THE JOURNAL OF SYMBOLIC LOGIC
Volume 85, Number 3, September 2020

WHAT THE ŁUKASIEWICZ AXIOMS MEAN

DANIELE MUNDICI

Abstract. Let $\to$ be a continuous $[0,1]$-valued function defined on the unit square $[0,1]^2$, having the following properties: (i) $x \to (y \to z) = y \to (x \to z)$ and (ii) $x \to y = 1$ iff $x \leq y$. Let $\neg x = x \to 0$. Then the algebra $W = ([0,1], 1, \neg, \to)$ satisfies the time-honored Łukasiewicz axioms of his infinite-valued calculus. Let $x \to_Ł y = \min(1, 1-x+y)$ and $\neg_Ł x = x \to_Ł 0 = 1-x$. Then there is precisely one isomorphism ϕ of W onto the standard Wajsberg algebra $W_Ł = ([0,1], 1, \neg_Ł, \to_Ł)$. Thus $x \to y = \phi^{-1}(\min(1, 1-\phi(x)+\phi(y)))$.

§1. Foreword. We present an original itinerary to infinite-valued Łukasiewicz propositional logic. The only prerequisite is knowledge of the most basic properties of continuous $[0,1]$-valued functions on $[0,1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ and on the square $[0,1]^2$.

The Łukasiewicz axioms for his infinite-valued calculus have the following algebraic reformulation [2, p. 144]: $1 \to x = x$, $(x \to y) \to ((y \to z) \to (x \to z)) = 1$, $((x \to y) \to y) = ((y \to x) \to x)$, $(\neg x \to \neg y) \to (y \to x) = 1$.

Theorem 2.3 shows that every $[0,1]$-valued function $\to$ defined on $[0,1]^2$ determines an algebra W satisfying the Łukasiewicz axioms, provided $\to$ is continuous and satisfies the following two conditions: (i) $x \to (y \to z) = y \to (x \to z)$, and (ii) $x \to y = 1$ iff $x \leq y$. This theorem sheds new light on the meaning of the Łukasiewicz axioms, notably the intriguing axiom $((x \to y) \to y) = ((y \to x) \to x)$.

Corollary 4.2 yields a unique isomorphism ϕ between W and the standard Wajsberg algebra $W_Ł = ([0,1], 1, \neg_Ł, \to_Ł)$. So this paper may serve as an introduction to Łukasiewicz logic for college mathematics students. Historical remarks and further motivation will be given in a final section. The symbol " $\Rightarrow$" is to be read as "implies". The symbol " $\Leftrightarrow$" is an abbreviation of "iff", meaning "if and only if".

§2. Order, exchange, and continuity.

DEFINITION 2.1. Let $\to$ be a $[0,1]$-valued function defined on the real unit square $[0,1]^2$. We then say that $\to$ has the *order property* if for all $x, y \in [0,1]$, $x \to y = 1 \Leftrightarrow x \leq y$. Further, $\to$ has the *exchange property* if for all $x, y, z \in [0,1]$, $x \to (y \to z) = y \to (x \to z)$. A $[0,1]$-valued function $\to$ defined on the real unit square $[0,1]^2$ is called *implicative* if it has both the exchange and the order property.

Received May 13, 2018.
2020 *Mathematics Subject Classification*. 03B50, 06D35.
Keywords and phrases. Łukasiewicz logic, Łukasiewicz calculus, MV-algebra, Wajsberg algebra, Łukasiewicz axioms, Łukasiewicz implication, [0,1]-valued logic, [0,1]-valued observable, [0,1]-valued event, Modus Ponens, exchange property, order determining implication.

0022-4812/20/8503-0002
DOI:10.1017/jsl.2020.74

PROPOSITION 2.2. *Let* $\rightarrow$ *be an implicative function. Let the functions* $\neg$, $\oplus$ *and* $\odot$ *be defined by* $\neg x = x \rightarrow 0$, $x \oplus y = \neg x \rightarrow y$ *and* $x \odot y = \neg(x \rightarrow \neg y)$. *Then for all* $x, y, z \in [0,1]$ *we have*:

$$x \rightarrow y \;\geq\; y, \tag{1}$$

$$1 \rightarrow x = x, \tag{2}$$

$$x \leq y \;\Rightarrow\; y \rightarrow z \leq x \rightarrow z, \tag{3}$$

$$(x \rightarrow y) \rightarrow y \;\geq\; \max(x,y), \tag{4}$$

$$x \leq y \Rightarrow \neg x \;\geq\; \neg y. \tag{5}$$

If, in addition, $\rightarrow$ *is continuous then so are* $\neg, \oplus$ *and* $\odot$. *Considering the* $\neg$ *function more binding than any binary function, we have for all* $x, y, z \in [0,1]$:

$$\neg\neg x = x, \tag{6}$$

$$x \rightarrow y = \neg y \rightarrow \neg x, \tag{7}$$

$$x \oplus y = y \oplus x, \tag{8}$$

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z, \tag{9}$$

$$x \odot y \leq z \;\Leftrightarrow\; x \leq y \rightarrow z \;\Leftrightarrow\; y \leq x \rightarrow z \;\Leftrightarrow\; y \odot x \leq z, \tag{10}$$

$$x \odot y = y \odot x, \tag{11}$$

$$x \odot (x \rightarrow y) \;\leq\; y, \tag{12}$$

$$y \leq z \Rightarrow x \rightarrow y \;\leq\; x \rightarrow z, \tag{13}$$

$$y \leq z \Rightarrow x \odot y \;\leq\; x \odot z, \tag{14}$$

$$x \odot (y \odot z) = (x \odot y) \odot z, \tag{15}$$

$$x \rightarrow y = \max\{t \mid x \odot t \leq y\}, \tag{16}$$

$$x \odot (x \rightarrow y) = \min(x,y), \tag{17}$$

$$(y \rightarrow x) \rightarrow x = \max(x,y). \tag{18}$$

PROOF. The implicative property of the $\rightarrow$ function will be applied throughout without explicit mention: the reader will always be able to discern which one of the exchange or the order property is being applied.

(1) is proved by writing $1 = x \rightarrow 1 = x \rightarrow (y \rightarrow y) = y \rightarrow (x \rightarrow y)$. For a proof of (2), from $1 = (1 \rightarrow x) \rightarrow (1 \rightarrow x) = 1 \rightarrow ((1 \rightarrow x) \rightarrow x)$ we get $1 \rightarrow x \leq x$. The converse inequality is proved in (1). To prove (3) we first write $1 = (y \rightarrow z) \rightarrow (y \rightarrow z) = y \rightarrow ((y \rightarrow z) \rightarrow z)$, whence $y \leq ((y \rightarrow z) \rightarrow z)$. From $x \leq y$ we get $x \leq (y \rightarrow z) \rightarrow z$. Therefore, $1 = x \rightarrow ((y \rightarrow z) \rightarrow z) = (y \rightarrow z) \rightarrow (x \rightarrow z)$, whence $y \rightarrow z \leq x \rightarrow z$. For (4), from $1 = (x \rightarrow y) \rightarrow (x \rightarrow y) = x \rightarrow ((x \rightarrow y) \rightarrow y)$ we have $x \leq (x \rightarrow y) \rightarrow y$. On the other hand, the identity $y \leq (x \rightarrow y) \rightarrow y$ follows from $y \rightarrow ((x \rightarrow y) \rightarrow y) =$

https://doi.org/10.1017/jsl.2020.74 Published online by Cambridge University Press

$(x \to y) \to (y \to y) = 1$. From $x \leq y$ and (3) we get $\neg x = x \to 0 \geq y \to 0 = \neg y$, which settles (5).

We now assume that the implicative function $\to$ is continuous, whence so are the derived functions $\neg$, $\oplus$ and $\odot$.

The proof of (6) is in two steps. From (4) we obtain $x \leq (x \to 0) \to 0 = \neg\neg x$, whence by (5), $\neg x \geq \neg\neg\neg x$. Conversely, from $\neg x \to \neg\neg\neg x = \neg x \to (((x \to 0) \to 0) \to 0) = ((x \to 0) \to 0) \to (\neg x \to 0) = ((x \to 0) \to 0) \to ((x \to 0) \to 0) = 1$ we get $\neg x \leq \neg\neg\neg x$, whence $\neg x = \neg\neg\neg x$. Since by (2), $\neg 0 = 0 \to 0 = 1$ and $\neg 1 = 1 \to 0 = 0$, the range of the continuous function $\neg$ coincides with $[0,1]$, whence for every $x \in [0,1]$ there is $a = a_x \in [0,1]$ such that $x = \neg a$. Thus, $\neg\neg x = \neg\neg\neg a = \neg a = x$. This settles (6). For a proof of (7), using (6) we can write $(x \to y) \to (\neg y \to \neg x) = (x \to y) \to (\neg y \to (x \to 0)) = (x \to y) \to (x \to (\neg y \to 0)) = (x \to y) \to (x \to \neg\neg y) = (x \to y) \to (x \to y) = 1$, thus showing that $x \to y \leq \neg y \to \neg x$. The converse inequality now follows from (6). By (6) and (7) we can write $x \oplus y = \neg x \to y = \neg y \to \neg\neg x = \neg y \to x = y \oplus x$, which settles (8). Using (6), (7) and (8), the identity (9) follows by writing $(x \oplus y) \oplus z = \neg(\neg x \to y) \to z = \neg z \to \neg\neg(\neg x \to y) = \neg z \to (\neg x \to y) = \neg x \to (\neg z \to y) = x \oplus (z \oplus y) = x \oplus (y \oplus z)$. To prove (10), using (3) and (6)–(7) we can write $x \odot y \leq z \Leftrightarrow \neg(x \to \neg y) \leq z \Leftrightarrow x \to \neg y \geq \neg z \Leftrightarrow \neg z \to (x \to \neg y) = 1 \Leftrightarrow x \to (\neg z \to \neg y) = 1 \Leftrightarrow x \to (y \to z) = 1 \Leftrightarrow x \leq (y \to z)$. Since $x \to (y \to z) = 1 \Leftrightarrow y \to (x \to z) = 1$, the proof of (10) is completed by interchanging the roles of x and y. From (10) it follows that $x \odot y$ and $y \odot x$ have the same upper bounds, so they must coincide. This settles (11). Now (11) yields $x \odot (x \to y) = (x \to y) \odot x$. From (10) we have $(x \to y) \odot x \leq y \Leftrightarrow x \to y \leq x \to y$. Since the latter inequality holds for all $x, y \in [0,1]$, then so does the former, and (12) is proved. The proof of (13) is as follows: by (11) and (12), $(x \to y) \odot x = x \odot (x \to y) \leq y$, whence $y \leq z \Rightarrow x \odot (x \to y) \leq z$. Further, by (11) and (10), $x \odot (x \to y) \leq z \Leftrightarrow (x \to y) \odot x \leq z \Leftrightarrow x \to y \leq x \to z$. (14) now follows by combining (13) with (6)–(7). (15) follows from (6)–(7) and (11) by writing $x \odot (y \odot z) = \neg(x \to \neg(y \odot z)) = \neg(x \to \neg\neg(y \to \neg z)) = \neg(x \to (y \to \neg z)) = \neg(x \to (z \to \neg y)) = \neg(z \to (x \to \neg y)) = \neg(z \to \neg\neg(x \to \neg y)) = \neg(z \to \neg(x \odot y)) = z \odot (x \odot y) = (x \odot y) \odot z$. To prove (16), let $\mathcal{W} = \{t \mid x \odot t \leq y\}$. Since $\odot$ is continuous, $\sup \mathcal{W} = \max \mathcal{W}$. Since $x \odot 0 = 0$ then, by (14), $\mathcal{W}$ is a nonempty interval. By (10), $\max(\mathcal{W}) \geq z \Leftrightarrow z \in \mathcal{W} \Leftrightarrow x \odot z \leq y \Leftrightarrow x \to y \geq z$. As a consequence, $x \to y = \max(\mathcal{W})$. Proof of (17). Since $\odot$ is continuous and monotonic, a number $w \in [0,1]$ is of the form $x \odot t$ for some $t \in [0,1]$ iff $w \in [0,x]$. By (16), $x \odot (x \to y) = x \odot \max\{t \mid x \odot t \leq y\} = \max\{x \odot t \mid x \odot t \leq y\} = \max\{w \mid w \leq y,\ w \in [0,x]\} = \min(x,y)$. Finally, to prove (18), iterated application of (7) and (6) using (17) yields: $\max(x,y) = \neg\min(\neg x, \neg y) = \neg(\neg x \odot (\neg x \to \neg y)) = \neg\neg(\neg x \to \neg(\neg x \to \neg y)) = \neg x \to \neg(\neg x \to \neg y) = (\neg x \to \neg y) \to x = (y \to x) \to x$. ⊣

THEOREM 2.3. *Suppose the implicative function $\to$ is continuous. Let the functions $\neg$ and $\oplus$ be as in Proposition 2.2. Let the algebras $A = A_\to$ and $W = W_\to$ be defined by $A = ([0,1], 0, \neg, \oplus)$ and $W = ([0,1], 1, \neg, \to)$. We then have:*

(a) *A is an MV-algebra, that is, A satisfies the axioms*
MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
MV2) $x \oplus y = y \oplus x$,
MV3) $x \oplus 0 = x$,
MV4) $\neg\neg x = x$,

https://doi.org/10.1017/jsl.2020.74 Published online by Cambridge University Press

MV5) $x \oplus \neg 0 = \neg 0$,
MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

(b) *W is a Wajsberg algebra, that is, W satisfies the axioms*
Ł1) $1 \to x = x$,
Ł2) $(z \to x) \to ((x \to y) \to (z \to y)) = 1$,
Ł3) $((x \to y) \to y) = ((y \to x) \to x)$,
Ł4) $(\neg x \to \neg y) \to (y \to x) = 1$.

Proof.

(a) MV1) is (9). MV2) is (8). MV4) is (6). MV6) is a reformulation of (18). Using MV2) and (2), MV3) is proved by writing $x \oplus 0 = 0 \oplus x = \neg 0 \to x = 1 \to x = x$. Using MV2) and (6), MV5) is proved by writing $x \oplus \neg 0 = \neg 0 \oplus x = \neg\neg 0 \to x = 0 \to x = 1$.

(b) Ł1) is (2). Ł3) follows from (18). Ł4) is (7). To settle Ł2) in view of the continuity of the functions $\to, \neg, \odot$, using the monotonicity, associativity and commutativity of $\odot$, we first prove the inequality

$$(u \to v) \odot w \leq u \to (v \odot w), \text{ for all } u, v, w \in [0,1]. \tag{19}$$

As a matter of fact, from (16) we have the equivalences $(u \to v) \odot w \leq u \to (v \odot w) \Leftrightarrow w \odot (u \to v) \leq u \to (v \odot w) \Leftrightarrow w \odot \max\{s \mid u \odot s \leq v\} \leq \max\{t \mid u \odot t \leq v \odot w\} \Leftrightarrow \max\{w \odot s \mid u \odot s \leq v\} \leq \max\{t \mid u \odot t \leq v \odot w\} \Leftrightarrow \max\{t \mid t = w \odot s \text{ and } u \odot s \leq v\} \leq \max\{t \mid u \odot t \leq v \odot w\}$. Thus, for the proof of (19) it suffices to prove

$$(t = w \odot s \text{ and } u \odot s \leq v) \Rightarrow u \odot t \leq v \odot w, \quad \text{for all } s, t, u, v, w \in [0,1]. \tag{20}$$

Commutativity (11) and associativity (15) yield $u \odot (w \odot s) = (u \odot s) \odot w$. Monotonicity (14) yields $u \odot s \leq v \Rightarrow (u \odot s) \odot w \leq v \odot w$. A fortiori,

$$(t = w \odot s \text{ and } u \odot s \leq v) \Rightarrow u \odot t = u \odot (w \odot s) = (u \odot s) \odot w \leq v \odot w,$$

which settles (20) and completes the proof of (19).

The inequality $(x \to y) \odot t \leq x \to (y \odot t)$ is an instance of (19). Thus by (13), $y \odot t \leq z \Rightarrow (x \to y) \odot t \leq x \to z$, whence $\max\{t \mid y \odot t \leq z\} \leq \max\{t \mid (x \to y) \odot t \leq x \to z\}$. By (16), $y \to z \leq (x \to y) \to (x \to z)$, that is, $1 = (y \to z) \to ((x \to y) \to (x \to z)) = (x \to y) \to ((y \to z) \to (x \to z))$, which settles Ł2). ⊣

§3. Further properties of continuous implicative functions. In this section a *self-contained* proof is given of the existence of a unique isomorphism ϕ of the MV-algebra $A = A_\to$ onto the standard MV-algebra $A_Ł = ([0,1], 0, \neg_Ł, \oplus_Ł)$ where $\neg_Ł = 1 - x$ and $x \oplus_Ł y = \min(1, x + y)$. This strengthening of Theorem 2.3 is obtained as a corollary of the key Theorem 4.1, where an isomorphism ψ is constructed from a dense subalgebra of A onto the subalgebra of $A_Ł$ consisting of all rational numbers in $[0,1]$.

Proposition 3.1. *For* $\to$ *an arbitrary continuous implicative function, let* $A = A_\to = ([0,1], 0, \neg, \oplus)$ *be the MV-algebra of Theorem* 2.3. *With* $x \odot y = \neg(x \to \neg y)$ *as*

https://doi.org/10.1017/jsl.2020.74 Published online by Cambridge University Press

defined in Proposition 2.2, let the $[0,1]$*-valued function* $\ominus$ *be defined by*

$$x \ominus y = x \odot \neg y \quad \textit{for all } x, y \in [0,1].$$

For all $x, y, z, \varepsilon \in [0,1]$ *we then have*:

$$x \ominus y = \neg(x \to y), \tag{21}$$

$$x \to y = \neg x \oplus y, \tag{22}$$

$$y \leq z \ \Rightarrow \ x \oplus y \leq x \oplus z, \tag{23}$$

$$x \odot y = \neg(\neg x \oplus \neg y), \quad x \oplus y = \neg(\neg x \odot \neg y), \tag{24}$$

$$x < y \Rightarrow \neg x > \neg y, \tag{25}$$

$$\textit{If } \ 0 \leq y < z \leq 1 \quad \textit{then} \quad z \ominus y > 0 \quad \textit{and} \quad z = y \oplus (z \ominus y), \tag{26}$$

$$\textit{If } \ y < 1 \quad \textit{and} \quad \varepsilon > 0 \quad \textit{then} \quad y \oplus \varepsilon > y, \tag{27}$$

$$\textit{If } \ z > 0 \quad \textit{and} \quad \varepsilon < 1 \quad \textit{then} \quad z \odot \varepsilon < z. \tag{28}$$

Proof. Both identities (21) and (22) immediately follow from (6). (23) then follows from (22) and (13). For (24), from (7) and (6) we get $x \oplus y = \neg x \to y = \neg y \to x = \neg(\neg y \odot \neg x)$. The identity $x \odot y = \neg(\neg x \oplus \neg y)$ follows in a similar way. To prove (25), suppose $x < y$ and $\neg x \leq \neg y$ (absurdum hypothesis). By (5), $\neg x \geq \neg y$, whence $\neg x = \neg y$. By (6) $y = \neg\neg y = \neg\neg x = x$, a contradiction. For the first statement of (26), arguing by way of contradiction, if $z \odot \neg y = 0$ then $1 = \neg(z \odot \neg y) = z \to y$, whence $z \leq y$, which is impossible. For the second statement, by hypothesis together with (7) and (18) we get $y \oplus (z \odot \neg y) = \neg y \to \neg(z \to y) = (z \to y) \to y = \max(z, y) = z$. We are now in a position to prove (27). By (14), $\odot$ is a monotonically increasing continuous function. The range of the function $z \in [y, 1] \mapsto z \ominus y$ is the nonsingleton interval $[0, \neg y]$. Thus for any $0 < \varepsilon \leq \neg y$ there is $z = z_\varepsilon$ with $z \ominus y = \varepsilon$. Necessarily, $z > y$. By (26), $y \oplus \varepsilon = y \oplus (z \ominus y) = z > y$. The inequality $y \oplus \varepsilon > y$ actually holds for all $0 < \varepsilon \in [0,1]$, because of the monotonicity and commutativity properties (8) and (23) of the $\oplus$ function. This settles (27). To prove (28), by (24) we can write $z \odot \varepsilon = \neg(\neg z \oplus \neg \varepsilon)$. For all $\varepsilon < 1$, (27) yields $\neg z \oplus \neg \varepsilon > \neg z$, because $\neg z < 1$ and $\neg \varepsilon > 0$. The desired conclusion now follows from (25). ⊣

Proposition 3.2. *For* $\to$ *an arbitrary continuous implicative function, let* $A = A_{\to} = ([0,1], 0, \neg, \oplus)$ *be the MV-algebra of Theorem 2.3. We then have*:

(a) *For all* x, y *in the open interval* $[0,1] \setminus \{0,1\}$ *we have the following equivalence*: $x = \neg y \Leftrightarrow (x \oplus y = 1$ *and* $x \oplus (y \ominus \varepsilon) < 1$ *for all* $\varepsilon > 0)$.

(b) *Since* $\oplus$ *is associative, for any* $x \in A$ *and* $k = 1, 2, \ldots$ *we may use the notation* $k.x = \underbrace{x \oplus \cdots \oplus x}_{k \text{ summands}}$ *and* $0.x = 0$. *Then for all* $n = 1, 2, \ldots$ *there is precisely one* $z \in [0,1]$ *satisfying the equation*

$$(n-1).z = \neg z. \tag{29}$$

https://doi.org/10.1017/jsl.2020.74 Published online by Cambridge University Press

(c) *Let* $a, b, \varepsilon \in [0,1]$ *and* $k = 0, 1, \ldots$. *Then the following identities hold*:

$$(b \oplus \varepsilon) \ominus \varepsilon = b, \quad (\textit{for all } b < 1 \textit{ and } \varepsilon < \neg b). \tag{30}$$

$$((k+1).a) \ominus \varepsilon \leq k.a \oplus (a \ominus \varepsilon), \quad (\textit{provided } (k+1).a < 1 \textit{ and } 0 < \varepsilon < a). \tag{31}$$

PROOF.

(a) ($\Rightarrow$) By hypothesis, $x \oplus y = x \oplus \neg x = \neg x \to \neg x = 1$. For all $\varepsilon > 0$ with $\varepsilon \leq y$, (28) yields the inequality $\neg x \ominus \varepsilon = \neg x \odot \neg \varepsilon < \neg x$, because $\neg x > 0$ and $\neg \varepsilon < 1$. As a consequence, $x \oplus (y \ominus \varepsilon) = (x \oplus (\neg x \ominus \varepsilon)) = \neg x \to (\neg x \ominus \varepsilon) < 1$. ($\Leftarrow$) We will repeatedly use the monotonicity of $\oplus$, and the fact that $y \ominus \varepsilon$ is monotonically decreasing when ε increases. The hypothesis $x \oplus y = 1$ means $1 = \neg x \to y$, that is, $\neg x \leq y$. For the converse inequality, the assumption $x \oplus (y \ominus \varepsilon) = \neg x \to (y \ominus \varepsilon) < 1$ for all $\varepsilon > 0$ entails $\neg x > (y \ominus \varepsilon)$ for all $\varepsilon > 0$. From the continuity of $\ominus$ and $y \ominus 0 = y$ we then obtain $\neg x \geq y$. Having thus proved $\neg x = y$, the desired conclusion follows from (6).

(b) For $n = 1$ the only possible solution of (29) is $z = 1$. So assume $n = 2, 3, \ldots$. The continuous function $x \mapsto n.x$ increasingly covers the interval $[0,1]$ while $\neg x$ decreasingly covers $[0,1]$. Therefore, (29) has at least one solution $z \in [0,1]$. To prove uniqueness, arguing by way of contradiction, suppose both x and y satisfy (29), with $x < y$. Then $0 < x < y < 1$. By (a), $n.x = n.y = 1$, and for all $\varepsilon > 0$ both $(n-1).x \oplus (x \ominus \varepsilon)$ and $(n-1).y \oplus (y \ominus \varepsilon)$ are < 1. Since, a fortiori, $(n-1).x < 1$, (23) yields $\neg x = (n-1).x \leq (n-1).y = \neg y$, whence by (24), $x \geq y$, a contradiction.

(c) Using (18), the identity (30) is proved by writing $(b \oplus \varepsilon) \ominus \varepsilon = (\neg b \to \varepsilon) \odot \neg \varepsilon = \neg((\neg b \to \varepsilon) \to \varepsilon) = \neg \max(\neg b, \varepsilon) = \neg\neg b = b$. Finally let us prove (31). Recalling (26), for all $0 < \varepsilon < a$ we may write $a = \varepsilon \oplus \theta$, with $0 < \theta = a \ominus \varepsilon < a$. Upon setting $b = k.a \oplus \theta$ we obtain $((k+1).a) \ominus \varepsilon = (k.a \oplus a) \ominus \varepsilon = (k.a \oplus \theta \oplus \varepsilon) \ominus \varepsilon = (b \oplus \varepsilon) \ominus \varepsilon$. By hypothesis and (27), $b = k.a \oplus \theta < k.a \oplus a = (k+1).a < 1$, whence an application of (30) yields $((k+1).a) \ominus \varepsilon = (b \oplus \varepsilon) \ominus \varepsilon = b = k.a \oplus \theta = k.a \oplus (a \ominus \varepsilon)$, as desired. ⊣

3.1. Ideals and the underlying order of an MV-algebra. The *underlying order* $\leq_A$ of an MV-algebra $A = (A, 0, \neg, \oplus)$ is defined by $x \leq_A y \Leftrightarrow \neg x \oplus y = \neg 0 = 1$. An *ideal* I of A is a proper subset of A containing 0, *closed under minorants* (i.e., $y \in I$ and $x \leq_A y \Rightarrow x \in I$), and under the $\oplus$ function (i.e., $x, y \in I \Rightarrow x \oplus y \in I$).

Let us denote by $n^{-\circ 1}$ the uniquely determined z satisfying $(n-1).z = \neg z$ in (29).

PROPOSITION 3.3. *For* $\to$ *an arbitrary continuous implicative function, let* $A = A_{\to} = ([0,1], 0, \neg, \oplus)$ *be the MV-algebra of Theorem* 2.3. *We then have*:

(a) *The order* $\leq_A$ *coincides with the natural order* $\leq$ *of* $[0,1]$, *whence it is complete* (*every nonempty subset* T *of* $[0,1]$ *has a least upper bound* $\sup T$ *and a greatest lower bound* $\inf T$ *with respect to* $\leq_A$). *Accordingly, we will write* $\leq$ *instead of* $\leq_A$.

(b) *A is simple, in the sense that its only ideal is the singleton* $\{0\}$.

(c) *For every* $0 < x \leq 1$ *there is* $n = 1, 2, \ldots$ *with* $n.x = 1$.

(d) *For every* $0 < x \leq 1$ *there is* $m = 1, 2, \ldots$ *with* $m^{-\circ 1} < x$. *Thus in particular,* $\inf\{n^{-\circ 1} \mid n = 1, 2, \ldots\} = 0$.

https://doi.org/10.1017/jsl.2020.74 Published online by Cambridge University Press

PROOF.

(a) By (22), $x \leq_A y \Leftrightarrow x \to y = 1 \Leftrightarrow x \leq y$.

(b) Arguing by way of contradiction, suppose A has an ideal I different from $\{0\}$. There is $a > 0$ such that either $I = [0,a]$ or $I = [0,a] \setminus \{a\}$. Let $s = \sup I$ (such s existing by (a)). Then $s > 0$. Since $0 \oplus 0 = 0$ and $2^{-\circ 1} \oplus 2^{-\circ 1} = 1$, the monotonicity property (27) of the continuous function $x \mapsto x \oplus x$ defined on $[0, 2^{-\circ 1}]$ yields an element $h \leq 2^{-\circ 1}$ such that $h \oplus h = s$. Necessarily, $h > 0$ and, by (27), $h < s$. By definition of s, there is $j \in I$ with $h \leq j$. Thus h belongs to I and so does $s = h \oplus h$. Since I is a proper subset of $[0,1]$ then $s < 1$. Another application of (27) yields $s < s \oplus s \in I$, which contradicts the definition of s.

(c) Again by way of contradiction, let $x > 0$ satisfy $n.x < 1$ for all $n = 1,2,\dots$. Let $J = \{y \in [0,1] \mid n.x \geq y \text{ for some } n = 1,2,\dots\}$. Then $1 \notin J$ and $x \in J$. It is easy to see that whenever $z \leq y$ and $y \in J$ then $z \in J$. Further, $y,z \in J \Rightarrow z \oplus y \in J$. So J is an ideal of A other than $\{0\}$, against (b).

(d) We must find $m = 1,2,\dots$ with $m^{-\circ 1} < x$. If $x = 1$, then $2^{-\circ 1}$ will do. So let us assume $x < 1$. Let n be the smallest integer satisfying $n.x = 1$ (such n existing by (c)). Then $n \geq 2$. Let $m = (n+1)$. By definition, $0 < m^{-\circ 1}$. If $m^{-\circ 1} \geq x$ (absurdum hypothesis) then $n.m^{-\circ 1} \geq n.x = 1$. However, $n.m^{-\circ 1} = (m-1).m^{-\circ 1} = \neg m^{-\circ 1} < 1$, a contradiction. ⊣

PROPOSITION 3.4. *For $\to$ an arbitrary continuous implicative function, let $A = A_\to = ([0,1], 0, \neg, \oplus)$ be the MV-algebra of Theorem 2.3. With the notation of Proposition 3.2(b) let the set $A_{\mathbb{Q}} \subseteq [0,1]$ be defined by*

$$A_{\mathbb{Q}} = \{0,1\} \cup \{h.n^{-\circ 1} \mid n = 2,3,\dots \text{ and } h = 1,2,\dots,n-1\}.$$

*For any $h.n^{-\circ 1} \in A_{\mathbb{Q}} \setminus \{0,1\}$ let us agree to say that h is its A-*numerator *and n its A-*denominator. *Then for each $n = 2,3,\dots$ we have:*

$$\neg\, l.n^{-\circ 1} = (n-l).n^{-\circ 1}, \; (l = 1,2,\dots,n-1), \text{ and} \tag{32}$$

$$n^{-\circ 1} = k.(kn)^{-\circ 1}, \; (k = 1,2,\dots). \tag{33}$$

Further, for all $m,n \in \{2,3,\dots\}$, $p = 1,2,\dots,m-1$ and $q = 1,2,\dots,n-1$,

$$p.m^{-\circ 1} \oplus q.n^{-\circ 1} = \min(mn, (qm+pn)).(mn)^{-\circ 1}, \text{ and} \tag{34}$$

$$p.m^{-\circ 1} = q.n^{-\circ 1} \;\Leftrightarrow\; pn = qm. \tag{35}$$

PROOF. The proof of (32) in case $l = 1$ immediately follows by definition. So assume $l > 1$. We have $(n-l).n^{-\circ 1} \oplus l.n^{-\circ 1} = 1$. By Proposition 3.2(a), there remains to be proved $(n-l).n^{-\circ 1} \oplus (l.n^{-\circ 1} \ominus \varepsilon) < 1$ for all $\varepsilon > 0$, that is,

$$(n-l).n^{-\circ 1} \oplus (((l-1).n^{-\circ 1} \oplus n^{-\circ 1}) \ominus \varepsilon) < 1 \text{ for all } \varepsilon > 0. \tag{36}$$

Since $l.n^{-\circ 1} \leq (n-2).n^{-\circ 1} < 1$, from (31) it follows that (36) amounts to saying that for all $\varepsilon > 0$, $(n-l).n^{-\circ 1} \oplus ((l-1).n^{-\circ 1}) \oplus (n^{-\circ 1} \ominus \varepsilon) < 1$. This latter inequality holds because $(n-1).n^{-\circ 1} \oplus (n^{-\circ 1} \ominus \varepsilon) < 1$.

Proof of (33). By Proposition 3.2(b) we must show $(n-1).(k.(kn)^{-\circ 1}) = \neg k.(kn)^{-\circ 1}$. Proposition 3.2(a) immediately yields $(n-1).k.(kn)^{-\circ 1} \oplus k.(kn)^{-\circ 1} = nk.(kn)^{-\circ 1} = 1$. Next let us turn to the inequality $(n-1).k.(kn)^{-\circ 1} \oplus (k.(kn)^{-\circ 1} \ominus \varepsilon) < 1$

https://doi.org/10.1017/jsl.2020.74 Published online by Cambridge University Press

(for all $\varepsilon > 0$). By (31), this is equivalent to $(n-1).k.(kn)^{-\circ 1} \oplus (k-1).(kn)^{-\circ 1} \oplus ((kn)^{-\circ 1} \ominus \varepsilon) < 1$. Now the left hand term equals $(kn-k+k-1).(kn)^{-\circ 1} \oplus ((kn)^{-\circ 1} \ominus \varepsilon)$, which is < 1 by definition of $(kn)^{-\circ 1}$. This settles (33).

To prove (34), using (33) and passing to the common A-denominator mn of $p.m^{-\circ 1}$ and $q.n^{-\circ 1}$, we may write $p.m^{-\circ 1} \oplus q.n^{-\circ 1} = np.(nm)^{-\circ 1} \oplus mq.(mn)^{-\circ 1} = (np+mq).(mn)^{-\circ 1}$. This latter term equals $mn.(mn)^{-\circ 1} = 1$ whenever $np + mq \geq nm$.

For (35), mimicking the proof of (34) we rewrite $p.m^{-\circ 1}$ and $q.n^{-\circ 1}$ into their equivalents with common A-denominator mn and A-numerators respectively given by pn and qm. By (27), $p.m^{-\circ 1}$ and $q.n^{-\circ 1}$ are equal iff qm is equal to pn. ⊣

COROLLARY 3.5. *For $\to$ an arbitrary continuous implicative function, let $A = A_{\to} = ([0,1], 0, \neg, \oplus)$ be the MV-algebra of Theorem 2.3. Then $A_{\mathbb{Q}}$ is the underlying set of a subalgebra of A, also denoted $A_{\mathbb{Q}}$.*

§4. The standard Wajsberg algebra and the standard MV-algebra.

For all $x, y \in [0,1]$ let $\neg_Ł x = 1 - x$ and $x \to_Ł y = \min(1, 1 - x + y)$. The continuous implicative function $\to_Ł$ is known as the *Łukasiewicz implication*. The *standard Wajsberg algebra* $W_Ł = ([0,1], 0, \neg_Ł, \to_Ł)$ satisfies axioms Ł1)–Ł4).

Similarly, for all $x, y \in [0,1]$ let $x \oplus_Ł y = \min(1, x + y) = \neg_Ł x \to_Ł y$. The *standard MV-algebra* $A_Ł = ([0,1], 0, \neg_Ł, \oplus_Ł)$ satisfies axioms MV1)–MV6).

The subalgebra $A_{Ł,\mathbb{Q}}$ of $A_Ł$ whose universe are the rational numbers in $[0,1]$ is called the *standard rational MV-algebra*.

THEOREM 4.1. *For $\to$ an arbitrary continuous implicative function, let $A = A_{\to} = ([0,1], 0, \neg, \oplus)$ be the MV-algebra of Theorem 2.3. Let $A_{\mathbb{Q}}$ be the subalgebra of A given by Corollary 3.5.*

(i) *There is an isomorphism ψ of $A_{\mathbb{Q}}$ onto $A_{Ł,\mathbb{Q}}$. Thus in particular,*

$$\psi(x \to y) = \psi(x) \to_Ł \psi(y) = \min(1, 1 - \psi(x) + \psi(y)). \tag{37}$$

(ii) *For all $p, q \in A_{\mathbb{Q}}$, $p \leq q \;\Leftrightarrow\; \psi(p) \leq \psi(q)$.*

(iii) *For all p, q with $0 \leq p < q \leq 1$ there exists $a \in A_{\mathbb{Q}}$ with $p < a < q$.*

(iv) *For every $x \in [0,1]$, let us set $L_x = \{\psi(z) \mid z \leq x,\, z \in A_{\mathbb{Q}}\}$ and $R_x = \{\psi(y) \mid y \geq x,\, y \in A_{\mathbb{Q}}\}$. Let further $l = \sup L_x$ and $r = \inf R_x$, as given by Proposition 3.3(a). Then $l = r$.*

PROOF.

(i) For $x \in \{0,1\}$ let us set $\psi(x) = x$. For each $k.m^{-\circ 1} \in A_{\mathbb{Q}}$, let $\psi(k.m^{-\circ 1})$ be the rational $k/m \in [0,1]$. If $l.n^{-\circ 1}$ belongs to $A_{\mathbb{Q}}$ and $k.m^{-\circ 1} = l.n^{-\circ 1}$, then from (35) it follows that $\psi(k.m^{-\circ 1}) = k/m = l/n = \psi(l.n^{-\circ 1})$. Thus ψ is a *function* from $A_{\mathbb{Q}}$ into $[0,1]$. Evidently, ψ is *onto* $[0,1] \cap \mathbb{Q}$. To prove that ψ is *one–one*, first observe that $\psi(x) = 0$ implies $x = 0$ and $\psi(y) = 1$ implies $y = 1$. Next suppose $k.m^{-\circ 1}$ and $l.n^{-\circ 1}$ are distinct elements of $A_{\mathbb{Q}}$. By (33) and (35), the equivalents of $k.m^{-\circ 1}$ and $l.n^{-\circ 1}$ with common A-denominator mn have different A-numerators. Then $\psi(k.m^{-\circ 1})$ and $\psi(l.n^{-\circ 1})$ are rational numbers with equal denominators and different numerators, which shows that ψ is one–one. In view of Corollary 3.5, the proof that ψ is an isomorphism of $A_{\mathbb{Q}}$ onto $A_{Ł,\mathbb{Q}}$ proceeds as follows: On the one hand, by (32), $\psi(\neg k.m^{-\circ 1}) = \psi((m-k).m^{-\circ 1}) = (m-k).\psi(m^{-\circ 1}) =$

https://doi.org/10.1017/jsl.2020.74 Published online by Cambridge University Press

$\frac{m-k}{m} = \neg_{Ł}\frac{k}{m} = \neg_{Ł}\psi(k.m^{-\circ 1})$. On the other hand, by (34), $\psi(p.m^{-\circ 1} \oplus q.n^{-\circ 1}) = \psi(\min(mn, pn+qm).(mn)^{-\circ 1}) = \min(mn, pn+qm).\psi((mn)^{-\circ 1}) = \min(mn, pn+qm).(\frac{1}{mn}) = \frac{p}{m} \oplus_{Ł} \frac{q}{n} = \psi(p.m^{-\circ 1}) \oplus \psi(q.n^{-\circ 1})$. Having thus shown that ψ is the desired isomorphism, (37) follows from (22).

(ii) For all $p, q \in A_{\mathbb{Q}}$, by (22) we have $p \leq q \Leftrightarrow p \rightarrow q = 1 \Leftrightarrow \psi(p \rightarrow q) = 1 \Leftrightarrow \psi(p) \rightarrow_{Ł} \psi(q) = 1 \Leftrightarrow \psi(p) \leq \psi(q)$.

(iii) If either $p = 0$ or $q = 1$, Proposition 3.3(d) provides the desired a. If both p and q are A-rationals other than 0 or 1, by (33) we may assume they have the same *even* denominator $2d$, and *even* A-numerators, say, $p = 2h.(2d)^{-\circ 1} < q = 2k.(2d)^{-\circ 1}$. By (27), $p < (h+k)(2d)^{-\circ 1} < q$ and we are done. Next let us assume $0 < p < q < 1$ and $p \notin A_{\mathbb{Q}}$. (The case $q \notin A_{\mathbb{Q}}$ is similar.) For each $m = 2, 3, \ldots$, let $n_m =$ the largest integer n such that $n.m^{-\circ 1} < p$, whose existence is established by Proposition 3.3(d). Thus in particular, $A_{\mathbb{Q}} \ni (n_m+1).m^{-\circ 1} > p \notin A_{\mathbb{Q}}$. As m tends to ∞, $(m-1).m^{-\circ 1} = \neg m^{-\circ 1}$ tends to 1, because $m^{-\circ 1}$ tends to 0, by Proposition 3.3(d). Further, $n_m.m^{-\circ 1} < p$ by construction. Thus for all large integers m we have $\neg m^{-\circ 1} > n_m.m^{-\circ 1}$. Using (18) we get

$$\begin{aligned}
(n_m+1).m^{-\circ 1} \odot \neg n_m.m^{-\circ 1} &= \neg((n_m+1).m^{-\circ 1} \rightarrow n_m.m^{-\circ 1}) \\
&= \neg((m^{-\circ 1} \oplus n_m.m^{-\circ 1}) \rightarrow n_m.m^{-\circ 1}) \\
&= \neg((\neg m^{-\circ 1} \rightarrow n_m.m^{-\circ 1}) \rightarrow n_m.m^{-\circ 1}) \\
&= \neg \max(\neg m^{-\circ 1}, n_m.m^{-\circ 1}) \\
&= \neg\neg m^{-\circ 1} \\
&= m^{-\circ 1}.
\end{aligned}$$

Therefore, $\lim_{n\to\infty}((n_m+1).m^{-\circ 1} \ominus n_m.m^{-\circ 1}) = \lim_{n\to\infty} m^{-\circ 1} = 0$. Next, in view of Proposition 3.3(a), let $l = \sup\{n_m.m^{-\circ 1} \mid m = 2, 3, \ldots\} \leq p$ and $r = \inf\{(n_m+1).m^{-\circ 1} \mid m = 2, 3, \ldots\} \geq p$. From $\lim_{n\to\infty}(n_m+1).m^{-\circ 1} \ominus n_m.m^{-\circ 1} = 0$ and $(n_m+1).m^{-\circ 1} \ominus n_m.m^{-\circ 1} \geq r \ominus l = 0$, we obtain $l = r$ by (26). Thus some integer $\tilde{m}$ will satisfy $(n_{\tilde{m}}+1).\tilde{m}^{-\circ 1} < q$. Since $n_{\tilde{m}}.\tilde{m}^{-\circ 1} > p$, letting $a = n_{\tilde{m}}.\tilde{m}^{-\circ 1}$ the proof of (iii) is complete.

(iv) If $x \in A_{\mathbb{Q}}$ then $l = \psi(x) = r$ by (i), and we are done. In case $x \notin A_{\mathbb{Q}}$ by way of contradiction, suppose $l \neq r$, whence $l < r$. The denseness of the rationals yields $q \in \mathbb{Q}$ satisfying $l < q < r$. Since $\psi^{-1}(q)$ is a member of $A_{\mathbb{Q}}$ and x is not, then $\psi^{-1}(q) \neq x$, say, $\psi^{-1}(q) < x$ (the proof for the case $\psi^{-1}(q) > x$ is similar). By (iii) there is $a \in A_{\mathbb{Q}}$ satisfying $\psi^{-1}(q) < a < x$. Since a is a lower bound for x, then $\psi(a) \in L_x$, and hence, $\psi(a) \leq l$. Thus $\psi(\psi^{-1}(q)) = q < \psi(a) \leq l$, whence $q < l$, a contradiction. ⊣

COROLLARY 4.2. *For $\rightarrow$ an arbitrary continuous implicative function, let $A = A_{\rightarrow} = ([0,1], 0, \neg, \oplus)$ be the MV-algebra of Theorem 2.3. Let ψ be the isomorphism of Theorem 4.1(i). Let the $[0,1]$-valued function ϕ be defined by*

$$\phi(x) = \sup\{\psi(l) \mid l \leq x,\ l \in A_{\mathbb{Q}}\}, \ (x \in [0,1]).$$

https://doi.org/10.1017/jsl.2020.74 Published online by Cambridge University Press

(i) *ϕ is a one–one order preserving (whence, a continuous) function of* $[0,1]$ *onto* $[0,1]$ *extending ψ, and for all $x, y \in [0,1]$,*

$$x \to y \;=\; \phi^{-1}(\phi(x) \to_{Ł} \phi(y)) \;=\; \phi^{-1}(\min(1, 1-\phi(x)+\phi(y))). \qquad (38)$$

(ii) *ϕ is an isomorphism of A onto the standard MV-algebra $A_{Ł}$.*
(iii) *ϕ is the only isomorphism of A onto $A_{Ł}$.*

PROOF.

(i) First note that ϕ agrees with ψ over $A_{\mathbb{Q}}$. Applying Theorem 4.1(iii) twice, for all $x, y \in [0,1]$ with $x < y$ there are $l, r \in A_{\mathbb{Q}}$ such that $x < l < r < y$. Thus, by Theorem 4.1(ii), $\phi(x) \le \phi(l) = \psi(l) < \psi(r) = \phi(r) \le \phi(y)$, which shows that ϕ is one–one and order preserving. As a consequence, for all $x \in [0,1]$, $\phi(x) = \inf\{\psi(r) \mid r \ge x,\ r \in A_{\mathbb{Q}}\}$. To see that every $y \in [0,1]$ belongs to the range of ϕ, write $y = \sup\{l \in [0,1] \cap \mathbb{Q} \mid l \le y\}$. Let $Q = \{\psi^{-1}(l) \mid l \in [0,1] \cap \mathbb{Q},\ l \le y\}$. Since ϕ is order preserving, then by Theorem 4.1(iv), $\phi(\sup Q) = y$, thus showing that ϕ is onto $[0,1]$. The continuity of ϕ follows upon noting that the inverse image of any closed interval $I \subseteq [0,1]$ is a closed interval in $[0,1]$. To prove (38), by Theorem 4.1(i), the two *continuous* functions $(x,y) \mapsto x \to y$ and $(x,y) \mapsto \phi^{-1}(\phi(x) \to_{Ł} \phi(y)) = \phi^{-1}(\min(1, 1-\phi(x)+\phi(y)))$ agree on $A_{\mathbb{Q}}$, the latter being a dense subset of $[0,1]$. So these two functions agree over $[0,1]^2$.

(ii) By (38), for all $x, y \in [0,1]$ we have $\phi(x \to y) = \phi(x) \to_{Ł} \phi(y)$. It follows that $\phi(\neg x) = \phi(x \to 0) = \phi(x) \to_{Ł} \phi(0) = \phi(x) \to_{Ł} 0 = \neg_{Ł}\phi(x)$ and $\phi(x \oplus y) = \phi(\neg x \to y) = \phi(\neg x) \to_{Ł} \phi(y) = \neg_{Ł}\phi(x) \to_{Ł} \phi(y) = \phi(x) \oplus_{Ł} \phi(y)$. Therefore, ϕ is an isomorphism of A onto $A_{Ł}$.

(iii) Suppose $\varphi \ne \phi$ is another isomorphism of A onto $A_{Ł}$ (absurdum hypothesis). Then the composite function $\alpha = \varphi(\phi^{-1})$ is an automorphism of $A_{Ł}$ different from the identity. Fix $n = 1, 2, \ldots$. By direct inspection (or by applying Proposition 3.2(b) to $A_{Ł}$), for each $n = 2, 3, \ldots$ the rational $\frac{1}{n}$ is the unique solution z of the equation $(n-1) \cdot z = z \oplus_{Ł} \cdots \oplus_{Ł} z$ $(n-1$ times$) = \neg_{Ł} z = 1 - z$. The set $A_{Ł,\mathbb{Q}} = \{0,1\} \cup \{\frac{h}{n} \mid n = 2, 3, \ldots;\ h = 1, 2, \ldots, n-1\}$ coincides with the set of rational numbers in $[0,1]$. The automorphism α satisfies $\alpha(\frac{1}{n}) = \frac{1}{n}$, because α fixes unique solutions of equations. As a consequence, $\alpha(\frac{k}{n}) = \frac{k}{n}$ for all rationals $\frac{k}{n} \in [0,1]$, $k = 0, \ldots, n$. By our absurdum hypothesis, $\alpha(z) \ne z$ for some $z \in [0,1]$, say, $\alpha(z) < z$ (otherwise use $\neg z$ in place of z). Pick a rational d with $z < d < \alpha(z)$. Since α preserves order, $\alpha(z) < \alpha(d) = d$, which is impossible. ⊣

§5. Concluding remarks.

5.1. Historical and bibliographical remarks. The Łukasiewicz implication was originally introduced by Łukasiewicz in 1922 (see [2, pp. 129–130] for details). The Łukasiewicz axioms for his infinite-valued calculus are as follows [2, p. 144]: $A \to (B \to A)$, $(A \to B) \to ((B \to C) \to (A \to C))$, $((A \to B) \to B) \to ((B \to A) \to A)$, $(\neg B \to \neg A) \to (A \to B)$. An additional axiom in his original list was proved to follow from these four, by Chang and Meredith (see [3] for bibliographical details).

The axioms of Wajsberg algebras, as well as those of MV-algebras, are an algebraic counterpart of these axioms [3].

Readers familiar with the first chapters of [3] will immediately get a proof of Corollary 4.2 from Theorem 2.3 and Proposition 3.3, using the fact that every simple MV-algebra A is isomorphic to a subalgebra S of the standard MV-algebra $A_Ł$ [3, Theorem 3.5.1]. In particular, when the underlying order of A is also complete and dense, so is S, whence S coincides with $A_Ł$ [3, 3.5.3]. In this paper, however, a short direct proof has been given, with no MV-algebraic prerequisites. In this way the reader has been introduced to the beautiful theory of MV-algebras and Łukasiewicz logic without any prior mathematical background beyond knowledge of the basic properties of continuous real-valued functions on $[0,1]$ and $[0,1]^2$.

The function $\odot$ is an example of a continuous t-norm [4]. The identity (38) in Corollary 4.2 is known as the Smets–Magrez theorem. Under the minimal hypotheses of continuity and implicativeness used in the present paper, Baczyński proved (38) building on earlier results by several people (see [1, p. 65] for details). His proof in [1, Theorem 2.4.20] depends on the theory of t-norms, for which the reader of [1] is referred to [4]. Our shorter elementary proof follows a different path, via Theorems 2.3 and 4.1, without requiring any prior knowledge of t-norms.

5.2. The gist of continuity and implicativeness. Closing a circle of ideas, let us spend some words on the significance of the continuity and implicativeness hypotheses for $[0,1]$-valued logics. Our results in this paper will then automatically show the pivotal role of Łukasiewicz logic among all infinite-valued logics.

While boolean logic L_2 deals with facts that can only be true or false, with yes–no events, and ultimately with $\{0,1\}$-observables, most observables in physics, as well as most random variables in real life, are not $\{0,1\}$-valued, but have a continuous spectrum of values. Since no assessment α of these observables can be infinitely precise, α is specified by a real number together with an error interval. And yet, all physical laws are formulated in terms of relations between real-valued quantities rather than using relations between intervals. For this to make sense, small errors in the measurement of the basic observables (e.g., mass, speed) must have *initially* [1] small effects on the evaluation of compound observables (e.g., energy, momentum). In precise terms, any compound observable varies continuously with the basic observables.

For any bounded observable $\mathcal{O}$ one may rescale the minimum value and the measurement unit of $\mathcal{O}$ in such a way that the result of any measurement of $\mathcal{O}$ fits into the real interval $[0,1]$. This makes $\mathcal{O}$ "dimensionless", like angle amplitude. Suppose a $[0,1]$-valued logic L is devised to deal with $[0,1]$-valued observables, just as boolean logic L_2 does for their $\{0,1\}$-valued counterparts. Compound L-observables will be represented by applying the connectives of L to the variables, which stand for the basic L-observables. In dealing with $[0,1]$-observables we do not aim at discovering new physical laws. Mimicking what boolean logic does for yes–no observables/events, our aim is the construction of an apparatus L where consequences can be computed from premises concerning $[0,1]$-observables.

[1] In chaotic systems, arbitrarily small errors can have maximally large effects after some finite time.

https://doi.org/10.1017/jsl.2020.74 Published online by Cambridge University Press

Once L_2 is equipped with the negation and implication connectives, the functional completeness of L_2 is to the effect that all $\{0,1\}$-valued functions on $\{0,1\}^n$ are obtainable from the identity functions $x_1,\dots,x_n$ defined on $\{0,1\}^n$ by applying these connectives. By contrast, no $[0,1]$-valued logic L with finitely many connectives can express all continuous $[0,1]$-valued functions on $[0,1]^n$ starting from the identity functions. Therefore, one must make a careful selection of the most appropriate connectives of L. If Modus Ponens is to have a role in the formulation of the L-consequence relation, then L must be equipped with a connective for a $[0,1]$-valued "implication" function $\rightarrow$ defined on $[0,1]^2$. The fault tolerance of L in dealing with $[0,1]$-observables requires $\rightarrow$ to be continuous. If, as is often the case, the order of premises x and y in L is irrelevant in drawing a conclusion z, then $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$. For $\rightarrow$ to (minimally) abide by the total order of the unit interval $[0,1]$, it is natural to assume that $x \rightarrow y = 1$ iff $x \le y$. Thus $\rightarrow$ must be implicative in the sense of Definition 2.1. Upon defining the derived function $\neg$ by $\neg x = x \rightarrow 0$, Theorem 2.3 shows that the algebra $W = ([0,1], 0, \neg, \rightarrow)$ satisfies the Łukasiewicz axioms. Indeed, Corollary 4.2 shows that W is uniquely isomorphic to the standard Wajsberg algebra.

Having now understood the *meaning* of the Łukasiewicz axioms, our young readers who have followed us thus far are encouraged to explore their deep *consequences*. Proofs of some of the most important results of the theory of MV-algebras, and their equivalents for Wajsberg algebras, can be found in the introductory monograph [3]. Advanced topics are the subject matter of [5].

REFERENCES

[1] M. Baczyński and J. Balasubramaniam, ***Fuzzy Implications***, Springer, Berlin, 2008.

[2] L. Borkowski (ed.), ***Jan Łukasiewicz, Selected Works***, Studies in Logic and the Foundations of Mathematics, North Holland, Amsterdam and Polish Scientific Publishers, Warsaw, 1970.

[3] R. Cignoli, I. M. L. D'Ottaviano, and D. Mundici, ***Algebraic Foundations of Many-Valued Reasoning, vol. 7***, Trends in Logic, Kluwer, Dordrecht, 2000.

[4] E. P. Klement, R. Mesiar, and E. Pap, ***Triangular Norms***, Kluwer, Dordrecht, 2000.

[5] D. Mundici, ***Advanced Łukasiewicz Calculus and MV-Algebras***, Trends in Logic, vol. 35, Springer, Berlin, 2011.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE "ULISSE DINI"
UNIVERSITY OF FLORENCE
VIALE MORGAGNI 67/A, I-50134 FLORENCE, ITALY
E-mail: daniele.mundici@unifi.it

https://doi.org/10.1017/jsl.2020.74 Published online by Cambridge University Press