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Diffusive-dispersive travelling waves and kinetic relations. II A hyperbolic–elliptic model of phase-transition dynamics

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We deal here with a mixed (hyperbolic–elliptic) system of two conservation laws modelling phase-transition dynamics in solids undergoing phase transformations. These equations include nonlinear viscosity and capillarity terms. We establish general results concerning the existence, uniqueness and asymptotic properties of the corresponding travelling wave solutions. In particular, we determine their behaviour in the limits of dominant diffusion, dominant dispersion or asymptotically small or large shock strength. As the viscosity and capillarity parameters tend to zero, the travelling waves converge to propagating discontinuities, which are either classical shock waves or supersonic phase boundaries satisfying the Lax and Liu entropy criteria, or else are *undercompressive subsonic phase boundaries*. The latter are uniquely characterized by the so-called *kinetic function*, whose properties are investigated in detail here.

1. Introduction

Consider the following system of two conservation laws in one space dimension arising in nonlinear elastodynamics:

$$\left.\begin{aligned} \delta_t v - \delta_x \sigma(w) &= \beta \delta_x (b(w)\delta_x v) - \alpha \delta_x (a(w)\delta_x (a(w)\delta_x w)),\\ \delta_t w - \delta_x v &= 0. \end{aligned}\right\} \tag{1.1}$$

Here, $v \in \mathbb{R}$ and $w > -1$ represent the velocity and the deformation gradient of some solid material or fluid, respectively. The viscosity function $b(w)$ and the

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capillarity function $a(w)$ are assumed to be smooth and bounded below by some positive constants b_0 and a_0, respectively,

$$b(w) \geqslant b_0 > 0, \quad a(w) \geqslant a_0 > 0 \quad \text{for all } w > -1. \tag{1.2}$$

The parameters $\beta, \alpha > 0$ measure the relative strengths of the viscosity and capillarity terms in the material.

The stress function σ depends on the material under consideration; for materials undergoing phase transformations, a typical shape is determined by the following conditions:

$$\left.\begin{aligned} &w\sigma''(w) > 0 \quad \text{for all } w \neq 0, \quad \sigma'(0) < 0, \\ &\lim_{w\to -1} \sigma(w) = -\infty, \qquad \lim_{w\to+\infty} \sigma'(w) = +\infty. \end{aligned}\right\} \tag{1.3}$$

As a consequence of (1.3), outside some interval (w_*, w^*) defined by

$$\sigma'(w_*) = \sigma'(w^*) = 0, \quad \text{with } -1 < w_* < 0 < w^*,$$

equation (1.1) forms a hyperbolic system of partial differential equations, having two real and distinct wave speeds, $-c(w)$ and $c(w)$, where c is the sound speed, defined by

$$c(w) := \sqrt{\sigma'(w)} \quad \text{for all } w \in (-1, w_*] \cup [w^*, \infty).$$

Throughout, we restrict our attention to values in the hyperbolic region $\mathcal{H}_- \cup \mathcal{H}_+$, where

$$\mathcal{H}_- = (-1, w_*], \qquad \mathcal{H}_+ = [w^*, \infty).$$

Equations (1.1) arise in continuum physics in the following 'variational form',

$$\begin{gathered} \delta_t v - \delta_x \Sigma(w, w_x, w_{xx}) = \beta\delta_x(b(w)\delta_x v), \\ \delta_t w - \delta_x v = 0, \end{gathered}$$

where the stress Σ is defined from the following nonlinear internal energy,

$$e(w, w_x) = \epsilon(w) + \tfrac{1}{2}\alpha\lambda(w)w_x^2,$$

where $\lambda(w)$ is a positive function and $\epsilon(w)$ is a smooth function. Precisely, we have

$$\Sigma(w, w_x, w_{xx}) = \frac{\delta e}{\delta w}(w, w_x) - \left(\frac{\delta e}{\delta w_x}(w, w_x)\right)_x = \epsilon'(w) + \tfrac{1}{2}\alpha\lambda'(w)w_x^2 - \alpha(\lambda(w)w_x)_x.$$

Setting $\sigma(w) = \epsilon'(w)$ and $a(w) = \sqrt{\lambda(w)}$ yields exactly the model (1.1) under consideration.

This paper is the second part of a series (see [3,4]) devoted to travelling solutions associated with diffusive-dispersive conservation laws. We search for solutions of the system (1.1) depending only on the variable $y := x - \lambda t$ for some speed λ and connecting two constant states at infinity. Precisely, a travelling wave solution $y \mapsto (v(y), w(y))$ satisfies

$$\left.\begin{aligned} \lambda v_y + \sigma(w)_y &= -\beta(b(w)v_y)_y + \alpha(a(w)(a(w)w_y)_y)_y, \\ \lambda w_y + v_y &= 0. \end{aligned}\right\} \tag{1.4}$$

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It is also required that

$$\left.\begin{aligned} v_y(y), w_y(y), w_{yy}(y) \to 0 &\quad \text{when } |y| \to \infty, \\ v(y) \to v_-, \quad w(y) \to w_- &\quad \text{when } y \to -\infty, \\ v(y) \to v_+, \quad w(y) \to w_+ &\quad \text{when } y \to +\infty, \end{aligned}\right\} \tag{1.5}$$

where v_-, w_-, v_+, w_+ are given constants. To describe the travelling wave solutions, we set

$$v_0 := v_-, \qquad w_0 := w_-$$

and we search for all of the right-hand states (v_+, w_+) that can be attained through a travelling wave initiating at (v_0, w_0). By integration of (1.4) over some interval $(-\infty, y]$ and using (1.5), we obtain

$$\left.\begin{aligned} \lambda(v - v_0) + \sigma(w) - \sigma(w_0) = -\beta b(w) v_y + \alpha a(w)(a(w) w_y)_y, \\ \lambda(w - w_0) + v - v_0 = 0. \end{aligned}\right\} \tag{1.6}$$

This is a system of second-order ordinary differential equations.

We point out that the shock speed λ is determined by the Rankine–Hugoniot relation

$$\lambda(w - w_0) + v - v_0 = \lambda(v - v_0) + \sigma(w) - \sigma(w_0) = 0,$$

hence

$$\lambda^2 = \begin{cases} \dfrac{\sigma(w) - \sigma(w_0)}{w - w_0} & \text{if } w \neq w_0, \\ \sigma'(w_0) & \text{if } w = w_0. \end{cases} \tag{1.7}$$

Indeed, this follows immediately by letting $y \to \infty$ in (1.6) and using (1.5). Obviously, λ must be real, which implies that $(\sigma(w) - \sigma(w_0))(w - w_0) \geqslant 0$.

Observe that we can eliminate the variable v in (1.6) and derive an equation in w only,

$$-\lambda^2(w - w_0) + \sigma(w) - \sigma(w_0) = \lambda\beta b(w) w_y + \alpha a(w)(a(w) w_y)_y. \tag{1.8}$$

Setting $z = a(w) w_y$, we reformulate (1.8) in the form of a first-order system in two variables w and z,

$$\left.\begin{aligned} a(w) w_y &= z, \\ \alpha a(w) z_y &= -\lambda\beta \frac{b(w)}{a(w)} z + g(w, \lambda) - g(w_0, \lambda), \end{aligned}\right\} \tag{1.9}$$

with a right-hand side given by

$$g(w, \lambda) := \sigma(w) - \lambda^2 w.$$

The boundary conditions (1.5) now read

$$\left.\begin{aligned} w_y(y), z_y(y) \to 0 &\quad \text{when } |y| \to \infty, \\ w(y) \to w_-, \quad z(y) \to 0 &\quad \text{when } y \to -\infty, \\ w(y) \to w_+, \quad z(y) \to 0 &\quad \text{when } y \to +\infty, \end{aligned}\right\} \tag{1.10}$$

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Our objective in this paper is to establish general existence and uniqueness results concerning the travelling wave solutions of (1.9), (1.10) for viscosity and capillarity functions and stress functions satisfying solely (1.2), (1.3). We are also interested in investigating their asymptotic behaviour in the viscosity dominant and capillarity dominant limits.

Recall that the existence of travelling wave solutions was established by Shearer and Yang [24] in the case of the cubic stress function

$$\sigma(w) = w(w^2 - 1) \quad \text{for all } w \tag{1.11}$$

and for constant viscosity and capillarity $a(w) := b(w) := 1$ for all w. Importantly, it was observed therein that some of the trajectories of the system do not satisfy the standard Lax and Liu entropy conditions, and correspond to (*undercompressive*) *subsonic phase boundaries*. By definition, the propagating discontinuities associated with such travelling waves have fewer incoming characteristics than observed with classical compressive waves.

Subsonic propagating phase boundaries (for hyperbolic–elliptic systems), as well as non-classical undercompressive shock waves (for hyperbolic but non-genuinely nonlinear systems), have drawn a lot of attention in recent years. For references, see [1,2,4,13,14,16–18,24–30]. In particular, the importance of the so-called *kinetic relation* in characterizing propagating phase transitions was recognized by Abeyaratne and Knowles [1,2] and Truskinovsky [29,30]. The mathematical formulation of the kinetic relation is due to LeFloch [17]. Hayes and LeFloch [13,14] extended the concept of kinetic relation to general (strictly hyperbolic, but not genuinely nonlinear) systems of conservation laws and determined kinetic functions numerically [15]. See also [18,19,30] for further background material and references. Related works can be found in [6–9,11,12,20–23].

Our main results are as follows.

(1) The existence of *subsonic phase boundaries* and of the corresponding *kinetic function* are established by relying on the approach of the authors in [3].

(2) The existence and properties of the *classical shocks* and *supersonic phase boundaries* are also established.

(3) These results provide a description of the *shock curve* generated by the model (1.1)–(1.3).

(4) Moreover, *asymptotic properties* of the kinetic function (small speeds, large amplitude, vanishing viscosity, vanishing capillarity) are determined.

Furthermore, some examples of systems will be studied for which some important parameters or functions can be determined explicitly. The case when only the viscosity is taken into account is covered by our analysis (by letting $\alpha = 0$).

An outline of this paper follows. In § 2, we state our main results concerning the kinetic function (theorem 2.3), the shock curve (theorem 2.4) and the asymptotic properties (theorem 2.5). In § 3, we prove the main existence results relying on some of the results in [3]. Section 4 is devoted to asymptotic properties of the kinetic function. Finally, in § 5, we treat a few examples for which explicit formulae are available.

2. Statements of the main results

Throughout the paper, we sometimes first disregard the constraint that the speed defined in (1.7) be real, and, next, enforce this condition by restricting the interval of values under consideration. We begin by discussing some general properties of the solutions of (1.9).

By definition, an *equilibrium point* is a pair (w, z) for which the vector field in the right-hand side of (1.9) vanishes. Clearly, $z = 0$ for such a point and we can focus on the component w. In view of (1.3), if a left-hand state w_0 and a speed λ are fixed, there exist at most three equilibria w (including w_0 itself) satisfying

$$g(w, \lambda) = g(w_0, \lambda). \tag{2.1}$$

Assume first in the presentation that

$$w_0 > w^*$$

and that the speed remains in the range where three equilibria exist (precise conditions being introduced below). Denote them by w_2, w_1 and w_0, with the convention that

$$-1 < w_2 \leqslant w_1 \leqslant w_0. \tag{2.2}$$

Observe that one (at most) among the points w_2 and w_1 may well be in the elliptic region. We want to study the system (1.9), (1.10) for a fixed left-hand state w_0, by using the speed λ or the right-hand states $w_+ = w_1$ or $w_+ = w_2$ as parameters. Throughout this paper, for definiteness, we focus our attention on waves propagating to the right, that is,

$$\lambda > 0.$$

We will need some notation concerning the graph of the function σ. In view of (1.3), for any $w \neq 0$, there exists a unique line passing through the point of the graph with coordinate w and being tangent to the graph at some other point, whose coordinate is denoted by $\varphi^\natural(w) \neq w$. In other words, we have

$$\sigma'(\varphi^\natural(w)) = \frac{\sigma(w) - \sigma(\varphi^\natural(w))}{w - \varphi^\natural(w)} \quad \text{for all } w \neq 0,\ w > -1. \tag{2.3}$$

Note that $w\varphi^\natural(w) < 0$ and, by continuity, $\varphi^\natural(0) = 0$. Thanks to (1.3), the map $\varphi^\natural : (-1, \infty) \to (-1, \infty)$ is monotone decreasing and onto, and so is invertible. Its inverse function, denoted by $\varphi^{-\natural}$, satisfies

$$\sigma'(w) = \frac{\sigma(w) - \sigma(\varphi^{-\natural}(w))}{w - \varphi^{-\natural}(w)} \quad \text{for all } w \neq 0,\ w > -1. \tag{2.4}$$

For each $w_0 > -1$, we also set

$$\lambda^\natural(w_0) = \sqrt{\sigma'(\varphi^\natural(w_0))}, \qquad \lambda^{-\natural}(w_0) = \sqrt{\sigma'(w_0)}.$$

Loosely speaking, for each fixed w_0, these values are lower and upper bounds, respectively, among all shock speeds λ in (1.7). More precisely, this makes sense only in the intervals where the speeds $\lambda^\natural(w_0)$ and $\lambda^{-\natural}(w_0)$ are real, i.e. only if $\varphi^\natural(w_0) \notin (w_*, w^*)$ and $w_0 \notin (w_*, w^*)$, respectively.

Defining first the speed by

$$\lambda = \lambda(w_0, w) := \begin{cases} \sqrt{\dfrac{\sigma(w) - \sigma(w_0)}{w - w_0}} & \text{if } w \neq w_0, \\ \sqrt{\sigma'(w_0)} & \text{if } w = w_0, \end{cases}$$

consider then the function

$$H(w_0, w) = \int_{w_0}^{w} (g(s, \lambda(w_0, w)) - g(w_0, \lambda(w_0, w)))\,\mathrm{d}s, \quad w, w_0 \in (-1, \infty). \tag{2.5}$$

Multiplying (1.8) by w_y and integrating over $\mathbb{R}$, we obtain the following result.

LEMMA 2.1. *If there exists a travelling wave solution of (1.9), (1.10) connecting $w_- = w_0$ to some $w_+ = w$, then, necessarily,*

$$H(w_0, w) \geqslant H(w_0, w_0) = 0,$$

where the inequality is strict if $w \neq w_0$ and $\beta > 0$.

LEMMA 2.2. *There exists a function $\varphi^\flat_\infty : (-1, \infty) \to (-1, \infty)$, strictly monotone decreasing and onto, such that, for all $w_0 > 0$ (and, similarly, for $w_0 < 0$),*

$$\varphi^{-\natural}(w_0) \leqslant \varphi^\flat_\infty(w_0) < \varphi^\natural(w_0)$$

and

$$H(w_0, w) = 0 \quad \textit{and} \quad w \neq w_0 \quad \textit{if and only if } w = \varphi^\flat_\infty(w_0).$$

Moreover, for all $w_0 > 0$ and all w, we have

$$H(w_0, w) > 0 \quad \textit{if and only if } \varphi^\flat_\infty(w_0) < w < w_0.$$

Geometrically, the function $\varphi^\flat_\infty$ corresponds to the 'equal-area' condition (the line connecting w_0 to w cuts the graph of σ in two equal areas). Observe that $\varphi^\flat_\infty$ is its own inverse; indeed, $\varphi^\flat_\infty \circ \varphi^\flat_\infty = \mathrm{id}$.

Combining the above two lemmas, we deduce, for instance, that if there exists a travelling wave connecting $w_0 > w^*$ to w, then

$$\varphi^\flat_\infty(w_0) \leqslant w \leqslant w_0.$$

Among these travelling waves, some correspond to *classical shock waves* ($w \geqslant w^*$) and *supersonic phase boundaries* ($w \leqslant w_*$), which satisfy the standard Liu entropy condition, that is, for which the line connecting w_0 to w does not intersect the graph of σ (except, of course, at the end points),

$$w \in [\varphi^{-\natural}(w_0), w_0]. \tag{2.6}$$

On the other hand, by definition, *subsonic phase boundaries* satisfy

$$w \in [\varphi^\flat_\infty(w_0), \varphi^{-\natural}(w_0)]. \tag{2.7}$$

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Based on the function $\varphi^\flat_\infty$, we also define a unique function $\varphi^\sharp_\infty$ by the two conditions (here, $w_0 \neq 0$)

$$\varphi^\sharp_\infty(w_0) \neq \varphi^\flat_\infty(w_0), \qquad \frac{\sigma(w_0) - \sigma(\varphi^\sharp_\infty(w_0))}{w_0 - \varphi^\sharp_\infty(w_0)} = \frac{\sigma(w_0) - \sigma(\varphi^\flat_\infty(w_0))}{w_0 - \varphi^\flat_\infty(w_0)}.$$

Let us also set $\lambda_\infty(0) = 0$ and, for $w_0 \neq 0$,

$$\lambda_\infty(w_0) = \sqrt{\frac{\sigma(w_0) - \sigma(\varphi^\flat_\infty(w_0))}{w_0 - \varphi^\flat_\infty(w_0)}}$$

(as long an the quantity under the square-root is non-negative), which is the *maximal admissible speed* for the range of right-hand states w comprised between $\varphi^\sharp_\infty(w_0)$ and $\varphi^\natural(w_0)$, at least. Recall that $\lambda^\natural(w_0)$ is a lower bound for the speeds.

To clearly express the constraint that *the speed must be real,* we introduce the monotone increasing one-to-one function

$$\tilde{\varphi} : [\tilde{w}^*, w_*] \cup [w^*, \tilde{w}_*] \to [\tilde{w}^*, w_*] \cup [w^*, \tilde{w}_*]$$

by

$$w \neq \tilde{\varphi}(w), \quad \sigma(w) = \sigma(\tilde{\varphi}(w)) \quad \text{with } \tilde{\varphi}(w^*) = \tilde{w}^* \text{ and } \tilde{\varphi}(w_*) = \tilde{w}_*.$$

In other words, $\tilde{\varphi}(w)$ is connected to w by a stationary phase boundary. Observe that $\tilde{\varphi}$ is its own inverse; indeed, $\tilde{\varphi} \circ \tilde{\varphi} = \mathrm{id}$. A wave connecting the left-hand state w_0 to some right-hand state w satisfies the condition

$$\text{if } w_0 \in [w^*, \tilde{w}_*), \quad \text{then } w \notin (\tilde{\varphi}(w), w_*],$$

which precisely excludes imaginary speeds. Now, the so-called *Maxwell states*, $\underline{w} < 0 < \overline{w}$, are uniquely defined as the intersection points of the monotone functions $\varphi^\flat_\infty$ and $\tilde{\varphi}$, as follows:

$$\varphi^\flat_\infty(\underline{w}) = \overline{w}, \qquad \tilde{\varphi}(\underline{w}) = \overline{w}.$$

Modulo some trivial rescaling, the travelling trajectories depend only upon the ratio

$$\delta := \frac{\sqrt{\alpha}}{\beta}.$$

To state our result, for each left-hand state w_0, we define the 2-*shock set* generated by equations (1.9), (1.10) by

$$S^2_\delta(w_0) := \{w \mid \text{there is a travelling wave satisfying (1.9), (1.10),} \\ \text{with } w_- = w_0 \text{ and } w_+ = w\}.$$

When searching for travelling wave solutions, one first identifies all of the states $w_+ = w_2$ associated with subsonic phase boundaries.

THEOREM 2.3 (The kinetic function). *Consider the travelling wave solutions of (1.9), (1.10) under the assumptions (1.2), (1.3) and for a given diffusion-dispersion ratio $\delta = \sqrt{\alpha}/\beta \in [0, \infty)$. There exists a continuous* kinetic function

$$\varphi^\flat_\delta : [\overline{w}, \infty) \to (-1, w_*),$$

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which satisfies

$$\left.\begin{aligned}&\varphi^\flat_\infty(w_0) < \varphi^\flat_\delta(w_0) \leqslant \varphi^\natural(w_0) \quad \textit{for all } w \in [\tilde{w}_*, \infty),\\&\varphi^\flat_\infty(w_0) < \varphi^\flat_\delta(w_0) \leqslant \tilde{\varphi}(w_0) \quad \textit{for all } w_0 \in [\overline{w}, \tilde{w}_*].\end{aligned}\right\} \tag{2.8}$$

For each right-hand side $w \in [\overline{w}, \infty)$, *there exists a* (*unique*) *subsonic phase boundary connecting the left-hand state* w_0 *to the right-hand state* $\varphi^\flat_\delta(w_0)$.

The function $\varphi^\flat_\delta$ completely characterizes the dynamics of the subsonic phase boundaries. Of course, a kinetic function can also be defined on the interval $(-1, \underline{w}]$ with similar properties. In view of theorem 2.3, a function

$$\varphi^\sharp_\delta : [\overline{w}, \infty) \to (-1, \infty)$$

can be uniquely characterized by the two conditions

$$\frac{\sigma(w_0) - \sigma(\varphi^\sharp_\delta(w_0))}{w_0 - \varphi^\sharp_\delta(w_0)} = \frac{\sigma(w_0) - \sigma(\varphi^\flat_\delta(w_0))}{w_0 - \varphi^\flat_\delta(w_0)} \tag{2.9 a}$$

and

$$\varphi^\natural(w_0) \leqslant \varphi^\sharp_\delta(w_0) \leqslant w_0 \quad \text{for } w_0 > \overline{w}. \tag{2.9 b}$$

Observe that (2.8) and (2.9) also imply

$$\varphi^\sharp_\delta(w_0) < \varphi^\sharp_\infty(w_0) \leqslant w_0. \tag{2.10}$$

THEOREM 2.4 (The 2-shock curve). *Under the same assumptions as in theorem 2.3, we have* (*tacitly* excluding *all states in the elliptic interval* (w_*, w^*))

$$S^2_\delta(w_0) = \begin{cases}\{\varphi^\flat_\delta(w_0)\} \cup (\varphi^\sharp_\delta(w_0), w_0] & \textit{for } w_0 > \overline{w},\\ [w^*, w_0] & \textit{for } w_0 \in (w^*, \overline{w}].\end{cases} \tag{2.11}$$

THEOREM 2.5 (Asymptotic properties).

(1) *There exists a continuous function*

$$\kappa^\natural : (-1, \tilde{w}^*) \cup (\tilde{w}_*, +\infty) \to [0, \infty), \quad w_0 \mapsto \kappa^\natural(w_0),$$

such that

$$\left.\begin{aligned}&\varphi^\flat_\delta(w_0) = \varphi^\natural(w_0), \quad \textit{provided } \delta\kappa^\natural(w_0) \leqslant 1,\\&\kappa^\natural(w_0) \to +\infty \quad \textit{as } w_0 \to \tilde{w}_*.\end{aligned}\right\} \tag{2.12}$$

(2) *For each* $w > \overline{w}$, *we have*

$$\varphi^\flat_\delta(w_0) \to \varphi^\flat_\infty(w_0) \quad \textit{as } \delta \to \infty.$$

The function $\kappa^\natural$ can also be defined on the interval $(-1, \tilde{w}^*)$. The proofs of the above results will be given in § 3 below. Further asymptotic properties will be discussed in § 4.

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3. Existence of travelling waves

We will rely on the observation that system (1.9) *formally* reduces to the (scalar) model studied in [3], provided the following transformation is applied, in which the index s refers to the 'scalar case',

$$\lambda = \sqrt{\lambda_{\rm s}}, \qquad \lambda\beta = \beta_{\rm s}, \tag{3.1}$$

with obvious notation. Indeed, using (3.1), equation (1.8) transforms into

$$\sigma(w) - \sigma(w_0) - \lambda_{\rm s}(w - w_0) = \beta_{\rm s} b(w) w_y + \alpha a(w)(a(w)w_y)_y. \tag{3.2}$$

The existence and properties of the travelling wave solutions of (3.2) were investigated in [3]. However, one source of difficulty in applying [3] is the fact that the diffusion parameter $\beta_{\rm s}$ in (3.2) *depends on the shock speed* ($\beta_{\rm s} = \lambda\beta$). Another new feature is the fact that the speed $\lambda^2 = \lambda_{\rm s}$ must remain non-negative, a condition that need not hold in the scalar case.

First of all, without loss of generality, we take

$$a(w) \equiv 1.$$

(Use the change of variable $y \mapsto \xi(y)$ with $\mathrm{d}\xi = a(w)\,\mathrm{d}y$ and redefine the viscosity accordingly.) By a straightforward rescaling of the travelling wave, we can also assume that

$$\alpha = 1.$$

Hence system (1.9) becomes

$$\left.\begin{aligned} w_y &= z, \\ z_y &= -\lambda\beta b(w) z + g(w,\lambda) - g(w_0,\lambda). \end{aligned}\right\} \tag{3.3}$$

Define the function

$$G(w; w_0, \lambda) := \int_{w_0}^{w} (g(s,\lambda) - g(w_0,\lambda))\,\mathrm{d}s. \tag{3.4}$$

Observe that $\delta_w G(w; w_0, \lambda) = 0$ if and only if (2.1) holds, i.e. if and only if w is an equilibrium point. Recall that to each w_0 and speed λ (in some interval) we associate (see (2.1), (2.2)) two other equilibria w_1 and w_2.

LEMMA 3.1. *Given $w_0 > w^*$ and $\lambda > 0$ in the interval*

$$\lambda \in \begin{cases} (\lambda^{\natural}(w_0), \lambda^{-\natural}(w_0)) & \text{if } w_0 \geqslant \tilde{w}_*, \\ (0, \lambda^{-\natural}(w_0)) & \text{if } w_0 \in (w^*, \tilde{w}_*), \end{cases} \tag{3.5}$$

the function $J(w) := G(w, w_0, \lambda)$ satisfies

$$\begin{aligned} J'(w) &< 0 \quad \text{for all } w < w_2 \text{ or } w \in (w_1, w_0), \\ J'(w) &> 0 \quad \text{for all } w \in (w_2, w_1) \text{ or } w > w_0. \end{aligned}$$

Moreover, if

$$\lambda \in \begin{cases} (\lambda^{\natural}(w_0), \lambda_{\infty}(w_0)) & \text{if } w_0 \geqslant \tilde{w}_*, \\ (0, \lambda_{\infty}(w_0)) & \text{if } w_0 \in (\overline{w}, \tilde{w}_*), \end{cases}$$

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we have

$$J(w_0) = 0 < J(w_2) < J(w_1).$$

If $\lambda = \lambda_\infty(w_0)$, *then*

$$J(w_0) = J(w_2) = 0 < J(w_1).$$

If $\lambda \in (\lambda_\infty(w_0), \lambda^{-\natural}(w_0))$, *then*

$$J(w_2) < 0 = J(w_0) < J(w_1).$$

Observe that the functions G and H are related in the following way:

$$G(w; w_0, \lambda) = H(w_0, w) \quad \text{if and only if } \lambda = \lambda(w_0, w). \tag{3.6}$$

In view of lemma 2.1, we must have $H(w_0, w) \geqslant 0$ for the existence of a travelling wave connecting w_0 to w. Thus, from lemma 3.1,

$$\left.\begin{aligned} &\text{if there exists a trajectory connecting } w_0 \text{ to } w_2,\\ &\qquad\text{then } \lambda \in \begin{cases} [\lambda^\natural(w_0), \lambda_\infty(w_0)] & \text{if } w_0 > \tilde{w}_*,\\ [0, \lambda_\infty(w_0)] & \text{if } w_0 \in (\overline{w}, \tilde{w}_*). \end{cases}\end{aligned}\right\} \tag{3.7}$$

Fix a propagation speed $\lambda > 0$ and a left-hand state $w_0 > w^*$, and search for trajectories connecting w_0 to the associated equilibrium w_2 introduced in § 2. According to our earlier discussion in § 2 and to (3.7), we can conclude that

$$\left.\begin{aligned} &w_2 \in [\varphi^\flat_\infty(w_0), \varphi^\natural(w_0)], \quad \lambda \in [\lambda^\natural(w_0), \lambda_\infty(w_0)] \quad \text{if } w_0 \geqslant \tilde{w}_*,\\ &w_2 \in [\varphi^\flat_\infty(w_0), \tilde{\varphi}(w_0)], \quad \lambda \in [0, \lambda_\infty(w_0)] \quad \text{if } w_0 \in (\overline{w}, \tilde{w}_*);\end{aligned}\right\} \tag{3.8}$$

conditions to be assumed throughout this section.

On the other hand, the eigenvalues of system (3.3) at the equilibrium point are found to be

$$\mu = \tfrac{1}{2}(-\beta\lambda b(w) \pm \sqrt{\beta^2\lambda^2 b(w)^2 + 4(\sigma'(u) - \lambda^2)}).$$

Specifically (for $\beta \neq 0$), we set

$$\left.\begin{aligned} \underline{\mu}(w, \lambda, \beta) &= \tfrac{1}{2}\beta\lambda b(w)\biggl(-1 - \sqrt{1 + 4\frac{\sigma'(w) - \lambda^2}{\beta^2\lambda^2 b(w)^2}}\biggr),\\ \overline{\mu}(w, \lambda, \beta) &= \tfrac{1}{2}\beta\lambda b(w)\biggl(-1 + \sqrt{1 + 4\frac{\sigma'(w) - \lambda^2}{\beta^2\lambda^2 b(w)^2}}\biggr).\end{aligned}\right\} \tag{3.9}$$

Lemma 3.2 (Equilibrium points). *Fix some state* w_0 *and speed* λ *and let* w *be any equilibrium point.*

If $\sigma'(w) - \lambda^2 > 0$, *then* w *is a saddle point having two real eigenvalues,* $\underline{\mu} < 0 < \overline{\mu}$.

If $\sigma'(w) - \lambda^2 < 0$, *then* $\mathrm{Re}(\underline{\mu})$ *and* $\mathrm{Re}(\overline{\mu})$ *are both negative and* w *is referred to as* a stable point. *Furthermore, if* $\beta^2\lambda^2 b(w)^2 + 4(\sigma'(w) - \lambda^2) > 0$, *then* w *corresponds to a* stable node *with two real negative eigenvalues* $\underline{\mu} < \overline{\mu} < 0$. *Otherwise, if* $\beta^2 b(w)^2 + 4(\sigma'(w) - \lambda^2) > 0$, w *is a* stable spiral *with two complex conjugate eigenvalues with negative real parts.*

In view of the transformation (3.1), (3.2) given above, and using the results of existence of non-classical trajectories derived in [3], we obtain the following result.

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THEOREM 3.3 (Existence of subsonic phase boundaries). *Given two states $w_0 > \overline{w}$ and $w_2 < w_*$, corresponding to a (real) propagation speed λ, also satisfying*

$$\lambda = \sqrt{\frac{\sigma(w_2) - \sigma(w_0)}{w_2 - w_0}} \in \begin{cases} (\lambda^\natural(w_0), \lambda_\infty(w_0)] & \text{if } w_0 > \tilde{w}_*, \\ [0, \lambda_\infty(w_0)] & \text{if } w_0 \in (\overline{w}, \tilde{w}_*), \end{cases}$$

there is a unique diffusion $\beta = \beta(w_0, w_2) \geqslant 0$ such that w_0 can be connected to w_2 by a travelling wave solution of (3.3).

LEMMA 3.4. *Define*

$$\Delta = \{(w_0, w_2) \in \mathcal{H}_+ \times \mathcal{H}_- \mid w_0 \geqslant \overline{w},\ w_2 \text{ satisfies (3.8)}\}$$

and consider the function

$$\Delta \ni (w_0, w_2) \mapsto \beta(w_0, w_2),$$

which associates the (unique) value β such that there is a travelling wave connecting w_0 to w_2 (theorem 3.3).

Then, for each fixed $w_0 > \overline{w}$, $\beta(w_0, w_2)$ is a strictly monotone increasing function of w_2, mapping the interval given by (3.8) onto some interval $[0, \beta^\natural(w_0))$, where the upper bound $\beta^\natural(w_0)$ is finite if $w_0 > \tilde{w}_*$, but $\beta^\natural(w_0) = \infty$ if $w_0 \in (\overline{w}, \tilde{w}_*)$.

Following the terminology in [3], the value $\beta^\natural(w_0)$ is called the *critical diffusion* at w_0. Subsonic phase boundaries leaving from w_0 exist only when $\beta < \beta^\natural(w_0)$.

Proof. Based on the transformation (3.1) and with an obvious notation, we have

$$\beta_{\mathrm{s}}(w_0, w_2) = \lambda(w_0, w_2)\beta(w_0, w_2). \tag{3.10}$$

Fixing $w_0 > \overline{w}$, let w_2 and w_2' be two reals in the interval given by (3.8) with $w_2 < w_2'$, associated with some speeds λ and λ', respectively. Then theorem 3.7 in [3] gives

$$\beta_{\mathrm{s}}(w_0, w_2) < \beta_{\mathrm{s}}(w_0, w_2').$$

On the other hand,

$$\lambda > \lambda' > 0,$$

and so, using (3.10), we conclude that

$$\beta(w_0, w_2) < \beta(w_0, w_2').$$

Now, since the functions $w_2 \mapsto \beta_{\mathrm{s}}(w_0, w_2)$ and $w_2 \mapsto \lambda(w_0, w_2)$ are increasing and decreasing, respectively, and using the boundedness of these functions, we deduce that β remains in an interval of the form $(0, \beta^\natural(w_0))$, where $\beta^\natural(w_0)$ is finite if $w_0 > \tilde{w}_*$ and $\beta^\natural(w_0) = \infty$ if $w_0 \in (\overline{w}, \tilde{w}_*)$. □

In view of lemma 3.4, for all $\beta < \beta^\natural(w_0)$, there exists a unique subsonic travelling wave of (3.3) connecting w_0 to some point $w_2 = \varphi^\flat(w_0, \beta)$ and associated with a speed $\lambda = \lambda(w_0, \beta)$. Equivalently, with the notation (3.1), $\lambda_{\mathrm{s}} = \lambda_{\mathrm{s}}(\beta_{\mathrm{s}}, w_0)$. The proof of theorem 2.3 is then completed.

Note also that lemma 3.4 implies immediately statement (1) in theorem 2.5.

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We now discuss the proof of theorem 2.4 decomposed in the two lemmas below. Given $w_0 > w^*$ and $\beta > 0$, we study the existence of the (classical and supersonic) travelling waves of (3.3) connecting $w_- = w_0$ to $w_+ = w_1$. Recall that the shock speed λ must lie in the interval (3.5).

LEMMA 3.5. *If $w_0 > \overline{w}$, then, for each $\beta < \beta^{\natural}(w_0)$ and each speed λ satisfying $\lambda(\beta, w_0) < \lambda < \lambda^{-\natural}(w_0)$, there exists a travelling wave connecting $w_- = w_0$ to $w_+ = w_1$.*

If $w_0 > \tilde{w}_*$, then, for each $\beta \geqslant \beta^{\natural}(w_0)$ and each speed $\lambda^{\natural}(w_0) \leqslant \lambda < \lambda^{-\natural}(w_0)$, there exists a travelling wave connecting $w_- = w_0$ to $w_+ = w_1$.

Proof. We first treat the case where $w_0 > \overline{w}$ and $\beta < \beta^{\natural}(w_0)$. The region $\lambda(\beta, w_0) < \lambda < \lambda^{-\natural}(w_0)$ corresponds to the region $\lambda_{\rm s}(\beta_{\rm s}, w_0) < \lambda_{\rm s} < \lambda_{\rm s}^{-\natural}(w_0)$. On the other hand, the parameter $\beta_{\rm s} = \beta\lambda = \beta\sqrt{\lambda_{\rm s}}$ satisfies

$$\beta_{\rm s} > \beta\sqrt{\lambda_{\rm s}(\beta_{\rm s}, w_0)} = \beta_{\rm s}(w_0, \varphi^{\flat}(w_0, \beta)).$$

Thus, in view of the monotonicity of the function $w_0 \mapsto \beta_{\rm s}(w_0, w_2)$ and theorem 5.1 in [3] applied to (3.2), there exists a travelling wave connecting w_0 to w_1.

Now, if $w_0 > \tilde{w}_*$ and $\beta \geqslant \beta^{\natural}(w_0)$, then, for all $\lambda_{\rm s}^{\natural}(w_0) \leqslant \lambda_{\rm s} < \lambda^{-\natural}(w_0)$, we have

$$\beta_{\rm s} = \beta\sqrt{\lambda_{\rm s}} \geqslant \beta^{\natural}(w_0)\sqrt{\lambda_{\rm s}} \geqslant \beta^{\natural}(w_0)\sqrt{\lambda_{\rm s}^{\natural}(w_0)} = \beta_{\rm s}^{\natural}(w_0).$$

Applying again theorem 5.1 in [3], there exists a travelling wave of (1.9), (1.10) connecting w_0 to w_1. □

LEMMA 3.6. *If*

$$\lambda \in \begin{cases} (\lambda^{\natural}(w_0), \lambda(w_0, \beta)) & \text{if } w_0 > \tilde{w}_0, \\ (0, \lambda(w_0, \beta)) & \text{if } w_0 \in (\overline{w}, \tilde{w}_0), \end{cases} \tag{3.11}$$

then there is no travelling wave connecting $w_- = w_0$ to $w_+ = w_1$.

Proof. The proof is similar to the one of the previous lemma. Suppose that $\beta < \beta^{\natural}(w_0)$. Then the interval given by (3.11) corresponds to the interval

$$\lambda_{\rm s} \in \begin{cases} (\lambda_{\rm s}^{\natural}(w_0), \lambda_{\rm s}(\beta_{\rm s}, w_0)) & \text{if } w_0 > \tilde{w}_0, \\ (0, \lambda_{\rm s}(\beta_{\rm s}, w_0)) & \text{if } w_0 \in (\overline{w}, \tilde{w}_0), \end{cases}$$

and the parameter $\beta_{\rm s} = \beta\sqrt{\lambda_{\rm s}}$ satisfies

$$\beta_{\rm s} < \beta\sqrt{\lambda_{\rm s}(\beta_{\rm s}, w_0)} = \beta_{\rm s}(w_0, \varphi^{\flat}(w_0, \beta)) < \beta_{\rm s}(w_0, w_2).$$

The proof is completed by relying on the monotonicity of the function $w_2 \mapsto \beta_{\rm s}(w_0, w_2)$ and on theorem 5.2 in [3] applied to (3.2). □

4. Asymptotic properties of the kinetic functions

This section is devoted to deriving various asymptotic properties of the kinetic function.

Theorem 4.1. *For each $\beta > 0$, the kinetic function $w_0 \to \varphi^\flat(w_0, \beta)$ defined above on the interval $(\overline{w}, +\infty)$ can be extended by continuity up to $w_0 = \overline{w}$ by setting $\varphi^\flat_\delta(\overline{w}) = \underline{w}$. Moreover, it is differentiable at this point and*

$$\frac{\delta\varphi^\flat}{\delta w_0}(\overline{w}, \beta) = \frac{\sigma'(\underline{w})}{\sigma'(\overline{w})} > 0. \tag{4.1}$$

Hence the kinetic function is strictly monotone increasing in a neighbourhood of $w_0 = \overline{w}$, but is not globally monotone. Indeed, for large values of w, we have $\lim_{w_0 \to +\infty} \varphi^\flat(w_0, \beta) = -1$, while $-1 < \underline{w} = \varphi^\flat_\delta(\overline{w})$.

Proof. The kinetic function is immediately extended by continuity in view of the inequalities $0 < \lambda(w_0, \beta) \leqslant \lambda_\infty(w_0)$, in which $\lambda_\infty(w_0) \to 0$ as $w_0 \to \overline{w}$. Consider now the implicit relation relating w_0 and $\varphi^\flat(w_0, \beta)$, that is,

$$\sigma(\varphi^\flat(w_0, \beta)) - \sigma(w_0) - \lambda(w_0, \beta)^2(\varphi^\flat(w_0, \beta) - w_0) = 0. \tag{4.2}$$

By differentiating equation (4.2) with respect to w_0, we obtain

$$\begin{aligned}(\sigma'(\varphi^\flat(w_0, \beta)) - \lambda(w_0, \beta)^2)&\frac{\delta\varphi^\flat}{\delta w_0}(w_0, \beta) \\ &= \sigma'(w_0) - \lambda(w_0, \beta)^2 - 2\lambda(w_0, \beta)(\varphi^\flat(w_0, \beta) - w_0)\frac{\delta\lambda}{\delta w_0}(w_0, \beta).\end{aligned} \tag{4.3}$$

On the other hand, the inequalities

$$\lambda(\overline{w}, \beta) = \lambda_\infty(\overline{w}) = 0 \leqslant \lambda(w_0, \beta) \leqslant \lambda_\infty(w_0)$$

for $w_0 > \overline{w}$ clearly imply that $\delta\lambda/\delta w_0(\overline{w}, \beta)$ is finite and

$$0 \leqslant \frac{\delta\lambda}{\delta w_0}(\overline{w}, \beta) \leqslant \frac{\delta\lambda_\infty}{\delta w_0}(\overline{w}).$$

By letting $w_0 \to \overline{w}$ in (4.3), we obtain (4.1). □

Theorem 4.2.

(1) *The critical diffusion is bounded below as follows (for all $w_0 > \overline{w}$),*

$$\beta^\natural(w_0) \geqslant T(w_0) := \frac{1}{\sqrt{2}} \frac{\sqrt{G(\varphi^\natural(w_0) - G(w_0))}}{\lambda^\natural(w_0)(D(w_0) - D(\varphi^\natural(w_0)))}, \tag{4.4}$$

where $D'(w) = d(w)$ and G was defined in (3.4).

(2) *In particular, suppose, for instance, that $\sigma'(w) \sim Aw^{\gamma-1}$ as $w \to +\infty$, $\sigma(w) = o(\sigma'(w))$ as $w \to -1$ and $d(w) \leqslant d_M$, where A and d_M are positive constants and $\gamma > 1$. Then we have*

$$\liminf_{w_0 \to +\infty} \beta^\natural(w_0) \geqslant \frac{1}{2d_M}\sqrt{\frac{\gamma - 1}{\gamma + 1}}. \tag{4.5}$$

Proof. We will rely on equation (1.9) written in the phase plane (w, z),

$$z(w)\frac{\mathrm{d}z}{\mathrm{d}w}(w) = -\lambda^\natural(w_0)\beta^\natural(w_0)d(w)z(w) + g(w, \lambda^\natural(w_0)) - g(w_0, \lambda^\natural(w_0)). \tag{4.6}$$

We fix some values $w_0 > \overline{w}$ and $\beta = \beta^\natural(w_0)$ and consider the trajectory $z = z(w)$ connecting w_0 to $w_2 = \varphi^\natural(w_0)$. The maximal negative value of the function $w \to z(w)$ is achieved at some point $w_3 \in (w_2, w_0)$. So we have

$$z_3 := z(w_3) = -\max_w |z(w)|.$$

Integrating (4.6) over the interval $[w_2, w_3]$, we get

$$\tfrac{1}{2} z_3^2 - \lambda^\natural(w_0)\beta^\natural(w_0) \int_{w_2}^{w_3} |z(w)| d(w) \, \mathrm{d}w = G(w_3) - G(w_2).$$

Since $G(w_3) - G(w_2) \leqslant 0$, we deduce that

$$\begin{aligned} \tfrac{1}{2} z_3^2 &\leqslant \lambda^\natural(w_0)\beta^\natural(w_0)|z_3|(D(w_3) - D(w_2)) \\ &\leqslant \lambda^\natural(w_0)\beta^\natural(w_0)|z_3|(D(w_0) - D(w_2)). \end{aligned} \tag{4.7}$$

In other words, we have the following upper bound for the maximal value z_*:

$$|z_3| \leqslant 2\lambda^\natural(w_0)\beta^\natural(w_0)(D(w_0) - D(w_2)). \tag{4.8}$$

Next we integrate (4.5) again, but now on the interval $[w_2, w_0]$,

$$\begin{aligned} 0 &\leqslant G(w_2) - G(w_0) \\ &= \lambda^\natural(w_0)\beta^\natural(w_0) \int_{w_2}^{w_0} |z(w)| d(w) \, \mathrm{d}w \qquad (4.9) \\ &\leqslant \lambda^\natural(w_0)\beta^\natural(w_0)|z_3|(D(w_0) - D(w_2)). \qquad (4.10) \end{aligned}$$

Combining (4.8) and (4.10), we conclude that

$$\beta^\natural(w_0) \geqslant T(w_0),$$

which establishes the first item of the theorem.

To prove the second claim, we observe that

$$2T(w_0)^2 \geqslant \frac{1}{\sigma'(w_2) d_M^2 (w_0 - w_2)^2} \int_{w_2}^{w_0} (\sigma(w_0) + \sigma'(w_2)(w - w_0) - \sigma(w)) \, \mathrm{d}w. \tag{4.11}$$

On the other hand, w_0 and $w_2 = \varphi^\natural(w_0)$, by definition, are related by

$$\sigma(w_2) - \sigma(w_0) - \sigma'(w_2)(w_2 - w_0) = 0. \tag{4.12}$$

When $w_0 \to +\infty$, we also have $w_2 = \varphi^\natural(w_0) \to -1$. By contradiction, if $\sigma(w_2)$ and $\sigma'(w_2)$ remain bounded, then (4.12) would imply $\sigma(w_0) \sim c w_0$, which contradicts our assumption that $\sigma'(w_0) \sim A w^{\gamma-1}$ with $\gamma > 1$. Therefore, for all w_0 large enough, we deduce from (4.12) that

$$\sigma'(w_2) \sim \frac{\sigma(w_0)}{w_0} \sim \frac{A}{\gamma} w_0^{\gamma-1}. \tag{4.13}$$

We now estimate the right-hand side of (4.11),

$$\int_{w_2}^{w_0} (\sigma(w_0) + \sigma'(w_2)(w - w_0) - \sigma(w)) \, \mathrm{d}w \geqslant \int_0^{w_0} (\sigma(w_0) + \sigma'(w_2)(w - w_0) - \sigma(w)) \, \mathrm{d}w. \tag{4.14}$$

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By using (4.13) and the behaviour of σ when $w \to +\infty$, we get

$$\int_0^{w_0} (\sigma(w_0) + \sigma'(w_2)(w - w_0) - \sigma(w))\,\mathrm{d}w$$
$$\sim \sigma(w_0)w_0 - \tfrac{1}{2}\sigma'(w_2)w_0^2 - A\frac{w_0^{\gamma+1}}{(\gamma+1)(\gamma)}$$
$$\sim \frac{(\gamma-1)Aw_0^{\gamma+1}}{2\gamma(\gamma+1)}. \tag{4.15}$$

Finally, combining (4.15), (4.14), (4.13) and (4.4), we obtain (4.5). □

COROLLARY 4.3. *Under the asymptotic assumptions made in theorem 4.2, there exists a constant $C > 0$ such that, for all $\beta \leqslant C$ and for all $w_0 > \overline{w}$, there is a (unique) subsonic phase boundary connecting w_0 to some right-hand state $w_2 = \varphi^\flat(w_0, \beta)$.*

5. Examples of kinetic functions

This section focuses on a class of polynomial stress-functions (see (5.1) below). On one hand, in theorem 5.1, the critical diffusion introduced in our analysis in § 3 is determined explicitly. On the other hand, in the cubic case, following Shearer and Yang [24], we explicitly compute the kinetic function. We reformulate Shearer and Yang's result in a convenient form by relying on the framework introduced in § 2. For simplicity, the condition $\lim_{w\to-1}\sigma(w) = -\infty$ is no longer imposed and problem (1.9), (1.10) is considered on the real line, that is, $w \in \mathbb{R}$.

Consider the polynomial stress function

$$\sigma_{k,\gamma}(w) = w|w|^{\gamma-1} - k^2 w, \quad \gamma > 1, \quad k > 0. \tag{5.1}$$

It satisfies the scaling property

$$\sigma_{0,\gamma}(tw) = t^\gamma \sigma_{0,\gamma}(w), \quad t > 0, \tag{5.2}$$

which is the key to proving the following result.

THEOREM 5.1. *Consider the stress function (4.1) and constant diffusion and dispersion $a(w) = b(w) \equiv 1$. Then the corresponding critical diffusion $\beta^\natural_{k,\gamma}(w_0)$ (for $w_0 > \tilde{w}_*$) is given by*

$$\beta^\natural_{k,\gamma}(w_0) = C_\gamma \frac{w_0^{(\gamma-1)/2}}{\sqrt{\gamma(D_\gamma w_0)^{\gamma-1} - k^2}}, \tag{5.3}$$

where C_γ and D_γ are positive constants depending on γ only. In particular, D_γ is the positive solution D of

$$(\gamma-1)D^\gamma + \gamma D^{\gamma-1} - 1 = 0.$$

For instance, theorem 5.1 implies that, for all

$$\beta > \frac{C_\gamma}{\sqrt{\gamma {D_\gamma}^{(\gamma-1)}}},$$

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model (1.9) admits *no subsonic phase boundaries* leaving from $w_0 \in (\beta_{k,\gamma}^{-\natural}(\beta), +\infty)$, where $\beta_{k,\gamma}^{-\natural}$ is the inverse function of $\beta_{k,\gamma}^{\natural}$,

$$\beta_{k,\gamma}^{-\natural}(\beta) = \left(\frac{k^2\beta^2}{\gamma D_\gamma{}^{(\gamma-1)}\beta^2 - C_\gamma^2}\right)^{1/(\gamma-1)}.$$

Proof. Given the function $\sigma_{k,\gamma}$ and $w_0 > \tilde{w}_*$, one sees easily that $w_2 = \varphi^\natural(w_0)$ is independent of k. We want to determine the viscosity $\beta = \beta_{k,\gamma}^\natural(w_0)$ for which w_0 can be connected to $w_2 = \varphi^\flat(w_0)$. Rewrite equation (1.8) in the form

$$-\sigma_{k,\gamma}'(w_2)(w - w_0) + \sigma_{k,\gamma}(w) - \sigma_{k,\gamma}(w_0) = \lambda_{k,\gamma}^\natural(w_0)\beta_{k,\gamma}^\natural(w_0)w_y + \alpha w_{yy}. \quad (5.4)$$

For $\gamma > 1$ fixed, we set

$$\beta_k^\natural = \lambda_{k,\gamma}^\natural(w_0)\beta_{k,\gamma}^\natural(w_0), \quad (5.5)$$

so that

$$-\sigma_{k,\gamma}'(w_2)(w - w_0) + \sigma_{k,\gamma}(w) - \sigma_{k,\gamma}(w_0) = \beta_k^\natural w_y + \alpha w_{yy}. \quad (5.6)$$

Our main observation is that for all $w \in \mathbb{R}$

$$-\sigma_{k,\gamma}'(w_2)(w-w_0)+\sigma_{k,\gamma}(w)-\sigma_{k,\gamma}(w_0) = -\sigma_{0,\gamma}'(w_2)(w-w_0)+\sigma_{0,\gamma}(w)-\sigma_{0,\gamma}(w_0),$$

which, in view of (5.6), implies

$$\beta_k^\natural = \beta_0^\natural. \quad (5.7)$$

Consider now the transformation $w \mapsto \tilde{w} = w/w_0$ and set $\sigma = \sigma_{0,\gamma}$. In view of the scaling property (5.2), we have

$$\sigma(w) = w_0^\gamma \sigma(\tilde{w}),$$

and (5.6) becomes

$$-\sigma'(\tilde{w}_2)(\tilde{w} - 1) + \sigma(\tilde{w}) - \sigma(1) = w_0^{1-\gamma}(\beta_k^\natural \tilde{w}_y + \alpha \tilde{w}_{yy}). \quad (5.8)$$

By the transformation $y \mapsto \xi := y w_0^{(\gamma-1)/2}$, the last equation becomes

$$-\sigma'(\tilde{w}_2)(\tilde{w} - 1) + \sigma(\tilde{w}) - \sigma(1) = \beta_k^\natural w_0^{(1-\gamma)/2}\tilde{w}_\xi + \alpha \tilde{w}_{\xi\xi}. \quad (5.9)$$

On the other hand, the parameter $\tilde{w_2} = w_2/w_0 = \varphi^\flat(w_0)/w_0$ is a negative constant independent of w_0; it is the (unique) negative solution of

$$\frac{|x|^\gamma + 1}{|x| + 1} = \gamma |x|^{\gamma-1}. \quad (5.10)$$

We deduce that $\beta_k^\natural w_0^{(1-\gamma)/2}$ is a constant C_γ independent of w_0, i.e.

$$\beta_k^\natural = C_\gamma w_0^{(\gamma-1)/2}. \quad (5.11)$$

Finally, using (5.5), (5.7), (5.11) and the expression of $\sigma_{k,\gamma}'(w_2)$, we obtain (5.3). □

Consider next the case $\gamma = 3$ and $k = 1$ of the cubic function

$$\sigma(w) = w(w^2 - 1).$$

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From the definitions in § 2, we find

$$-w_* = w^* = \frac{1}{\sqrt{3}}, \qquad \tilde{w}^* = -\tilde{w}_* = \frac{2}{\sqrt{3}}, \qquad -\underline{w} = \overline{w} = 1.$$

We also consider the following functions (defined on the intervals of interest only):

$$\begin{aligned} \varphi^{\natural}(w) &= -\tfrac{1}{2}w && \text{if } w \in (-\infty, -2/\sqrt{3}] \cup [2/\sqrt{3}, +\infty), \\ \tilde{\varphi}(w) &= -\tfrac{1}{2}(w + \sqrt{4 - 3w^2}) && \text{if } w \in (-2/\sqrt{3}, -1) \cup (1, 2/\sqrt{3}). \end{aligned}$$

We also have

$$\lambda^{\natural}(w) = \sqrt{\tfrac{3}{4}w^2 - 1} \quad \text{for } w \in (-\infty, -2, \sqrt{3}] \cup [2/\sqrt{3}, +\infty),$$
$$\lambda^{-\natural}(w) = \sqrt{3w^2 - 1}, \qquad \varphi^{-\natural}(w) = -2w$$

and

$$\varphi^{\flat}_{\infty}(w) = -w, \qquad \varphi^{\sharp}_{\infty}(w) = 0, \qquad \lambda_{\infty}(w) = \sqrt{w^2 - 1}.$$

First, we exclude the area $0 \leqslant w_0 < 1$ for which the condition

$$\lambda^{\natural}(w_0) = \sqrt{\tfrac{3}{4}w_0^2 - 1} < \lambda < \sqrt{w_0^2 - 1} = \lambda_{\infty}(w_0)$$

is impossible.

For $w_0 > 1$ and λ such that

$$\sqrt{\tfrac{3}{4}w_0^2 - 1} < \lambda < \sqrt{w_0^2 - 1},$$

there exist exactly two solutions $w_2 < w_1 \leqslant 0$ (plus w_0) of the cubic equation

$$\lambda^2 = \frac{\sigma(w) - \sigma(w_0)}{w - w_0} = w^2 + w_0 w + w_0^2 - 1,$$

given explicitly by

$$w_1 = \tfrac{1}{2}\Big(-w_0 + \sqrt{4\lambda^2 - 3w_0^2 + 4}\Big), \qquad w_2 = -\tfrac{1}{2}\Big(w_0 + \sqrt{4\lambda^2 - 3w_0^2 + 4}\Big). \tag{5.12}$$

To derive the kinetic function, we start from the equation in the phase plane (w, z),

$$\alpha z(w)\frac{\mathrm{d}z}{\mathrm{d}w}(w) + \beta\lambda z(w) = (w - w_0)(w - w_1)(w - w_2). \tag{5.13}$$

(This is possible from the monotonicity of the non-classical travelling waves, established in [3].) For simplicity in the notation, we take $\alpha = 1$. Then, following [24], the travelling wave is sought in the (parabolic) form $z(w) = M(w - w_0)(w - w_2)$. After some simplification, we see that, necessarily, $M = 1/\sqrt{2}$ and

$$\beta\lambda = M(w_0 + w_2) - \frac{w_1}{M} = -\bigg(M + \frac{1}{M}\bigg)w_1 = -\frac{3}{\sqrt{2}}w_1.$$

Using the expressions (5.12) for w_1 and w_2, we arrive at an explicit relation between the diffusion, the left-hand state and the shock speed,

$$\beta = \beta(w_0, \lambda) := \frac{3}{2\sqrt{2}\lambda}\Big(w_0 - \sqrt{4\lambda^2 - 3w_0^2 + 4}\Big), \tag{5.14}$$

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which is relevant as long as

$$\lambda^\flat(w_0)^2 = \tfrac{3}{4}w_0^2 - 1 \leqslant \lambda^2 < w_0^2 - 1 = \lambda_\infty(w_0)^2. \tag{5.15}$$

One can also express the shock speed λ in term of the left- and right-hand states. Using (5.12) in (5.14), the diffusion is expressed as a function of w_0 and w_2,

$$\beta := \beta(w_0, w_2) := \frac{3}{2\sqrt{2}} \frac{w_0 - \sqrt{4w_2^2 + 4w_0 w_2 + w_0^2}}{\sqrt{w_0^2 + w_0 w_2 + w_2^2 - 1}}. \tag{5.16}$$

This represents the diffusion for which two given states can be connected by a subsonic phase boundary.

To describe the kinetic function, we fix some diffusion (recall that the capillarity has been normalized to be 1) and we distinguish between two regimes.

First of all, considering equation (5.14), the range of β (for which a subsonic phase boundary exists) is determined by letting the speed λ vary in the relevant interval given by (5.15). Precisely, we find that

$$0 \leqslant \beta < \beta^\natural(w_0) := \begin{cases} 3w_0/\sqrt{2(3w_0^2 - 4)} & \text{if } w_0 \geqslant 2/\sqrt{3}, \\ +\infty & \text{if } 1 < w_0 < 2/\sqrt{3}. \end{cases} \tag{5.17}$$

On the other hand, for β fixed in the range (5.17), we can inverse the relation (5.14) and obtain the quadratic equation

$$(9 - 2\beta^2)\lambda^2 + 3\sqrt{2}\beta w_0 \lambda - 9(w_0^2 - 1) = 0,$$

and so

$$\lambda = \lambda(w_0, \beta) := \begin{cases} -\dfrac{3}{\sqrt{2}} \dfrac{\beta w_0 - \sqrt{(18 - 3\beta^2)w_0^2 + 4\beta^2 - 18}}{9 - 2\beta^2} & \text{if } \beta^2 \neq \tfrac{9}{2}, \\ w_0 - 1/w_0 & \text{if } \beta^2 = \tfrac{9}{2}. \end{cases} \tag{5.18}$$

Using that $w_0 + w_1 + w_2 = 0$, we also obtain

$$w_2 = -w_0 - w_1 = -w_0 + \tfrac{1}{3}\sqrt{2}\beta\lambda.$$

Finally, in the case $\beta^2 \neq \frac{9}{2}$, the kinetic relation is found to be

$$\varphi^\flat(w_0, \beta) := \frac{w_0(9 - \beta^2) - \beta\sqrt{(18 - 3\beta^2)w_0^2 + 4\beta^2 - 18}}{2\beta^2 - 9}$$
$$\text{when } \begin{cases} 1 < w_0 < \dfrac{2\sqrt{2}\beta}{\sqrt{6\beta^2 - 9}} & \text{if } \beta^2 > \tfrac{3}{2}, \\ w_0 > 1 & \text{if } \beta^2 \leqslant \tfrac{3}{2}. \end{cases} \tag{5.19}$$

Now, if $\beta^2 = \frac{9}{2}$, we find that

$$\varphi^\flat(w_0, \tfrac{9}{2}) := -\frac{1}{w_0} \quad \text{when } 1 < w_0 < \sqrt{2}. \tag{5.19$'$}$$

Second, for values of the diffusion parameter for which (5.17) does not hold, there is actually a connection between w_0 and $w_2 = w_1 = -\frac{1}{2}w_0$ (see [24] and the discussion on classical trajectories in § 3). In this case, the kinetic function is trivial,

$$\varphi^\flat(w_0,\beta) = -\tfrac{1}{2}w_0 \quad \text{when } \beta^2 > \tfrac{3}{2} \text{ and } w_0 \geqslant \frac{2\sqrt{2}\beta}{\sqrt{6\beta^2-9}}. \tag{5.20}$$

Note that for all fixed $\beta > \frac{3}{2}$, the extension of the kinetic function defined by (5.20) is continuous, but there is a discontinuity in its derivative at the point

$$w_0 = \frac{2\sqrt{2}\beta}{\sqrt{6\beta^2-9}}.$$

Indeed, a simple calculation using the expression of $\varphi^\flat$ given by (5.19) and (5.20) shows that, for $\beta^2 \neq \frac{9}{2}$, the continuity of $\delta_{w_0}\varphi^\flat$ would be equivalent to

$$4\beta^4 - 24\beta^2 + 27 = 0,$$

which has two solutions, $\beta^2 = \frac{9}{2}$ and $\beta^2 = \frac{3}{2}$, both outside the interval under consideration.

In the case $\beta^2 = \frac{9}{2}$, by using (5.19′) and (5.20), we get

$$\lim_{w_0\to\sqrt{2}^-} \delta_{w_0}\varphi^\flat(w_0,\beta) = \tfrac{1}{2},$$

which is different from

$$\lim_{w_0\to\sqrt{2}^+} \delta_{w_0}\varphi^\flat(w_0,\beta) = -\tfrac{1}{2}.$$

On the other hand, observe that the kinetic function (5.19), (5.20) is defined on the whole interval $w_0 > 1$. Also note that, by continuity, thanks to (5.15), we can take, for all fixed $\beta > 0$, $\varphi^\flat(1,\beta) = -1$.

Now, in all the cases, one easily checks that

$$\delta_{w_0}\varphi^\flat(1,\beta) = 1 \tag{5.21}$$

In addition, since $\varphi^\flat(w_0,\beta) \geqslant -\frac{1}{2}w_0$, we deduce that $\varphi^\flat(\cdot,\beta)$ is not monotone. More precisely, by considering the expressions of $\varphi^\flat$ given in (5.19), the equation $\delta_{w_0}\varphi^\flat(w_0,\beta) = 0$ has one solution $w_0 > 1$ at most. Finally, by the arguments given above, we deduce that the function $\varphi^\flat(\cdot,\beta)$ is strictly monotone increasing in some interval of the form $(1,\hat{w}_0(\beta))$ and strictly monotone decreasing in $(\hat{w}_0(\beta),+\infty)$, where the value $\hat{w}_0(\beta)$ only depends on β.

Furthermore, we can check directly that, when $\beta \to \infty$,

$$\varphi^\flat(w_0,\beta) \to \begin{cases} \tilde{\varphi}(w_0) = -\frac{1}{2}(w+\sqrt{4-3w^2}) & \text{if } 1 < w_0 < 2/\sqrt{3}, \\ \varphi^\natural(w_0) = -\frac{1}{2}w_0 & \text{if } w_0 \geqslant 2/\sqrt{3} \end{cases}$$

and

$$\varphi^\flat(w_0,\beta) \to \varphi^\flat_\infty(w_0) = -w_0 \quad \text{when } \beta \to 0.$$

References

1 R. Abeyaratne and J. K. Knowles. Kinetic relations and the propagation of phase boundaries in solids. *Arch. Ration. Mech. Analysis* **114** (1991), 119–154.

https://doi.org/10.1017/S0308210500001773 Published online by Cambridge University Press

2 R. Abeyaratne and J. K. Knowles. Implications of viscosity and strain gradient effects for the kinetics of propagating phase boundaries in solids. *SIAM J. Appl. Math.* **51** (1991), 1205–1221.

3 N. Bedjaoui and P. G. LeFloch. Diffusive-dispersive travelling waves and kinetic relations. I. Nonconvex hyperbolic conservation laws. *J. Diff. Eqns* **178** (2002), 574–607.

4 N. Bedjaoui and P. G. LeFloch. Diffusive-dispersive travelling waves and kinetic relations. III. An hyperbolic model from nonlinear elastodynamics. *Ann. Univ. Ferrara Sez.* **47** (2001), 117–144.

5 J. Bona and M. E. Schonbek. Travelling wave solutions to the Korteweg–de Vries–Burgers equation. *Proc. R. Soc. Edinb.* A **101** (1985), 207–226.

6 C. Chalons and P. G. LeFloch. High-order entropy conservative schemes and kinetic relations for van der Waals-type fluids. *J. Computat. Phys.* **167** (2001), 1–23.

7 H.-T. Fan. A limiting 'viscosity' approach to the Riemann problem for materials exhibiting a change of phase II. *Arch. Ration. Mech. Analysis* **116** (1992), 317–337.

8 H.-T. Fan. A vanishing viscosity approach on the dynamics of phase transitions in van der Waals fluids. *J. Diff. Eqns* **103** (1993), 179–204.

9 H.-T. Fan. Travelling waves, Riemann problems and computations of a model of the dynamics of liquid/vapor phase transitions. *J. Diff. Eqns* **150** (1998), 385–437.

10 J. Guckenheimer and P. Holmes. *Nonlinear oscillations, dynamical systems and bifurcations of vector fields.* Applied Mathematical Science, vol. 42 (Springer, 1983).

11 H. Hattori. The Riemann problem for a van der Waals fluid with the entropy rate admissibility criterion: isothermal case. *Arch. Ration. Mech. Analysis* **92** (1986), 247–263.

12 H. Hattori H. and K. Mischaikow. A dynamical system approach to a phase transition problem. *J. Diff. Eqns* **94** (1991), 340–378.

13 B. T. Hayes and P. G. LeFloch. Nonclassical shocks and kinetic relations: scalar conservation laws. *Arch. Ration. Mech. Analysis* **139** (1997), 1–56.

14 B. T. Hayes and P. G. LeFloch. Nonclassical shocks and kinetic relations: finite difference schemes. *SIAM J. Numer. Analysis* **35** (1998), 2169–2194.

15 B. Hayes and P. G. LeFloch. Nonclassical shock waves and kinetic relations: systems of conservation laws. *SIAM J. Math. Analysis* **31** (2000), 941–991.

16 D. Jacobs, W. R. McKinney and M. Shearer. Travelling wave solutions of the modified Korteweg–de Vries Burgers equation. *J. Diff. Eqns* **116** (1995), 448–467.

17 P. G. LeFloch. Propagating phase boundaries: formulation of the problem and existence via the Glimm scheme. *Arch. Ration. Mech. Analysis* **123** (1993), 153–197.

18 P. G. LeFloch. An introduction to nonclassical shocks of systems of conservation laws. In *Proc. International School on Theory and Numerics for Conservation Laws, Freiburg/Littenweiler, Germany, October 1997* (ed. D. Kröner, M. Ohlberger and C. Rohde), pp. 20–24. Lecture Notes in Computational Science and Engineering, pp. 28–72 (1998).

19 P. G. LeFloch. *Hyperbolic systems of conservation laws: the theory of classical and nonclassical shock waves.* ETH Lecture Notes Series (Basel: Birkhaüser, 2002). (In the press.)

20 P. G. LeFloch and M. D. Thanh. Nonclassical Riemann solvers and kinetic relations. III. A non-convex hyperbolic model for van der Waals fluids. *Electron. J. Diff. Eqns* **72** (2000), 19.

21 P. G. LeFloch and M. D. Thanh. Nonclassical Riemann solvers and kinetic relations. I. An hyperbolic model of phase transitions dynamics. *Z. Angew. Math. Phys.* **52** (2001), 597–619.

22 P. G. LeFloch and M. D. Thanh. Non-classical Riemann solvers and kinetic relations. II. A hyperbolic–elliptic model of phase transition dynamics. *Proc. R. Soc. Edinb.* A **132** (2002), 181–219.

23 S. Schulze and M. Shearer. Undercompressive shocks for a system of hyperbolic conservation laws with cubic nonlinearity. *J. Math. Analysis Applic.* **229** (1999), 344–362.

24 M. Shearer and Y. Yang. The Riemann problem for a system of conservation laws of mixed type with a cubic nonlinearity. *Proc. R. Soc. Edinb.* A **125** (1995), 675–699.

25 M. Slemrod. Admissibility criteria for propagating phase boundaries in a van der Waals fluid. *Arch. Ration. Mech. Analysis* **81** (1983), 301–315.

26 M. Slemrod. The viscosity–capillarity criterion for shocks and phase transitions. *Arch. Ration. Mech. Analysis* **83** (1983), 333–361.

27 M. Slemrod. Vanishing viscosity–capillarity approach to the Riemann problem for a van der Waals fluid. In *Non-classical continuum mechanics*. London Mathematical Society Lecture Note Series, vol. 122, pp. 325–335 (Cambridge University Press, 1987).

28 M. Slemrod. A limiting 'viscosity' approach to the Riemann problem for materials exhibiting change of phase. *Arch. Ration. Mech. Analysis* **105** (1989), 327–365.

29 L. Truskinovsky. Dynamics of non-equilibrium phase boundaries in a heat conducting non-linear elastic medium. *J. Appl. Math. Mech.* **51** (1987), 777–784.

30 L. Truskinovsky. Kinks versus shocks. In *Shock induced transitions and phase structures in general media* (ed. R. Fosdick, E. Dunn and H. Slemrod). IMA Volumes in Mathematics Applications, vol. 52, pp. 185–229 (Springer, 1993).

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