mathfrak{m}$*.*

Proof. Since $\mathfrak{X}$ is assumed to be quasi-compact, we can find a finite formally étale covering

$$\coprod_{j \in J} \mathfrak{U}_j \to \mathfrak{X},$$

where for each $j \in J$, $\mathfrak{U}_j$ are formally affine. Suppose that the assertion of the lemma holds for each restriction

$$\mathcal{F}_{|\mathfrak{U}_j} \in \mathrm{Coh}^-(\mathfrak{U}_j), \quad j \in J.$$

In other words, for every $i \in \mathbb{Z}$, there exists $n_{ij} \in \mathbb{N}$ such that

$$t^{n_{ij}} \cdot \pi_i(\mathcal{F}_{|\mathfrak{U}_j}) = 0.$$

Let $n_i = \max_{j \in J} n_{ij}$. Étale descent for $\mathrm{Coh}^-(\mathfrak{X})$ implies therefore that $t^{n_i} \cdot \pi_i(\mathcal{F}) = 0$ in $\mathrm{Coh}^-(\mathfrak{X})$. In particular, we can assume from the start that $\mathfrak{X}$ is a derived formal affine scheme. Write

$$A := \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{alg}}).$$

Let $X := \mathfrak{X}^{\mathrm{rig}}$. Then [2, Corollary 4.1.3] shows that

$$\mathrm{t}_0(\mathfrak{X}^{\mathrm{rig}}) \simeq (\mathrm{t}_0(\mathfrak{X}))^{\mathrm{rig}}.$$

In particular, we deduce that X is a derived k-affinoid space. Write

$$B := \Gamma(X, \mathcal{O}_X^{\mathrm{alg}}).$$

We can therefore use Proposition 2.12 to obtain canonical equivalences

$$\mathrm{Coh}^-(\mathfrak{X}) \simeq \mathrm{Coh}^-(A), \quad \mathrm{Coh}^-(X) \simeq \mathrm{Coh}^-(B).$$

Under these identifications, the functor $(-)^{\mathrm{rig}}$ becomes equivalent to the base change functor

$$- \otimes_A B \colon \mathrm{Coh}^-(A) \longrightarrow \mathrm{Coh}^-(B).$$

Moreover, it follows from [2, Proposition A.1.4] that there is a canonical identification

$$B \simeq A \otimes_{k^\circ} k.$$

In particular, $(-)^{\mathrm{rig}} \colon \mathrm{Coh}^-(\mathfrak{X}) \to \mathrm{Coh}^-(X)$ is t-exact. The conclusion is now straightforward. □

Definition 3.3. Let $\mathfrak{X}$ be a derived k°-adic Deligne–Mumford stack topologically almost of finite presentation. Let $\mathcal{F} \in \mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}})$. A *formal model for* $\mathcal{F}$ consists of a pair $(\mathfrak{F}, \alpha)$, where $\mathfrak{F} \in \mathrm{Coh}^-(\mathfrak{X})$ and $\alpha \colon \mathfrak{F}^{\mathrm{rig}} \xrightarrow{\sim} \mathcal{F}$ is an equivalence in $\mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}})$. We let $\mathrm{FM}(\mathcal{F})$ denote the full subcategory of

$$\mathrm{Coh}^-(\mathfrak{X})_{/\mathcal{F}} := \mathrm{Coh}^-(\mathfrak{X}) \times_{\mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}})} \mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}})_{/\mathcal{F}}$$

spanned by formal models of $\mathcal{F}$.

Our goal in this section is to study the structure of $\mathrm{FM}(\mathcal{F})$, and in particular to establish that it is non-empty and filtered when $\mathfrak{X}$ is a derived k°-adic scheme topologically almost of finite presentation. Note that saying that $\mathrm{FM}(\mathcal{F})$ is non-empty for every choice of $\mathcal{F} \in \mathrm{Coh}^-(X)$ is equivalent to asserting that the functor (1.1)

$$(-)^{\mathrm{rig}} \colon \mathrm{Coh}^-(\mathfrak{X}) \longrightarrow \mathrm{Coh}^-(X)$$

is essentially surjective.

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To complete the proof of the non-emptiness of $\mathbf{FM}(\mathcal{F})$, it would be enough to know that the essential image of the functor $\mathrm{Coh}^-(\mathfrak{X}) \to \mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}})$ satisfies descent. This is analogous to [8, Theorem 7.3].

Definition 3.4. Let $\mathfrak{X}$ be a derived k°-adic Deligne–Mumford stack topologically almost of finite presentation. We define the stable ∞-category $\mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X})$ of $\mathfrak{m}$-*local almost perfect complexes* as the cofiber

$$\mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X}) \coloneqq \mathrm{cofib}\left(\mathrm{Coh}^-_{\mathrm{nil}}(\mathfrak{X}) \hookrightarrow \mathrm{Coh}^-(\mathfrak{X})\right).$$

We denote by $\mathrm{L}\colon \mathrm{Coh}^-(\mathfrak{X}) \to \mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X})$ the canonical functor. We refer to L as the *localization functor*.

We summarize below the formal properties of $\mathfrak{m}$-local almost perfect complexes.

Proposition 3.5. *Let $\mathfrak{X}$ be a derived k°-adic Deligne–Mumford stack topologically almost of finite presentation. Then, we have the following:*

(1) *There exists a unique t-structure on the stable ∞-category $\mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X})$ having the property of making the localization functor*

$$\mathrm{L}\colon \mathrm{Coh}^-(\mathfrak{X}) \longrightarrow \mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X})$$

t-exact.

(2) *The functor $(-)^{\mathrm{rig}}\colon \mathrm{Coh}^-(\mathfrak{X}) \to \mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}})$ factors through*

$$\Lambda\colon \mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X}) \longrightarrow \mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}}).$$

Moreover, the essential images of $(-)^{\mathrm{rig}}$ and Λ coincide.

(3) *If $\mathfrak{X}$ is affine, then the functor Λ is an equivalence.*

Proof. We start by proving (1). Using [8, Corollary 2.9], we have to check that the t-structure on $\mathrm{Coh}^-(\mathfrak{X})$ restricts to a t-structure on $\mathrm{Coh}^-_{\mathrm{nil}}(\mathfrak{X})$ and that the inclusion

$$i\colon \mathrm{Coh}^\heartsuit_{\mathrm{nil}}(\mathfrak{X}) \longrightarrow \mathrm{Coh}^\heartsuit(\mathfrak{X})$$

admits a right adjoint R whose counit $i(R(\mathcal{F})) \to \mathcal{F}$ is a monomorphism for every $\mathcal{F} \in \mathrm{Coh}^\heartsuit(\mathfrak{X})$. For the first statement, we remark that it is enough to check that the functor $(-)^{\mathrm{rig}}\colon \mathrm{Coh}^-(\mathfrak{X}) \to \mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}})$ is t-exact. As both $\mathrm{Coh}^-(\mathfrak{X})$ and $\mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}})$ satisfy étale descent in $\mathfrak{X}$, we can test this locally on $\mathfrak{X}$. When $\mathfrak{X}$ is affine, the assertion follows directly from Proposition 2.12. As for the second statement, we first observe that one as an equivalence of abelian categories

$$\mathrm{Coh}^\heartsuit(\mathfrak{X}) \simeq \mathrm{Coh}^\heartsuit(\mathrm{t}_0(\mathfrak{X})), \quad \mathrm{Coh}^\heartsuit(\mathfrak{X}^{\mathrm{rig}}) \simeq \mathrm{Coh}^\heartsuit(\mathrm{t}_0(\mathfrak{X}^{\mathrm{rig}})).$$

Moreover, Lemma 3.2 implies that the first equivalence is compatible with the inclusion i of nilpotent almost perfect complexes. We can therefore assume that $\mathfrak{X}$ is underived. At this point, the functor R can be explicitly described as the functor sending $\mathfrak{F} \in \mathrm{Coh}^\heartsuit(\mathfrak{X})$ to the subsheaf of $\mathfrak{F}$ spanned by $\mathfrak{m}$-nilpotent sections. The proof of (1) is thus complete.

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We now turn to the proof of (2). The existence of Λ and the factorization $(-)^{\mathrm{rig}} \simeq \Lambda \circ \mathrm{L}$ follow from the definitions. Moreover, $\mathrm{L}\colon \mathrm{Coh}^-(\mathfrak{X}) \to \mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X})$ is essentially surjective (cf. [8, Lemma 2.3]). It follows that the essential images of $(-)^{\mathrm{rig}}$ and of Λ coincide.

Finally, (3) follows directly from Proposition 2.12 and [8, Theorem 2.12]. □

The commutativity of (3.1) together with Proposition 3.5 implies that a morphism $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ in $\mathrm{dfDM}^{\mathrm{tafp}}_{k^\circ}$ induces a well-defined t-exact functor

$$\mathfrak{f}^{\circ *}\colon \mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{Y}) \longrightarrow \mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X}).$$

It is a simple exercise in ∞-categories to promote this construction to an actual functor

$$\mathrm{Coh}^-_{\mathrm{loc}}\colon \big(\mathrm{dfDM}^{\mathrm{tafp}}_{k^\circ}\big)^{\mathrm{op}} \longrightarrow \mathrm{Cat}^{\mathrm{st}}_\infty.$$

Having Remark 2.13 and Proposition 3.5 at our disposal, the question of the non-emptiness of $\mathrm{FM}(\mathcal{F})$ is essentially reduced to the following.

Theorem 3.6. *Let* $\mathrm{dfSch}^{\mathrm{tafp}}_{k^\circ}$ *denote the* ∞*-category of derived* k°*-adic schemes, which are topologically almost of finite presentation. Then the functor*

$$\mathrm{Coh}^-_{\mathrm{loc}}\colon \big(\mathrm{dfSch}^{\mathrm{tafp}}_{k^\circ}\big)^{\mathrm{op}} \longrightarrow \mathrm{Cat}^{\mathrm{st}}_\infty$$

is a hypercomplete sheaf for the formal Zariski topology.

Proof. It is enough to prove that for every $\mathfrak{X} \in \mathrm{dfSch}^{\mathrm{tafp}}_{k^\circ}$, the restriction of $\mathrm{Coh}^-_{\mathrm{loc}}$ to the Zariski site $\mathfrak{X}_{\mathrm{Zar}}$ is a hypercomplete sheaf. Let $\mathfrak{X}_s$ denote the special fiber of $\mathfrak{X}$. Then there is a canonical equivalence

$$\mathfrak{X}_{\mathrm{Zar}} \simeq (\mathfrak{X}_s)_{\mathrm{Zar}},$$

and since $\mathfrak{X}$ is quasi-compact and quasi-separated, the ∞-topos $\mathrm{Sh}((\mathfrak{X}_s)_{\mathrm{Zar}}, \tau_{\mathrm{Zar}})$ is hypercomplete (combine Propositions 7.2.1.0, 7.2.4.7 and Corollary 7.2.4.17 in [10]). It is therefore sufficient to deal with Zariski descent, rather than hyperdescent.

A standard argument reduces us to proving the following statement: let $\mathfrak{f}_\bullet\colon \mathfrak{U}_\bullet \to \mathfrak{X}$ be the Čech nerve of a derived affine k°-adic Zariski cover. Then the canonical map

$$\mathfrak{f}^{\circ *}_\bullet\colon \mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X}) \longrightarrow \lim_{[n]\in\Delta} \mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{U}_\bullet) \tag{3.2}$$

is an equivalence. Using [8, Lemma 3.20], we can endow the right hand side with a canonical t-structure. It follows from the characterization of the t-structure on $\mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X})$ given in Proposition 3.5 that $\mathfrak{f}^{\circ *}_\bullet$ is t-exact.

We will prove in Corollary 3.12 that $\mathfrak{f}^{\circ *}_\bullet$ is fully faithful. Assuming this fact, we can complete the proof as follows. We only need to check that $\mathfrak{f}^{\circ *}_\bullet$ is essentially surjective. Let $\mathcal{C}$ be the essential image of $\mathfrak{f}^{\circ *}_\bullet$. We now make the following observations:

(1) The heart of $\lim_\Delta \mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{U}_\bullet)$ is contained in $\mathcal{C}$. Indeed, Remark 2.13 implies that

$$\Lambda_n\colon \mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{U}_n) \longrightarrow \mathrm{Coh}^-(\mathfrak{U}_n^{\mathrm{rig}})$$

is an equivalence. These equivalences induce a t-exact equivalence

$$\mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}}) \simeq \lim_{[n]\in\Delta} \mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{U}_\bullet). \tag{3.3}$$

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Passing to the heart and using the canonical equivalences

$$\mathrm{Coh}^{\heartsuit}_{\mathrm{loc}}(\mathfrak{X}) \simeq \mathrm{Coh}^{\heartsuit}_{\mathrm{loc}}(\mathrm{t}_0(\mathfrak{X})), \quad \mathrm{Coh}^{\heartsuit}(\mathfrak{X}^{\mathrm{rig}}) \simeq \mathrm{Coh}^{\heartsuit}(\mathrm{t}_0(\mathfrak{X}^{\mathrm{rig}})),$$

we can invoke the classical Raynaud theorem on formal models of coherent sheaves (cf. [6, Theorem 4.1]) to deduce that the heart of the target of $\mathfrak{f}_\bullet^{\circ *}$ is contained in its essential image.

(2) The subcategory $\mathcal{C}$ is stable. Indeed, let

$$\mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''$$

be a fiber sequence in $\mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}}) \simeq \lim_\Delta \mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{U}_\bullet)$ and suppose that two among $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{F}''$ belong to $\mathcal{C}$. Without loss of generality, we can assume that $\mathcal{F}$ and $\mathcal{F}''$ belong to $\mathcal{C}$. Then choose elements $\mathfrak{F}$ and $\mathfrak{F}''$ in $\mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X})$ representing $\mathcal{F}$ and $\mathcal{F}''$. Since $\mathfrak{f}_\bullet^{\circ *}$ is fully faithful, we can find a morphism $\widetilde{\psi}\colon \mathfrak{F} \to \mathfrak{F}''$ lifting ψ. Set

$$\mathfrak{F}' \coloneqq \mathrm{fib}(\widetilde{\psi}\colon \mathfrak{F} \to \mathfrak{F}'').$$

Then $\Lambda(\mathfrak{F}') \simeq \mathcal{F}'$ since Λ is an exact functor between stable ∞-categories. The latter means that under equivalence (3.3), the object $\mathcal{F}'$ belongs to $\mathcal{C}$.

These two points together imply that $\mathfrak{f}_\bullet^{\circ *}$ is essentially surjective on cohomologically bounded elements. As both the t-structures on the source and the target of $\mathfrak{f}_\bullet^{*}$ are left t-complete and the functor t-exact, we conclude that $\mathfrak{f}_\bullet^{\circ *}$ commutes with the limit of Postnikov towers. The conclusion follows. □

Corollary 3.7. *Let* $\mathfrak{X} \in \mathrm{dfSch}^{\mathrm{tafp}}_{k^\circ}$. *Then the canonical map*

$$\Lambda\colon \mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X}) \longrightarrow \mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}})$$

introduced in Proposition 3.5 is an equivalence.

Proof. Let $\mathfrak{f}_\bullet\colon \mathfrak{U}_\bullet \to \mathfrak{X}$ be a derived affine k°-adic Zariski hypercover. Consider the induced commutative diagram

$$\begin{array}{ccc} \mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X}) & \xrightarrow{\mathfrak{f}_\bullet^*} & \lim_{[n]\in\Delta} \mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{U}_n) \\ \downarrow{\scriptstyle\Lambda} & & \downarrow{\scriptstyle\Lambda_\bullet} \\ \mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}}) & \xrightarrow{f_\bullet^*} & \lim_{[n]\in\Delta} \mathrm{Coh}^-(\mathfrak{U}_n^{\mathrm{rig}}), \end{array}$$

where we set $f_\bullet \coloneqq (\mathfrak{f}_\bullet)^{\mathrm{rig}}$. Since we chose an affine hypercover, Proposition 3.5(3) implies that the right vertical map is an equivalence. On the other hand, $\mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}})$ satisfies descent in $\mathfrak{X}$, and therefore the bottom horizontal map is also an equivalence. Finally, Theorem 3.6 implies that the top horizontal map is an equivalence as well. We thus conclude that $\Lambda\colon \mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X}) \to \mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}})$ is an equivalence in this case. □

Corollary 3.8. *Let* $\mathfrak{X} \in \mathrm{dfSch}^{\mathrm{tafp}}_{k^\circ}$ *and assume moreover that it is quasi-compact and quasi-separated. For any* $\mathcal{F} \in \mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}})$, *the* ∞*-category* $\mathrm{FM}(\mathcal{F})$ *is non-empty.*

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Proof. The localization functor $\mathrm{L}\colon \mathrm{Coh}^-(\mathfrak{X}) \to \mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X})$ is essentially surjective by construction. Since $\mathfrak{X}$ is a quasi-compact and quasi-separated derived k°-adic scheme topologically of finite presentation, Corollary 3.7 implies that $\Lambda\colon \mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X}) \to \mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}})$ is an equivalence. The conclusion follows. □

3.2. Proof of Theorem 3.6: full faithfulness

The only missing step in the proof of Theorem 3.6 is the full faithfulness of functor (3.2). We will address this question by passing to the ∞-categories of ind-objects. Let $\mathfrak{X}$ be a quasi-compact and quasi-separated derived k°-adic scheme locally topologically almost of finite presentation. Let

$$\mathfrak{f}\colon \mathfrak{U} \longrightarrow \mathfrak{X}$$

be a formally étale morphism. Then $\mathfrak{f}$ induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{X})) & \xrightarrow{\mathrm{L}_{\mathfrak{X}}} & \mathrm{Ind}(\mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X})) \\ \downarrow{\scriptstyle \mathfrak{f}^*} & & \downarrow{\scriptstyle \mathfrak{f}^{\circ *}} \\ \mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{U})) & \xrightarrow{\mathrm{L}_{\mathfrak{U}}} & \mathrm{Ind}(\mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{U})). \end{array}$$

The functors $\mathfrak{f}^*$ and $\mathfrak{f}^{\circ *}$ commute with colimits, and therefore they admit right adjoints $\mathfrak{f}_*$ and $\mathfrak{f}^\circ_*$. In particular, we obtain a Beck–Chevalley transformation

$$\theta\colon \mathrm{L}_{\mathfrak{X}} \circ \mathfrak{f}_* \longrightarrow \mathfrak{f}^\circ_* \circ \mathrm{L}_{\mathfrak{U}}. \tag{3.4}$$

A key step in the proof of the full faithfulness of functor (3.2) is to verify that θ is an equivalence when evaluated on objects in $\mathrm{Coh}^\heartsuit(\mathfrak{U})$. Let us start with the following variation of [8, Lemma 7.14].

Lemma 3.9. *Let*

$$\begin{array}{ccccc} \mathcal{K}_{\mathcal{C}} & \xhookrightarrow{i_{\mathcal{C}}} & \mathcal{C} & \xrightarrow{L_{\mathcal{C}}} & \mathcal{Q}_{\mathcal{C}} \\ \downarrow{\scriptstyle F_{\mathcal{K}}} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F_{\mathcal{Q}}} \\ \mathcal{K}_{\mathcal{D}} & \xhookrightarrow{i_{\mathcal{D}}} & \mathcal{D} & \xrightarrow{L_{\mathcal{D}}} & \mathcal{Q}_{\mathcal{D}} \end{array} \tag{3.5}$$

be a diagram of stable ∞-categories and exact functors between them. Assume the following:

(1) *The functors $i_{\mathcal{C}}$ and $i_{\mathcal{D}}$ are fully faithful and admit right adjoints $R_{\mathcal{C}}$ and $R_{\mathcal{D}}$, respectively.*

(2) *The functors $L_{\mathcal{C}}$ and $L_{\mathcal{D}}$ admit fully faithful right adjoints $j_{\mathcal{C}}$ and $j_{\mathcal{D}}$, respectively.*

(3) *The rows are fiber and cofiber sequences in $\mathrm{Cat}^{\mathrm{st}}_\infty$.*

(4) *The functors F, $F_{\mathcal{K}}$ and $F_{\mathcal{Q}}$ admit right adjoints G, $G_{\mathcal{K}}$ and $G_{\mathcal{Q}}$, respectively.*

Let $X \in \mathcal{D}$ be an object. Then the following statements are equivalent:

(1) *The Beck–Chevalley transformation*

$$q_X\colon L_{\mathcal{C}}(G(X)) \longrightarrow G_{\mathcal{Q}}(L_{\mathcal{D}}(X))$$

is an equivalence.

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(2) *The Beck–Chevalley transformation*

$$\kappa_{R_{\mathcal{D}}(X)} \colon i_{\mathcal{C}}(G_{\mathcal{K}}(R_{\mathcal{D}}(X))) \longrightarrow G(i_{\mathcal{D}}(R_{\mathcal{D}}(X)))$$

is an equivalence.

Proof. Since $j_{\mathcal{C}}$ and $i_{\mathcal{C}}$ are fully faithful, it is equivalent to check that

$$j_{\mathcal{C}}(L_{\mathcal{C}}(G(X))) \longrightarrow j_{\mathcal{C}}(G_{\mathcal{Q}}(L_{\mathcal{D}}(X)))$$

is an equivalence if and only if $\kappa_{R_{\mathcal{D}}(X)}$ is an equivalence. Using the natural equivalences

$$j_{\mathcal{C}} \circ G \simeq G j_{\mathcal{D}}, \quad G_{\mathcal{K}} \circ R_{\mathcal{D}} \simeq R_{\mathcal{C}} \circ G$$

, we obtain the following commutative diagram

$$\begin{array}{ccccc} i_{\mathcal{C}}(R_{\mathcal{C}}(G(X))) & \longrightarrow & G(X) & \longrightarrow & j_{\mathcal{C}}(L_{\mathcal{C}}(G(X))) \\ \downarrow & & \| & & \downarrow \\ G(i_{\mathcal{D}}(R_{\mathcal{D}}(X))) & \longrightarrow & G(X) & \longrightarrow & G(j_{\mathcal{D}}(L_{\mathcal{D}}(X))). \end{array}$$

Moreover, since the rows of diagram (3.5) are Verdier quotients, we conclude that the rows in the above diagram are fiber sequences. Therefore, the leftmost vertical arrow is an equivalence if and only if the rightmost one is. □

Lemma 3.10. *The Beck–Chevalley transformation (3.4) is an equivalence whenever evaluated on objects in* $\mathrm{Coh}^{\heartsuit}(\mathfrak{U})$.

Proof. Using Lemma 3.9, we see that it is enough to prove that the Beck–Chevalley transformation associated with the square

$$\begin{array}{ccc} \mathrm{Ind}(\mathrm{Coh}^{-}_{\mathrm{nil}}(\mathfrak{X})) & \longrightarrow & \mathrm{Ind}(\mathrm{Coh}^{-}(\mathfrak{X})) \\ \downarrow{\scriptstyle \mathfrak{f}^*} & & \downarrow{\scriptstyle \mathfrak{f}^*} \\ \mathrm{Ind}(\mathrm{Coh}^{-}_{\mathrm{nil}}(\mathfrak{U})) & \longrightarrow & \mathrm{Ind}(\mathrm{Coh}^{-}(\mathfrak{U})) \end{array}$$

is an equivalence when evaluated on objects of $\mathrm{Coh}^{\heartsuit}_{\mathrm{nil}}(\mathfrak{U})$. As the horizontal functors are fully faithful, it is enough to check that the functor

$$\mathfrak{f}_* \colon \mathrm{Ind}(\mathrm{Coh}^{-}(U)) \longrightarrow \mathrm{Ind}(\mathrm{Coh}^{-}(\mathfrak{X}))$$

takes $\mathrm{Coh}^{\heartsuit}_{\mathrm{nil}}(\mathfrak{U})$ to $\mathrm{Ind}(\mathrm{Coh}^{-}_{\mathrm{nil}}(\mathfrak{X}))$. Let $\mathfrak{F} \in \mathrm{Coh}^{\heartsuit}_{\mathrm{nil}}(\mathfrak{U})$. We have to verify that $(\mathfrak{f}_*(\mathfrak{F}))^{\mathrm{rig}} \simeq 0$. Since $\mathfrak{F}$ is coherent and in the heart and since $\mathfrak{U}$ is quasi-compact, we see that there exists an element $a \in \mathfrak{m}$ such that the map $\mu_a \colon \mathfrak{F} \to \mathfrak{F}$ given by multiplication by a is zero. Therefore $\mathfrak{f}_*(\mu_a) \colon \mathfrak{f}_*(\mathfrak{F}) \to \mathfrak{f}_*(\mathfrak{F})$ is homotopic to zero. Since $\mathfrak{f}_*(\mu_a)$ is equivalent to the endomorphism $\mathfrak{f}_*(\mathfrak{F})$ given by multiplication by a, we conclude that $(\mathfrak{f}_*(\mathfrak{F}))^{\mathrm{rig}} \simeq 0$. The conclusion follows. □

Having these adjointability statements at our disposal, we turn to the actual study of the full faithfulness of functor (3.2). Let

$$\mathfrak{U}_\bullet \colon \Delta^{\mathrm{op}} \longrightarrow \mathrm{dfSch}^{\mathrm{tafp}}_{k^\circ}$$

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be an affine k°-adic Zariski hypercovering of $\mathfrak{X}$ and let $\mathfrak{f}_\bullet \colon \mathfrak{U}_\bullet \to \mathfrak{X}$ be the augmentation morphism. The morphism $\mathfrak{f}_\bullet$ induces functors

$$\mathfrak{f}_\bullet^* \colon \mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{X})) \longrightarrow \lim_{[n]\in\Delta} \mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{U}_n))$$

and

$$\mathfrak{f}_\bullet^{\circ *} \colon \mathrm{Ind}(\mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X})) \longrightarrow \lim_{[n]\in\Delta} \mathrm{Ind}(\mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{U}_n)).$$

These functors commute by construction with filtered colimits, and therefore they admit right adjoints, which we denote, respectively, as

$$\mathfrak{f}_{\bullet *} \colon \lim_{[n]\in\Delta} \mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{U}_n)) \longrightarrow \mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{X}))$$

and

$$\mathfrak{f}^\circ_{\bullet *} \colon \lim_{[n]\in\Delta} \mathrm{Ind}(\mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{U}_n)) \longrightarrow \mathrm{Ind}(\mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X})).$$

Moreover, the functors $\mathfrak{f}_\bullet^*$ and $\mathfrak{f}_\bullet^{\circ *}$ fit in the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{X})) & \xrightarrow{\mathfrak{f}_\bullet^*} & \lim_{[n]\in\Delta} \mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{U}_\bullet)) \\ \downarrow{\scriptstyle \mathrm{L}} & & \downarrow{\scriptstyle \mathrm{L}_\bullet} \\ \mathrm{Ind}(\mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X})) & \xrightarrow{\mathfrak{f}_\bullet^{\circ *}} & \lim_{[n]\in\Delta} \mathrm{Ind}(\mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{U}_\bullet)). \end{array}$$

In particular, we have an associated Beck–Chevalley transformation

$$\theta \colon \mathrm{L} \circ \mathfrak{f}_{\bullet *} \longrightarrow \mathfrak{f}^\circ_{\bullet *} \circ \mathrm{L}_\bullet. \tag{3.6}$$

Proposition 3.11. *The Beck–Chevalley transformation (3.6) is an equivalence when restricted to the full subcategory* $\lim_\Delta \mathrm{Coh}^\heartsuit(\mathfrak{U}_\bullet)$ *of* $\lim_\Delta \mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{U}_\bullet))$.

Proof. The discussion right after [15, Corollary 8.6] allows us to identify the functor

$$\mathfrak{f}_{\bullet *} \colon \lim_{[n]\in\Delta} \mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{U}_n)) \longrightarrow \mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{X}))$$

with
the functor informally described by sending a descent datum $\mathfrak{F}_\bullet \in \lim_\Delta \mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{U}_\bullet))$ to

$$\lim_{[n]\in\Delta} \mathfrak{f}_{n*}\mathfrak{F}_n \in \mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{X})).$$

Similarly, the functor $\mathfrak{f}^\circ_{\bullet *}$ sends a descent datum $\mathcal{F}_\bullet \in \lim_\Delta \mathrm{Ind}(\mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{U}_\bullet)$ to

$$\lim_{[n]\in\Delta} \mathfrak{f}^\circ_{n*}\mathcal{F}_n \in \mathrm{Ind}(\mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X})).$$

We therefore have to show that the Beck–Chevalley transformation

$$\theta \colon \mathrm{L}\Big(\lim_{[n]\in\Delta} \mathfrak{f}_{n*}\mathfrak{F}_n\Big) \longrightarrow \lim_{[n]\in\Delta} \mathfrak{f}^\circ_{n*}(\mathrm{L}_n\mathfrak{F}_n)$$

is an equivalence whenever each $\mathfrak{F}_n$ belongs to $\mathrm{Coh}^\heartsuit(\mathfrak{U}_n)$. First, note that the functors $\mathfrak{f}_{\bullet *}$ and $\mathfrak{f}^\circ_{\bullet *}$ are left t-exact. In particular, if $\mathfrak{F}_\bullet \in \lim_\Delta \mathrm{Ind}(\mathrm{Coh}^\heartsuit(\mathfrak{U}_\bullet))$, then both $\mathrm{L}\mathfrak{f}_{\bullet *}(\mathfrak{F}_\bullet)$ and $\mathfrak{f}^\circ_{\bullet *}(\mathfrak{F}_\bullet)$ are coconnective. As the t-structures on $\lim_\Delta \mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{U}_\bullet))$ and on $\lim_\Delta \mathrm{Ind}(\mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{U}_\bullet))$ are right t-complete, we conclude that it is enough to prove that $\pi_i(\theta)$ is an isomorphism for every $i \in \mathbb{Z}$. We now observe that for $m \geqslant i+2$, we have

$$\pi_i\left(\lim_{[n]\in\Delta} \mathfrak{f}^\circ_{n*}(\mathrm{L}_n\mathfrak{F}_n)\right) \simeq \pi_i\left(\lim_{[n]\in\Delta_{\leqslant m}} \mathfrak{f}^\circ_{n*}(\mathrm{L}_n\mathfrak{F}_n)\right),$$

and similarly

$$\pi_i\left(\mathrm{L}\left(\lim_{[n]\in\Delta} \mathfrak{f}_{n*}\mathfrak{F}_n\right)\right) \simeq \mathrm{L}\left(\pi_i\left(\lim_{[n]\in\Delta} \mathfrak{f}_{n*}\mathfrak{F}_n\right)\right) \simeq \mathrm{L}\left(\pi_i\left(\lim_{[n]\in\Delta_{\leqslant m}} \mathfrak{f}_{n*}\mathfrak{F}_{n*}\right)\right).$$

It is therefore enough to prove that for every $m \geqslant 0$, the canonical map

$$\mathrm{L}\left(\lim_{[n]\in\Delta_{\leqslant m}} \mathfrak{f}_{n*}\mathfrak{F}_n\right) \longrightarrow \lim_{[n]\in\Delta_{\leqslant m}} \mathfrak{f}^\circ_{n*}(\mathrm{L}_n\mathfrak{F}_n)$$

is an equivalence. As L commutes with finite limits, we are reduced to showing that the canonical map

$$\mathrm{L}(\mathfrak{f}_{n*}\mathfrak{F}_n) \longrightarrow \mathfrak{f}^\circ_{n*}(\mathrm{L}_n\mathfrak{F}_n)$$

is an equivalence whenever $\mathfrak{F}_n \in \mathrm{Coh}^\heartsuit(\mathfrak{U}_n)$, which follows from Lemma 3.10. □

Corollary 3.12. *Let $\mathfrak{X}$ and $\mathfrak{f}_\bullet\colon \mathfrak{U}_\bullet \to \mathfrak{X}$ be as in the above discussion. Then the functor*

$$\mathfrak{f}^{\circ *}_\bullet\colon \mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X}) \longrightarrow \lim_{[n]\in\Delta} \mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{U}_n)$$

is fully faithful.

Proof. Observe that the t-structure on both categories is left complete. Since $\mathfrak{f}^{\circ *}_\bullet$ is t-exact, it is therefore forced to commute with Postnikov towers. Furthermore, since $\mathfrak{X}$ is quasi-compact, the t-structure on both categories is right bounded as well. Let $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X})$. We have to prove that the natural map

$$\mathrm{Map}_{\mathrm{Coh}^-_{\mathrm{loc}}}(\mathcal{F}, \mathcal{G}) \longrightarrow \lim_{[n]\in\Delta} \mathrm{Map}_{\mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{U}_n)}(\mathfrak{f}^{\circ *}_n\mathcal{F}, \mathfrak{f}^{\circ *}_n\mathcal{G})$$

is an equivalence. Write

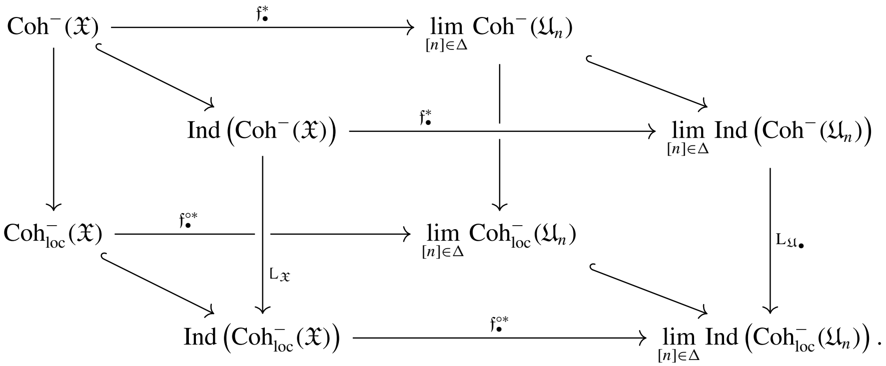
$$\mathcal{G} \simeq \lim_{m\in\mathbb{N}} \tau_{\leqslant m}\mathcal{G}.$$

Using the fact that $\mathfrak{f}^{\circ *}_n$ commutes with Postnikov towers and the fact that limits commute with limits, we reduce ourselves to proving that the above morphism is an equivalence when $\mathcal{G}$ is bounded. However, if $\mathcal{G} \in \mathrm{Coh}^{\leqslant m}_{\mathrm{loc}}(\mathfrak{X})$, then

$$\mathrm{Map}_{\mathrm{Coh}^-_{\mathrm{loc}}}(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Map}_{\mathrm{Coh}^-_{\mathrm{loc}}}(\tau_{\leqslant m}\mathcal{F}, \mathcal{G}).$$

Using once more t-exactness of $\mathfrak{f}^{\circ *}_\bullet$, we see that we can replace $\mathcal{F}$ by its truncation. In other words, we are reduced to proving that $\mathfrak{f}^{\circ *}_\bullet$ is fully faithful when restricted to $\mathrm{Coh}^{\mathrm{b}}_{\mathrm{loc}}(\mathfrak{X})$.

Consider now the following commutative cube:



(3.7)

First of all, we observe that the diagonal functors are all fully faithful. It is therefore enough to prove that the functor

$$\mathfrak{f}_\bullet^{\circ*}\colon \mathrm{Ind}(\mathrm{Coh}^-_{\mathrm{loc}}((\mathfrak{X})) \longrightarrow \lim_{[n]\in\Delta} \mathrm{Ind}(\mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{U}_n))$$

is fully faithful when restricted to $\mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X})$. As this functor admits a right adjoint $\mathfrak{f}^\circ_{\bullet*}$, it is in turn enough to verify that for every $\mathcal{F} \in \mathrm{Coh}^{\mathrm{b}}_{\mathrm{loc}}(\mathfrak{X})$, the unit transformation

$$\eta\colon \mathcal{F} \longrightarrow \mathfrak{f}^\circ_{\bullet*}\mathfrak{f}_\bullet^{\circ*}(\mathcal{F})$$

is an equivalence. Proceeding by induction on the number of nonvanishing homotopy groups of $\mathcal{F}$, we see that it is enough to deal with the case of $\mathcal{F} \in \mathrm{Coh}^\heartsuit_{\mathrm{loc}}(\mathfrak{X})$.

As the functor $\mathrm{L}_{\mathfrak{X}}\colon \mathrm{Coh}^-(\mathfrak{X}) \to \mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X})$ is essentially surjective and t-exact, we can choose $\mathfrak{F} \in \mathrm{Coh}^\heartsuit(\mathfrak{X})$ and an equivalence

$$\mathrm{L}_{\mathfrak{X}}(\mathfrak{F}) \simeq \mathcal{F}.$$

Moreover, the unit transformation

$$\mathfrak{F} \longrightarrow \mathfrak{f}_{\bullet*}\mathfrak{f}_\bullet^*\mathfrak{F}$$

is an equivalence. It is therefore enough to check that the Beck–Chevalley transformation associated with the front square is an equivalence when evaluated on objects in $\lim_\Delta \mathrm{Coh}^\heartsuit(\mathfrak{U}_n)$. This is exactly the content of Proposition 3.11. □

3.3. Categories of formal models

Let $\mathfrak{X} \in \mathrm{dfSch}^{\mathrm{tafp}}_{k^\circ}$ be a quasi-compact and quasi-separated derived k°-adic scheme topologically almost of finite presentation. We established in Corollary 3.8 that for any $\mathcal{F} \in \mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}})$, the ∞-category of formal models $\mathrm{FM}(\mathcal{F})$ is non-empty. Actually, we can use Corollary 3.7 to be more precise about the structure of $\mathrm{FM}(\mathcal{F})$. We are in particular interested in showing that it is filtered. We start by recording the following immediate consequence of Corollary 3.7.

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Lemma 3.13. *Let* $\mathfrak{X} \in \mathrm{dfSch}_{k^\circ}^{\mathrm{tafp}}$ *be a derived* k°*-adic scheme topologically almost of finite presentation. Then the functor*

$$(-)^{\mathrm{rig}} \colon \mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{X})) \longrightarrow \mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}}))$$

admits a right adjoint

$$j \colon \mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}})) \longrightarrow \mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{X})),$$

which is furthermore fully faithful.

Proof. Corollary 3.7 implies that the functor $(-)^{\mathrm{rig}}$ induces the equivalence

$$\Lambda \colon \mathrm{Coh}^-_{\mathrm{loc}}(\mathfrak{X}) \xrightarrow{\sim} \mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}}).$$

In other words, we see that the diagram

$$\begin{array}{ccc} \mathrm{Coh}^-_{\mathrm{nil}}(\mathfrak{X}) & \longrightarrow & \mathrm{Coh}^-(\mathfrak{X}) \\ \downarrow & & \downarrow{\scriptstyle (-)^{\mathrm{rig}}} \\ 0 & \longrightarrow & \mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}}) \end{array}$$

is a pushout diagram in $\mathrm{Cat}_\infty^{\mathrm{st}}$. Passing to ind-completions, we deduce that $\mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}}))$ is a Verdier quotient of $\mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{X}))$. Applying [8, Lemma 2.5 and Remark 2.6] we conclude that $\mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{X}^{\mathrm{rig}}))$ is an accessible localization of $\mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{X}))$. As these categories are presentable, we deduce that the localization functor $(-)^{\mathrm{rig}}$ admits a fully faithful right adjoint, as desired. □

Notation 3.14. Let $\mathfrak{X} \in \mathrm{dfDM}_{k^\circ}$. Given $\mathcal{F}, \mathcal{G} \in \mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{X}))$, we write $\mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, \mathcal{G}) \in \mathrm{Mod}_{k^\circ}$ for the k°-enriched stable mapping space in $\mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{X}))$.

Lemma 3.15. *Let* $\mathfrak{X} \in \mathrm{dfSch}_{k^\circ}^{\mathrm{tafp}}$ *be a derived* k°*-adic scheme topologically almost of finite presentation. Let* $\mathcal{F} \in \mathrm{Coh}^-(\mathfrak{X})$ *and* $\mathcal{G} \in \mathrm{Coh}^-_{\mathrm{nil}}(\mathfrak{X})$*. Then*

$$\mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, \mathcal{G}) \otimes_{k^\circ} k \simeq 0.$$

In other words, $\mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, \mathcal{G})$ *is* $\mathfrak{m}$*-nilpotent in* Mod_{k°*.*

Proof. Since $\mathfrak{X}$ is quasi-compact, we can find a finite formal Zariski cover $\{\mathfrak{U}_i = \mathrm{Spf}(A_i)\}_{i=0,\dots,n}$ by formal affine schemes. Consider the Zariski site $\mathfrak{X}_{\mathrm{Zar}}$ as a poset and let I be the subposet generated by the opens $\mathfrak{U}_i$ and all their possible intersections. Note that I is a finite category. Given $m \in I$ we denote by $\mathfrak{U}_m$ the corresponding formal Zariski open subset of $\mathfrak{X}$. Induction on n shows that

$$\mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, \mathcal{G}) \simeq \lim_{m \in I} \mathrm{Hom}_{\mathfrak{U}_m}(\mathcal{F}|_{\mathfrak{U}_m}, \mathcal{G}|_{\mathfrak{U}_m}).$$

Since the functor $- \otimes_{k^\circ} k \colon \mathrm{Mod}_{k^\circ} \to \mathrm{Mod}_k$ is exact, it commutes with finite limits. Therefore, we see that it is enough to prove that the conclusion holds after replacing $\mathfrak{X}$ by $\mathfrak{U}_m$. Since $\mathfrak{X}$ is quasi-compact and quasi-separated, we see that each $\mathfrak{U}_m$ is quasi-compact and separated. In other words, we can assume from the very beginning that $\mathfrak{X}$ is quasi-compact and separated. In this case, each $\mathfrak{U}_m$ will be formal affine, and therefore we can further reduce to the case where $\mathfrak{X}$ is formal affine itself.

Assume therefore $\mathfrak{X} = \mathrm{Spf}(A)$. In this case, $\mathrm{Coh}^-(\mathfrak{X}) \simeq \mathrm{Coh}^-(A)$ lives fully faithfully inside Mod_A. Note that $A \to A \otimes_{k^\circ} k$ is a Zariski open immersion. Therefore,

$$\mathrm{Hom}_A(\mathcal{F}, \mathcal{G}) \otimes_{k^\circ} k \simeq \mathrm{Hom}_A(\mathcal{F}, \mathcal{G}) \otimes_A (A \otimes_{k^\circ} k) \simeq \mathrm{Hom}_A(\mathcal{F} \otimes_A k^\circ, \mathcal{G} \otimes_A k^\circ) \simeq 0.$$

Thus, the proof is complete. □

Corollary 3.16. *Let $\mathfrak{X}$ be as in the previous lemma. Given $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}^-(\mathfrak{X})$, the canonical map*

$$\mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, \mathcal{G}) \otimes_{k^\circ} k \longrightarrow \mathrm{Hom}_{\mathfrak{X}^{\mathrm{rig}}}(\mathcal{F}^{\mathrm{rig}}, \mathcal{G}^{\mathrm{rig}})$$

is an equivalence.

Proof. Denote by $R\colon \mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{X})) \to \mathrm{Ind}(\mathrm{Coh}^-_{\mathrm{nil}}(\mathfrak{X}))$ the right adjoint to the inclusion

$$i\colon \mathrm{Ind}(\mathrm{Coh}^-_{\mathrm{nil}}(\mathfrak{X})) \hookrightarrow \mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{X})).$$

Then for any $\mathcal{G} \in \mathrm{Coh}^-(\mathfrak{X})$, we have a fiber sequence

$$iR(\mathcal{G}) \longrightarrow \mathcal{G} \longrightarrow j(\mathcal{G}^{\mathrm{rig}}).$$

In particular, we obtain a fiber sequence

$$\mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, iR(\mathcal{G})) \longrightarrow \mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, j(\mathcal{G}^{\mathrm{rig}})).$$

Now observe that

$$\mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, j(\mathcal{G}^{\mathrm{rig}})) \simeq \mathrm{Hom}_{\mathfrak{X}^{\mathrm{rig}}}(\mathcal{F}^{\mathrm{rig}}, \mathcal{G}^{\mathrm{rig}}).$$

Note also that since $k^\circ \to k$ is an open Zariski immersion, $\mathrm{Hom}_{\mathfrak{X}^{\mathrm{rig}}}(\mathcal{F}^{\mathrm{rig}}, \mathcal{G}^{\mathrm{rig}}) \otimes_{k^\circ} k \simeq \mathrm{Hom}_{\mathfrak{X}^{\mathrm{rig}}}(\mathcal{F}^{\mathrm{rig}}, \mathcal{G}^{\mathrm{rig}})$. In particular, applying $- \otimes_{k^\circ} k\colon \mathrm{Mod}_{k^\circ} \to \mathrm{Mod}_k$, we find a fiber sequence

$$\mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, iR(\mathcal{G})) \otimes_{k^\circ} k \longrightarrow \mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, \mathcal{G}) \otimes_{k^\circ} k \longrightarrow \mathrm{Hom}_{\mathfrak{X}^{\mathrm{rig}}}(\mathcal{F}^{\mathrm{rig}}, \mathcal{G}^{\mathrm{rig}}).$$

It is therefore enough to check that $\mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, iR(\mathcal{G})) \otimes_{k^\circ} k \simeq 0$. Since i is a left adjoint, we can write

$$iR(\mathcal{G}) \simeq \operatorname*{colim}_{\alpha \in I} \mathcal{G}_\alpha,$$

where I is filtered and $\mathcal{G}_\alpha \in \mathrm{Coh}^-_{\mathrm{nil}}(\mathfrak{X})$. As $\mathcal{F}$ is compact in $\mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{X}))$, we find

$$\mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, iR(\mathcal{G})) \otimes_{k^\circ} k \simeq \left(\operatorname*{colim}_{\alpha \in I} \mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, \mathcal{G}_\alpha) \right) \otimes_{k^\circ} k \simeq \operatorname*{colim}_{\alpha \in I} \mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, \mathcal{G}_\alpha) \otimes_{k^\circ} k.$$

Since each $\mathcal{G}_\alpha$ belongs to $\mathrm{Coh}^-_{\mathrm{nil}}(\mathfrak{X})$, Lemma 3.15 implies that $\mathrm{Hom}_{\mathfrak{X}}(\mathcal{F}, \mathcal{G}_\alpha) \otimes_{k^\circ} k \simeq 0$. The conclusion follows. □

Construction 3.17. Recall that t is a fixed pseudo-uniformizer for $\mathfrak{m}$. We consider $\mathbb{N}$ as a poset with its natural ordering. Introduce the functor

$$K\colon \mathbb{N} \longrightarrow \mathrm{Ind}(\mathrm{Coh}^{\heartsuit}(\mathrm{Spf}(k^{\circ})))$$

defined as follows: K sends every integer to k°, and it sends the morphism $m \leqslant m'$ to multiplication by $t^{m'-m}$. By abuse of notation, we still denote the composition of K with the inclusion $\mathrm{Ind}(\mathrm{Coh}^{\heartsuit}(k^{\circ})) \to \mathrm{Ind}(\mathrm{Coh}^{-}(k^{\circ}))$ by K.

Let now $\mathfrak{X} \in \mathrm{dfSch}_{k^{\circ}}^{\mathrm{tafp}}$ be a quasi-compact and quasi-separated derived k°-adic scheme topologically almost of finite presentation. Let $\mathcal{F} \in \mathrm{Coh}^{-}(\mathfrak{X})$. The natural morphism $q\colon \mathfrak{X} \to \mathrm{Spf}(k^{\circ})$ induces a functor

$$q^{*}\colon \mathrm{Ind}(\mathrm{Coh}^{-}(\mathrm{Spf}(k^{\circ}))) \longrightarrow \mathrm{Ind}(\mathrm{Coh}^{-}(\mathfrak{X})).$$

We define the functor $K_{\mathcal{F}}$ as

$$K_{\mathcal{F}} := q^{*}(K(-)) \otimes \mathcal{F}\colon \mathbb{N} \longrightarrow \mathrm{Ind}(\mathrm{Coh}^{-}(\mathfrak{X})).$$

We let $\mathcal{F}^{\mathrm{loc}}$ denote the colimit of the functor $K_{\mathcal{F}}$.

Let $\mathcal{G} \in \mathrm{Coh}^{-}(\mathfrak{X}^{\mathrm{rig}})$ and let $\alpha\colon \mathcal{F}^{\mathrm{rig}} \to \mathcal{G}$ be a given map. Note that the natural map

$$\mathcal{F}^{\mathrm{rig}} \longrightarrow \operatorname*{colim}_{\mathbb{N}} (K_{\mathcal{F}}(-))^{\mathrm{rig}}$$

is an equivalence. Therefore α induces a cone

$$(K_{\mathcal{F}}(-))^{\mathrm{rig}} \longrightarrow \mathcal{G},$$

which is equivalent to the given of a natural transformation

$$K_{\mathcal{F}}(-) \longrightarrow j(\mathcal{G}).$$

Specializing this construction for $\alpha = \mathrm{id}_{\mathcal{F}^{\mathrm{rig}}}$, we obtain a canonical map

$$\gamma_{\mathcal{F}}\colon \mathcal{F}^{\mathrm{loc}} \longrightarrow j(\mathcal{F}^{\mathrm{rig}}).$$

Lemma 3.18. *Let $\mathfrak{X} \in \mathrm{dfSch}_{k^{\circ}}^{\mathrm{tafp}}$ be a derived k°-adic scheme topologically almost of finite presentation. Let $\mathcal{F} \in \mathrm{Coh}^{-}_{\mathrm{nil}}(\mathfrak{X})$. Then $\mathcal{F}^{\mathrm{loc}} \simeq 0$.*

Proof. For any $\mathcal{G} \in \mathrm{Coh}^{-}(\mathfrak{X})$, we write $\mathrm{Hom}_{\mathfrak{X}}(\mathcal{G}, \mathcal{F}) \in \mathrm{Mod}_{k^{\circ}}$ for the k°-enriched mapping space. As $\mathcal{G}$ is compact in $\mathrm{Ind}(\mathrm{Coh}^{-}(\mathfrak{X}))$, we have

$$\mathrm{Hom}_{\mathfrak{X}}(\mathcal{G}, \mathcal{F}^{\mathrm{loc}}) \simeq \operatorname*{colim}_{\mathbb{N}} \mathrm{Hom}_{\mathfrak{X}}(\mathcal{G}, K_{\mathcal{F}}(-)) \simeq \mathrm{Hom}_{\mathfrak{X}}(\mathcal{G}, \mathcal{F}) \otimes_{k^{\circ}} k.$$

Corollary 3.16 implies that

$$\mathrm{Hom}_{\mathfrak{X}}(\mathcal{G}, \mathcal{F}) \otimes_{k^{\circ}} k \simeq \mathrm{Hom}_{\mathfrak{X}^{\mathrm{rig}}}(\mathcal{G}^{\mathrm{rig}}, \mathcal{F}^{\mathrm{rig}}) \simeq 0.$$

It follows that $\mathcal{F}^{\mathrm{loc}} \simeq 0$. □

Lemma 3.19. *Let $\mathfrak{X} \in \mathrm{dfSch}_{k^{\circ}}^{\mathrm{tafp}}$ be a derived k°-adic scheme topologically almost of finite presentation. Let $\mathcal{F} \in \mathrm{Coh}^{-}(\mathfrak{X})$. Then for any $\mathcal{G} \in \mathrm{Coh}^{-}_{\mathrm{nil}}(\mathfrak{X})$, one has*

$$\mathrm{Map}_{\mathrm{Ind}(\mathrm{Coh}^{-}(\mathfrak{X}))}(\mathcal{G}, \mathcal{F}^{\mathrm{loc}}) \simeq 0.$$

Proof. It is enough to prove that for every $i \geqslant 0$, we have

$$\pi_i \operatorname{Map}_{\operatorname{Ind}(\operatorname{Coh}^-(\mathfrak{X}))}(\mathcal{G}, \mathcal{F}^{\mathrm{loc}}) \simeq 0.$$

Up to replacing $\mathcal{F}$ by $\mathcal{F}[i]$, we see that it is enough to deal with the case $i = 0$. Let therefore $\alpha \colon \mathcal{G} \to \mathcal{F}^{\mathrm{loc}}$ be a representative for an element in $\pi_0 \operatorname{Map}_{\operatorname{Ind}(\operatorname{Coh}^-(\mathfrak{X}))}(\mathcal{G}, \mathcal{F}^{\mathrm{loc}})$. As $\mathcal{G}$ is compact in $\operatorname{Ind}(\operatorname{Coh}^-(\mathfrak{X}))$, the map α factors as $\alpha' \colon \mathcal{G} \to \mathcal{F}$, and therefore it induces a map $\widetilde{\alpha} \colon \mathcal{G}^{\mathrm{loc}} \to \mathcal{F}^{\mathrm{loc}}$ making the diagram

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\alpha'} & \mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{G}^{\mathrm{loc}} & \xrightarrow{\widetilde{\alpha}} & \mathcal{F}^{\mathrm{loc}} \end{array}$$

commutative, where both compositions are equivalent to α. Now, Lemma 3.18 implies that $\mathcal{G}^{\mathrm{loc}} \simeq 0$, and therefore α is null-homotopic, completing the proof. □

Lemma 3.20. *Let $\mathfrak{X} \in \mathrm{dfSch}_{k^\circ}^{\mathrm{tafp}}$ be a derived k°-adic scheme topologically almost of finite presentation. Let $\mathcal{F} \in \operatorname{Coh}^-(\mathfrak{X})$. Then the canonical map*

$$\gamma_{\mathcal{F}} \colon \mathcal{F}^{\mathrm{loc}} \longrightarrow j(\mathcal{F}^{\mathrm{rig}})$$

is an equivalence.

Proof. Let $\mathcal{G} \in \operatorname{Coh}^-_{\mathrm{nil}}(\mathfrak{X})$. Then

$$\operatorname{Map}_{\operatorname{Ind}(\operatorname{Coh}^-(\mathfrak{X}))}(\mathcal{G}, j(\mathcal{F}^{\mathrm{rig}})) \simeq \operatorname{Map}_{\operatorname{Ind}(\operatorname{Coh}^-(\mathfrak{X}^{\mathrm{rig}}))}(\mathcal{G}^{\mathrm{rig}}, \mathcal{F}^{\mathrm{rig}}) \simeq 0.$$

Lemma 3.19 implies that the same holds true replacing $j(\mathcal{F}^{\mathrm{rig}})$ with $\mathcal{F}^{\mathrm{loc}}$. As $\operatorname{Coh}^-_{\mathrm{nil}}(\mathfrak{X})$ is a stable full subcategory of $\operatorname{Coh}^-(\mathfrak{X})$, it follows that

$$\operatorname{Hom}_{\mathfrak{X}}(\mathcal{G}, j(\mathcal{F}^{\mathrm{rig}})) \simeq \operatorname{Hom}_{\mathfrak{X}}(\mathcal{G}, \mathcal{F}^{\mathrm{loc}}) \simeq 0.$$

Let $\mathcal{H} \coloneqq \operatorname{fib}(\gamma_{\mathcal{F}})$. Then for any $\mathcal{G} \in \operatorname{Coh}^-_{\mathrm{nil}}(\mathfrak{X})$, one has

$$\operatorname{Hom}_{\mathfrak{X}}(\mathcal{G}, \mathcal{H}) \simeq 0.$$

On the other hand,

$$\mathcal{H}^{\mathrm{rig}} \simeq \operatorname{fib}(\gamma_{\mathcal{F}}^{\mathrm{rig}}) \simeq 0.$$

It follows that $\mathcal{H} \in \operatorname{Ind}(\operatorname{Coh}^-_{\mathrm{nil}}(\mathfrak{X}))$, and hence that $\mathcal{H} \simeq 0$. Thus, $\gamma_{\mathcal{F}}$ is an equivalence. □

Theorem 3.21. *Let $\mathfrak{X} \in \mathrm{dfSch}_{k^\circ}$ be a derived k°-adic scheme. Let $\mathcal{F} \in \operatorname{Coh}^-(\mathfrak{X}^{\mathrm{rig}})$. Then the ∞-category $\mathrm{FM}(\mathcal{F})$ of formal models for $\mathcal{F}$ is non-empty and filtered.*

Proof. We know that $\mathrm{FM}(\mathcal{F})$ is non-empty thanks to Corollary 3.8. Pick one formal model $\mathfrak{F} \in \mathrm{FM}(\mathcal{F})$. Then Lemma 3.20 implies that the canonical map

$$\gamma_{\mathcal{F}} \colon \mathfrak{F}^{\mathrm{loc}} \longrightarrow j(\mathcal{F})$$

is an equivalence. We now observe that $\mathrm{FM}(\mathcal{F})$ is by definition a full subcategory of

$$\operatorname{Coh}^-(\mathfrak{X})_{/\mathcal{F}} \coloneqq \operatorname{Coh}^-(\mathfrak{X}) \times_{\operatorname{Ind}(\operatorname{Coh}^-(\mathfrak{X}))} \operatorname{Ind}(\operatorname{Coh}^-(\mathfrak{X}))_{/j(\mathcal{F})}.$$

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As this ∞-category is filtered, it is enough to prove that every object $\mathcal{G} \in \mathrm{Coh}^-(\mathfrak{X})_{/\mathcal{F}}$ admits a map to an object in $\mathrm{FM}(\mathcal{F})$. Let $\alpha \colon \mathcal{G} \to j(\mathcal{F})$ be the structural map. Using the equivalence $\gamma_{\mathcal{F}}$ and the fact that $\mathcal{G}$ is compact in $\mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{X}))$, we see that α factors as $\mathcal{G} \to \mathfrak{F}$, which belongs to $\mathrm{FM}(\mathcal{F})$ by construction. □

Corollary 3.22. *Let $X \in \mathrm{dAn}_k$ and $f \colon \mathcal{F} \to \mathcal{G}$ be a morphism $\mathrm{Coh}^-(X)$. Suppose we are given a formal model $\mathfrak{X}$ for X together with formal models $\mathfrak{F}, \mathfrak{G} \in \mathrm{Coh}^-(\mathfrak{X})$ for $\mathcal{F}$ and $\mathcal{G}$, respectively. Then there exists a morphism $\mathfrak{f} \colon \mathfrak{F} \to \mathfrak{G}$ in the ∞-category $\mathrm{Coh}^-(\mathfrak{X})$ lifting*

$$t^m f \colon \mathcal{F} \to \mathcal{G}, \quad \text{in } \mathrm{Coh}^-(X)$$

for a suitable non-negative integer $m \geqslant 0$.

Proof. Any map $\mathcal{F} \to \mathcal{G}$ induces a map $\mathfrak{F} \to j(\mathcal{F}) \to j(\mathcal{G})$. Using the equivalence $j(\mathcal{G}) \simeq \mathfrak{G}^{\mathrm{loc}}$ and the fact that $\mathfrak{F}$ is compact in $\mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{X}))$, we see that the map $\mathfrak{F} \to j(\mathcal{G})$ factors as $\mathfrak{F} \to \mathfrak{G}$. Unraveling the definition of the functor $K_{\mathcal{G}}(-)$, we see that the conclusion follows. □

For later use, let us record the following consequence of Lemma 3.20.

Corollary 3.23. *Let $\mathfrak{X} \in \mathrm{dfSch}_{k^\circ}^{\mathrm{tafp}}$ be a derived k°-adic scheme topologically almost of finite presentation. Let $\mathcal{F} \in \mathrm{Coh}^-(\mathfrak{X})$. Then $\mathcal{F}$ is $\mathfrak{m}$-nilpotent if and only if $\mathcal{F}^{\mathrm{loc}} \simeq 0$.*

Proof. If $\mathcal{F}$ is $\mathfrak{m}$-nilpotent, the conclusion follows from Lemma 3.18. Suppose vice versa that $\mathcal{F}^{\mathrm{loc}} \simeq 0$. Then Lemma 3.20 implies that

$$j(\mathcal{F}^{\mathrm{rig}}) \simeq \mathcal{F}^{\mathrm{loc}} \simeq 0.$$

Now, Lemma 3.13 shows that j is fully faithful. In particular, it is conservative and therefore $\mathcal{F}^{\mathrm{rig}} \simeq 0$. In other words, $\mathcal{F}$ belongs to $\mathrm{Coh}^-_{\mathrm{nil}}(\mathfrak{X})$. □

4. Flat models for morphisms of derived analytic spaces

Using the study of formal models for almost perfect complexes carried out in the previous section, we can prove the following derived version of [5, Theorem 5.2].

Theorem 4.1. *Let $f \colon X \to Y$ be a proper map of quasi-paracompact derived k-analytic spaces. Assume the following:*

(1) *The truncations of X and Y are k-analytic spaces.*[4]
(2) *The map f is flat.*

Then there exists a proper flat formal model $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{Y}$ in $\mathrm{dfSch}_{k^\circ}^{\mathrm{tafp}}$ for f.

Proof. We construct, by induction on n, the following data:

(1) Derived k°-adic schemes $\mathfrak{X}_n$ and $\mathfrak{Y}_n$ equipped with equivalences

$$\mathfrak{X}_n^{\mathrm{rig}} \simeq \mathrm{t}_{\leqslant n}(X), \quad \mathfrak{Y}_n^{\mathrm{rig}} \simeq \mathrm{t}_{\leqslant n}(Y).$$

[4]As opposed to k-analytic Deligne–Mumford stacks.

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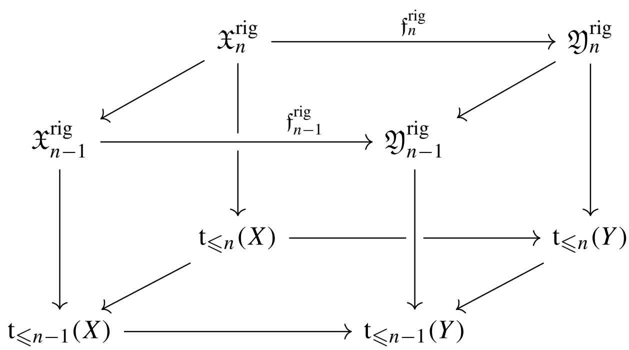
(2) Morphisms $\mathfrak{X}_n \to \mathfrak{X}_{n-1}$ and $\mathfrak{Y}_n \to \mathfrak{Y}_{n-1}$ exhibiting $\mathfrak{X}_{n-1}$ and $\mathfrak{Y}_{n-1}$ as $(n-1)$-truncations of $\mathfrak{X}_n$ and $\mathfrak{Y}_n$, respectively.

(3) A proper flat morphism $\mathfrak{f}_n \colon \mathfrak{X}_n \to \mathfrak{Y}_n$ and homotopies making the cube



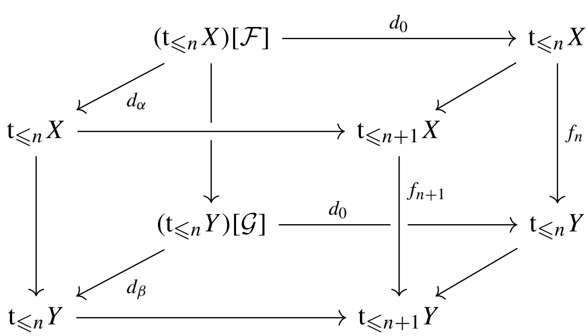
commutative.

Having these data at our disposal, we set

$$\mathfrak{X} \coloneqq \operatorname*{colim}_n \mathfrak{X}_n, \quad \mathfrak{Y} \coloneqq \operatorname*{colim}_n \mathfrak{Y}_n,$$

and we let $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{Y}$ be a map induced by the morphisms $\mathfrak{f}_n$. The properties listed above imply that $\mathfrak{f}$ is proper and flat and that its generic fiber is equivalent to f.

We are therefore left to construct the data listed above. When $n = 0$, we can apply the flattening technique of Raynaud–Gruson (see [5, Theorem 5.2]) to produce a proper flat formal model $\mathfrak{f}_0 \colon \mathfrak{X}_0 \to \mathfrak{Y}_0$ for $\mathrm{t}_0(f) \colon \mathrm{t}_0(X) \to \mathrm{t}_0(Y)$. Assume now that we constructed the above data up to n and let us construct it for $n+1$. Set $\mathcal{F} \coloneqq \pi_{n+1}(\mathcal{O}_X)[n+2]$ and $\mathcal{G} \coloneqq \pi_{n+1}(\mathcal{O}_Y)[n+2]$. Using [16, Corollary 5.44], we can find analytic derivations $d_\alpha \colon (\mathrm{t}_{\leqslant n} X)[\mathcal{F}] \to \mathrm{t}_{\leqslant n} X$ and $d_\beta \colon (\mathrm{t}_{\leqslant n} Y)[\mathcal{G}] \to \mathrm{t}_{\leqslant n} Y$ making the following cube



(4.1)

commutative. Here d_0 denotes the zero derivation, and we set $f_n \coloneqq \mathrm{t}_{\leqslant n}(f)$, $f_{n+1} \coloneqq \mathrm{t}_{\leqslant n+1}(f)$. The derivations d_α and d_β correspond to morphisms $\alpha \colon \mathbb{L}^{\mathrm{an}}_{\mathrm{t}_{\leqslant n} X} \to \mathcal{F}$ and $\beta \colon \mathbb{L}^{\mathrm{an}}_{\mathrm{t}_{\leqslant n} Y} \to \mathcal{G}$, respectively. Moreover, the commutativity of the left side square in (4.1)

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is equivalent to the commutativity of

$$\begin{array}{ccc} f_n^*\mathbb{L}^{\mathrm{an}}_{\mathrm{t}_{\leqslant n}Y} & \xrightarrow{f_n^*\beta} & f_n^*\mathcal{G} \\ \downarrow & & \downarrow \\ \mathbb{L}^{\mathrm{an}}_{\mathrm{t}_{\leqslant n}X} & \xrightarrow{\alpha} & \mathcal{F} \end{array}$$

in $\mathrm{Coh}^-(\mathrm{t}_{\leqslant n}X)$. Note that, since f is flat, the morphism $f_n^*\mathcal{F} \to \mathcal{G}$ is an equivalence. Using Theorem 2.16 and the induction hypothesis, we know that $\mathbb{L}^{\mathrm{ad}}_{\mathfrak{Y}_n}$ is a canonical formal model for $\mathbb{L}^{\mathrm{an}}_{\mathrm{t}_{\leqslant n}X}$. Using Theorem 3.21, we can therefore find a formal model $\overline{\beta}\colon \mathbb{L}^{\mathrm{ad}}_{\mathfrak{Y}_n} \to \mathfrak{G}$ for the map β. We now set

$$\mathfrak{F} \coloneqq \mathfrak{f}_n^*\mathfrak{G}.$$

Using Corollary 3.22, we can find $m \in \mathbb{N}$ and a formal model $\widetilde{\alpha}\colon \mathbb{L}^{\mathrm{ad}}_{\mathfrak{X}_n} \to \mathfrak{F}$ for $t^m\alpha$ together with a homotopy making the diagram

$$\begin{array}{ccc} \mathfrak{f}_n^*\mathbb{L}^{\mathrm{ad}}_{\mathfrak{Y}_n} & \xrightarrow{t^m\mathfrak{f}_n^*\overline{\beta}} & \mathfrak{f}_n^*\mathfrak{G} \\ \downarrow & & \| \\ \mathbb{L}^{\mathrm{ad}}_{\mathfrak{X}_n} & \xrightarrow{\tilde{\alpha}} & \mathfrak{F} \end{array}$$

commutative. Set $\widetilde{\beta} \coloneqq t^m\overline{\beta}\colon \mathbb{L}^{\mathrm{ad}}_{\mathfrak{Y}_n} \to \mathfrak{G}$. Then $\widetilde{\alpha}$ and $\widetilde{\beta}$ induce a commutative square

$$\begin{array}{ccc} \mathfrak{X}_n[\mathfrak{F}] & \xrightarrow{d_{\tilde{\alpha}}} & \mathfrak{X}_n \\ \downarrow & & \downarrow \mathfrak{f}_n \\ \mathfrak{Y}_n[\mathfrak{G}] & \xrightarrow{d_{\tilde{\beta}}} & \mathfrak{Y}_n. \end{array} \tag{4.2}$$

We now define $\mathfrak{X}_{n+1}$ and $\mathfrak{Y}_{n+1}$ as the square-zero extensions associated with $\widetilde{\alpha}$ and $\widetilde{\beta}$. In other words, they are defined by the following pushout diagrams:

$$\begin{array}{ccc} \mathfrak{X}_n[\mathfrak{F}] & \xrightarrow{d_0} & \mathfrak{X}_n \\ \downarrow d_{\tilde{\alpha}} & & \downarrow \\ \mathfrak{X}_n & \longrightarrow & \mathfrak{X}_{n+1} \end{array}, \qquad \begin{array}{ccc} \mathfrak{Y}_n[\mathfrak{G}] & \xrightarrow{d_0} & \mathfrak{Y}_n \\ \downarrow d_{\tilde{\beta}} & & \downarrow \\ \mathfrak{Y}_n & \longrightarrow & \mathfrak{Y}_{n+1}. \end{array}$$

The commutativity of (4.2) provides a canonical map $\mathfrak{f}_{n+1}\colon \mathfrak{X}_{n+1} \to \mathfrak{Y}_{n+1}$, which is readily verified to be proper and flat. We are therefore left to verify that $\mathfrak{f}_{n+1}$ is a formal model for f_{n+1}. Unraveling the definitions, we see that it is enough to produce equivalences $a\colon (\mathrm{t}_{\leqslant n}X)[\mathcal{F}] \xrightarrow{\sim} (\mathrm{t}_{\leqslant n}X)[\mathcal{F}]$ and $b\colon (\mathrm{t}_{\leqslant n}Y)[\mathcal{G}] \xrightarrow{\sim} (\mathrm{t}_{\leqslant n}Y)[\mathcal{G}]$ making the following diagrams

$$\begin{array}{ccc} (\mathrm{t}_{\leqslant n}X)[\mathcal{F}] & \xrightarrow{d_{t^m\alpha}} & \mathrm{t}_{\leqslant n}X \\ \downarrow a & & \| \\ (\mathrm{t}_{\leqslant n}X)[\mathcal{F}] & \xrightarrow{d_\alpha} & \mathrm{t}_{\leqslant n}X \end{array}, \qquad \begin{array}{ccc} (\mathrm{t}_{\leqslant n}Y)[\mathcal{G}] & \xrightarrow{d_{t^m\beta}} & \mathrm{t}_{\leqslant n}Y \\ \downarrow b & & \| \\ (\mathrm{t}_{\leqslant n}Y)[\mathcal{G}] & \xrightarrow{d_\beta} & \mathrm{t}_{\leqslant n}Y \end{array} \tag{4.3}$$

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commutative. The situation is symmetric, so it is enough to deal with $\mathrm{t}_{\leqslant n}X$. Consider the morphism

$$t^{-m}\colon \mathcal{F} \longrightarrow \mathcal{F},$$

which exists because all the elements $t_i \in \mathfrak{m}$ are invertible in k. For the same reason, it is an equivalence, with inverse given by multiplication by t^m. This morphism induces a map

$$a\colon (\mathrm{t}_{\leqslant n}X)[\mathcal{F}] \longrightarrow (\mathrm{t}_{\leqslant n}X)[\mathcal{F}],$$

which by functoriality is an equivalence. We now observe that the commutativity of (4.3) is equivalent to the commutativity of

$$\begin{array}{ccc} \mathbb{L}^{\mathrm{an}}_{\mathrm{t}_{\leqslant n}X} & \xrightarrow{t^m\alpha} & \mathcal{F} \\ \| & & \downarrow{\scriptstyle t^{-m}} \\ \mathbb{L}^{\mathrm{an}}_{\mathrm{t}_{\leqslant n}X} & \xrightarrow{\alpha} & \mathcal{F}, \end{array}$$

which is immediate. The proof is therefore achieved. □

5. The plus pushforward for almost perfect sheaves

Let $f\colon X \to Y$ be a proper map between derived k-analytic spaces of finite tor amplitude. In [17, Definition 7.9], it is introduced a functor

$$f_+\colon \mathrm{Perf}(X) \longrightarrow \mathrm{Perf}(Y),$$

and it is shown in Proposition 7.11 in loc. cit. that for every $\mathcal{G} \in \mathrm{Coh}^-(Y)$, there is a natural equivalence

$$\mathrm{Map}_{\mathrm{Coh}^-(X)}(\mathcal{F}, f^*\mathcal{G}) \simeq \mathrm{Map}_{\mathrm{Coh}^-(Y)}(f_+(\mathcal{F}), \mathcal{G}).$$

In this section, we extend the definition of f_+ to the entire $\mathrm{Coh}^-(X)$, at least under the stronger assumption of f being flat.

Remark 5.1. In algebraic geometry, the extension of f_+ to $\mathrm{Coh}^-(X)$ passes through the extension to $\mathrm{QCoh}(X) \simeq \mathrm{Ind}(\mathrm{Perf}(X))$. This ultimately requires being able to describe every element in $\mathrm{Coh}^-(X)$ as a filtered colimit of elements in $\mathrm{Perf}(X)$, which in analytic geometry is possible only locally.

Therefore, this technique cannot be applied in analytic geometry. When dealing with non-archimedean analytic geometry, formal models can be used to circumvent this problem.

Proposition 5.2. *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a proper map between derived k°-adic schemes. Assume that f has finite tor amplitude. Then the functor*

$$f^*\colon \mathrm{Coh}^-(\mathfrak{Y}) \to \mathrm{Coh}^-(\mathfrak{X})$$

admits a left adjoint

$$f_+\colon \mathrm{Coh}^-(\mathfrak{X}) \to \mathrm{Coh}^-(\mathfrak{Y}).$$

Proof. Let $\mathfrak{X}_n \coloneqq \mathfrak{X} \times_{\mathrm{Spf}(k^\circ)} \mathrm{Spec}(k^\circ/\mathfrak{m}^n)$ and define similarly $\mathfrak{Y}_n$. Let $f_n \colon \mathfrak{X}_n \to \mathfrak{Y}_n$ be the induced morphism. Then by the definition of k°-adic schemes, we have

$$\mathfrak{X} \simeq \operatorname*{colim}_{n\in\mathbb{N}} \mathfrak{X}_n, \quad \mathfrak{Y} \simeq \operatorname*{colim}_{n\in\mathbb{N}} \mathfrak{Y}_n,$$

and therefore

$$\mathrm{Coh}^-(\mathfrak{X}) \simeq \lim_{n\in\mathbb{N}} \mathrm{Coh}^-(\mathfrak{X}_n), \quad \mathrm{Coh}^-(\mathfrak{Y}) \simeq \lim_{n\in\mathbb{N}} \mathrm{Coh}^-(\mathfrak{Y}_n).$$

Combining [9, Remark 6.4.5.2(b) and Proposition 6.4.5.4(1)], we see that each functor

$$f_n^* \colon \mathrm{Coh}^-(\mathfrak{Y}_n) \longrightarrow \mathrm{Coh}^-(\mathfrak{X}_n)$$

admits a left adjoint f_{n+}. Moreover, Proposition 6.4.5.4(2) in loc. cit. implies that these functors f_{n+} can be assembled into a natural transformation, and that therefore they induce a well-defined functor

$$f_+ \colon \mathrm{Coh}^-(\mathfrak{X}) \longrightarrow \mathrm{Coh}^-(\mathfrak{Y}).$$

Now let $\mathcal{F} \in \mathrm{Coh}^-(\mathfrak{X})$ and $\mathcal{G} \in \mathrm{Coh}^-(\mathfrak{Y})$. Let $\mathcal{F}_n$ and $\mathcal{G}_n$ be the pullbacks of $\mathcal{F}$ and $\mathcal{G}$ to $\mathcal{X}_n$ and $\mathcal{Y}_n$, respectively. Then

$$\begin{aligned} \mathrm{Map}_{\mathrm{Coh}^-(\mathfrak{X})}(\mathcal{F}, f^*(\mathcal{G})) &\simeq \lim_{n\in\mathbb{N}} \mathrm{Map}_{\mathrm{Coh}^-(\mathfrak{X}_n)}(\mathcal{F}_n, f_n^*(\mathcal{G}_n)) \\ &\simeq \lim_{n\in\mathbb{N}} \mathrm{Map}_{\mathrm{Coh}^-(\mathfrak{Y}_n)}(f_{n+}(\mathcal{F}_n), \mathcal{G}_n) \\ &\simeq \mathrm{Map}_{\mathrm{Coh}^-(\mathfrak{Y})}(f_+(\mathcal{F}), \mathcal{G}), \end{aligned}$$

which completes the proof. □

Corollary 5.3. *Let $f \colon X \to Y$ be a proper map between derived analytic spaces. Assume that f is flat. Then the functor*

$$f^* \colon \mathrm{Coh}^-(Y) \to \mathrm{Coh}^-(X)$$

admits a left adjoint

$$f_+ \colon \mathrm{Coh}^-(X) \to \mathrm{Coh}^-(Y).$$

Proof. Using Theorem 4.1, we can choose a proper flat formal model $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{Y}$ for f. Thanks to Proposition 5.2, we have a well-defined functor

$$\mathfrak{f}_+ \colon \mathrm{Coh}^-(\mathfrak{X}) \longrightarrow \mathrm{Coh}^-(\mathfrak{Y}).$$

We claim that it restricts to a functor

$$\mathfrak{f}_+ \colon \mathrm{Coh}^-_{\mathrm{nil}}(\mathfrak{X}) \longrightarrow \mathrm{Coh}^-_{\mathrm{nil}}(\mathfrak{Y}).$$

Using Corollary 3.23, it is enough to prove that

$$\mathfrak{f}_+(\mathcal{F})^{\mathrm{loc}} \simeq 0.$$

Extending $\mathfrak{f}_+$ to a functor $\mathfrak{f}_+ \colon \mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{X})) \to \mathrm{Ind}(\mathrm{Coh}^-(\mathfrak{Y}))$, we see that

$$\mathfrak{f}_+(\mathcal{F})^{\mathrm{loc}} \simeq \mathfrak{f}_+(\mathcal{F}^{\mathrm{loc}}) \simeq 0.$$

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Using Corollary 3.7, we get a well-defined functor

$$f_+ \colon \mathrm{Coh}^-(X) \longrightarrow \mathrm{Coh}^-(Y).$$

We only have to prove that it is left adjoint to f^*. Let $\mathcal{F} \in \mathrm{Coh}^-(X)$ and $\mathcal{G} \in \mathrm{Coh}^-(Y)$. Choose a formal model $\mathfrak{F} \in \mathrm{Coh}^-(\mathfrak{X})$. Then unraveling the construction of f_+, we find a canonical equivalence

$$f_+(\mathcal{F}) \simeq \mathfrak{f}_+(\mathfrak{F})^{\mathrm{rig}}.$$

We now have the following sequence of natural equivalences:

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Coh}^-(Y)}(f_+(\mathcal{F}), \mathcal{G}) &\simeq \mathrm{Map}_{\mathrm{Coh}^-(Y)}((\mathfrak{f}_+(\mathfrak{F}))^{\mathrm{rig}}, \mathfrak{G}^{\mathrm{rig}}) \\
&\simeq \mathrm{Map}_{\mathrm{Coh}^-(\mathfrak{X})}(\mathfrak{f}_+(\mathfrak{F}), \mathfrak{G}) \otimes_{k^\circ} k \quad \text{by Corollary 3.16} \\
&\simeq \mathrm{Map}_{\mathrm{Coh}^-(\mathfrak{X})}(\mathfrak{F}, \mathfrak{f}^*\mathfrak{G}) \otimes_{k^\circ} k \\
&\simeq \mathrm{Map}_{\mathrm{Coh}^-(\mathfrak{X})}(\mathfrak{F}^{\mathrm{rig}}, (\mathfrak{f}^*\mathfrak{G})^{\mathrm{rig}}) \quad \text{by Corollary 3.16} \\
&\simeq \mathrm{Map}_{\mathrm{Coh}^-(\mathfrak{X})}(\mathcal{F}, f^*\mathcal{G}).
\end{aligned}$$

The proof is therefore complete. □

Corollary 5.4. *Let $f \colon X \to Y$ be a proper and flat map between derived analytic spaces. Let $p \colon Z \to Y$ be any other map and consider the pullback square*

$$\begin{array}{ccc}
W & \xrightarrow{q} & X \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
Z & \xrightarrow{p} & Y.
\end{array}$$

Then for any $\mathcal{F} \in \mathrm{Coh}^-(X)$, the canonical map

$$g_+(q^*(\mathcal{F})) \longrightarrow p^*(f_+(\mathcal{F}))$$

is an equivalence.

Proof.
Using Theorem 4.1, we find a proper and flat formal model $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{Y}$ for $f \colon X \to Y$. Choose a formal model $\mathfrak{p} \colon \mathfrak{Z} \to \mathfrak{Y}$ for $p \colon Z \to Y$, and form the pullback square

$$\begin{array}{ccc}
\mathfrak{W} & \xrightarrow{\mathfrak{q}} & \mathfrak{X} \\
\downarrow{\scriptstyle \mathfrak{g}} & & \downarrow{\scriptstyle \mathfrak{f}} \\
\mathfrak{Z} & \xrightarrow{\mathfrak{p}} & \mathfrak{Y}.
\end{array}$$

Choose also a formal model $\mathfrak{F} \in \mathrm{Coh}^-(\mathfrak{X})$ for $\mathcal{F}$. It is then enough to prove that the canonical map

$$\mathfrak{g}_+(\mathfrak{q}^*(\mathfrak{F})) \longrightarrow \mathfrak{p}^*(\mathfrak{f}_+(\mathfrak{F}))$$

is an equivalence. This follows at once by [9, Proposition 6.4.5.4(2)]. □

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6. Representability of $\mathbf{R}\mathrm{Hilb}(X)$

Let $p\colon X \to S$ be a proper and flat morphism of underived k-analytic spaces. We define the functor

$$\mathrm{RHilb}(X/S)\colon \mathrm{dAfd}_S^{\mathrm{op}} \longrightarrow \mathcal{S}$$

by sending $T \to S$ to the space of diagrams

$$\begin{array}{ccc} Y & \xhookrightarrow{\;i\;} & T \times_S X \\ & {\scriptstyle q_T}\searrow \quad \swarrow{\scriptstyle p_T} & \\ & T & \end{array} \tag{6.1}$$

where i is a closed immersion of derived k-analytic spaces, and q_T is flat.

Proposition 6.1. *Keeping the above notation and assumptions,* $\mathrm{RHilb}(X/S)$ *admits a global analytic cotangent complex.*

Proof. Let $x\colon T \to \mathbf{R}\mathrm{Hilb}(X/S)$ be a morphism from a derived k-affinoid space $T \in \mathrm{dAfd}_S$. It classifies a diagram of form (6.1). Unraveling the definitions, we see that the functor

$$\mathrm{Der}^{\mathrm{an}}_{\mathbf{R}\mathrm{Hilb}(X/S),x}(T;-)\colon \mathrm{Coh}^-(T) \longrightarrow \mathbf{R}\mathrm{Hilb}(X/S)$$

can be explicitly written as

$$\mathrm{Der}^{\mathrm{an}}_{\mathbf{R}\mathrm{Hilb}(X/S),x}(T;\mathcal{F}) \simeq \mathrm{Map}_{\mathrm{Coh}^-(Y)}(\mathbb{L}^{\mathrm{an}}_{Y/T\times_S X}, q_T^*(\mathcal{F})).$$

Since $q_T\colon Y \to T$ is proper and flat, Corollary 5.3 implies the existence of a left adjoint $q_{T+}\colon \mathrm{Coh}^-(Y) \to \mathrm{Coh}^-(T)$ for q_T^*. Moreover, [16, Corollary 5.40] implies that $\mathbb{L}^{\mathrm{an}}_{Y/T\times_S X} \in \mathrm{Coh}\geqslant 0(Y)$. Therefore, we find

$$\mathrm{Der}^{\mathrm{an}}_{\mathbf{R}\mathrm{Hilb}(X/S),x}(T;\mathcal{F}) \simeq \mathrm{Map}_{\mathrm{Coh}^-(T)}(q_{T+}(\mathbb{L}^{\mathrm{an}}_{Y/T\times_S X}), \mathcal{F}),$$

and therefore $\mathrm{RHilb}(X/S)$ admits an analytic cotangent complex at x. Using Corollary 5.4, we see that it admits as well a global analytic cotangent complex. □

Proposition 6.2 (Conrad–Gabber; see [7, Theorem 5.3.2]). *Keeping the above notation and assumptions, the (underived) functor of points* $\mathrm{Hilb}(X/S)$ *is representable by a k-analytic space.*

Proof. In [7], this result is deduced from the representability of the Quot functor. Let $Y \to S$ be a separated map of k-analytic spaces and let $\mathcal{F} \in \mathrm{Coh}^{\heartsuit}(Y)$. Then $\mathrm{Quot}(Y/S, \mathcal{F})$ is proven in Theorem B.1.2 in loc. cit. to be representable by an S-separated k-analytic space. The proof goes in three major steps:

(1) Let $\mathfrak{Y} \to \mathfrak{S}$ be a separated formal model for $Y \to S$ and let $\mathfrak{F} \in \mathrm{Coh}^{\heartsuit}(\mathfrak{Y})$ be a formal model for $\mathcal{F}$. Then it is shown that $\mathrm{Quot}(\mathfrak{Y}/\mathfrak{S}, \mathfrak{F})$ is representable by formal algebraic spaces. This is done by considering the reductions modulo the powers of the pseudo-uniformizers, where the result of Artin applies.

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(2) It is next proven in §B.2.2 that the generic fiber of a quasi-compact and quasi-separated formal algebraic space is a (quasi-compact and quasi-separated) k-analytic space. The main idea is attributed to M. Temkin, and it relies on using the Raynaud–Gruson theory to prove that up to an admissible blowup, every quasi-compact and quasi-separated formal algebraic space is a formal scheme.

(3) It is finally proven in Proposition B.4.1 that $\mathrm{Quot}(\mathfrak{Y}/\mathfrak{S}, \mathfrak{F})^{\mathrm{rig}}$ represents the functor $\mathrm{Quot}(Y/S, \mathcal{F})$. This is once again achieved via the Raynaud–Gruson theory.

□

Theorem 6.3. *Keeping the above notation and assumptions,* $\mathbf{R}\mathrm{Hilb}(X/S)$ *is a derived* k*-analytic space.*

Proof. We only need to check the hypotheses of [16, Theorem 7.1]. The representability of the truncation is guaranteed by Proposition 6.2. The existence of the global analytic cotangent complex has been dealt with in Proposition 6.1. Convergence and infinitesimal cohesiveness are straightforward checks. The theorem follows. □

As the second concluding application, let us mention that the theory of the plus pushforward developed in this paper allows us to remove the lci assumption in [17, Theorem 8.6].

Theorem 6.4. *Let* S *be a rigid* k*-analytic space. Let* X, Y *be rigid* k*-analytic spaces over* S*. Assume that* X *is proper and flat over* S *and that* Y *is separated over* S*. Then the* ∞*-functor* $\mathbf{Map}_S(X, Y)$ *is representable by a derived* k*-analytic space separated over* S*.*

Proof. The same proof of [17, Theorem 8.6] applies. It is enough to observe that Corollaries 5.3 and 5.4 allow us to prove Lemma 8.4 in loc. cit. by removing the assumption of $Y \to S$ being locally of finite presentation. □

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References

1. J. António, Moduli of p-adic representations of a profinite group, preprint, 2017, arXiv:1709.04275.
2. J. António, p-adic derived formal geometry and derived Raynaud localization theorem, preprint, 2018, arXiv:1805.03302.
3. J. António, Moduli of ℓ-adic pro-étale local systems for smooth non-proper schemes, preprint, 2019, arXiv:1904.08001.
4. S. Bosch, *Lectures on Formal and Rigid Geometry*, Lecture Notes in Mathematics, (Springer International Publishing, 2014). ISBN: 978-3-319-04416-3; 978-3-319-04417-0.
5. S. Bosch and W. Lütkebohmert, Formal and rigid geometry. II. Flattening techniques, *Math. Ann.* **296**(3) (1993), 403–429.

6. S. Bosch, W. Lütkebohmert and M. Raynaud, Formal and rigid geometry. IV. The reduced fibre theorem, *Invent. Math.* **119**(2) (1995), 361–398.
7. B. Conrad and O. Gabber, Spreading-out for rigid-analytic families, preprint, 2016.
8. B. Hennion, M. Porta and G. Vezzosi, Formal gluing along non-linear flags, preprint, 2016, arXiv:1607.04503.
9. J. Lurie, Spectral algebraic geometry, available at http://www.math.harvard.edu/~lurie/papers/SAG-rootfile.pdf, 2018.
10. J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, Volume 170 (Princeton University Press, Princeton, NJ, 2009).
11. J. Lurie, DAG IX: Closed immersions, preprint, 2011.
12. J. Lurie, DAG V: Structured spaces, preprint, 2011.
13. J. Lurie, DAG VII: Spectral schemes, preprint, 2011.
14. M. Porta and T. Y. Yu, Derived non-archimedean analytic spaces, preprint, 2016, arXiv:1601.00859. To appear in *Selecta Mathematica*: MR 3782411.
15. M. Porta and T. Y. Yu, Higher analytic stacks and GAGA theorems, *Adv. Math.* **302** (2016), 351–409.
16. M. Porta and T. Y. Yu., Representability theorem in derived analytic geometry, preprint, 2017, arXiv:1704.01683. To appear in *J. Eur. Math. Soc.*
17. M. Porta and T. Y. Yu, Derived Hom spaces in rigid analytic geometry, preprint, 2018, arXiv:1801.07730.

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