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THE ONTO MAPPING OF SIERPINSKI AND NONMEAGER SETS

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Abstract. The principle $(*)$ of Sierpinski is the assertion that there is a family of functions $\{\varphi_n : \omega_1 \longrightarrow \omega_1 \mid n \in \omega\}$ such that for every $I \in [\omega_1]^{\omega_1}$ there is $n \in \omega$ such that $\varphi_n[I] = \omega_1$. We prove that this principle holds if there is a nonmeager set of size ω_1, answering question of Arnold W. Miller. Combining our result with a theorem of Miller it then follows that $(*)$ is equivalent to $non(\mathcal{M}) = \omega_1$. Miller also proved that the principle of Sierpinki is equivalent to the existence of a weak version of a Luzin set, we will construct a model where all of these sets are meager yet $non(\mathcal{M}) = \omega_1$.

§1. Introduction.

The *principle* $(*)$ *of Sierpinski* is the following statement: There is a family of functions $\{\varphi_n : \omega_1 \longrightarrow \omega_1 \mid n \in \omega\}$ such that for every $I \in [\omega_1]^{\omega_1}$ there is $n \in \omega$ for which $\varphi_n[I] = \omega_1$. It was introduced by Sierpinski and he proved that it is a consequence of the Continuum Hypothesis. It was recently studied by Arnold W. Miller in [6], which was the motivation for this work. This principle is related to the following type of sets:

DEFINITION 1.1. Let $\mathcal{I}$ be a σ-ideal on ω^ω. We say $X = \{f_\alpha \mid \alpha < \omega_1\} \subseteq \omega^\omega$ is an *$\mathcal{I}$-Luzin set* if $X \cap A$ is at most countable for every $A \in \mathcal{I}$.

In this terminology, the Luzin sets are $\mathcal{M}$-Luzin sets (where $\mathcal{M}$ denotes the ideal of all meager sets) and the Sierpinski sets are the $\mathcal{N}$-Luzin sets (where $\mathcal{N}$ denotes the ideal of all sets with Lebesgue measure zero). Given a σ-ideal $\mathcal{I}$, its *uniformity number* $non(\mathcal{I})$ is the smallest size of a set that is not an element of $\mathcal{I}$. Clearly the existence of an $\mathcal{I}$-Luzin set implies $non(\mathcal{I}) = \omega_1$, but the converse is usually not true. For example, it was shown by Shelah and Judah in [4] that there are no Luzin or Sierpinski sets in the Miller model while $non(\mathcal{M}) = non(\mathcal{N}) = \omega_1$ holds.

DEFINITION 1.2.

1. Given $f \in \omega^\omega$ we define $ED(f) = \{g \in \omega^\omega \mid |f \cap g| < \omega\}$.
2. $\mathcal{IE}$ is the σ-ideal generated by $\{ED(f) \mid f \in \omega^\omega\}$.

It is easy to see that each $ED(f)$ is a meager set so $\mathcal{IE} \subseteq \mathcal{M}$. It is well known that $non(\mathcal{IE}) = non(\mathcal{M})$ (see [3]). In [6], Miller proved the following result:

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PROPOSITION 1.3 (Miller [6]). *The following are equivalent*:

1. *The principle* $(*)$ *of Sierpinski*.
2. *There is a family* $\{g_\alpha : \omega \longrightarrow \omega_1 \mid \alpha < \omega_1\}$ *with the property that for every* $g : \omega \longrightarrow \omega_1$ *there is* $\alpha < \omega_1$ *such that if* $\beta > \alpha$ *then* $g_\beta \cap g$ *is infinite.*
3. *There is an* $\mathcal{IE}$*-Luzin set*.

The implication from 3 to 1 is not explicit in [6] (it is implicitly proved in the lemma 6 of [6]). The referee found a very elegant and short proof of this result which we reproduce here. We are grateful with the referee for allowing us to include his proof.

PROPOSITION 1.4. *The existence of an* $\mathcal{IE}$*-Luzin set implies the principle* $(*)$ *of Sierpinski*.

PROOF. Let $\mathcal{A} = \{A_\alpha \mid \omega \leq \alpha < \omega_1\}$ be an almost disjoint family. Since there is an $\mathcal{IE}$-Luzin, for each α, we can find a family $\mathcal{F}_\alpha = \left\{f_{\alpha\beta} : A_\alpha \longrightarrow \alpha \mid \beta < \omega_1\right\}$ such that for every $g : A_\alpha \longrightarrow \alpha$ there is δ such that if $\beta > \delta$ then $f_{\alpha\beta} \cap g$ is infinite. Since $\mathcal{A}$ is an almost disjoint family, we can then construct a family $\mathcal{G} = \left\{g_\beta : \omega \longrightarrow \omega_1 \mid \omega \leq \beta < \omega_1\right\}$ such that $f_{\alpha\beta} =^* g_\beta \restriction A_\alpha$ for every $\alpha < \beta < \omega_1$.

By the previous proposition, we need to prove that for every $g : \omega \longrightarrow \omega_1$ there is $\alpha < \omega_1$ such that if $\beta > \alpha$ then $g_\beta \cap g$ is infinite. First we find δ such that $g : \omega \longrightarrow \delta$ and then we know there is γ such that if $\beta > \gamma$ then $f_{\delta\beta} \cap (g \restriction A_\delta)$ is infinite. It then follows that if $\beta > max\,\{\delta, \gamma\}$ then $g_\beta \restriction A_\delta =^* f_{\delta\beta}$ so $\left|g_\beta \cap g\right| = \omega$. ⊣

It then follows that the existence of a Luzin set implies the principle $(*)$ of Sierpinski while it implies $non\,(\mathcal{M}) = \omega_1$. Miller then asked if the principle $(*)$ of Sierpinski is a consequence of $non\,(\mathcal{M}) = \omega_1$ and we will show that this is indeed the case. In the second part of the paper, we will prove (with the aid of an inaccessible cardinal) that while $non\,(\mathcal{M}) = \omega_1$ implies the existence of a $\mathcal{IE}$-Luzin set, it does not imply the existence of a nonmeager $\mathcal{IE}$-Luzin set.

§2. ***non*** $(\mathcal{M}) = \omega_1$ **implies the existence of an** $\mathcal{IE}$**-Luzin set.** We will now show that the principle $(*)$ of Sierpinski follows by $non\,(\mathcal{M}) = \omega_1$, answering the question of Miller. By $Partial\,(\omega^\omega)$, we shall denote the set of all infinite partial functions from ω to ω. We start with the following lemma:

LEMMA 2.1. *If* $non\,(\mathcal{M}) = \omega_1$ *then there is a family* $X = \{f_\alpha \mid \alpha < \omega_1\}$ *with the following properties*:

1. *Each* f_α *is an infinite partial function from* ω *to* ω.
2. *The set* $\{dom\,(f_\alpha) \mid \alpha < \omega_1\}$ *is an almost disjoint family.*
3. *For every* $g : \omega \longrightarrow \omega$, *there is* $\alpha < \omega_1$ *such that* $f_\alpha \cap g$ *is infinite.*

PROOF. Let $\omega^{<\omega} = \{s_n \mid n \in \omega\}$ and we define $H : \omega^\omega \longrightarrow Partial\,(\omega^\omega)$ where the domain of $H\,(f)$ is $\{n \mid s_n \sqsubseteq f\}$ and if $n \in dom\,(H\,(f))$ then $H\,(f)\,(n) = f\,(|s_n|)$. It is easy to see that if $f \neq g$ then $dom\,(H\,(f))$ and $dom\,(H\,(g))$ are almost disjoint.

Given $g : \omega \longrightarrow \omega$, we define $N\,(g) = \{f \in \omega^\omega \mid |H\,(f) \cap g| < \omega\}$. It then follows that $N\,(g)$ is a meager set since $N\,(g) = \bigcup_{k \in \omega} N_k\,(g)$, where $N_k\,(g) = \{f \in \omega^\omega \mid |H\,(f) \cap g| < k\}$, and it is easy to see that each $N_k\,(g)$ is a nowhere dense set. Finally, if $X = \{h_\alpha \mid \alpha < \omega_1\}$ is a nonmeager set then $H\,[X]$ is the family we were looking for. ⊣

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With the previous lemma we can prove the following:

PROPOSITION 2.2. *If* $non(\mathcal{M}) = \omega_1$ *then the principle* $(*)$ *of Sierpinski is true.*

PROOF. Let $X = \{f_\alpha \mid \alpha < \omega_1\}$ be a family as in the previous lemma. We will build a $\mathcal{IE}$-Luzin set $Y = \{h_\alpha \mid \alpha < \omega_1\}$. For simplicity, we may assume $\{dom(f_n) \mid n \in \omega\}$ is a partition of ω.

For each $n \in \omega$, let h_n be any constant function. Given $\alpha \geq \omega$, enumerate it as $\alpha = \{\alpha_n \mid n \in \omega\}$ and then we recursively define $B_0 = dom(f_{\alpha_0})$ and $B_{n+1} = dom(f_{\alpha_n}) \setminus (B_0 \cup \cdots \cup B_n)$. Clearly $\{B_n \mid n \in \omega\}$ is a partition of ω. Let $h_\alpha = \bigcup_{n\in\omega} f_{\alpha_n} \restriction B_n$, it then follows that $Y = \{h_\alpha \mid \alpha < \omega_1\}$ is an $\mathcal{IE}$-Luzin set. ⊣

§3. $non(\mathcal{M}) = \omega_1$ **does not imply the existence of a nonmeager** $\mathcal{IE}$**-Luzin set.** It is not hard to see that the $\mathcal{IE}$-Luzin set constructed in the previous proof is meager. One may then wonder if it is possible to construct a nonmeager $\mathcal{IE}$-Luzin set from $non(\mathcal{M}) = \omega_1$. We will prove that this is not the case. This will be achieved by using Todorcevic's method of forcing with models as side conditions (see [8] for more on this very useful technique).

DEFINITION 3.1. We define the forcing $\mathbb{P}_{cat}$ as the set of all $p = (s_p, \overline{M}_p, F_p)$ with the following properties:

1. $s_p \in \omega^{<\omega}$ (this is usually referred as *the stem* of p).
2. $\overline{M}_p = \{M_0, \ldots, M_n\}$ is an $\in$-chain of countable elementary submodels of $\mathsf{H}((2^{\mathfrak{c}})^{++})$.
3. $F_p : \overline{M}_p \longrightarrow \omega^\omega$.
4. $s_p \cap F_p(M_i) = \emptyset$, for every $i \leq n$.
5. $F_p(M_i) \notin M_i$ and if $i < n$ then $F_p(M_i) \in M_{i+1}$.
6. $F_p(M_i)$ is a Cohen real over M_i (i.e., if $Y \in M_i$ is a meager set then $F_p(M_i) \notin Y$).

Finally, if $p, q \in \mathbb{P}_{cat}$ then $p \leq q$ if $s_q \subseteq s_p$, $\overline{M}_q \subseteq \overline{M}_p$ and $F_q \subseteq F_p$.

The following lemma is easy and it is left to the reader:

LEMMA 3.2.

1. *If* $M \preceq \mathsf{H}((2^{\mathfrak{c}})^{+++})$ *is countable and* $p \in M \cap \mathbb{P}_{cat}$ *then there is* $f \in \omega^\omega$ *such that if* $N = M \cap \mathsf{H}((2^{\mathfrak{c}})^{++})$ *then* $\overline{p} = (s_p, \overline{M}_p \cup \{N\}, F_p \cup \{(N, f)\})$ *is a condition of* $\mathbb{P}_{cat}$ *and it extends* p.
2. *If* $n \in \omega$ *then* $D_n = \{p \in \mathbb{P}_{cat} \mid n \subseteq dom(s_p)\}$ *is an open dense subset of* $\mathbb{P}_{cat}$.

We will now prove that $\mathbb{P}_{cat}$ is a proper forcing by applying the usual "side conditions trick".

LEMMA 3.3. $\mathbb{P}_{cat}$ *is a proper forcing.*

PROOF. Let $p \in \mathbb{P}_{cat}$ and M a countable elementary submodel of $\mathsf{H}((2^{\mathfrak{c}})^{+++})$ such that $p \in M$. By the previous lemma, we know there is $f \in \omega^\omega$ such that $\overline{p} = (s_p, \overline{M}_p \cup \{N\}, F_p \cup \{(N, f)\}) \in \mathbb{P}_{cat}$ (where $N = M \cap \mathsf{H}((2^{\mathfrak{c}})^{++})$). We will now prove that $\overline{p}$ is an $(M, \mathbb{P}_{cat})$-generic condition.

Let $D \in M$ be an open dense subset of $\mathbb{P}_{cat}$ and $q \leq \overline{p}$ (we may even assume $q \in D$). We must prove that q is compatible with an element of $M \cap D$. In order to achieve this, let $q_M = (s_q, \overline{M}_q \cap M, F_q \cap M)$ it is easy to see it is a condition as well as an element of M. By elementarity, we can find $r \in M \cap D$ such that $r \leq q_M$ and $s_r = s_q$. It is then easy to see that r and q are compatible (this is easy since r and q share the same stem). ⊣

The next lemma shows that $\mathbb{P}_{cat}$ destroys all the ground model nonmeager $\mathcal{IE}$-Luzin families.

LEMMA 3.4. *If $X = \{f_\alpha \mid \alpha < \omega_1\} \subseteq \omega^\omega$ is a nonmeager set then $\mathbb{P}_{cat}$ adds a function that is almost disjoint with uncountably many elements of X.*

PROOF. Given a generic filter $G \subseteq \mathbb{P}_{cat}$, we denote the *generic real* by f_{gen} i.e., f_{gen} is the union of all the stems of the elements in G. We will show that f_{gen} is forced to be almost disjoint with uncountably many elements of X. Let $p \in \mathbb{P}_{cat}$ with stem s_p and $\alpha < \omega_1$. Choose $t \in \omega^{<\omega}$ with the same length as s_p but disjoint with it. Let $Y = \{g_\beta \mid \alpha < \beta < \omega_1\}$, where $g_\beta = t \cup (f_\beta \restriction [|t|, \omega))$. It is easy to see that Y is a nonmeager set and then we can find $\beta > \alpha$ and $q \leq p$ such that g_β is in the image of F_q. In this way, f_{gen} is forced by q to be disjoint from g_β, so it will be almost disjoint with f_β. ⊣

We say a forcing notion $\mathbb{P}$ *destroys category* if there is $p \in \mathbb{P}$ such that $p \Vdash$ "$\omega^\omega \cap V \in \mathcal{M}$". It is a well known fact that a partial order $\mathbb{P}$ does not destroy category if and only if $\mathbb{P}$ does not add an eventually different real (under any condition). Given a polish space X, we denote by $NWD(X)$ the ideal of all nowhere dense subsets of X. We will need the following result of Kuratowski and Ulam (see [5]):

PROPOSITION 3.5 (Kuratowski–Ulam). *Let X and Y be two polish spaces. If $N \subseteq X \times Y$ is a nowhere dense set, then $\{x \in X \mid N_x \in nwd(Y)\}$ is comeager (where $N_x = \{y \mid (x, y) \in N\}$).*

As a consequence of the Kuratowski–Ulam, we get the following result:

LEMMA 3.6. *Let $p \in \mathbb{P}_{cat}$, $\overline{M}_p = \{M_0, \ldots, M_n\}$ and $i \leq n$. Let $g_j = F_p(M_{i+j})$ and $m = n - i$. If $D \in M_i$ and $D \subseteq (\omega^\omega)^{m+1}$ is a nowhere dense set, then $(g_0, \ldots, g_m) \notin D$.*

PROOF. We prove it by induction over m. If $m = 0$, this is true just by the definition of $\mathbb{P}_{cat}$. Assume this is true for m and we will show it is also true for $m + 1$. Since $D \subseteq (\omega^\omega)^{m+2}$ is a nowhere dense set, then by the Kuratowski–Ulam we conclude that $A = \{h \in \omega^\omega \mid D_h \in nwd((\omega^\omega)^{m+1})\}$ is comeager and note it is an element of M_i. In this way, $g_0 \in A$ so $D_{g_0} \in nwd((\omega^\omega)^{m+1})$ and it is an element of M_{i+1}. By the inductive hypothesis, we know $(g_1, \ldots, g_{m+1}) \notin D_{g_0}$ which implies $(g_0, \ldots, g_m) \notin D$. ⊣

We will prove that $\mathbb{P}_{cat}$ does not destroy category and it is a consequence of the following result:

LEMMA 3.7. *Let $p \in \mathbb{P}_{cat}$ and $\dot{g}$ a $\mathbb{P}_{cat}$-name for an element of ω^ω. Let $\langle M_n \mid n \in \omega\rangle$ be an $\in$-chain of elementary submodels of $\mathsf{H}((2^{\mathfrak{c}})^{+++})$, $h : \omega \longrightarrow \omega$ and $\{A_n \mid n \in \omega\} \subseteq [\omega]^\omega$ a family of pairwise infinite disjoint sets with the following properties*:

1. $p, \dot{g} \in M_0$.
2. $h \restriction A_n \in M_{n+1}$.
3. *If $f \in M_n \cap \omega^\omega$ then $f \cap (h \restriction A_n)$ is infinite.*

Then there is a condition $q \leq p$ such that $q \Vdash$ "$|h \cap \dot{g}| = \omega$".

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PROOF. Let $M = \bigcup_{n\in\omega} M_n$ and define $h_n = h \upharpoonright A_n \in M_n$. We know that there is some $f \in \omega^\omega$ such that $\overline{p} = \left(s_p, \overline{M}_p \cup \{N\}, F_p \cup \{(N, f)\}\right) \in \mathbb{P}_{cat}$ (where $N = M \cap \mathsf{H}((2^{\mathfrak{c}})^{++})$). We will now prove that $\overline{p}$ forces that $\dot{g}$ and h will have infinite intersection. We may assume $A_n \cap n = \emptyset$ for every $n \in \omega$.

Pick any $q \leq \overline{p}$ and $k \in \omega$, we must find an extension of q that forces that $\dot{g}$ and h share a common value bigger than k. We first find $n > k$ such that $q' = \left(s_q, \overline{M}_q \cap M, F_q \cap M\right) \in M_n$. Let $m = \left|\overline{M}_q \setminus \overline{M}_{q'}\right|$ and now we define D as the set of all $t \in \omega^{<\omega}$ such that there are $l \in A_n$ and $r \in \mathbb{P}_{cat}$ with the following properties:

1. $r \leq q'$.
2. $r \in M_n$.
3. $s_q \subseteq t$ and the stem of r is t.
4. $r \Vdash$ "$\dot{g}(l) = h_n(l)$".

It is easy to see that D is an element of M_{n+1}. We now define $N(D) \subseteq (\omega^\omega)^m$ as the set of all $(f_1, \dots, f_m) \in (\omega^\omega)^m$ such that $(f_1 \cup \dots \cup f_m) \cap t \nsubseteq s_q$ for every $t \in D$. We claim that $N(D)$ is a nowhere dense set.

Let $z_1, \dots, z_m \in \omega^{<\omega}$, and we may assume all of them have the same length and it is bigger than the length of s_q. We know $q' = \left(s_q, \overline{M}_{q'}, F_{q'}\right)$ and let $im(F_{q'}) = \{f_1, \dots, f_k\}$ (where *im* denotes the image of the function). Let t_0 be any extension of s_q such that $t_0 \cap (f_{\alpha_1} \cup \dots \cup f_{\alpha_k} \cup z_1 \cup \dots z_m) \subseteq s_q$ and $|t_0| = |z_1|$. In this way, $q_0 = \left(t_0, \overline{M}_{q'}, F_{q'}\right)$ is a condition and is an element of M_n. Inside M_n, we build a decreasing sequence $\langle q_i \rangle_{i\in\omega}$ (starting from the q_0 we just constructed) in such a way that q_i determines $\dot{g} \upharpoonright i$. In this way, there is a function $u : \omega \longrightarrow \omega \in M_n$ such that $q_i \Vdash$ "$\dot{g} \upharpoonright i = u \upharpoonright i$". Since $u \in M_n$, we may then find $l \in A_n$ such that $u(l) = h_n(l)$. Let $t = t_{l+1}$ and $r = q_{l+1}$, we may then find $z_i' \supseteq z_i$ such that $t \cap (z_1' \cup \dots \cup z_m') \subseteq s_q$ and $|z_i'| = |t|$. In this way, we conclude that $\langle z_1', \dots, z_m' \rangle \cap N(D) = \emptyset$ (where $\langle z_1', \dots, z_m' \rangle = \{(g_1, \dots, g_m) \mid \forall i \leq m\, (z_i' \subseteq g_i)\}$), so we conclude $N(D)$ is a nowhere dense set.

Let $g_1, \dots, g_m$ be the elements of $im(F_q)$ that are not in M. Since $D \in N$ then by the previous lemma, we know that $(g_1, \dots, g_m) \notin N(D)$. This means there are $l \in A_n$, $t \in \omega^{<\omega}$ and $r \in M_n$ such that $r \leq q'$, whose stem is t and $r \Vdash$ "$\dot{g}(l) = h_n(l)$" with the property that $t \cap (g_1 \cup \dots \cup g_m) \subseteq s_q$, but since q is a condition, it follows that $t \cap (g_1 \cup \dots \cup g_m) = \emptyset$. In this way, r and q are compatible, which finishes the proof. ⊣

As a corollary we get the following:

COROLLARY 3.8. *$\mathbb{P}_{cat}$ does not destroy category.*

Unfortunately, the iteration of forcings that does not destroy category may destroy category (this may even occur at a two step iteration, see [1]). Luckily for us, the iteration of the $\mathbb{P}_{cat}$ forcing does not destroy category as we will prove soon. First we need a couple of lemmas,

LEMMA 3.9. *Let $\mathbb{P}$ be a proper forcing that does not destroy category and $p \in \mathbb{P}$. If $\dot{S}$ is a $\mathbb{P}$-name for a countable set of reals, then there is $q \leq p$ and $h : \omega \longrightarrow \omega$ such that $q \Vdash$ "$\forall f \in \dot{S}\, (|f \cap h| = \omega)$".*

PROOF. First note that if $\dot{f}_0, \dots, \dot{f}_n$ are $\mathbb{P}$-names for reals, then there is $q \leq p$ and $h : \omega \longrightarrow \omega$ such that q forces $\dot{f}_i$ and h have infinite intersection for every

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$i \leq n$. To prove this, we choose a partition $\{A_{0,}, \dots, A_n\}$ of ω in infinite sets and let $\dot{g}_i$ be the $\mathbb{P}$-name of $\dot{f}_i \upharpoonright A_i$. Since $\mathbb{P}$ does not destroy category, there are $q \leq p$ and $h_i : A_i \longrightarrow \omega$ such that q forces that h_i and $\dot{f}_i$ have infinite intersection. Clearly q and $h = \bigcup h_i$ have the desired properties.

To prove the lemma, let $\dot{S} = \{\dot{g}_n \mid n \in \omega\}$ and fix $\{A_n \mid n \in \omega\}$ a partition of ω in infinite sets. By the previous remark, we know there is a $\mathbb{P}$-name $\dot{F}$ such that $p \Vdash$ "$\dot{F} : \omega \longrightarrow Partial\,(\omega^\omega) \cap V$" such that every $\dot{F}(n)$ is forced to be a function with domain A_n and intersects infinitely $\dot{g}_0 \upharpoonright A_0, \dots, \dot{g}_n \upharpoonright A_n$. Since $\mathbb{P}$ is a proper forcing, we can find $q \leq p$ and $M \in V$ a countable subset of $Partial\,(\omega^\omega)$ such that $q \Vdash$ "$\dot{F} : \omega \longrightarrow M$". We know that $\mathbb{P}$ does not destroy category and M is countable, so there must be $r \leq q$ and $H : \omega \longrightarrow M$ such that $r \Vdash$ "$\exists^\infty n(\dot{F}(n) = H(n))$". We may assume that the domain of $H(n)$ is A_n for every $n \in \omega$. Finally, we define $h = \bigcup_{n\in\omega} H(n)$ and it is easy to see that r forces that h has infinite intersection with every element of $\dot{S}$. ⊣

We will also need the following lemma,

Lemma 3.10. *Let $\mathbb{P}$ be a proper forcing that does not destroy category, $G \subseteq \mathbb{P}$ a generic filter and X any set. Then there are $\overline{M} = \{M_n \mid n \in \omega\} \subseteq V$, $P = \{A_n \mid n \in \omega\} \subseteq V$ and $h : \omega \longrightarrow \omega \in V$ with the following properties*:

1. *Each M_n is a countable elementary submodel of $H(\kappa)$ for some big enough κ (in V).*
2. *$X \in M_0$ and $M_n \in M_{n+1}$, for every $n \in \omega$.*
3. *P is a family of pairwise infinite disjoint sets of ω.*
4. *$P, \overline{M} \in V[G]$ (while $\overline{M}$ is a subset of V, in general it will not be a ground model set, the same is true for P).*
5. *$G \cap M_n$ is a $(M_n, \mathbb{P})$-generic filter, for every $n \in \omega$.*
6. *$h \upharpoonright A_n \in M_{n+1}$ and if $f \in M_n[G]$ then $h \upharpoonright A_n \cap f$ is infinite.*

Proof. Let r be any condition of $\mathbb{P}$, we will prove that there is an extension of r that forces the existence of the desired objects. Let $\{B_n \mid n \in \omega\}$ be any definable partition of ω into infinite sets.

Claim 3.11. *If $G \subseteq \mathbb{P}$ is a generic filter with $r \in G$ then (in $V[G]$) there a sequence $\langle (N_i, p_i, h_i) \mid i \in \omega \rangle$ such that for every $i \in \omega$ the following holds*:

1. *$N_i \in V$ is a countable elementary submodel of $H(\kappa)$ (the $H(\kappa)$ of the ground model).*
2. *$r, X \in N_0$ and $N_i \in N_{i+1}$.*
3. *$p_0 \leq r$ and $\langle p_k \rangle_{k\in\omega}$ is a decreasing sequence contained in G.*
4. *p_i is $(N_i, \mathbb{P})$-generic.*
5. *$h_i : B_i \longrightarrow \omega \in N_{i+1}$.*
6. *$p_i \Vdash$ "$\forall f \in N_i[\dot{G}] \cap \omega^\omega\, (|f \cap h_i| = \omega)$".*

Assume the claim is false, so we can find $n \in \omega$ and a sequence $R = \langle (N_i, p_i, h_i) \mid i \leq n \rangle$ that is maximal with the previous properties (the point 5 is only demanded for $i < n$). Let $p \in G$ be a condition forcing R has all this features (including the maximality). Back in V, let M be a countable elementary submodel such that $\mathbb{P}, p, R \in M$. By the previous lemma, there is an $(M, \mathbb{P})$-generic condition $q \leq p$ and $g : B_{n+1} \longrightarrow \omega$ such that g is forced by q to intersect infinitely every real of $M[G]$. In this way, q forces that R could be extended by adding (M, q, g)

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but this is a contradiction since $q \leq p$ so it forces R was maximal. This finishes the proof of the claim.

Fix $\langle (\dot{N}_i, \dot{p}_i, \dot{h}_i) \mid i \in \omega \rangle$ to be the name of a sequence as in the claim. We can now define a name for a function $\dot{F}$ from ω to $Partial\,(\omega^\omega) \cap V$ such that $r \Vdash$ "$\forall n(\dot{F}\,(n) = \dot{h}_n)$". As in the previous lemma, we can find a condition $p \leq r$ and $H : \omega \longrightarrow Partial\,(\omega^\omega)$ such that $p \Vdash$ "$\exists^\infty n(\dot{F}\,(n) = H\,(n))$". We may assume the domain of $H\,(n)$ is B_n and let $h = \bigcup_{n\in\omega} H\,(n)$. Let $\dot{Z} = \{\dot{z}_n \mid n \in \omega\}$ be a name for a subset of ω such that $p \Vdash$ "$\forall n\,(F\,(\dot{z}_n) = H\,(\dot{z}_n))$". If $G \subseteq \mathbb{P}$ is a generic filter such that $p \in G$ then we define $M_n = N_{\dot{z}_n[G]}$ and $A_n = B_{\dot{z}_n[G]}$, it is clear that this sets have the desired properties. ⊣

From this we can conclude the following,

Corollary 3.12. *If $\mathbb{P}$ is a proper forcing that does not destroy category then $\mathbb{P} * \mathbb{P}_{cat}$ does not destroy category.*

Proof. Let $\dot{p}$ be a $\mathbb{P}$-name for a condition of $\mathbb{P}_{cat}$ and $\dot{f}$ a $\mathbb{P}$-name for a $\mathbb{P}_{cat}$-name for a real. Let $G \subseteq \mathbb{P}$ be a generic filter. By the previous lemma, there are $h : \omega \longrightarrow \omega \in V$, an $\in$-chain of elementary submodels $\{M_n\,[G] \mid n \in \omega\}$ and a pairwise disjoint family $\{A_n \mid n \in \omega\}$ of infinite subsets of ω such that $\dot{p}\,[G]\,, \dot{f}\,[G] \in M_0\,[G]$ and $h \restriction A_n \in M_{n+1}\,[G]$ has infinite intersection with every real in $M_n\,[G]$. Then by lemma 13, we can extend $\dot{p}\,[G]$ to a condition forcing that $\dot{f}\,[G]$ and h will have infinite intersection. ⊣

As commented before, the iteration of forcings that does not destroy category may destroy category, but the following preservation result of Dilip Raghavan shows this can only happen at the successor steps of the iteration:

Proposition 3.13 (Raghavan [7]). *Let δ be a limit ordinal and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta \rangle$ a countable support iteration of proper forcings. If $\mathbb{P}_\alpha$ does not destroy category for every $\alpha < \delta$ then $\mathbb{P}_\delta$ does not destroy category.*

With the aid of the previous preservation theorem, we conclude the following:

Corollary 3.14. *The countable support iteration of $\mathbb{P}_{cat}$ does not destroy category.*

Putting all the pieces together, we can finally prove our theorem:

Proposition 3.15. *If the existence of an inaccessible cardinal is consistent, then so it is the following statement: non $(\mathcal{M}) = \omega_1$ and every $\mathcal{IE}$-Luzin set is meager.*

Proof. Let μ be an inaccessible cardinal, we perform a countable support iteration $\{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \mu\}$ in which $\dot{\mathbb{Q}}_\alpha$ is forced by $\mathbb{P}_\alpha$ to be the $\mathbb{P}_{cat}$ forcing. It is easy to see that if $\alpha < \mu$ then $\mathbb{P}_\alpha$ has size less than μ so it has the μ-chain condition and then $\mathbb{P}_\mu$ has the μ-chain condition (see [2]). The result then follows by the previous results. ⊣

We would like to finish with some questions:

Problem 3.16. *Does $\mathbb{P}_{cat}$ preserve $\sqsubseteq^{Cohen}$? (see [1] chapter 6).*

Problem 3.17. *Does $\mathbb{P}_{cat}$ preserve every nonmeager set as a nonmeager set? (we only know that it preserves the ground model as a nonmeager set).*

Problem 3.18. *Is the inaccessible cardinal really needed for the last result?*

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