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CHROMATIC SUMS FOR ROOTED PLANAR TRIANGULATIONS: THE CASES $\lambda = 1$ AND $\lambda = 2$

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Summary. In this paper we derive a functional equation whose solution would give the sum of the chromatic polynomial $P(M, \lambda)$ over certain classes of rooted planar maps M called "triangulations" and "near-triangulations". For an integral colour-number λ this sum is the number of λ-coloured rooted maps of the kind considered, but the sum can also be discussed for non-integral λ.

The cases $\lambda = 0$, 1 and 2 are trivial. However it is an interesting and non-trivial problem to determine the sum of $(\partial/\partial\lambda)P(M, \lambda)$, at $\lambda = 1$ and $\lambda = 2$, for our rooted planar maps, and a graph-theoretical interpretation of this derivative is available, at least in the case $\lambda = 1$. The problem is solved in the last two sections of the paper.

1. Planar near-triangulations. A *planar map* M is the figure obtained by embedding a finite connected graph G, having at least one vertex, in the Euclidean plane. The graph separates its complement in the plane into a finite number of disjoint connected domains called the *faces* of M. The edges and vertices of G are called the *edges* and *vertices* of M respectively. An edge and a vertex are said to be *incident* in M if they are incident in G. An edge or vertex is said to be incident with a face if it is contained in the boundary of that face.

In a planar map one face is unbounded and all the other faces, if there are any, are bounded. The bounded and unbounded faces are the *inner* and *outer* faces of M respectively.

A planar map M is a *near-triangulation* if it has at least one edge, its graph G is non-separable, and every inner face is a triangle, in the sense that its boundary is a simple closed curve made up of exactly three edges of G. If in addition the outer face is a triangle we say simply that M is a *triangulation* (of the plane). We note that a near-triangulation can have no *loop*, that is no edge whose two ends coincide.

We note the special case of the *link-map*. Here G consists of a single link, with its two ends. There are no inner faces. With the author's definition of non-separability [**4**] the link-map satisfies the definition of a near-triangulation. We distinguish it from the other near-triangulations by calling it *degenerate*.

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In any non-degenerate near-triangulation there are three or more edges, the boundary of the outer face is a simple closed curve determined by a circuit of G, and there is at least one inner face.

We *root* a near-triangulation M by choosing some directed edge R in the boundary of the outer face F as the *root*. The negative end V of the root is the *root-vertex*, the undirected edge corresponding to R is the *root-edge* E, and F may be called the *root-face*. Two rooted near-triangulations M_1 and M_2 are *combinatorially equivalent* if there is a homeomorphism of the plane onto itself which transforms M_1 into M_2, with preservation of the root-vertex, root-edge and root-face. In what follows we do not distinguish between combinatorially equivalent near-triangulations. We are thus able to say that the number of rooted near-triangulations with a given number of inner faces is finite.

2. The chromatic equation for near-triangulations. Let M be a rooted near-triangulation. We write $m(M)$ for the valency of its root-face. This *valency* is the number of incident edges, isthmuses being counted twice. Thus $m(M) = 2$ for the link-map. Since we have not excluded "multiple joins" there are other near-triangulations with $m(M) = 2$. But in every case $m(M) \geqq 2$.

We write $n(M)$ for the valency of the root-vertex of M, that is the number of incident edges. For the link-map we have $n(M) = 1$, but in every other case $n(M) \geqq 2$.

We write $t(M)$ for the number of inner faces of M. This is zero for the link-map, but non-zero in every other case.

Finally we write $P(M, \lambda)$ for the chromial of M, that is the chromatic polynomial of the corresponding graph G. When λ is a positive integer this is the number of ways of colouring the vertices of M in λ colours so that no two of the same colour are joined by an edge. It is a polynomial in λ, and as such it can be extended to all real or complex numbers λ.

We introduce the generating series

$$(1)\qquad \begin{aligned} g &= g(x, y, z, \lambda) \\ &= \sum_M x^{m(M)} y^{n(M)} z^{t(M)} P(M, \lambda). \end{aligned}$$

Here the sum is over all rooted near-triangulations M. We can explain g as follows. If λ is a positive integer then the coefficient of $x^m y^n z^t$ in g is the number of λ-coloured rooted near-triangulations M with $m(M) = m$, $n(M) = n$ and $t(M) = t$.

We now write

$$(2)\qquad q = q(x, z, \lambda) = g(x, 1, z, \lambda).$$

We also write $l = l(y, z, \lambda)$ for the coefficient of x^2 in g, and we put $h = h(z, \lambda) = l(1, z, \lambda)$. Evidently h is the coefficient of x^2 in q. The coefficients in q give the sums of chromials for rooted near-triangulations with specified values of $m(M)$ and $t(M)$. Those of l give such sums for rooted near-triangulations with $m(M) = 2$, and with specified values of $n(M)$ and $t(M)$.

In the power series h each contributing non-degenerate rooted map M has an outer digon. It can be transformed into a true triangulation T by deleting the edge other than E in this digon. The coefficient of z^{2k} in h is thus the sum of the chromials of the rooted triangulations with exactly $2k$ faces, provided that $k > 0$. It is clear that only even powers of z can occur in h.

We proceed to determine a functional equation for the generating series g. The contribution of the link-map to the sum (1) is

$$x^2y\lambda(\lambda - 1).$$

Consider however a non-degenerate rooted near-triangulation M. Its root-edge E is incident with exactly one inner triangle T. The vertices of T can be listed as V, the other end W of E, and a third vertex X. Let M' be the planar map formed from M by deleting the edge E, that is by fusing this edge with T and F to form a new outer face F'. Let M'' be the planar map formed from M by contracting E into a single vertex V'' and correspondingly contracting the triangle T into a single segment with ends V'' and X. We discuss the question of rooting M' and M'' later. We observe now that by a fundamental property of chromials

$$P(M, \lambda) = P(M', \lambda) - P(M'', \lambda). \tag{3}$$

(See for example [**5**, Section 1].) The chromial of a map, whether rooted or unrooted, is to be identified with that of the corresponding graph.

We can now write

$$g = x^2y\lambda(\lambda - 1) + \sum' - \sum'', \tag{4}$$

where

$$\sum{}' = \sum_M x^{m(M)}y^{n(M)}z^{t(M)}P(M', \lambda), \tag{5}$$

$$\sum{}'' = \sum_M x^{m(M)}y^{n(M)}z^{t(M)}P(M'', \lambda) \tag{6}$$

and the summations are over all non-degenerate rooted near-triangulations M.

It is convenient to split $\sum'$ into two partial sums $\sum_1'$ and $\sum_2'$ as follows. Let R_1 be the class of all non-degenerate rooted near-triangulations M such that X is not incident with F, and let R_2 be the class of those for which X is incident with F. We denote by $\sum'_1$ and $\sum'_2$ the sum $\sum'$ restricted to members of R_1 and R_2 respectively. Thus

$$\sum{}' = \sum{}'_1 + \sum{}'_2. \tag{7}$$

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The case $M \in R_1$ is illustrated in Figure 1.

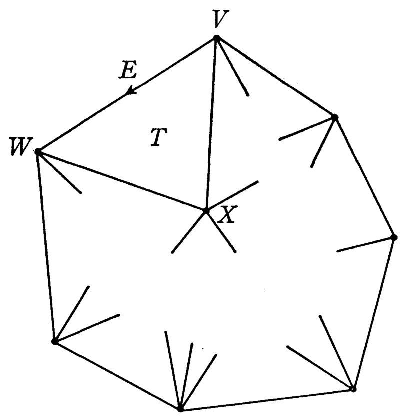

FIGURE 1

In this case M' is a near-triangulation. We take V as its root-vertex, and the edge VX incident with its outer face as its root-edge. Thus the rooted map M' is uniquely defined by M. It satisfies $m(M') \geqq 3$. Moreover any rooted near-triangulation N such that $m(N) \geqq 3$ can appear as M'. For, let V be the root-vertex of N and X the other end of the root-edge. Let W be the next vertex after X in the cyclic sequence $(V, X, \ldots)$ of vertices in the boundary of the outer face of N. We note that W and V are distinct. We can obtain a rooted near-triangulation M such that $M' = N$ by adjoining a new root-edge E, crossing the outer face, with ends V and W, and retaining V as root-vertex. Moreover this construction gives the only such map M.

Expressing $\sum'_1$ in terms of the possible maps N we find that

$$\sum{}'_1 = x^{-1}yz \sum_N x^{m(N)} y^{n(N)} z^{t(N)} P(N, \lambda) \tag{8}$$
$$= x^{-1}yz(g - x^2 l),$$

since the sum is over all rooted near-triangulations N except those for which $m(N) = 2$.

The case $M \in R_2$ is shown in Figure 2. Here M' is best regarded as a combination of two rooted near-triangulations M_1 and M_2, whose graphs have only the vertex X in common. V is a vertex of M_1 and W a vertex of M_2. For M_1 we take V to be the root-vertex, and the edge VX incident with the outer face to be the root-edge. For M_2 we take X to be the root-vertex, and the edge XW incident with the outer face to be the root-edge. Thus M_1 and M_2 are uniquely determined as rooted near-triangulations, possibly degenerate, by M. Moreover if M_1 and M_2 are chosen arbitrarily as rooted near-triangulations, then there is a unique rooted triangulation M giving rise to them by the above construction.

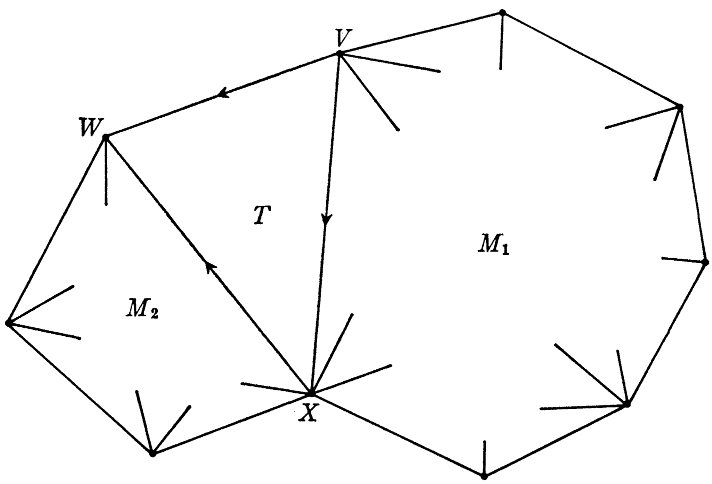

FIGURE 2

We note that, by the elementary properties of chromials,

$$(9) \qquad P(M', \lambda) = \lambda^{-1}P(M_1, \lambda)P(M_2, \lambda).$$

We can now express $\sum'_2$ in terms of M_1 and M_2 as follows.

$$\sum{}'_2 = \sum_{(M_1,M_2)} \{x^{m(M_1)+m(M_2)-1}y^{n(M_1)+1}z^{t(M_1)+t(M_2)+1}\lambda^{-1}P(M_1, \lambda)P(M_2, \lambda)\}$$

$$= \lambda^{-1}x^{-1}yz\left\{\sum_{M_1} x^{m(M_1)}y^{n(M_1)}z^{t(M_1)}P(M_1, \lambda)\right\}\left\{\sum_{M_2} x^{m(M_2)}z^{t(M_2)}P(M_2, \lambda)\right\}.$$

Thus,

$$(10) \qquad \sum{}'_2 = \lambda^{-1}x^{-1}yzgq.$$

It remains to discuss the sum $\sum''$. To begin with we may remark that in M the vertices V and W might be joined by an edge A other than E. When this happens A is a loop in M'' and so M'' is not a near-triangulation. However a map with a loop has a zero chromial. (The two ends of a loop are coincident, and so cannot have distinct colours.) Maps M of this kind make a zero contribution to $\sum''$ and can now be ignored. We can regard $\sum''$ as a sum over those non-degenerate rooted near-triangulations M in which V and W are joined by no edge other than E. For such maps M'' is always a near-triangulation. We adopt the following convention for rooting it. Let Y be the vertex of M coming next after W in the cyclic sequence $(V, W, \ldots)$ of vertices in the boundary of the outer face F of M. By the restriction now imposed on M the

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vertices V and Y are distinct. In M'' we take V'' as the root-vertex, and the edge $V''Y$ incident with the outer face as the root-edge. (See Figure 3.)

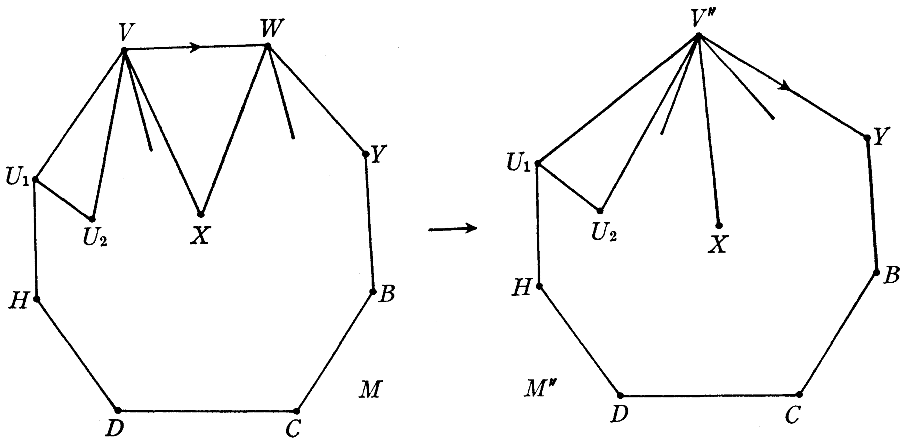

FIGURE 3

We must now consider the following problem. If M'' is given as an arbitrarily chosen rooted near-triangulation, what are the possible corresponding rooted near-triangulations M? Evidently any such M is to be obtained by reversing the construction of Figure 3. Some edge XV'' incident with V'' is to be expanded into a triangle XVW. Any one of the $n(M'')$ edges incident with V'' can be thus expanded, and we conclude that M'' has exactly $n(M'')$ corresponding maps M. If we expand the edge U_1V'' of Figure 3 we obtain a map M with $n(M) = 2$, if we expand U_2V'' we find that $n(M) = 3$, and so on. Thus $n(M)$ takes all values from 2 to $n(M'') + 1$.

Expressing $\sum''$ in terms of the arbitrary rooted near-triangulations M'' we find that

$$\begin{aligned}\sum{}'' &= \sum_{M''} \{x^{m(M'')+1}(y^2 + y^3 + \ldots + y^{n(M'')+1})z^{t(M'')+1}P(M'', \lambda)\} \\ &= xy^2z\sum_{M''} x^{m(M'')}\left[\frac{y^{n(M'')} - 1}{y - 1}\right]z^{t(M'')}P(M'', \lambda) \\ &= \frac{xy^2z(g - q)}{y - 1}.\end{aligned}$$

It is convenient to introduce an operator Δ, operating on functions $F(y)$ of y, defined as follows:

$$\Delta F(y) = (F(y) - F(1))/(y - 1). \tag{11}$$

Using this operator we can write

$$\sum{}'' = xy^2z \, \Delta \, g. \tag{12}$$

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Substituting our expressions for $\sum'_1$, $\sum'_2$ and $\sum''$ in (7) and (4) we obtain the following identity

$$xg = x^3y\lambda(\lambda - 1) + \lambda^{-1}yzgq + yz(g - x^2l) - x^2y^2z \triangle g. \tag{13}$$

We call this the *chromatic equation for near-triangulations.*

3. Uniqueness of the solution. Let us choose a particular value of λ and consider (13) as an equation for a formal power series g in the indeterminates x, y and z. The solution g has then to be a power series without negative indices. We know that one such solution exists; it is our generating series for chromatic sums. We can also show that this solution is unique.

To do this we first agree to write an arbitrary solution g as

$$g = g_0 + zg_1 + z^2g_2 + \ldots,$$

where the g_j are power series in x and y, and to use an analogous notation for the corresponding series q and l.

Considering the power z^0 in (13) we find that $xg_0 = x^3y\lambda(\lambda - 1)$, that is

$$g_0 = x^2y\lambda(\lambda - 1).$$

Thus g_0 is uniquely determined. From it we deduce that

$$\begin{aligned} q_0 &= x^2\lambda(\lambda - 1), \\ l_0 &= y\lambda(\lambda - 1), \\ \triangle g_0 &= x^2\lambda(\lambda - 1). \end{aligned}$$

Now we pick out the coefficients of z^1 in (13).

$$\begin{aligned} xg_1 &= \lambda^{-1}yg_0q_0 + y(g_0 - x^2l_0) - x^2y^2 \triangle g_0 \\ &= x^4y^2\lambda(\lambda - 1)^2 - x^4y^2\lambda(\lambda - 1), \\ g_1 &= x^3y^2\lambda(\lambda - 1)(\lambda - 2). \end{aligned}$$

Thus g_1 is uniquely determined, and with it q_1, l_1 and $\triangle g_1$. Now we can pick out the coefficients of z^2 in (13) and so determine g_2, and so on. In general if we find the sequence (g_j) to be uniquely determined up to, say, g_k, then by picking out the terms in z^{k+1} in (13) we obtain an equation that gives g_{k+1} uniquely. The uniqueness of the complete solution g follows by induction.

We can recognize g_1 as the contribution to g of the rooted triangle, a rooted triangulation with $m(M) = 3$, $n(M) = 2$ and $t(M) = 1$. It satisfies $P(M, \lambda) = \lambda(\lambda - 1)(\lambda - 2)$.

In a sense the above process of recursion gives a solution of the chromatic equation (13). We can compute the coefficients in g, in the order of ascending powers of z. Such computations have been carried out by D. H. Younger at Waterloo. From a theoretical point of view however a solution is not satisfactory unless it enables us to make at least asymptotic estimates of coefficients, such as those in h, for arbitrary large values of $t(M)$. In this sense there is as yet no satisfactory solution of (13) for general λ.

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There are however some special values of λ for which at least the series h can be completely and explicitly determined. These values are 0, 1, 2, $\tau + 1$ and 3, where τ is the golden ratio. The author has also obtained some partial results in the case $\lambda = 4$ [**6**]. The sequence $(4, 0, 1, 2, \tau + 1, 3)$ is now becoming familiar to students of chromials as the beginning of the infinite sequence of "Beraha numbers". The nth Beraha number is

$$2 + 2\cos(2\pi/n).$$

S. Beraha has pointed out in lectures an apparent tendency for the chromials of triangulations to have zeros close to these numbers.

The solution of the chromatic equation in the cases $\lambda = 0$ and $\lambda = 1$ is trivial; g is identically zero. This is to be expected since no near-triangulation can be coloured in fewer than two colours.

The case $\lambda = 2$ is almost equally trivial. Here $g = 2x^2y$ since the link-map is the only near-triangulation with a 2-colouring. It is an easy but interesting exercise to verify this result by substituting in (13).

We may also claim $\lambda = \infty$ as a special value of λ for which h and q have been determined. We interpret the problem in this case as that of summing the leading coefficient of $P(M, \lambda)$ over the appropriate classes of rooted near-triangulations M. Since this coefficient always has the value 1 the problem is that of enumerating the rooted near-triangulations. This problem is solved for triangulations, in their dual form of trivalent maps, in [**2**]. The function corresponding to q is determined by R. C. Mullin in [**1**].

It is convenient to note here some limitations on the coefficients in g, valid for all λ. Writing $e(M)$ for the number of edges of a near-triangulation M we have

$$m(M) + 3t(M) = 2e(M), \tag{14}$$

$$m(M) \equiv t(M) \bmod 2. \tag{15}$$

From (14) and the Euler polyhedron formula we can deduce also that

$$m(M) \leqq t(M) + 2. \tag{16}$$

Hence the coefficient of $x^m y^n z^t$ in g must be zero unless m and t have the same parity and $2 \leqq m \leqq t + 2$.

4. The case $\lambda = 1$. We have seen that g is identically zero when $\lambda = 1$. At this value of λ we can however study g' instead of g, where the prime denotes differentiation with respect to λ. At $\lambda = 1$ the function g' is non-trivial, yet its coefficients can be determined explicitly for all powers of x, y and z. We shall be concerned with the evaluation of these coefficients in the remainder of this Section.

The theoretical justification for this investigation lies mainly in the hope that by amassing information about g and its derivatives we may be helped towards a complete solution of (13). Nevertheless g', $\lambda = 1$, does have its own graph-theoretical significance, as we proceed to explain.

In [**3**] we study the spanning trees T of a connected graph G. We suppose the edges of G to be enumerated as $A_1, A_2, \ldots, A_m$. We denote the number of vertices of G by α_0.

Consider an edge A_j of a spanning tree T. When it is deleted T falls apart into two components C and D. It may happen that all the remaining edges of G with one end in C and one in D have suffixes $<j$. If so we say that A_j is *internally active* in T. We denote by $r(T)$ the number of edges of T that are internally active.

Now consider an edge A_k not in T. The two ends of A_k are joined by a unique arc in T (degenerating to a single vertex if A_k is a loop). It may happen that each edge in this arc has a suffix $<k$. If so we say that A_k is *externally active* in T. We denote by $s(T)$ the number of edges of G, not in T, that are externally active in T.

Considering the edge with the greatest suffix we see that $r(T)$ and $s(T)$ cannot both be zero.

It follows from the theory of [**3**] that

$$P(G, \lambda) = -(-1)^{\alpha_0}\lambda \sum_T (1-\lambda)^{r(T)}, \tag{17}$$

where the summation is over all spanning trees T of G such that $s(T) = 0$. The formula is valid for an arbitrary initial enumeration of the edges. Differentiating and setting $\lambda = 1$ we obtain

$$P'(G, 1) = (-1)^{\alpha_0} J_1(G), \tag{18}$$

where $J_1(G)$ is the number of spanning trees T of G such that $r(T) = 1$ and $s(T) = 0$.

If G is the graph of a near-triangulation M the equation can be written as

$$P'(M, 1) = -(-1)^{(m(M)+t(M))/2} J_1(G). \tag{19}$$

(We use (14) and the Euler polyhedron formula.) Thus the coefficients in $g'(x, y, z, 1)$ sum $J_1(G)$ over the appropriate families of rooted near-triangulations.

Differentiating (13) with respect to λ we obtain

$$xg' = x^3y(2\lambda - 1) - \lambda^{-2}yzgq + \lambda^{-1}yz(gq' + g'q) + yz(g' - x^2l') - x^2y^2z \triangle g'. \tag{20}$$

From now on we assume that λ is set equal to 1. For this value of λ (20) reduces to

$$xg' = x^3y + yz(g' - x^2l') - x^2y^2z \triangle g', \tag{21}$$

since g, q and l are identically zero for $\lambda = 1$. Write G_j, Q_j, L_j and H_j for the coefficient of z^j in g', q', l' and h' respectively (with of course $\lambda = 1$). These

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coefficients become zero for negative values of j. Considering coefficients of z^{j+1}, where $j \geqq 0$, in (21) we find that

$$xG_{j+1} = yG_j - x^2yL_j - x^2y^2 \triangle G_j. \tag{22}$$

This is valid for all non-negative integers j. Moreover by considering coefficients of z^0 in (21) we find that

$$G_0 = x^2y, \qquad L_0 = y, \qquad \triangle G_0 = x^2. \tag{23}$$

Next we write $j = 0$ in (22) and obtain

$$\begin{aligned} xG_1 &= yG_0 - x^2yL_0 - x^2y^2 \triangle G_0 = -x^4y^2, \\ G_1 &= -x^3y^2, L_1 = 0, \triangle G_1 = -x^3(y+1). \end{aligned} \tag{24}$$

Let us consider the coefficient of x^r in G_t. It is zero unless r and t are non-negative, have the same parity, and satisfy $2 \leqq r \leqq t + 2$, by (15) and (16). Thus non-negative values of the coefficient can occur only for values of r satisfying $0 \leqq r = t + 2 - 2k$, where k is an integer such that $0 \leqq k \leqq t/2$.

If the coefficient of x^{t+2-2k} in G_t is non-zero its sign is given by $(-1)^{t+k}$, by (19). Accordingly we denote this coefficient by

$$(-1)^{t+k}G_{t,k}.$$

Here $G_{t,k}$ is a power-series in y, and by (19) its coefficients are non-negative. Actually it is a polynomial in y. For, in any near-triangulation M we have

$$1 \leqq n(M) \leqq t(M) + 1. \tag{25}$$

The first equality holds only if M is degenerate and the second equality holds if and only if the root-vertex is incident with every face. Hence $G_{t,k}$ is a polynomial in y with degree not exceeding $t + 1$, and with no constant term. Moreover if $t > 0$ then $G_{t,k}$ has no term of the first degree in y.

For $t = 0$, 1 the polynomials $G_{t,k}$ (with $0 \leqq k \leqq t/2$) can be read off from (23) and (24). They are as follows:

$$G_{0,0} = y, \qquad G_{1,0} = y^2. \tag{26}$$

Considering the coefficients of x^{t+2-2k} in (22) we obtain the following recursion formula.

$$G_{t+1,k+1} = yG_{t,k} - y\delta_{t,2k}G_{2k,k} + y^2 \triangle G_{t,k+1}. \tag{27}$$

This is valid for $t \geqq 0$ and all integers k, positive, negative or zero. Putting $k = -1$, $k = 0$ in it we find

$$G_{2,0} = y^3 + y^2, \qquad G_{2,1} = y^3. \tag{28}$$

Let us denote by $A_u(t, k)$ the polynomial

$$y \sum_{r=0}^{t-u} \binom{2t-k-r-1}{t-r-u} y^r$$

in y, where $u = 0$, 1 or 2, and the integers k and t satisfy $t \geqq u$ and $k \leqq t + u - 1$. We note the identity

$$yA_u(t, k) = A_u(t + 1, k + 1) + yB_u(t, k), \tag{29}$$

where $B_u(t, k)$ is a constant. We observe also that $A_u(t, k)$ is the coefficient of θ^{t-u} in

$$\frac{y}{(1 - \theta)^{t-k+u}(1 - \theta y)}.$$

Accordingly $\triangle A_u(t, k)$ is the coefficient of θ^{t-u} in

$$\frac{(y/(1 - \theta y)) - (1/(1 - \theta))}{(y - 1)(1 - \theta)^{t-k+u}} = \frac{1}{(1 - \theta)^{t-k+u+1}(1 - \theta y)}.$$

Thus,

$$y \triangle A_u(t, k) = A_u(t, k - 1). \tag{30}$$

We use (29) and (30) to establish explicit formulae for $G_{t,0}$ and $G_{t,k}$.

4.1. THEOREM. *For* $t \geqq 1$, $G_{t,0} = A_0(t, 0) - A_1(t, 0)$.

Proof. From the definition of $A_u(t, k)$ we find that $A_0(1, 0) = y^2 + y$ and $A_1(1, 0) = y$. Hence the theorem holds for $t = 1$, by (26). Assume as an inductive hypothesis that the theorem holds whenever t is less than some integer $s > 1$, and consider the case $t = s$. Putting $k = -1$ in (27) we obtain

$$G_{s,0} = y^2 \triangle G_{s-1,0}.$$

Thus

$$G_{s,0} = y^2 \triangle (A_0(s - 1, 0) - A_1(s - 1, 0)),$$

by the inductive hypothesis,

$$= y(A_0(s - 1, -1) - A_1(s - 1, -1)),$$

by (30), and

$$= A_0(s, 0) - A_1(s, 0),$$

by (29) and the fact that in g and g' the coefficient of $z^s y$ is zero whenever $s > 0$. The theorem follows by induction.

Using the explicit formulae for $A_0(s, 0)$ and $A_1(s, 0)$ we can rewrite 4.1 as follows:

$$G_{t,0} = \sum_{r=1}^{t} \left\{\frac{(2t - r - 1)!}{(t - r)!t!}\right\} r y^{r+1}. \tag{31}$$

4.2. THEOREM. *If* $t \geqq 2$ *and* $0 \leqq k \leqq t/2$, *then*

$$G_{t,k} = \frac{(t - 2k + 1)}{k!} \left\{\frac{t!(A_0(t, k) - A_1(t, k))}{(t + 1 - k)!} + \frac{(t - 2)!kA_2(t, k)}{(t - k)!}\right\}.$$

Proof. When $k = 0$ the formula reduces to that of 4.1. The theorem is accordingly valid in this case.

If $t = 2$ and $k = 1$ the formula becomes

$$\begin{aligned} G_{2,1} &= A_0(2, 1) - A_1(2, 1) + A_2(2, 1) \\ &= (y + y^2 + y^3) - (2y + y^2) + y \\ &= y^3, \end{aligned}$$

i.e., it reduces to the second formula of (28).

These preliminary observations show that the theorem holds when $t = 2$. We assume as an inductive hypothesis that it is true whenever $t \leqq s$, for some integer $s \geqq 2$, and we consider the case $t = s + 1$. It now suffices to validate the formula for $G_{s+1,k+1}$, with $k \geqq 0$ and $s + 1 \geqq 2(k + 1)$, that is $s \geqq 2k + 1$. For the formula for $G_{s+1,0}$ is valid by 4.1.

Suppose first that $s = 2k + 1$. Then, by (27),

$$\begin{aligned} G_{2k+2,k+1} &= yG_{2k+1,k} \\ &= \frac{2y}{k!}\left\{\frac{(2k+1)!(A_0(2k+1, k) - A_1(2k+1, k))}{(k+2)!}\right. \\ &\quad \left. + \frac{(2k-1)!kA_2(2k+1, k)}{(k+1)!}\right\}, \end{aligned}$$

by the inductive hypothesis,

$$\begin{aligned} &= \frac{1}{(k+1)!}\left\{\frac{(2k+2)!(A_0(2k+2, k+1) - A_1(2k+2, k+1))}{(k+2)!}\right. \\ &\quad \left. + \frac{(2k)!(k+1)A_2(2k+2, k+1)}{(k+1)!}\right\}, \end{aligned}$$

by (29) and the fact that the coefficient of y^0 in g' is zero. Thus the theorem holds in this case.

It remains to consider the case $s \geqq 2k + 2$. Then, by (27),

$$\begin{aligned} G_{s+1,k+1} &= yG_{s,k} + y^2\Delta G_{s,k+1} \\ &= \frac{(s-2k+1)y}{k!}\left\{\frac{s!(A_0(s, k) - A_1(s, k))}{(s+1-k)!} + \frac{(s-2)!kA_2(s, k)}{(s-k)!}\right\} \\ &\quad + \frac{(s-2k-1)y}{(k+1)!}\left\{\frac{s!(A_0(s, k) - A_1(s, k))}{(s-k)!}\right. \\ &\quad \left. + \frac{(s-2)!(k+1)A_2(s, k)}{(s-k-1)!}\right\}, \end{aligned}$$

by (30) and the inductive hypothesis.

But

$$\frac{(s-2k+1)k}{k!(s-k)!}+\frac{(s-2k-1)(k+1)}{(k+1)!(s-k-1)!}$$
$$=\frac{(s-2k+1)k+(s-2k-1)(s-k)}{(s-k)!k!}$$
$$=\frac{s^2-(2k+1)s+2k}{k!(s-k)!}$$
$$=\frac{(s-2k)(s-1)}{k!(s-k)!}.$$

We can use this identity to simplify the coefficient of $A_2(s, k)$ in the preceding formula. To simplify the coefficient of $A_0(s, k) - A_1(s, k)$ we use the same identity with s replaced by $s + 2$ and k replaced by $k + 1$. Thus, with the help of (29), we obtain

$$G_{s+1,k+1}=\frac{(s-2k)}{(k+1)!}\left\{\frac{(s+1)!(A_0(s+1,k+1)-A_1(s+1,k+1))}{(s+1-k)!}+\frac{(s-1)!(k+1)A_2(s+1,k+1)}{(s-k)!}\right\}.$$

We observe that the theorem is valid in this case. The proof for the case $t = s + 1$ is now complete, and the general result follows by induction.

Using the explicit formula for $A_u(t, k)$ we can rewrite the result of 4.2 as follows.

$$(32)\quad G_{t,k}=\sum_{s=2}^{t+1}\left\{\frac{(t-2)!(2t-k-s)!(s-1)(t-2k+1)}{(t-s+1)!k!(t-k)!(t-k+1)!}\times(t^2-(2k+1)t+ks)y^s\right\},$$

$(t \geqq 2, 0 \leqq k \leqq t/2)$.

Now L_j is zero if j is odd, and if $j = 2k$, then $L_{2k} = (-1)^k G_{2k,k}$. So, by (32) we have

$$(33)\qquad (-1)^k L_{2k}=\sum_{s=3}^{2k+1}\left\{\frac{(2k-2)!(3k-s)!(s-1)(s-2)}{(2k-s+1)!(k-1)!k!(k+1)!}y^s\right\}$$

$(k \geqq 1)$.

Let $Q_{t,k}$ be the integer obtained by putting $y = 1$ in $G_{t,k}$. We observe that if y is set equal to 1, then $A_u(t, k)$ becomes the coefficient of θ^{t-u} in

$$1/(1-\theta)^{t-k+u+1},$$

that is,

$$\frac{(2t-k)!}{(t-u)!(t-k+u)!}.$$

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Substituting in 4.2 we find

$$Q_{t,k} = \frac{(t-2k+1)}{k!}\left\{\frac{(2t-k)!}{(t-k)!(t-k+1)!} - \frac{t(2t-k)!}{((t-k+1)!)^2} + \frac{k(2t-k)!}{(t-k)!(t-k+2)!}\right\}$$

$$= \frac{(2t-k)!(t-2k+1)}{k!(t-k+1)!(t-k+2)!}\{(t-k+1)(t-k+2) - t(t-k+2) + k(t-k+1)\},$$

(34) $$Q_{t,k} = \frac{(2t-k)!(t-2k+1)(t-2k+2)}{k!(t-k+1)!(t-k+2)!}.$$

This formula holds for $t \geqq 0$ and $0 \leqq k \leqq t/2$. For $t = 0$ and $t = 1$ we verify it from (23) and (24). To find Q_t we multiply by $(-1)^{t+k}x^{t+2-2k}$ and sum for $0 \leqq k \leqq t/2$. To find H_{2k} we then take the coefficient of x^2, that is we put $t = 2k$ in (34) and multiply by $(-1)^k$;

(35) $$H_{2k} = \frac{2(-1)^k(3k)!}{k!(k+1)!(k+2)!}.$$

Of course $H_j = 0$ when j is odd.

For large k we can apply Stirling's formula to (35) and obtain the following asymptotic expression.

(36) $$H_{2k} \sim \frac{\sqrt{3}(-27)^k}{\pi k^4}.$$

5. The case $\lambda = 2$**.** There is a similar theory for the case $\lambda = 2$, though as yet not for the case $\lambda = 0$.

When $\lambda = 2$ we have $g = 2x^2y$ and $q = 2x^2$. Hence Equation (20) takes the form

(37) $$xg' = 3x^3y - x^4y^2z + x^2y^2zq' + x^2yzg' + yzg' - x^2yzl' - x^2y^2z \triangle g'.$$

As before we write G_j, Q_j, L_j and H_j for the coefficients of z^j in g', q', l' and h' respectively. They become zero for negative values of j. Taking coefficients of z^{j+1}, where $j \geqq 0$, in (37) we find

(38) $$xG_{j+1} = \delta_{0,j}x^4y^2 + x^2y^2Q_j + (x^2+1)yG_j - x^2yL_j - x^2y^2 \triangle G_j.$$

This is valid for all non-negative integers j. Moreover by considering the coefficient of z^0 in (37) we find

(39) $$G_0 = 3x^2y,\ Q_0 = 3x^2,\ L_0 = 3y,\ \triangle G_0 = 3x^2.$$

Putting $j = 0$ in (38), and using (39), we get

$$xG_1 = -x^4y^2 + 3x^4y^2 + 3x^4y^2 + 3x^2y^2 - 3x^2y^2 - 3x^4y^2,$$

(40) $$G_1 = 2x^3y^2,\ Q_1 = 2x^3,\ L_1 = 0,\ \triangle G_1 = 2x^3(y+1).$$

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Continuing in this manner the reader may verify that $G_2 = 2x^2y^3$ and $G_3 = -2x^3y^3$.

As in Section 4 the coefficient of x^{t+2-2k} in G_t can be non-zero only if $0 \leqq k \leqq t/2$. We now denote this coefficient by $-(-1)^{t+k}G_{t,k}$. We shall find that this makes $G_{t,k}$ a polynomial in y with non-negative coefficients, except when $t = 0$.

Taking coefficients of like powers of x in (38) we obtain the following recursion formula, valid for $t \geqq 1$ and k arbitrary.

$$(41) \quad G_{t+1,k+1} = -y^2Q_{t,k+1} - yG_{t,k+1} + yG_{t,k} - y\delta_{t,2k}G_{2k,k} + y^2 \triangle G_{t,k+1},$$

where $Q_{t,k}$ denotes the number obtained by setting $y = 1$ in $G_{t,k}$.

Formulae are simpler for $\lambda = 2$ than for $\lambda = 1$. For example we have the following theorems.

5.1. THEOREM. *If* $t \geqq 2$, *then* $G_{t,0} = 0$.

5.2. THEOREM. *If* $t \geqq 2$, *then* $G_{t,1} = 2y^3$.

Proof of 5.1. By (41), with $k = -1$, $t \geqq 1$,

$$(42) \qquad G_{t+1,0} = -y^2Q_{t,0} - yG_{t,0} + y^2 \triangle G_{t,0}.$$

But by (40) $G_{1,0} = 2y^2$, $Q_{1,0} = 2$ and $\triangle G_{1,0} = 2y + 2$. Hence by (42) $G_{2,0} = 0$.

The theorem follows by induction over t. For if $G_{t,0} = 0$ for some $t \geqq 2$ we have also $Q_{t,0} = 0$, $\triangle G_{t,0} = 0$, and therefore $G_{t+1,0} = 0$, by (42).

Proof of 5.2. By (41) with $k = 0$, $t \geqq 1$,

$$(43) \qquad G_{t+1,1} = -y^2Q_{t,1} - yG_{t,1} + yG_{t,0} + y^2 \triangle G_{t,1}.$$

Now $G_{1,1} = 0$ by (40). Hence, by (40) and (43), $G_{2,1} = yG_{1,0} = 2y^3$. But suppose $G_{t,1} = 2y^3$ for some $t \geqq 2$. Then, by (43) and 5.1,

$$\begin{aligned} G_{t+1,1} &= -y^2 . 2 - 2y^4 + y . 0 + y^2(2y^2 + 2y + 2) \\ &= 2y^3. \end{aligned}$$

Thus the theorem follows by induction over t.

In the initial investigation (41) was used to calculate $G_{t,k}$ for $0 \leqq t \leqq 14$. The results were found to fit the general formula stated below as 5.3, and so suggested that theorem.

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5.3. THEOREM.

$$G_{t,k} = \sum_{s=3}^{k+2} \left\{ \frac{(t-2)!(t-2k+1)(t-2k-1+s)(s-2)(t-s)!}{(t-k)!(t-k+1)!k!(k+2-s)!} \times 2(t^2-(2k+1)t+k(s-1))y^s \right\},$$

where $t \geqq 3$ *and* $0 \leqq k \leqq t/2$.

Proof. Let us first observe that if $t = 2$ we have only to consider the cases $k = 0$ and $k = 1$, and these are covered by Theorems 5.1 and 5.2. For $t < 2$ we have Equations (39) and (40). Hence the present theorem completes the determination of g' for $\lambda = 2$.

If we put $k = 0$ we obtain an empty sum on the right. Then 5.3 reduces to $G_{t.0} = 0$, in agreement with 5.1. If instead we put $k = 1$ the formula reduces to

$$G_{t,1} = \frac{2.(t-2)!(t-1)t(t-3)!(t^2-3t+2)y^3}{(t-1)!t!1!0!} = 2y^3,$$

in agreement with 5.2. Thus the theorem is valid whenever $k = 0$ or $k = 1$. It is therefore valid for $t = 3$.

Assume as an inductive hypothesis that the theorem holds when $t \leqq n$, for some integer $n \geqq 3$, and consider the case $t = n + 1$.

We use (41) to determine $G_{n+1,k+1}$ for $2 \leqq k + 1 \leqq (n + 1)/2$. Then $n \geqq 2k + 1$.

Consider first the case $n = 2k + 1$. Then, by (41),

$$\begin{aligned} G_{n+1,k+1} &= yG_{n,k} \\ &= \sum_{s=3}^{k+2} \left\{ \frac{(2k-1)!2s(s-2)(2k+1-s)!2k(s-1)y^{s+1}}{(k+1)!(k+2)!k!(k+2-s)!} \right\} \\ &= 2\sum_{s=3}^{k+3} \left\{ \frac{(2k)!(s-1)(s-2)(s-3)(2k+2-s)!y^s}{(k+2)!(k+1)!k!(k+3-s)!} \right\}, \end{aligned}$$

by the inductive hypothesis. But if we write $n + 1 = 2k + 2$ for t, and $k + 1$ for k on the right of formula 5.3, we obtain

$$\sum_{s=3}^{k+3} \left\{ \frac{(2k)!(s-1)(s-2)(2k+2-s)!2(-2k-2+(k+1)(s-1))y^s}{(k+1)!(k+2)!(k+1)!(k+3-s)!} \right\},$$

which is equivalent to the above formula for $G_{n+1,k+1}$. We deduce that the theorem is valid also for $t = n + 1$ in the case $n = 2k + 1$.

Next we consider the remaining case $n \geqq 2k + 2$. Equation (41) now reduces to

$$(44) \qquad G_{n+1,k+1} = -y^2Q_{n,k+1} - yG_{n,k+1} + yG_{n,k} + y^2 \triangle G_{n,k+1}.$$

Let us denote the expression on the right of 5.3 by $N_{t,k}$. ($t \geqq 2k + 2$, $k \geqq 1$).

We have

$$\begin{aligned}(s-2)(t-2k-1+s)(t^2+k(s-1)-(2k+1)t) \\ &= (s-2)(t-k-1)(t-k)(t-2k-1+s) \\ &\quad - (s-2)k(t-2k-1+s)(k+2-s) \\ &= (s-2)(t-k+1)(t-k)(t-k-1) \\ &\quad - (s-2)(t-k)(t-1)(k+2-s) \\ &\quad + (s-2)k(k+1-s)(k+2-s) \\ &= k(t-k+1)(t-k)(t-k-1) \\ &\quad - (t-k+1)(t-k)(t-k-1)(k+2-s) \\ &\quad - (k-1)(t-k)(t-1)(k+2-s) \\ &\quad + (t-k)(t-1)(k+2-s)(k+1-s) \\ &\quad + (k-2)k(k+2-s)(k+1-s) \\ &\quad - k(k+2-s)(k+1-s)(k-s) \\ &= k(t-k+1)(t-k)(t-k-1) \\ &\quad - (t-k)(t^2-(k+1)t+k^2-k)(k+2-s) \\ &\quad + (t^2-(k+1)t+k^2-k)(k+2-s)(k+1-s) \\ &\quad - k(k+2-s)(k+1-s)(k-s).\end{aligned}$$

Using this identity we write $N_{t,k}$ as follows.

$$(45)\quad \tfrac{1}{2}N_{t,k} = \left\{\frac{(t-2)!(t-2k+1)}{(k-1)!(t-k-2)!(t-k)!}\right\}\sum_{s=3}^{k+2}\left\{\frac{(t-s)!y^s}{(k+2-s)!}\right\}$$
$$-\left\{\frac{(t-2)!(t-2k+1)(t^2-(k+1)t+k^2-k)}{k!(t-k+1)!(t-k-1)!}\right\}\sum_{s=3}^{k+1}\left\{\frac{(t-s)!y^s}{(k+1-s)!}\right\}$$
$$+\left\{\frac{(t-2)!(t-2k+1)(t^2-(k+1)t+k^2-k)}{k!(t-k+1)!(t-k)!}\right\}\sum_{s=3}^{k}\left\{\frac{(t-s)!y^s}{(k-s)!}\right\}$$
$$-\left\{\frac{(t-2)!(t-2k+1)}{(k-1)!(t-k+1)!(t-k)!}\right\}\sum_{s=3}^{k-1}\left\{\frac{(t-s)!y^s}{(k-s-1)!}\right\}.$$

We note that a factor $(s-2)$ occurs in each term of the sum of 5.3. Accordingly (45) remains valid when we replace the lower limit of summation, $s = 3$, by $s = 2$.

We proceed to find a similar formula for

$$J(t,k) = \tfrac{1}{2}\{N_{t+1,k+1} + yN_{t,k+1} - yN_{t,k}\}.$$

Our device for multiplying such an expression as the right of (45) by y is to replace s by $s-1$ throughout, except in the index of y. If we use $s = 2$ as the initial lower limit of summation we then end with the lower limit $s = 3$. Let us write

$$(46)\qquad B_m(u) = \sum_{s=3}^{k+u}\left\{\frac{(t+m-s)!}{(k+u-s)!}\right\}y^s.$$

The coefficient of $B_1(4)$ in $J(t, k)$ arises solely from $yN_{t,k+1}$; it is

$$\frac{(t-2)!(t-2k-1)}{k!(t-k-3)!(t-k-1)!}.$$

We note that since $t \geqq 2k+2$ and $k \geqq 1$ we have $t-k-3 \geqq k-1 \geqq 0$.

The coefficient of $B_1(3)$ in $J(t, k)$ is

$$\frac{(t-1)!(t-2k)}{k!(t-k-2)!(t-k)!} - \frac{(t-2)!(t-2k+1)}{(k-1)!(t-k-2)!(t-k)!} - \frac{(t-2)!(t-2k-1)(t^2-(k+2)t+k^2+k)}{(k+1)!(t-k)!(t-k-2)!}.$$

But

$$\begin{aligned}(t-1)(t-2k) - k(t-2k-1) &= t^2 - (3k+1)t + 2k^2 + k \\ &= (t-2k-1)(t-k).\end{aligned}$$

Hence the coefficient of $B_1(3)$ is

$$\begin{aligned}&-\frac{(t-2)!(t-2k-1)(-(t-k)(k+1)+t^2-(k+2)t+k^2+k)}{(k+1)!(t-k)!(t-k-2)!} \\ &\qquad = \frac{-(t-2)!(t-2k-1)(t^2-(2k+3)t+2k(k+1))}{(k+1)!(t-k)!(t-k-2)!}.\end{aligned}$$

The contribution of $N_{t+1,k+1}$ to the coefficient of $B_1(2)$ is

$$\frac{-(t-1)!(t-2k)(t^2-kt+k^2-1)}{(k+1)!(t-k+1)!(t-k-1)!}.$$

The contribution of $yN_{t,k+1} - yN_{t,k}$ is

$$\frac{(t-2)!(t-2k-1)(t^2-(k+2)t+k^2+k)}{(k+1)!(t-k)!(t-k-1)!} + \frac{(t-2)!(t-2k+1)(t^2-(k+1)t+k^2-k)}{k!(t-k+1)!(t-k-1)!}.$$

But

$$\begin{aligned}&(t-2k-1)(t-k+1)(t^2-(k+2)t+k^2+k) \\ &\quad + (t-2k+1)(k+1)(t^2-(k+1)t+k^2-k) \\ &= (t-2k-1)(t-k+1)(t^2-kt+k^2-1-2t+k+1) \\ &\quad + (t-2k+1)(k+1)(t^2-kt+k^2-1-t-k+1) \\ &= (t-2k)(t^2-kt+k^2-1)(t+2) \\ &\quad + (t^2-kt+k^2-1)(-t+k-1+k+1) \\ &\quad + (t-2k)((t-k+1)(-2t+k+1)+(k+1)(-t-k+1)) \\ &\quad + (-(t-k+1)(-2t+k+1)+(k+1)(-t-k+1)) \\ &= (t-2k)(t+1)(t^2-kt+k^2-1) \\ &\quad + (t-2k)(-2t^2+(2k-2)t-2(k^2-1)) \\ &\quad + (2t^2-4kt) \\ &= (t-2k)(t-1)(t^2-kt+k^2-1).\end{aligned}$$

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We deduce that the contribution of $yN_{t,k+1} - yN_{t,k}$ exactly cancels that of $N_{t+1,k+1}$; the coefficient of $B_1(2)$ in $J(t, k)$ is zero.

The contribution of $yN_{t.k+1}$ to the coefficient of $B_1(1)$ in $J(t, k)$ is

$$\frac{-(t-2)!(t-2k-1)}{k!(t-k)!(t-k-1)!}.$$

The contribution of $N_{t+1,k+1} - yN_{t,k}$ is

$$\frac{(t-1)!(t-2k)(t^2-kt+k^2-1)}{(k+1)!(t-k+1)!(t-k)!} - \frac{(t-2)!(t-2k+1)(t^2-(k+1)t+k^2-k)}{k!(t-k+1)!(t-k)!} = \frac{(t-2)!(t-2k-1)(t^2-(k+2)t+k^2+k)}{(k+1)!(t-k)!(t-k)!},$$

by the identity given by the vanishing of the coefficient of $B_1(2)$. Hence the coefficient of $B_1(1)$ in $J(t, k)$ is

$$\frac{(t-2)!(t-2k-1)(t^2-(2k+3)t+2k(k+1))}{(k+1)!(t-k)!(t-k)!}.$$

Finally, the coefficient of $B_1(0)$ in $J(t, k)$ is found to be

$$-\frac{(t-1)!(t-2k)}{k!(t-k+1)!(t-k)!} + \frac{(t-2)!(t-2k+1)}{(k-1)!(t-k+1)!(t-k)!}.$$

But

$$\begin{aligned}(t-1)(t-2k) - k(t-2k+1) &= (t-2k-1)(t-1-k) + (t-1) - 2k \\ &= (t-2k-1)(t-k).\end{aligned}$$

The coefficient of $B_1(0)$ is thus

$$\frac{-(t-2)!(t-2k-1)}{k!(t-k+1)!(t-k-1)!}.$$

We state the results of these calculations as follows.

$$\begin{aligned}(47)\quad \tfrac{1}{2}\{N_{t+1,k+1} + yN_{t,k+1} - yN_{t,k}\} &= \frac{(t-2)!(t-2k-1)B_1(4)}{k!(t-k-3)!(t-k-1)!} \\ &\quad - \frac{(t-2)!(t-2k-1)(t^2-(2k+3)t+2k(k+1))B_1(3)}{(k+1)!(t-k)!(t-k-2)!} \\ &\quad + \frac{(t-2)!(t-2k-1)(t^2-(2k+3)t+2k(k+1))B_1(1)}{(k+1)!(t-k)!(t-k)!} \\ &\quad - \frac{(t-2)!(t-2k-1)B_1(0)}{k!(t-k+1)!(t-k-1)!}.\end{aligned}$$

Our next step is to use (45) to obtain a formula for $\triangle N_{t,k+1}$. Write

$$(48)\qquad C(u) = \sum_{s=0}^{k+u} \frac{(t-s)!y^s}{(k+u-s)!},$$

where $u = -1, 0, 1$ or 2. We observe that

$$(49)\quad C(u) = (t-k-u)!\left[\text{Coeff } \theta^{k+u} \text{ in } \left\{\frac{1}{(1-\theta)^{t-k-u-1}(1-\theta y)}\right\}\right].$$

We derive $\triangle C(u)$ from this by applying the operator $\triangle$ to $1/(1-\theta y)$, and thus replacing this fraction in (49) by $\theta/\{(1-\theta)(1-\theta y)\}$. We deduce from the result that

$$(50)\qquad \triangle C(u) = \frac{C(u-1)}{(t-k-u+1)}.$$

Let us consider how to apply this to (45). The sums over s on the right of (45) are, in order, $B_0(2)$, $B_0(1)$, $B_0(0)$ and $B_0(-1)$. It is legitimate to replace the lower limit of summation by $s = 2$, throughout. But let us reduce this limit to $s = 0$, so that each $B_0(u)$ is replaced by $C(u)$. Any resulting error affects only the coefficients of y^1 and y^0.

Let us now apply the operator $\triangle$ to each side. We use (50) and so obtain a linear formula for $\frac{1}{2}\triangle N_{t,k}$ in terms of $C(1)$, $C(0)$, $C(-1)$ and $C(-2)$ that can be in error only in the coefficient of y^0. But let us write $P_{t,k}$ for the number obtained from $N_{t,k}$ by setting $y = 1$. Since $N_{t,k}$ has a zero coefficient for each power of y less than the third, $P_{t,k}$ is the coefficient of y^2, of y^1 and of y^0 in $\triangle N_{t,k}$. We can thus make our formula for $\frac{1}{2}\triangle N_{t,k}$ exact by subtracting $\frac{1}{2}P_{t,k}$ on the left, and raising each lower limit of summation from $s = 0$ to $s = 1$ on the right.

It is now convenient to multiply both sides by y^2. On the right we do this by replacing s by $s-2$ throughout, except in the index of y. We summarize the effect of the above operations as follows: (45) remains valid when we replace $N_{t,k}$ on the left by $y^2\{\triangle N_{t,k} - P_{t,k}\}$, and replace each sum $B_0(u)$ on the right by

$$\frac{B_2(u+1)}{(t-k-u+1)}.$$

For the immediate application we have to do all this and then write $k+1$ for k. The result is

$$(51)\quad \tfrac{1}{2}y^2(\Delta N_{t,k+1} - P_{t,k+1}) =$$
$$\frac{(t-2)!(t-2k-1)B_2(4)}{k!(t-k-1)!(t-k-2)!}$$
$$-\frac{(t-2)!(t-2k-1)(t^2-(k+2)t+k^2+k)B_2(3)}{(k+1)!(t-k)!(t-k-1)!}$$
$$+\frac{(t-2)!(t-2k-1)(t^2-(k+2)t+k^2+k)B_2(2)}{(k+1)!(t-k)!(t-k)!}$$
$$-\frac{(t-2)!(t-2k-1)B_2(1)}{k!(t-k+1)!(t-k-1)!}.$$

For a comparison with (47) we must express this in terms of the sums $B_1(u)$. It is clear from (46) that

$$(52) \qquad B_2(u) = (t-k-u+2)B_1(u) + B_1(u-1).$$

Let us substitute accordingly on the right of (51).

The coefficient of $B_1(4)$ now agrees with that on the right of (47). The coefficient of $B_1(3)$ is

$$-\frac{(t-2)!(t-2k-1)(-(k+1)(t-k)+t^2-(k+2)t+k^2+k)}{(k+1)!(t-k)!(t-k-2)!},$$

the same as in (47). Moreover the coefficient of $B_1(2)$ vanishes, the coefficient of $B_1(1)$ is the negative of that of $B_1(3)$ except for a division by $(t-k)(t-k-1)$, and the coefficient of $B_1(0)$ is that of $B_2(1)$ in (51). We observe that the expressions on the right of (47) and (51) are identical. Hence

$$(53) \qquad y^2(\triangle N_{t,k+1} - P_{t,k+1}) = N_{t+1,k+1} + yN_{t,k+1} - yN_{t,k},$$

where $t \geqq 2k+2$ and $k \geqq 1$.

Consider the case $t = n$. Using (44) and the inductive hypothesis we deduce from (53) that

$$G_{n+1,k+1} = N_{n+1,k+1}.$$

This completes our proof that 5.3 is valid also when $t = n+1$. It has already been established for all t when $k = 0$ and when $k = 1$. The theorem follows in general by induction.

Writing $t = 2k$ in 5.3 we obtain

$$(54) \qquad -(-1)^k L_{2k} = 2\sum_{s=4}^{k+2}\left\{\frac{(2k-2)!(2k-s)!(s-1)(s-2)(s-3)y^s}{(k-1)!k!(k+1)!(k+2-s)!}\right\}$$

for $k \geqq 2$.

It appears from (44) that $Q_{t,k}$, which is the coefficient of y^2, y^1 and y^0 in $\triangle G_{t,k}$ is also the coefficient of y^3 in $G_{t+1,k}$, for $k \geqq 2$ and $t \geqq 2k$. So, by 5.3,

$$(55) \qquad Q_{t,k} = 2\left\{\frac{t!(t-2)!(t-2k+1)(t-2k+2)(t-2k+3)}{(t+2-k)!(t+1-k)!k!(k-1)!}\right\}$$

for $t \geqq 3$ and $2 \leqq k \leqq t/2$. Using 5.2 we can extend the range of validity to $(t \geqq 2, 1 \leqq k \leqq t/2)$.

The coefficient H_{2k} of x^2 in Q_{2k} is $-(-1)^k Q_{2k,k}$. Hence, by (55),

$$(56) \qquad H_{2k} = -12\left\{\frac{(-1)^k(2k)!(2k-2)!}{(k-1)!k!(k+1)!(k+2)!}\right\}.$$

For large k we can replace this by the asymptotic approximation

$$(57) \qquad H_{2k} \sim -\frac{3\cdot(-16)^k}{\pi k^5}.$$

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