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Uniformly quasi-Hermitian groups are supramenable

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Abstract. Motivated by the recent result in Samei and Wiersma (2020, *Advances in Mathematics* 359, 106897) that quasi-Hermitian groups are amenable, we consider a generalization of this property on discrete groups associated to certain Roe-type algebras; we call it *uniformly quasi-Hermitian*. We show that the class of uniformly quasi-Hermitian groups is contained in the class of supramenable groups and includes all subexponential groups. We also show that they are invariant under quasi-isometry.

1 Introduction

Let G be a discrete group, and let $\ell^1(G)$ be its ℓ^1-group algebra. The group G is said to be *Hermitian* if

$$\mathrm{Sp}_{\ell^1(G)}(f) \subseteq \mathbb{R} \tag{1.1}$$

for every self-adjoint element $f = f^* \in \ell^1(G)$. Here, $\mathrm{Sp}_{\ell^1(G)}(f)$ denotes the spectrum of $f \in \ell^1(G)$ and f^* is the canonical involution of f. A weaker concept of G being *quasi-Hermitian* requires that (1.1) holds only for all self-adjoint functions on G with finite support. For several decades, classifying (quasi-)Hermitian groups has been a desirable task in harmonic analysis as they have powerful properties and applications. Among many results, we wish to highlight two important ones: finitely generated groups with polynomial growth are Hermitian [9] and groups with subexponential growth are quasi-Hermitian [10]. On the other hand, every group containing a free subsemigroup with two generators is not quasi-Hermitian [7].

A major open problem regarding (quasi-)Hermitian groups was that whether they are amenable. It is not clear in the literature what the main motivations behind this conjecture were. We speculate one reason could have been the fact that groups containing $\mathbb{F}_2$, the nonabelian free group on two generators, are not quasi-Hermitian. As it was believed that a nonamenable group contains $\mathbb{F}_2$ as a subgroup (this is known as von Neumann's conjecture), it was then reasonable to expect that (quasi-)Hermitian groups are amenable. Of course, it was later shown that von Neumann's conjecture is false and there are many torsion nonamenable groups. Nonetheless, the conjecture

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regarding the amenability of (quasi-)Hermitian groups remained as a reasonable one for many years which was recently solved in the affirmative by the second-named author and Wiersma [12].

Amenability for groups has many characterizations, one of them is that a group G is amenable if and only if G is nonparadoxical. We could consider a stronger property requiring *every* subset of G being nonparadoxical. In this case, we say that G is *supramenable* [3]. Supramenability is much stronger than amenability. It is known that all subexponential groups are supramenable, but groups containing a free subsemigroup with two generators fail to be supramenable. One very interesting property of supramenable groups is that, although von Neumann's conjecture fails for amenable groups, the appropriate analogous one holds for supramenable groups. More precisely, it is shown in [8, Proposition 3.4] that G is supramenable if and only if there is no injective Lipschitz map from $\mathbb{F}_2$ into G (see also Definition 3.3). This, in particular, gives a characterization of the supramenability of G in terms of its large-scale geometry. Now, because we know that quasi-Hermitian groups are amenable, a natural question is whether there is an analogous concept that would imply supramenability. In this paper, we wish to provide an affirmative answer to this question. Our approach is as follows.

In [8], it is shown that supramenability of a group G can be characterized by the C*-type properties of its uniform Roe algebra $C^*_u(G)$. As this algebra is the norm closure of bounded operators on $\ell^2(G)$ with finite propagation, instead of looking at "functions" with finite support on G, we consider "operators with finite propagation" on G. More precisely, we look at the normed algebra of operators with finite propagation that are bounded on both $\ell^1(G)$ and $\ell^\infty(G)$ with the norm given by the maximal norm coming from $B(\ell^1(G))$ and $B(\ell^\infty(G))$. We denote $L^1_u(G)$ to be the completion of this normed algebra. It is straightforward to see that $L^1_u(G)$ becomes a Banach $*$-algebra that contains $\ell^1(G)$ as a closed $*$-algebra and embeds into $C^*_u(G)$ contractively and $*$-homomorphically. We say that G is *uniformly quasi-Hermitian* if

$$\mathrm{Sp}_{L^1_u(G)}(T) \subseteq \mathbb{R} \tag{1.2}$$

for every self-adjoint operator $T = T^* \in L^1_u(G)$ with finite propagation. We will show that this property is preserved under quasi-isometry and also every uniformly quasi-Hermitian group is supramenable. For the latter result, we will use the fact that $\mathbb{F}_2$ is not quasi-Hermitian and there is an injective Lipschitz map from $\mathbb{F}_2$ into any nonsupramenable group [8]. We also show that every group with subexponential growth is uniformly quasi-Hermitian.

2 Preliminaries and background

2.1 Quasi-Hermitian Banach $*$-algebras

We briefly recall the definition of a quasi-Hermitian $*$-subalgebra of a Banach $*$-algebra and some theorems that are useful for us.

Definition 2.1 [12, Definition 2.8] A dense $*$-subalgebra $\mathcal{S}$ of a Banach $*$-algebra $\mathcal{A}$ is *quasi-Hermitian* in $\mathcal{A}$ if $\mathrm{Sp}_{\mathcal{A}}(a) \subseteq \mathbb{R}$ for every $a = a^* \in \mathcal{S}$.

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The following is a generalization of [11, Theorem 9.8.4] whose proof is identical to the one given there. Thus, we omit the proof.

Theorem 2.2 *Let $\mathcal{S}$ be a dense $*$-subalgebra of a Banach $*$-algebra $\mathcal{A}$. Then, the following are equivalent:*

(i) *$\mathcal{S}$ is quasi-Hermitian in $\mathcal{A}$;*
(ii) *$\mathcal{S} \otimes M_n$ is quasi-Hermitian in $\mathcal{A} \otimes M_n$, for some $n \in \mathbb{N}$;*
(iii) *$\mathcal{S} \otimes M_n$ is quasi-Hermitian in $\mathcal{A} \otimes M_n$, for all $n \in \mathbb{N}$.*

Finally, we recall the following theorem of Barnes that slightly generalizes a well-known and frequently used result of Hulanicki (see [6, Proposition 3.5]). A further generalization can be found in [5, Lemma 3.1]. For a Banach algebra $\mathcal{A}$ and some $a \in \mathcal{A}$, we let $r_{\mathcal{A}}(a)$ denote the spectral radius of a in $\mathcal{A}$.

Theorem 2.3 (Barnes–Hulanicki theorem [1]) *Let $\mathcal{A}$ be a Banach $*$-algebra, $\mathcal{S}$ a $*$-subalgebra of $\mathcal{A}$, and $\pi : \mathcal{A} \to B(\mathcal{H})$ a faithful $*$-representation ($\mathcal{H}$ is a Hilbert space). If $\mathcal{A}$ is unital, we assume that $\pi(1_{\mathcal{A}}) = id_{B(\mathcal{H})}$. If*

$$r_{\mathcal{A}}(a) = \|\pi(a)\|,$$

for all $a = a^* \in \mathcal{S}$, then

$$Sp_{\mathcal{A}}(a) = Sp_{B(\mathcal{H})}(\pi(a)),$$

for every $a \in \mathcal{S}$. In particular, $\mathcal{S}$ is quasi-Hermitian in $\mathcal{A}$.

2.2 Uniform Roe-type algebras

Let G be a group, and let $1 \le p \le \infty$. The left-regular representation of $\ell^1(G)$ on $\ell^p(G)$ is given by

$$\lambda_p : \ell^1(G) \to B(\ell^p(G)) \quad \lambda_p(f)g = f * g, \tag{2.1}$$

for all $f \in \ell^1(G)$ and $g \in \ell^p(G)$. For $p = 1$, this is nothing but an isometric embedding of $\ell^1(G)$ as convolution operators over itself. For $p = 2$, the norm-closure of $\lambda_2(\ell^1(G))$ inside $B(\ell^2(G))$ is the reduced C*-algebra $C_r^*(G)$. We could also have the isometric algebra homomorphism

$$\pi_p : \ell^\infty(G) \to B(\ell^p(G)) \;\; \pi_p(\theta)(g) = \theta g \;\; (\theta \in \ell^\infty(G), g \in \ell^p(G)). \tag{2.2}$$

Let $c_c(G, \ell^\infty(G)) \cong c_c(G) \otimes \ell^\infty(G)$ be the set of all functions with finite support from G into $\ell^\infty(G)$. It is clear that under pointwise addition, this is a vector space. Moreover, for $\mathbf{f}, \mathbf{g} \in c_c(G, \ell^\infty(G)), s \in G$, the operations

$$\mathbf{f} * \mathbf{g}(s) = \sum_{t \in G} \mathbf{f}(t)[\delta_t * \mathbf{g}(t^{-1}s)] \;\; \mathbf{f}^*(s) = \delta_s * \overline{\mathbf{f}(s^{-1})}$$

induce a unital $*$-algebra structure on $c_c(G, \ell^\infty(G))$. Furthermore, the mapping

$$\pi_p \rtimes \lambda_p(\mathbf{f}) := \sum_{s \in G} \pi_p(\mathbf{f}(s))\lambda_p(s) \tag{2.3}$$

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defines a representation of $c_c(G, \ell^\infty(G))$ on $\ell^p(G)$ (it becomes a $*$-representation when $p = 2$). We let $\ell^\infty(G) \rtimes_{r,p} G$ to be the completion of $\pi_p \rtimes \lambda_p(c_c(G, \ell^\infty(G)))$ inside $B(\ell^p(G))$.

Definition 2.4 Let G be a group. We say that an operator $A = [a_{st}] \in B(\ell^p(G))$ has *finite propagation* if there is a finite set $S \subseteq G$ such that $a_{st} = 0$ whenever $st^{-1} \notin S$. We let $C_u^p(G)$ be the set of all elements of $B(\ell^p(G))$ with finite propagation and $B_u^p(G)$ be the completion of $C_u^p(G)$ in $B(\ell^p(G))$. This is called the ℓ^p *uniform Roe algebra* of G.

It is easy to verify that $C_u^p(G)$ is exactly the unital subalgebra of $B(\ell^p(G))$ generated by $\lambda_p(\mathbb{C}(G))$ and $\pi_p(\ell^\infty(G))$, so that $B_u^p(G)$ becomes a unital closed subalgebra of $B(\ell^p(G))$. Moreover, it is also well known that (see, for example, [4, p. 14])

$$B_u^p(G) \cong \ell^\infty(G) \rtimes_{r,p} G. \tag{2.4}$$

When $p = 2$, $B_u^2(G)$ is a unital C^*-algebra, which is usually denoted by $C_u^*(G)$ and called the *uniform Roe algebra* of G.

If H is also a group, we may view $B_u^p(G) \otimes B_u^p(H)$ as bounded operators acting on $\ell^p(G \times H)$. If we let $B_u^p(G) \otimes_p B_u^p(H)$ to be its completion in $B(\ell^p(G \times H))$, then it is straightforward to verify that $B_u^p(G) \otimes_p B_u^p(H)$ is a closed subalgebra of $B_u^p(G \times H)$. Moreover, $C_u^p(G) \otimes C_u^p(H)$ maps into $C_u^p(G \times H)$. If H is finite with $|H| = n$, then it is known that $B_u^p(H) = B(\ell^p(H)) = \mathbb{M}_n^p$, where $\mathbb{M}_n^p$ is the algebra of all n-by-n matrices over $\mathbb{C}$ together with the norm coming from $B(\ell^p(H))$. Moreover, by applying the identification (2.4), it is straightforward to verify that

$$B_u^p(G) \otimes_p \mathbb{M}_n^p \cong B_u^p(G \times H) \tag{2.5}$$

and

$$C_u^p(G) \otimes \mathbb{M}_n^p \cong C_u^p(G \times H). \tag{2.6}$$

3 Uniformly quasi-Hermitian groups

Suppose that G is a discrete group and $A = [a_{st}]_{s,t \in G}$ is a bounded operator on $\ell^1(G)$. It is easy to see that

$$\|A\|_{B(\ell^1(G))} = \sup_{s \in G} \sum_{t \in G} |a_{st}|.$$

We can use the preceding formula to identify an involutive subalgebra of $B(\ell^1(G))$ consisting of operators with finite propagation.

Definition 3.1 Let G be a group. We define

$$C_u^{1,\infty}(G) := C_u^1(G) \cap C_u^\infty(G). \tag{3.1}$$

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We also let $L^1_u(G)$ to be the unital Banach $*$-algebra generated as the completion of $C^{1,\infty}_u(G)$ under the algebra norm defined as follows for $A = [a_{st}]_{s,t\in G} \in C^{1,\infty}_u(G)$:

(3.2)
$$\|A\|_{L^1_u(G)} := \max\left\{\sup_{s\in G}\sum_{t\in G}|a_{st}|\ \ \sup_{t\in G}\sum_{s\in G}|a_{st}|\right\} = \max\{\|A\|_{B(\ell^1(G))}, \|A^*\|_{B(\ell^1(G))}\},$$

where the involution $*$ is defined in the usual way:

(3.3)
$$A^* = [\overline{a_{ts}}]_{s,t\in G}.$$

We note that because every element of $L^1_u(G)$ acts as a bounded operator both on $\ell^1(G)$ and $\ell^\infty(G)$, by interpolation, it must also act boundedly on all $\ell^p(G)$ for all $p \in [1,\infty]$. In particular, $L^1_u(G)$ can be viewed as a unital $*$-subalgebra of $C^*_u(G)$ and this embedding will be contractive. Furthermore, by (2.3), we could embed $\ell^1(G)$ canonically into $L^1_u(G)$ as a closed $*$-subalgebra.

We are now ready to state the definition of a uniformly quasi-Hermitian group.

Definition 3.2 A group G is *uniformly quasi-Hermitian* if $C^{1,\infty}_u(G)$ is quasi-Hermitian in $L^1_u(G)$.

In order to state our main results, we need to recall the following definitions from [8] that allow us to relate the uniform quasi-Hermitian properties of groups with the same coarse structure.

Definition 3.3 Let G and H be groups, and let $\varphi : G \to H$ be a map. We say that φ is *Lipschitz* if for every finite set $S \subset G$, there is a finite set $T \subseteq H$ such that

(3.4)
$$\forall s,t\in G : st^{-1}\in S \Longrightarrow \varphi(s)\varphi(t)^{-1}\in T.$$

We say that φ is a *quasi-isometric embedding* if it is Lipschitz and we also have that for every finite set $T \subseteq H$, there is a finite set $S \in G$ such that

(3.5)
$$\forall s,t\in G : \varphi(s)\varphi(t)^{-1}\in T \Longrightarrow st^{-1}\in S.$$

Definition 3.4 Let G and H be groups. A function $\varphi : G \to H$ is said to be *uniformly quasi-injective* if there is $N > 0$ such that for every $h \in \operatorname{Im}\varphi$, the set $\varphi^{-1}(h)$ has at most N elements.

Remark 3.5 It is clear that every injective map is also uniformly quasi-injective. Another class of examples are quasi-isometric embeddings. To see this, suppose that G and H are groups and $\varphi : G \to H$ is a quasi-isometric embedding. Suppose that T is the singleton set consisting of the identity element e_H in H and S is the finite subset of G for which (3.5) holds for the pair (T,S). Then, it is an immediate consequence of (3.5) that for any $s,t \in G$ with $\varphi(s) = \varphi(t)$, we will have $st^{-1} \in S$. Hence, for any $h \in \operatorname{Im}\varphi$ and $t \in \varphi^{-1}(h)$, we have

$$\varphi^{-1}(h) \subseteq St.$$

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In particular, $|\varphi^{-1}(h)| \le |S|$.

Suppose that X, Y are sets and $\phi : X \to Y$ is an injective map. We could view $\ell^1(X)$ as an isometric subspace of $\ell^1(Y)$ via the mapping

$$I_\phi : \ell^1(X) \to \ell^1(Y), \tag{3.6}$$

given by

$$I_\phi(f)(y) = \begin{cases} f(\varphi^{-1}(y)) & \text{if } y \in \phi(X), \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we have the projection

$$P_\phi : \ell^1(Y) \to \ell^1(X) \;\; P_\phi(f) = f_{\phi(X)}, \tag{3.7}$$

where $f_{\phi(X)}$ is the restriction of f on $\phi(X)$. If we define the operator

$$\Lambda_\phi : B(\ell^1(X)) \to B(\ell^1(Y)) \;\; \Lambda_\phi(A) = I_\phi \circ A \circ P_\phi, \tag{3.8}$$

then it is straightforward to see that for every operator $A \in B(\ell^1(X))$ with matrix representation $[a_{xx'}]$, $\Lambda_\phi(A)$ is represented by a matrix $[b_{yy'}]$, where

$$b_{yy'} = \begin{cases} a_{xx'} & \text{if } \phi(x) = y, \phi(x') = y', \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

From this, it is easy to verify that Λ_ϕ is an isometric $*$–algebra homomorphism.

The following theorem is one of the main results of this section. Its proof is a modification of the argument given in [4, Theorem 3.4, (1)$\Longrightarrow$(2)], which in part is derived from the proof of [2, Theorem 4].

Theorem 3.6 *Let G and H be groups, and let $\varphi : G \to H$ be a Lipschitz map from G into H.*

(i) *If φ is injective, then there is an isometric $*$-algebra homomorphism from $L^1_u(G)$ into $L^1_u(H)$, so that $C^{1,\infty}_u(G)$ maps into $C^{1,\infty}_u(H)$.*

(ii) *If φ is uniformly quasi-injective, then, for every $n \in \mathbb{N}$, there is $m > n$ such that there is an isometric $*$-algebra homomorphism from $L^1_u(G) \otimes \mathbb{M}_n$ into $L^1_u(H) \otimes \mathbb{M}_m$, so that $C^{1,\infty}_u(G) \otimes \mathbb{M}_n$ maps into $C^{1,\infty}_u(H) \otimes \mathbb{M}_m$.*

Proof (i) Because φ is injective, by (3.8), it induces an isometric algebra homomorphism

$$\Lambda_\varphi : B(\ell^1(G)) \to B(\ell^1(H)).$$

Now, take $A = [a_{st}] \in C^{1,\infty}_u(G)$ and a finite set $S \subseteq G$ such that $a_{st} = 0$ if $st^{-1} \notin S$. Because φ is Lipschitz, there is a finite set $T \subseteq H$, so that (S, T) satisfies (3.4). Then, $\varphi(s)\varphi(t)^{-1} \notin T$ implies that $st^{-1} \notin S$, so that, by our assumption and the relation (3.9), we will have that

$$\Lambda_\varphi(A)_{\varphi(s)\varphi(t)} = a_{st} = 0.$$

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Thus, $\Lambda_\varphi(A) \in B(\ell^1(H))$ has finite propagation. The final result follows because the restriction of Λ_ϕ on $L^1_u(G)$ preserves the $*$-operation.

(ii) By our hypothesis, there is an $N \in \mathbb{N}$ such that for every $h \in \operatorname{Im} \varphi$, the cardinality of the set $\varphi^{-1}(h)$ is at most N. If we enumerate

$$\varphi^{-1}(h) := \{1, \dots, N(h)\},$$

so that $N(h) \leq N$, then we could identify G with a subset of $H \times \{1, \dots, N\}$. Let π denote the corresponding projection from G to $\{1, \dots, N\}$, so that the identification is given by $s \mapsto (\varphi(s), \pi(s))$. Define the mapping

$$\phi : G \times \mathbb{N} \to H \times \mathbb{N} \;\; \phi(s, j) = (\varphi(s), \pi(s) + jN(\varphi(s))).$$

Because for all $s \in G$, $N(\varphi(s)) \leq N$, we can easily see that ϕ is injective. For every $n \in \mathbb{N}$, we may identify (canonically) the set $\{1, \dots, n\}$ with $\mathbb{Z}_n$, where $\mathbb{Z}_n$ is the cyclic group of all positive integers mod n. If we let ϕ_n to be the restriction of ϕ to $G \times \mathbb{Z}_n$, then it is easy to verify that, with $m = (1 + n)N$,

$$\phi_n : G \times \mathbb{Z}_n \to H \times \mathbb{Z}_m \;\; \phi_n(s, j) = (\varphi(s), \pi(s) + jN(\varphi(s)))$$

is an injective Lipschitz map. Hence, by part (i), there is an isometric $*$-algebra homomorphism from $L^1_u(G \times \mathbb{Z}_n)$ into $L^1_u(G \times \mathbb{Z}_m)$ taking $C^{1,\infty}_n(G \times \mathbb{Z}_n)$ into $C^{1,\infty}_n(G \times \mathbb{Z}_m)$. The final result follows from (2.5) and (2.6). ∎

We can now state the following, which follows from Theorems 2.2 and 3.6 and the known fact that closed $*$-subalgebras of quasi-Hermitian algebras are quasi-Hermitian.

Corollary 3.7 *Let G and H be groups, so that there is a uniformly quasi-injective Lipschitz map φ from G into H. If H is uniformly quasi-Hermitian, then so is G. In particular, being uniformly quasi-Hermitian is invariant under quasi-isometry.*

Corollary 3.8 *A uniformly quasi-Hermitian group is supramenable.*

Proof Suppose that G is not supramenable. Then, by [8, Proposition 3.4], there is an injective Lipschitz map from $\mathbb{F}_2$ into G. Hence, by Corollary 3.7, it suffices to prove that $\mathbb{F}_2$ is not uniformly quasi-Hermitian.

Now, because $\mathbb{F}_2$ is not quasi-Hermitian, there is a self-adjoint function $f : \mathbb{F}_2 \to \mathbb{C}$ with finite support such that it has nonreal spectrum in $\ell^1(\mathbb{F}_2)$. By viewing f as an element of $L^1_u(\mathbb{F}_2)$, it follows that it also has a nonreal spectrum in $L^1_u(\mathbb{F}_2)$. Thus, $\mathbb{F}_2$ is not uniformly quasi-Hermitian. We point out the fact that $\ell^1(\mathbb{F}_2)$ is a closed $*$-subalgebra of $L^1_u(\mathbb{F}_2)$, so that the spectrum of f in $\ell^1(\mathbb{F}_2)$ is equal to the spectrum of f in $L^1_u(\mathbb{F}_2)$ if the latter space is a subset of the real line. ∎

We finish this section with the following theorem that gives us a reasonably large class of uniformly quasi-Hermitian groups. We recall that a group G has *subexponential growth* if for every finite subset S of G, we have $\lim_{n\to\infty} |S^n|^{1/n} = 1$.

Theorem 3.9 *A group with subexponential growth is uniformly quasi-Hermitian.*

Proof Suppose that $A^* = A = [a_{st}] \in C_u^{1,\infty}(G)$ and S is a finite subset of G for which $a_{st} = 0$ if $st^{-1} \notin S$. Fix $t \in G$, and define

$$f_t(s) = \operatorname{sgn}(a_{ts}) \quad (s \in G).$$

As $a_{st} \neq 0$ only if $s \in St$, we see that $f_t \in \ell^1(G)$ with $\|f_t\|_2 \leq |S|^{1/2}$. Furthermore,

$$(Af_t)(t) = \sum_{s\in G} a_{ts} f_t(s).$$

Hence, we have

$$\begin{aligned}\|A\|_{C_u^*(G)} &\geq \|A(f_t)\|_2 / \|f_t\|_2 \\ &\geq \left|\sum_{s\in G} a_{ts} f_t(s)\right| / |S|^{1/2} \\ &= \frac{\sum_{s\in G} |a_{ts}|}{|S|^{1/2}}.\end{aligned}$$

Thus, by taking the supremum and using the fact that $A = A^*$ ($\overline{a_{st}} = a_{ts}$), we will have

$$\|A\|_{L_u^1(G)} \leq |S|^{1/2} \|A\|_{C_u^*(G)}.$$

As A^n, for all $n \in \mathbb{N}$, is supported on $\{(s,t) : st^{-1} \in S^n\}$, we will have that

$$\|A^n\|_{L_u^1(G)} \leq |S^n|^{1/2} \|A^n\|_{C_u^*(G)}.$$

By taking first the nth root and then the limit as $n \to \infty$ and using the fact that G has subexponential growth, we have

$$r_{L_u^1(G)}(A) \leq r_{C_u^*(G)}(A) \lim_{n\to\infty} |S^n|^{1/2n} = r_{C_u^*(G)}(A) \leq r_{L_u^1(G)}(A).$$

Thus, by Theorem 2.3, G is uniformly quasi-Hermitian. ∎

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