Canad. Math. Bull. Vol. 62 (4), 2019 pp. 810–821
http://dx.doi.org/10.4153/S0008439519000092


# Rings whose Elements are the Sum of a Tripotent and an Element from the Jacobson Radical

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*Abstract.* This paper is about rings $R$ for which every element is a sum of a tripotent and an element from the Jacobson radical $J(R)$. These rings are called semi-tripotent rings. Examples include Boolean rings, strongly nil-clean rings, strongly 2-nil-clean rings, and semi-boolean rings. Here, many characterizations of semi-tripotent rings are obtained. Necessary and sufficient conditions for a Morita context (respectively, for a group ring of an abelian group or a locally finite nilpotent group) to be semi-tripotent are proved.

## 1 Introduction

A ring is called Boolean if each of its elements is an idempotent. As natural generalizations of Boolean rings, rings $R$ for which $R/J(R)$ is Boolean and $J(R)$ is nil and, respectively, rings $R$ for which $R/J(R)$ is Boolean and idempotents lift modulo $J(R)$ have been well studied in the literature. The former is the characterization of strongly nil-clean rings, where a ring is called strongly nil-clean if each of its elements is a sum of an idempotent and a nilpotent that commute (see [5] and [9]), and the latter is the notion of semi-boolean rings, which are exactly those rings $R$ whose elements are the sum of an idempotent and an element from $J(R)$ (see [15]). Here we can view semi-boolean rings as a natural generalization of strongly nil-clean rings, with "$J(R)$ is nil" being replaced by "idempotents lift modulo $J(R)$".

Let $p$ be a prime. The following questions are motivated.

(i) What can be said about rings $R$ for which $R/J(R)$ has identity $x^p = x$ and $J(R)$ is nil?

(ii) What can be said about rings $R$ for which $R/J(R)$ has identity $x^p = x$ and idempotents lift modulo $J(R)$?

Answers to these questions are known for $p = 2$, as mentioned above. Moreover, question (i) was answered for $p = 3$ in [3], and further for $p = 5$ in [19]. In this paper, we provide an answer to question (ii) for $p = 3$. Let $n > 1$ be an integer. An element $a \in R$ is called an $n$-potent if $a^n = a$, and a 3-potent is usually called a tripotent. We call a ring $R$ semi-$n$-potent if $R/J(R)$ has identity $x^n = x$ and $n$-potents lift modulo $J(R)$, or equivalently every element of $R$ is a sum of an $n$-potent and an element from $J(R)$; a semi-3-potent ring is called a semi-tripotent ring. In Section 2, some basic properties

Received by the editors September 25, 2018.
Published online on Cambridge Core August 6, 2019.
AMS subject classification: **16U99**, 16S34.
Keywords: idempotent, tripotent, Jacobson radical, idempotent lifting modulo Jacobson radical, Boolean ring, semi-boolean ring.

of semi-$n$-potent rings are proved. For instance, the corner rings of a semi-$n$-potent ring are again semi-$n$-potent, and a sufficient and necessary condition for a Morita context to be semi-$n$-potent is obtained. In Section 3, various characterizations of semi-tripotent rings are obtained. Some new equivalent conditions of a semi-boolean ring are also presented. In Section 4, we determine when the group ring of an abelian group or a locally finite nilpotent group is semi-tripotent.

Throughout, $R$ is an associative ring with unity. The Jacobson radical of $R$ is denoted by $J(R)$ or $J$. The group of units and the set of nilpotents of $R$ are denoted by $U(R)$ and $\mathrm{Nil}(R)$, respectively. We write $\mathbb{Z}_n$ for the ring of integers modulo $n$, $\mathbb{M}_n(R)$ for the ring of $n \times n$ matrices over $R$, and $R[x]$ (respectively, $R[[x]]$) for the ring of polynomials (respectively, power series) over $R$.

## 2 Semi-$n$-potent Rings

In this section, $n \ge 2$ is a fixed integer.

***Lemma 2.1*** *Let $R$ be a ring and $x \in R$. The following are equivalent:*

(1) $x^n = x$.
(2) $x^2 = vx$, *where* $v^{n-1} = 1$.
(3) $x^2 = \varphi(x)x$, *where* $\varphi(t) \in \mathbb{Z}[t]$ *with* $\varphi(x)^{n-1} = 1$.
(4) $x^2 = vx$, *where* $v^{n-1} = 1$ *and* $vx = xv$.
(5) $x = eu$, *where* $e^2 = e$, $u^{n-1} = 1$ *and* $eu = ue$.
(6) $x = eu$, *where* $e^2 = e$, $u^{n-1} = 1$ *and* $eue = ue$.

**Proof** (1) $\Rightarrow$ (3). If $x^n = x$, let $\varphi(t) = 1 + t - t^{n-1} \in \mathbb{Z}[t]$. Then $x^2 = \varphi(x)x$, and $\varphi(x)^k = 1 + x^k - x^{n-1}$ for $k = 1, 2, \dots, n$. In particular, $\varphi(x)^{n-1} = 1$.

(3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (6). They are trivial.

(4) $\Rightarrow$ (5). Given (4), we see $x = v^{-1}x^2 = v^{n-2}x^2 = v(v^{n-3}x^2)$ with $(v^{n-3}x^2)^2 = v^{n-3}x^2$.

(6) $\Rightarrow$ (2). Given (6), we have $x^2 = eueu = ux$.

(2) $\Rightarrow$ (1). Given (2), we have $x^n = vx \cdot x^{n-2} = vx^{n-1} = v \cdot vx \cdot x^{n-3} = v^2 x^{n-2} = \cdots = v^{n-1}x = x$. ■

***Definition 2.2*** Let $I$ be an ideal of a ring $R$. We say that $n$-potents lift modulo $I$ in $R$ if whenever $a^n - a \in I$, there exists $e^n = e \in R$ such that $a - e \in I$.

***Definition 2.3*** A ring $R$ is called a semi-$n$-potent ring if every element of $R$ is a sum of an $n$-potent and an element from $J(R)$, equivalently if $R/J(R)$ has identity $x^n = x$ and $n$-potents lift modulo $J(R)$ in $R$.

The next example can be easily verified.

***Example 2.4*** Let $R, S$ be rings, $V$ an $(R, S)$-bimodule, $M$ an $R$-bimodule, and $m \ge 1$.

(1) If $R$ is semi-$n$-potent, then so is every homomorphic image of $R$.
(2) A direct product $\prod R_\alpha$ of rings is semi-$n$-potent if and only if every $R_\alpha$ is semi-$n$-potent.

(3) The formal triangular matrix ring $\begin{pmatrix} R & V \\ 0 & S \end{pmatrix}$ is semi-$n$-potent if and only if $R$ and $S$ are semi-$n$-potent.
(4) $\mathbb{T}_m(R)$ is semi-$n$-potent if and only if $R$ is semi-$n$-potent.
(5) The trivial extension $R \propto M$ is semi-$n$-potent if and only if $R$ is semi-$n$-potent.
(6) $R[x]/(x^m)$ is semi-$n$-potent if and only if $R$ is semi-$n$-potent.
(7) $R[[x]]$ is semi-$n$-potent if and only if $R$ is semi-$n$-potent.

A subring of a semi-$n$-potent ring need not be semi-$n$-potent: $\mathbb{Z}_2[[x]]$ is semi-$n$-potent, but $\mathbb{Z}_2[x]$ is not semi-$n$-potent.

***Lemma 2.5*** *Let $I$ be an ideal of a ring $R$. The following are equivalent:*
(1) *If $a^n - a \in I$, then there exists $e^n = e \in aR$ such that $a - e \in I$.*
(2) *If $a^n - a \in I$, then there exists $e^n = e \in aRa$ such that $a - e \in I$.*
(3) *If $a^n - a \in I$, then there exists $e^n = e \in Ra$ such that $a - e \in I$.*

**Proof** Write $r \equiv s$ to mean that $r - s \in I$, so that $r \equiv s$ implies that $xry \equiv xsy$ for all $x, y \in R$. It suffices to show the implication "(1) $\Rightarrow$ (2)". Suppose that (1) holds. If $a^n \equiv a$, then $(a^n)^n \equiv a^n$. Choose $f^n = f \in a^n R$ such that $a^n \equiv f$, so $f \equiv a$. Write $f = a^n x$ with $x \in R$. We may assume that $x = xf^{n-1}$. Let $e = a^{n-1}xa \in aRa$. Then $e^n = a^{n-1}x(a^n x)^{n-1}a = a^{n-1}xf^{n-1}a = a^{n-1}xa = e$. Moreover, $e = a^{n-1}xa \equiv (a^n)^{n-1}xa = (a^n)^{n-2}(a^n x)a \equiv (a^n)^{n-2}fa \equiv a^{n-2}fa \equiv a^n \equiv a$. So (2) holds. ∎

We say that $n$-potents lift strongly modulo $I$ if the equivalent conditions of Lemma 2.5 hold. The following result is known when $n = 2$ (see [16, Lemma 5]).

***Proposition 2.6*** *Let $R$ be a ring. If $n$-potents lift modulo $J(R)$, then they lift strongly modulo $J(R)$.*

**Proof** Let $a^n - a \in J(R)$. Choose $f^n = f \in R$ such that $f - a \in J(R)$. Then $f^{n-1} - a^{n-1} \in J(R)$, so $u := 1 - (f^{n-1} - a^{n-1}) \in U(R)$, and $uf = a^{n-1}f \in aR$. So $e := ufu^{-1} \in aR$ and $e^n = e$. As $\overline{u} = \overline{1}$ in $R/J(R)$, $\overline{e} = \overline{u}\overline{f}\overline{u}^{-1} = \overline{f} = \overline{a}$. ∎

***Corollary 2.7*** *Let $R$ be a ring with $e^2 = e \in R$. If $n$-potents lift modulo $J(R)$ in $R$, then $n$-potents lift modulo $J(eRe)$ in $eRe$.*

**Proof** Let $a^n - a \in J(eRe)$ where $a \in eRe$. Then $a^n - a \in eJ(R)e \subseteq J(R)$, so, by Proposition 2.6, there exists $f^n = f \in aRa$ such that $a - f \in J(R)$. As $f \in aRa \subseteq eRe$, $a - f \in J(R) \cap eRe = J(eRe)$. ∎

***Corollary 2.8*** *Let $e^2 = e \in R$. If $R$ is semi-$n$-potent, then so is $eRe$.*

**Proof** As $n$-potents lift modulo $J(R)$ in $R$, $n$-potents lift modulo $eJe$ in $eRe$ by Corollary 2.7. Moreover, $eRe/J(eRe) = eRe/eJe \cong \overline{e}\overline{R}\overline{e} \subseteq \overline{R}$, where $\overline{R} = R/J(R)$. As $\overline{R}$ has identity $x^n = x$, $\overline{e}\overline{R}\overline{e}$, and hence $eRe/J(eRe)$ has identity $x^n = x$. So $eRe$ is semi-$n$-potent. ∎

It is easy to see that no proper matrix ring can be semi-$n$-potent. Next we consider when a Morita context is a semi-$n$-potent ring.

A Morita context is a 4-tuple $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$, where $A, B$ are rings, ${}_AM_B$ and ${}_BN_A$ are bimodules, and there exist context products $M \times N \to A$ and $N \times M \to B$ written multiplicatively as $(w,z) \mapsto wz$ and $(z,w) \mapsto zw$, such that $\begin{pmatrix} A & M \\ N & B \end{pmatrix}$ is an associative ring with the obvious matrix operations.

The next result (in fact, a more general result) can be found in [17].

***Lemma 2.9*** ([17]) *Let $R := \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ be a Morita context. Then $J(R) = \begin{pmatrix} J(A) & M_0 \\ N_0 & J(B) \end{pmatrix}$, where $M_0 = \{x \in M : xN \subseteq J(A)\}$ and $N_0 = \{y \in N : yM \subseteq J(B)\}$.*

***Theorem 2.10*** *Let $R := \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ be a Morita context. Then $R$ is semi-$n$-potent if and only if $A, B$ are semi-$n$-potent, $MN \subseteq J(A)$ and $NM \subseteq J(B)$.*

**Proof** $(\Rightarrow)$. As $R$ is semi-$n$-potent, $R/J(R)$ has identity $x^n = x$. Especially, $R/J(R)$ is reduced. So, by Lemma 2.9, $M = M_0$ and $N = N_0$, and it follows that $MN \subseteq J(A)$ and $NM \subseteq J(B)$. By Corollary 2.8, $A, B$ are semi-$n$-potent.

$(\Leftarrow)$. As $MN \subseteq J(A)$ and $NM \subseteq J(B)$, we have $J(R) = \begin{pmatrix} J(A) & M \\ N & J(B) \end{pmatrix}$ by Lemma 2.9. For $\alpha := \begin{pmatrix} a & x \\ y & b \end{pmatrix} \in R$, write $a = e + j_A$ and $b = f + j_B$ where $e^n = e \in A$, $f^n = f \in B$, $j_A \in J(A)$ and $j_B \in J(B)$. Then $\alpha = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} + \begin{pmatrix} j_A & x \\ y & j_B \end{pmatrix}$ is a sum of an $n$-potent and an element from $J(R)$. So $R$ is semi-$n$-potent. ∎

As a consequence of Theorem 2.10, Corollary 2.8 has the following improvement.

***Corollary 2.11*** *Let $e^2 = e \in R$. Then $R$ is semi-$n$-potent if and only if $eRe$ and $(1-e)R(1-e)$ are semi-$n$-potent, and $eR(1-e)$ and $(1-e)Re$ both are contained in $J(R)$.*

**Proof** Consider the Pierce decomposition $R = \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix}$. By Theorem 2.10, $R$ is semi-$n$-potent if and only if $eRe$ and $(1-e)R(1-e)$ are semi-$n$-potent, $eR(1-e)Re \subseteq eJ(R)e$ and $(1-e)ReR(1-e) \subseteq (1-e)J(R)(1-e)$. Note that $eR(1-e)Re \subseteq eJ(R)e$ if and only if $eR(1-e)Re \subseteq J(R)$, if and only if $\big(eR(1-e)R\big)^2 \subseteq J(R)$, if and only if $eR(1-e)R \subseteq J(R)$, if and only if $eR(1-e) \subseteq J(R)$. Similarly, $(1-e)ReR(1-e) \subseteq (1-e)J(R)(1-e)$ if and only if $(1-e)Re \subseteq J(R)$. ∎

## 3 Characterizations of Semi-tripotent Rings

For $n \geq 3$, if $n$-potents lift modulo $J(R)$ in a ring $R$, then idempotents lift modulo $J(R)$. Indeed, if $a^2 - a \in J(R)$, then $a - e \in J(R)$ where $e^n = e \in R$. So $a - e^{n-1} = (a - a^{n-1}) + (a^{n-1} - e^{n-1}) \in J(R)$ with $e^{n-1}$ an idempotent. This raises the question whether the converse holds. The next example shows that, for each integer $n \geq 4$, there exists a ring $R$ such that idempotents lift modulo $J(R)$ but $n$-potents do not.

***Example 3.1*** Let $n \geq 4$, and let $p(t) \in \mathbb{R}[t]$ be any irreducible polynomial of degree 2 which divides $t^{n-1} - 1$. (For example, if $n = 4$ then take $p(t) = t^2 + t + 1$.) Let $R = \mathbb{R}[t]_{(p(t))}$, the localization of $\mathbb{R}[t]$ at the maximal ideal generated by $p(t)$.

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Then $J(R)$ is generated by $p(t)$, so $R/J(R) \cong \mathbb{R}[t]/\big(p(t)\big) \cong \mathbb{C}$, the field of complex numbers. Hence idempotents trivially lift modulo $J(R)$.

The only $n$-potents in $R$ are $0, 1$, and possibly $-1$ when $n$ is odd. (This is because $R$ is a subring of the field of rational functions over $\mathbb{R}$.) Since neither $t$ nor $t-1$ nor $t+1$ lies in $J(R)$, it follows that $t$ is an $n$-potent modulo $J(R)$ (since $t^n - t \in J(R)$) which cannot be lifted to an $n$-potent in $R$.

For $n = 3$, the next lemma gives a partial answer to the question above. Note that square roots of 1 lift modulo the Jacobson radical exactly when idempotents lift, provided 2 is a unit (this is proved in [7]), and this result can be used to give a quick proof of the next lemma. But here we give a direct, self-contained proof.

***Lemma 3.2*** *Let $R$ be a ring with $2 \in U(R)$. Then idempotents lift modulo $J(R)$ if and only if tripotents lift modulo $J(R)$.*

**Proof** The sufficiency is noticed above. For the necessity, suppose $a^3 - a \in J(R)$. Let $b = \frac{1}{2}(a^2 + a)$ and $c = \frac{1}{2}(a^2 - a)$. Then $a = b - c$, $b - b^2 = \frac{1}{4}(a+2)(a - a^3) \in J(R)$ and $c - c^2 = \frac{1}{4}(a-2)(a - a^3) \in J(R)$. Thus, in $R/J(R)$, $\overline{a} = \overline{b} - \overline{c}$, $\overline{b}^2 = \overline{b}$ and $\overline{c}^2 = \overline{c}$. Moreover, $\overline{b}\,\overline{c} = \overline{c}\,\overline{b} = \frac{1}{4}(\overline{a}^4 - \overline{a}^2) = \overline{0}$. Since idempotents lift modulo $J(R)$, $\overline{b}$ and $\overline{c}$ can be lifted to orthogonal idempotents $f$ and $g$ in $R$. Let $e = f - g$. Then $e^3 = e$ and $\overline{a} = \overline{b} - \overline{c} = \overline{f} - \overline{g} = \overline{e}$. Hence, tripotents lift modulo $J(R)$. ∎

***Lemma 3.3*** *The following are equivalent for a ring $R$.*

(1) *For each $a \in R$, $a = ve$ where $e^2 = e$ and $v^2 = 1$.*
(2) *For each $a \in R$, $a = fw$ where $f^2 = f$ and $w^2 = 1$.*
(3) *$R$ has identity $x^3 = x$.*

**Proof** (1) $\Rightarrow$ (3). By (1), $R$ is a unit-regular ring. We show that $R$ is reduced. Assume that $r^2 = 0$ with $0 \neq r \in R$. By [11, Theorem 2.1], there exists $0 \neq e^2 = e \in RrR$ such that $eRe \cong \mathbb{M}_2(S)$ for a non-trivial ring $S$. Let $A = \left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right) \in \mathbb{M}_2(S)$ and then $A^2 = A + I_2 \neq I_2$. So, there exists $u \in U(eRe)$ such that $u^2 \neq e$. Thus $y := u + (1-e) \in U(R)$ and $y^2 = u^2 + (1-e) \neq 1$, contradicting (1). So $R$ is reduced, and hence is abelian. Therefore, for each $a \in R$, write $a = ve$ with $e^2 = e$ and $v^2 = 1$, and we see $a^3 = v^3 e^3 = ve = a$; this proves (3).

(3) $\Rightarrow$ (1). This is clear by Lemma 2.1.

(1) $\Leftrightarrow$ (2). The proof is similar. ∎

***Lemma 3.4*** *Suppose that, for each $a \in R$, $a = b + j_1 + ev = b + j_2 + ve$, where $j_1, j_2 \in J(R)$, $e^2 = e \in R$, $v^2 = 1$, $b \in \mathrm{Nil}(R)$ with $ab = ba$. Then $6 \in J(R)$, idempotents lift modulo $J(R)$, and $R/J(R)$ has identity $x^3 = x$.*

**Proof** Let $J = J(R)$, and $\overline{R} = R/J$. Assume that $a^2 - a \in J$, and write $a = b + j_1 + ev = b + j_2 + ve$, where $j_1, j_2 \in J$, $e^2 = e \in R$, $v^2 = 1$, $b \in \mathrm{Nil}(R)$ with $ab = ba$. Then

$$\begin{aligned} a^2 &= (b + j_1 + ev)(b + j_2 + ve) \\ &= [ba + (a-b)b] + [j_1(j_2 + ve) + evj_2] + e. \end{aligned}$$

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Let $c = ba + (a-b)b$. Then $c \in \mathrm{Nil}(R)$ and $ac = ca$. So, in $\overline{R}$, $\overline{a} = \overline{a}^2 = \overline{c} + \overline{e}$, and it follows that $\overline{c} + \overline{e} = (\overline{c} + \overline{e})^2 = \overline{c}^2 + 2\overline{c}\,\overline{e} + \overline{e}$. Thus, $\overline{c} = \overline{c}^2 + 2\overline{c}\,\overline{e}$, *i.e.*, $\overline{c}(\overline{1} - 2\overline{e}) = \overline{c}^2$. So $\overline{c} = \overline{c}^2(\overline{1} - 2\overline{e})^{-1} = \overline{c}^2(\overline{1} - 2\overline{e})$. As $\overline{c}$ is nilpotent, it must be that $\overline{c} = \overline{0}$. Hence $\overline{a} = \overline{e}$.

Write $2 = b + j_1 + ev = b + j_2 + ve$, where $j_1, j_2 \in J$, $e^2 = e \in R$, $v^2 = 1$ and $b \in \mathrm{Nil}(R)$. Then $\overline{2} = \overline{b} + \overline{e}\,\overline{v} = \overline{b} + \overline{ve}$, so

$$\overline{4} = (\overline{b} + \overline{e}\,\overline{v})(\overline{b} + \overline{ve}) = \overline{b}(\overline{b} + \overline{ve}) + \overline{e}\,\overline{v}\overline{b} + \overline{e} = \overline{c} + \overline{e},$$

where $\overline{c} = \overline{b}(\overline{b} + \overline{ve}) + \overline{e}\,\overline{v}\overline{b}$ is nilpotent. So, $\overline{4}^2 = (\overline{c} + \overline{e})^2 = \overline{c}^2 + 2\overline{c}\,\overline{e} + \overline{e}$, and hence

$$\overline{12} = \overline{4}^2 - \overline{4} = (\overline{c}^2 + 2\overline{c}\,\overline{e} + \overline{e}) - (\overline{c} + \overline{e}) = \overline{c}(\overline{c} + 2\overline{e} - \overline{1})$$

is nilpotent in $\overline{R}$. It follows that $\overline{6}$ is nilpotent in $\overline{R}$, so $\overline{6} = \overline{0}$, or $6 \in J$.

As $6 \in J$, $\overline{R} = R_1 \times R_2$, where $2 = 0$ in $R_1$ and $3 = 0$ in $R_2$. Note that, by hypothesis, for any $a \in R$, $\overline{a} = \overline{b} + \overline{e}\,\overline{v} = \overline{b} + \overline{v}\,\overline{e}$, where $\overline{e}^2 = \overline{e}$, $\overline{v}^2 = \overline{1}$ and $\overline{b}$ is nilpotent with $\overline{a}\,\overline{b} = \overline{b}\,\overline{a}$, so $\overline{v}\,\overline{e} = \overline{e}\,\overline{v}$, and hence $(\overline{v}\,\overline{e})^3 = \overline{v}\,\overline{e}$. So, $R_1$ is Boolean by [19, Proposition 2.5] and $R_2$ is zero or a subdirect product of $\mathbb{Z}_3$'s by [19, Proposition 2.8]. It follows that $R/J$ has identity $x^3 = x$. ∎

Some characterizations are obtained for semi-tripotent rings.

**Theorem 3.5** *The following are equivalent for a ring R:*

(1) *For each $a \in R$, $a = j + f$ where $j \in J(R)$ and $f^3 = f$, i.e., R is semi-tripotent.*
(2) *For each $a \in R$, $a = b + j + f$ where $j \in J(R)$, $f^3 = f$ and $b \in \mathrm{Nil}(R)$ with $ab = ba$.*
(3) *For each $a \in R$, $a = j + ev$, where $j \in J(R)$, $e^2 = e \in R$ and $v^2 = 1$.*
(4) *For each $a \in R$, $a = j + ve$, where $j \in J(R)$, $e^2 = e \in R$ and $v^2 = 1$.*
(5) *$R/J(R)$ has identity $x^3 = x$ and idempotents lift modulo $J(R)$.*
(6) *$R/J(R) = R_1 \times R_2$, where $R_1$ is zero or a Boolean ring, $R_2$ is zero or a subdirect product of $\mathbb{Z}_3$'s, and idempotents lift modulo $J(R)$.*

**Proof** The implication (1) ⇒ (2) is obvious. The implications (1) ⇒ (3) and (1) ⇒ (4) follow from Lemma 2.1.

$(i)$ ⇒ (5): $i = 2, 3, 4$. First assume (3) holds. For each $a \in R$, $\overline{a} = \overline{e}\,\overline{v}$ where $e^2 = e$ and $v^2 = 1$. By Lemma 3.3, (3) implies that $R/J$ has identity $x^3 = x$; so $R/J$ is abelian. Therefore, $\overline{a} = \overline{e}\,\overline{v} = \overline{v}\,\overline{e}$. It follows that $a = j_1 + ev = j_2 + ve$ for some $j_1, j_2 \in J$. Similarly, (4) implies that, for each $a \in R$, $a = j_1 + ev = j_2 + ve$ where $e^2 = e$, $v^2 = 1$ and $j_1, j_2 \in J$. Hence, in view of Lemma 2.1, either of (2), (3) and (4) implies that, each $a \in R$, $a = b + j_1 + ev = b + j_2 + ve$, where $j_1, j_2 \in J(R)$, $e^2 = e \in R$, $v^2 = 1$, $b \in \mathrm{Nil}(R)$ with $ab = ba$. Thus, (5) holds by Lemma 3.4.

(5) ⇔ (6). This is clear.

(5) ⇒ (1). Let $a \in R$. Since $\overline{R} := R/J(R)$ has identity $x^3 = x$, by [6, Theorem 1] $\overline{1} + \overline{a} = y + z$ for some commuting idempotents $y$ and $z$. By [13, Theorem 2.1], one can lift the idempotents $y, z$ to commuting idempotents $f, g$ in $R$. Thus, $1 + a = f + g + j$ for some $j \in J(R)$, and hence $a = f - (1-g) + j$ where, as a difference of two commuting idempotents, $f - (1-g)$ is a tripotent. ∎

We remark that an element that is a product of an idempotent and a square root of 1 in a ring $R$ may not be a sum of a tripotent and an element from $J(R)$. To see this,

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consider $R = \mathbb{M}_2(\mathbb{Q})$. Then, in $R$, $a := \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, a product of an idempotent and a square root of 1, but $a$ can not be a sum of a tripotent and an element from $J(R)$.

We also remark that Theorem 3.5(6) can not be replaced by the condition that $R = A \times B$, where $A/J(A)$ is a Boolean ring and idempotents lift modulo $J(A)$, and $B$ is zero or $B/J(B)$ is a subdirect product of $\mathbb{Z}_3$'s and idempotents lift modulo $J(B)$. To see this, consider the formal matrix ring $R = \begin{pmatrix} \mathbb{Z}_{(2)} & \mathbb{Q} \\ 0 & \mathbb{Z}_{(3)} \end{pmatrix}$, where $\mathbb{Z}_{(2)}, \mathbb{Z}_{(3)}$ are the localizations of $\mathbb{Z}$ at 2 and 3. Then $R$ is indecomposable, but idempotents lift modulo $J(R)$ and $R/J(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_3$.

A ring $R$ is called semi-boolean if every element of $R$ is a sum of an idempotent and an element from $J(R)$. It is known that a ring $R$ is semi-boolean if and only if $R/J(R)$ is Boolean and idempotents lift modulo $J(R)$ (see [15, Lemma 2.4]). A semi-boolean ring was also termed a $J$-clean ring in [3], [8] and [12]. We note that the term $J$-clean ring was used differently in [1]. We can add some new conditions to the equivalence list for semi-boolean rings.

***Corollary 3.6*** *The following are equivalent for a ring $R$:*

(1) *$R$ is semi-boolean.*
(2) *For each $a \in R$, $a = b + j + e$, where $b \in \mathrm{Nil}(R)$, $j \in J(R)$, $e^2 = e \in R$ and $ab = ba$.*
(3) *$R/J(R)$ is Boolean and idempotents lift modulo $J(R)$.*
(4) *$R$ is semi-tripotent and $2 \in J(R)$.*

**Proof** (1) ⇒ (2). This is obvious.

(2) ⇒ (3). By Theorem 3.5, we have $R/J = R_1 \times R_2$, where $R_1$ is a Boolean ring and $R_2$ is zero or a subdirect product of $\mathbb{Z}_3$'s. Assume on the contrary that $R/J$ is not Boolean. Then $R_2 \neq 0$, so $R$ has a quotient ring isomorphic to $\mathbb{Z}_3$. As any quotient ring of $R$ still satisfies (2), we may assume that $R = \mathbb{Z}_3$. So, since $\mathrm{Nil}(R) = J(R) = 0$, 2 is an idempotent. Thus, $2 = 1$, a contradiction.

(1) ⇔ (3). This is from [15, Lemma 2.4] (also see [8, Theorem 3.2]).

(1) ⇒ (4). This is easy to see.

(4) ⇒ (3). For $a \in R$, $a^3 - a \in J(R)$ as $R/J(R)$ has identity $x^3 = x$. So $(a^2 - a)^2 = a^4 - 2a^3 + a^2 = a(a^3 - a) + 2(a^2 - a^3) \in J(R)$, and hence $a^2 - a \in J(R)$. So $R/J(R)$ is Boolean. ∎

The assumption that "$ab = ba$" in Theorem 3.5 and Corollary 3.6 cannot be removed: For $k \geq 2$ and $n \geq 1$, the ring $R := \mathbb{M}_k(\mathbb{Z}_{2^n})$ is a nil-clean ring (see [5, Corollary 3.17 and Example 4.5]), that is, every element of $R$ is a sum of a nilpotent and an idempotent, but $R/J(R)$ is neither a Boolean ring nor a subdirect product of a Boolean ring and a direct product of $\mathbb{Z}_3$'s. We also remark that there exists a ring $R$ such that $R/J(R)$ is Boolean, but idempotents do not lift modulo $J(R)$ (see [10, Example 15]).

None of the conditions of Theorem 3.5 can be replaced by "For each $a \in R$, $a = b + j + ev$, where $j \in J(R)$, $e^2 = e$, $v^2 = 1$, and $b \in \mathrm{Nil}(R)$ with $ab = ba$": One can easily check that $R = \mathbb{M}_2(\mathbb{Z}_2)$ satisfies the latter condition, but $R$ is not semi-tripotent.

***Theorem 3.7*** *The following are equivalent for a ring $R$:*

(1) *$R$ is semi-tripotent.*
(2) *For each $a \in R$, $a = j + e + f$, where $j \in J(R)$, $e^2 = e$, $f^2 = f$ and $ef = fe$.*

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(3) *For each* $a \in R$, $a = j + e + f$, *where* $j \in J(R)$, $e^2 = e$, $f^3 = f$ *and* $ef = fe$.
(4) *For each* $a \in R$, $a = j + b + e + f$, *where* $j \in J(R)$, $b \in \mathrm{Nil}(R)$, $e^2 = e$, $f^2 = f$, *and* $b, e, f$ *commute with one another.*
(5) *For each* $a \in R$, $a = j + b + e + f$, *where* $j \in J(R)$, $b \in \mathrm{Nil}(R)$, $e^2 = e$, $f^3 = f$, *and* $b, e, f$ *commute with one another.*

**Proof** (1) $\Rightarrow$ (2). By Theorem 3.5, $R/J = A/J \oplus B/J$, where $A/J$ is a Boolean ring and $B/J$ is a subdirect product of $\mathbb{Z}_3$'s, and idempotens lift modulo $J$. Write $\overline{1} = \alpha + \beta$ where $\alpha \in A/J$ and $\beta \in B/J$. We may assume that, for some $e^2 = e \in R$, $\alpha = \overline{e}$ and $\beta = \overline{1-e}$. Let $f = 1 - e$. Then $A/J = (eRe + J)/J$ and $B/J = (fRf + J)/J$. Let $x \in R$ be an arbitrary element, and write $\overline{x} = \overline{a} + \overline{b}$ where $\overline{a} \in A/J$ and $\overline{b} \in B/J$. We may assume that $a \in eRe$ and $b \in fRf$.

As $A/J$ is Boolean, $a^2 - a \in J$. So, by [16, Lemma 5], there exists $e_1^2 = e_1 \in aRa \subseteq eRe$ such that $a - e_1 \in J$.

By Corollary 2.7, idempotents lift modulo $J(fRf)$ in $fRf$. As $fRf/J(fRf) \cong B/J$ is a subdirect product of $\mathbb{Z}_3$'s, $b - b^3 \in J(fRf)$, and $2 \in U(fRf)$. So, by Lemma 3.2, there exists $e_2^3 = e_2 \in fRf$ such that $b - e_2 \in J(fRf) \subseteq J$. Then $x = j + e_1 + e_2$ with $e_1e_2 = e_2e_1 = 0$ and, to finish the proof, let $g = e_2^2$. Then $g^2 = g$, and $(e_2 - g)^2 - (e_2 - g) = -3(e_2 - g) \in J(fRf) \subseteq J$. So by [16, Lemma 5], there exists $h^2 = h \in (e_2 - g)R(e_2 - g)$ such that $e_2 - g - h \in J$. Let $j' = e_2 - g - h$ and write $h = (e_2 - g)r(e_2 - g)$ with $r \in R$. Then $gh = h = hg$. As $h, g \in fRf$, $e_1g = ge_1 = 0$ and $e_1h = he_1 = 0$. It follows that $e_1 + g$ and $h$ are commuting idempotents and $x = (j + j') + (e_1 + g) + h$. This verifies (2).

(2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5). The implications are clear.

(5) $\Rightarrow$ (1). By (5), every element of $R/J(R)$ is a sum of a nilpotent, an idempotent and a tripotent that commute with one another other. So, by [19, Theorem 2.12], $R/J(R)$ has identity $x^3 = x$, and hence $\mathrm{Nil}(R) \subseteq J(R)$. By Theorem 3.5, to show (1) it remains to show that idempotents lift modulo $J(R)$. Assume that $a - a^2 \in J(R)$. Write $a = j + b + e + f$ as in (5). Then $j + b \in J(R)$, so we may assume that $b = 0$. Thus $a - a^2 = (f - f^2 - 2ef) + \big(j - ja - (e + f)j\big)$, so $f - f^2 - 2ef \in J(R)$, and hence $f^2 - f - 2ef^2 = (f - f^2 - 2ef)f \in J(R)$. Let $g = e + f^2 - 2ef^2$. Then $a - g = j + (f - f^2 + 2ef^2) \in J(R)$ and $g^2 = g$. ∎

## 4 Group Rings

Semi-tripotent rings can be constructed via Morita contexts by Theorem 2.10. In this section, we discuss when a group ring is semi-tripotent. A group $G$ is called locally finite if every finitely generated subgroup of $G$ is finite. For a prime number $p$, a group is called a $p$-group if the order of each of its elements is a power of $p$. A group of exponent $p$ is a non-trivial group in which every element has order $p$. We write $C_n$ for the cyclic group of order $n$.

If $R$ is a ring and $G$ is a group, $RG$ denotes the group ring of the group $G$ over $R$. The ring homomorphism $\omega\colon RG \to R$, $\Sigma r_g g \mapsto \Sigma r_g$ is called the augmentation map, and the kernel $\ker(\omega)$ is called the augmentation ideal of the group ring $RG$ and is denoted by $\triangle(RG)$. Note that $\triangle(RG)$ is an ideal of $RG$ generated by the set $\{1 - g : g \in G\}$.

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***Lemma 4.1*** *Let $R$ be a ring with $2 \in J(R)$ and $G$ a group. If $RG$ is semi-tripotent, then $G$ is a 2-group.*

**Proof** As $RG$ is semi-tripotent, $(R/J)G \cong RG/JG$ is semi-tripotent. As $2 \in J(R)$, $2 = 0$ in $(R/J)G$, so $(R/J)G$ is semi-boolean by Corollary 3.6. Therefore, $G$ is a 2-group by [9, Theorem 4.4]. ∎

***Lemma 4.2*** *Let $R$ be a ring with $3 \in J(R)$ and $G$ a locally finite $p$-group with $p$ a prime. If $RG$ is semi-tripotent, then either $G$ is a 3-group or $G$ is a group of exponent 2.*

**Proof** As $RG$ is semi-tripotent, $R$ is semi-tripotent, so by Theorem 3.5 $R/J = A \times B$ where $A$ is a Boolean ring and $B$ is zero or a subdirect product of $\mathbb{Z}_3$'s. As $3 \in J(R)$, $A = 0$, so $R/J = B$ which has identity $x^3 = x$. If $p \neq 3$, then by [4, Theorem 3], $BG$ is (von Neumann) regular, so $J(BG) = 0$. But, as an image of $RG$, $BG$ is semi-tripotent, so $BG$ has identity $x^3 = x$. In particular, $g^2 = 1$ for all $g \in G$. So, $G$ is a group of exponent 2. ∎

***Lemma 4.3*** *Let $R$ be a ring and $G$ a locally finite group. If $RG$ is semi-tripotent and $2 \notin J(R)$ and $3 \notin J(R)$, then $G$ is a group of exponent 2.*

**Proof** As $RG$ is semi-tripotent, $R$ is semi-tripotent, so by Theorem 3.5 $R/J = A \times B$ where $A$ is a Boolean ring and $B$ is zero or a subdirect product of $\mathbb{Z}_3$'s. As $2 \notin J(R)$ and $3 \notin J(R)$, $A \neq 0$ and $B \neq 0$. As $AG$ is semi-tripotent and $2 = 0$ in $A$, $G$ is a 2-group by Lemma 4.1. As $BG$ is semi-tripotent and $3 = 0$ in $B$, $G$ is a group of exponent 2 by Lemma 4.2. ∎

***Lemma 4.4*** ([4, Proposition 9]) *If $R$ is a ring and $G$ is a locally finite group, then $J(R) = J(RG) \cap R$. In particular, $J(R)(RG) \subseteq J(RG)$.*

***Lemma 4.5*** *If $R$ is a semi-tripotent ring with $3 \in J(R)$ and $G$ is a group of exponent 2, then $RG$ is semi-tripotent.*

**Proof** Let $J = J(R)$ and $\alpha \in RG$. Then there exists a finite subgroup $H$ of $G$ such that $\alpha \in RH$. Here $H$ is a direct product of finite copies of $C_2$. As $2 \in U(R)$, $RH$ is a direct sum of finite copies of $R$, so $RH$ is semi-tripotent. So, $\alpha$ is semi-tripotent in $RH$. We show that $\alpha$ is semi-tripotent in $RG$. By Lemma 4.4, $JH \subseteq J(RH)$. As $R$ is semi-tripotent with $3 \in J(R)$, $R/J$ has identity $x^3 = x$ with $2 \in U(R/J)$. So, $(R/J)H$ is a commutative von Neumann regular ring by [4, Theorem 3]. As $(R/J)H \cong RH/JH$, it follows that $JH = J(RH)$. So $J(RH) = JH \subseteq J(RG)$ by Lemma 4.4. Hence, $\alpha$ semi-tripotent in $RH$ implies that $\alpha$ is semi-tripotent in $RG$. ∎

***Lemma 4.6*** *If $R$ is a semi-tripotent ring with $3 \in J(R)$ and $G$ is a locally finite 3-group, then $RG$ is semi-tripotent.*

**Proof** By [18, Lemma 2], $\Delta(RG) \subseteq J(RG)$. As $RG/\Delta(RG) \cong R$ is semi-tripotent, idempotents lift modulo $J$ in $R$ by Theorem 3.5, and moreover $RG/J(RG)$ has identity $x^3 = x$. By [14, Proposition 1.5], $R$ is a clean ring, *i.e.*, every element is a sum of an

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idempotent and a unit. Hence, by [18, Theorem 4], $RG$ is a clean ring, so idempotent lift modulo $J(RG)$ in $RG$. Hence, $RG$ is semi-tripotent by Theorem 3.5. ■

A group is said to be nilpotent if it has a central series.

***Theorem 4.7*** *Let $R$ be a ring and $G$ be a locally finite, nilpotent group. Then $RG$ is semi-tripotent if and only if $R$ is semi-tripotent and one of the following holds:*

(1) $2 \in J(R)$ *and $G$ is a 2-group.*
(2) $3 \in J(R)$ *and $G$ is a direct product of a group of exponent 2 and a 3-group.*
(3) $2 \notin J(R)$ *and* $3 \notin J(R)$, *and $G$ is a group of exponent 2.*

**Proof** ($\Rightarrow$). It is known that every locally finite nilpotent group is a direct product of its $p$-subgroups. So $G$ is a direct product of $p$-groups $G_p$ where $p$ runs over all primes. Since $RG$ is semi-tripotent, $RG_p$ is also semi-tripotent for each $p$ and $R$ is semi-tripotent. So, by Theorem 3.5, $R/J = A \times B$, where $A$ is Boolean and $B$ is zero or a subdirect product of $\mathbb{Z}_3$'s.

If $2 \in J(R)$, then $G$ is a 2-group by Lemma 4.1.

If $3 \in J(R)$, then either $p = 3$ or $p = 2$ with $G_2$ a group of exponent 2 by Lemma 4.2. So $G$ is a direct product of a group of exponent 2 and a 3-group.

If $2 \notin J(R)$ and $3 \notin J(R)$, then $2 \neq 0$ in $R/J$ and $3 \neq 0$ in $R/J$. So $A \neq 0$ and $B \neq 0$. As images of $RG$, $AG$ and $BG$ are semi-tripotent. Since $2 = 0$ in $A$, $G$ is a 2-group by Lemma 4.1. As $3 = 0$ in $B$, $G$ must be of exponent 2 by Lemma 4.2.

($\Leftarrow$). Suppose that $R$ is semi-tripotent. If (1) holds, then $R$ is semi-boolean by Corollary 3.6. So $RG$ is semi-boolean (and hence semi-tripotent) by [9, Theorem 4.4].

If (2) holds, then $G = H_1 \times H_2$, where $H_1$ is a group of exponent 2 and $H_2$ is a 3-group. So $RG \cong (RH_2)H_1$. By Lemma 4.6, $RH_2$ is semi-tripotent, and so $(RH_2)H_1$ is semi-tripotent by Lemma 4.5.

Suppose that (3) holds. As $R$ is semi-tripotent, $R/J = (X/J) \oplus (Y/J)$, where $X/J$ is Boolean and $Y/J$ is zero or a subdirect product of $\mathbb{Z}_3$'s. As idempotents lift modulo $J$, there exist $e^2 = e \in X$ and $f^2 = f \in Y$ such that $e + f = 1$, $X/J = (eRe + J)/J$ and $Y/J = (fRf + J)/J$. So, $R = eRe + fRf + J$, and hence $RG = (eRe)G + (fRf)G + JG$. Moreover, by Corollary 2.8 $eRe$ and $fRf$ are semi-tripotent. So, by Corollary 3.6, $eRe$ is semi-boolean.

If $\alpha \in RG$, write $\alpha = y + z + w$ where $y \in (eRe)G$, $z \in (fRf)G$ and $w \in JG$. Write $y = \sum a_i g_i$ where $a_i \in eRe$ and $g_i \in G$ for each $i$. As $2 = 0$ in $eRe/eJe$, $2e \in eJe$, so $\big(e(1+g_i)\big)^2 = e(1+2g_i+g_i^2) = e(2+2g_i) = 2e(1+g_i) \in (eJe)G$. Hence $e(1+g_i) \in (eJe)G \subseteq JG$, as $e(1+g_i)$ is central in $(eRe)G$ (for all $i$). Thus, $y = \sum a_i g_i + \sum a_i + (-\sum a_i) = \sum a_i e(1+g_i) + (-\sum a_i)$, where $\sum a_i e(1+g_i) \in JG$. As $eRe$ is semi-boolean, write $-\sum a_i = j_1 + h_1$ where $j_1 \in eJe \subseteq J$ and $h_1^2 = h_1 \in eRe$. So $j_1 \in J(RG)$ by Lemma 4.5. As $3 \in J(fRf)$, $(fRf)G$ is semi-tripotent by Lemma 4.5. So $z = j_2 + h_2$ where $j_2 \in J\big((fRf)G\big)$ and $h_2^3 = h_2 \in (fRf)G$. As $3 = 0$ in $fRf/fJf$ and $fRf/fJf$ has identity $x^3 = x$, $(fRf)G/(fJf)G \cong (fRf/fJf)G$ is semi-primitive (indeed, commutative von Neumann regular, by [4, Theorem 3]). It follows that $J\big((fRf)G\big) \subseteq (fJf)G \subseteq J(RG)$. So $j_2 \in J(RG)$. Let $\beta = \sum a_i e(1+g_i) + j_1 + j_2 + w$ and $\gamma = h_1 + h_2$. Then $\beta \in J(RG)$. As $h_1 h_2 = h_2 h_1 = 0$, $\gamma^3 = h_1^3 + h_2^3 = h_1 + h_2 = \gamma$. So $\alpha = \beta + \gamma$ is semi-tripotent in $RG$. ■

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***Theorem 4.8*** *Let $R$ be a ring and $G$ be an abelian group. Then $RG$ is semi-tripotent if and only if $R$ is semi-tripotent and one of the following holds:*

(1) $2 \in J(R)$ *and $G$ is a* 2*-group.*
(2) $3 \in J(R)$ *and $G$ is a direct product of a group of exponent* 2 *and a* 3*-group.*
(3) $2 \notin J(R)$ *and* $3 \notin J(R)$*, and $G$ is a group of exponent* 2*.*

**Proof** ($\Rightarrow$). By Theorem 4.7, it suffices to show that $G$ is torsion. As $RG$ is semi-tripotent, $R$ and hence $R/J(R)$ are semi-tripotent. So, by Theorem 3.5, $R/J(R) = A\times B$, where $A$ is Boolean and $B$ is zero or a subdirect product of $\mathbb{Z}_3$'s.

If $2 \in J(R)$, then $G$ is a 2-group by Lemma 4.1.

If $3 \in J(R)$, then $3 = 0$ in $R/J(R)$, so $A = 0$. As $\mathbb{Z}_3$ is an image of $R/J(R)$, $\mathbb{Z}_3 G$ is an image of $RG$ and hence it is semi-tripotent. Assume that $G$ is not torsion. Then $G/T(G)$ is non-trivial torsion-free where $T(G)$ is the torsion subgroup of $G$, and $\mathbb{Z}_3\big(G/T(G)\big)$ is semi-tripotent (being an image of $\mathbb{Z}_3 G$). So we can assume that $G$ is torsion-free. If $G$ has rank greater than 1, then $G$ has a torsion-free quotient $G'$ of rank 1. Since $\mathbb{Z}_3 G'$ is semi-tripotent again, we can assume that $G$ is of rank 1. So $G$ is isomorphic to a subgroup of $(\mathbb{Q}, +)$. Take $g \in G$ such that $g^{-1} \neq g$. Since $g + g^{-1}$ is semi-tripotent in $\mathbb{Z}_3 G$, there exist $j \in J(\mathbb{Z}_3 G)$ and $b^3 = b \in \mathbb{Z}_3 G$ such that $g + g^{-1} = j + b$. There exists a finitely generated subgroup $G_1$ of $G$ such that $g, j, b, (1 + j)^{-1} \in \mathbb{Z}_3 G_1$. Because every finitely generated subgroup of $(\mathbb{Q}, +)$ is cyclic, $G_1$ is cyclic. Write $G_1 = \langle h\rangle$. Then $g = h^k, g^{-1} = h^{-k}$ for some positive integer $k$. There is a natural isomorphism $\mathbb{Z}_3\langle h\rangle \cong \mathbb{Z}_3[x, x^{-1}]$ with $h^k + h^{-k} \longleftrightarrow x^k + x^{-k}$. As $h^k + h^{-k} - b + 1 = 1 + j$ is a unit in $\mathbb{Z}_3\langle h\rangle$, $x^k + x^{-k} - f + 1$ is a unit in $\mathbb{Z}_3[x, x^{-1}]$, where $f^3 = f \in \mathbb{Z}_3[x, x^{-1}]$. But this is impossible because the tripotents of $\mathbb{Z}_3[x, x^{-1}]$ are in $\mathbb{Z}_3$ and the units of $\mathbb{Z}_3[x, x^{-1}]$ are in $\{ax^i : 0 \neq a \in \mathbb{Z}_3, i \in \mathbb{Z}\}$. The contradiction shows that $G$ is torsion.

If $2 \notin J(R)$ and $3 \notin J(R)$, then $2 \neq 0$ and $3 \neq 0$ in $R/J(R)$, so $A \neq 0$ and $B \neq 0$. As an image of $RG$, $AG$ is semi-tripotent. As $2 = 0$ in $A$, $G$ is a 2-group by Lemma 4.1. As $BG$ is semi-tripotent and $3 = 0$ in $B$, $G$ is a group of exponent 2 by Lemma 4.2.

($\Leftarrow$). This follows from Theorem 4.7. ■

**Acknowledgments** The research was supported by a grant (117F070) from TUBITAK of Turkey and a Discovery Grant from NSERC of Canada, and is part of the Ph.D. dissertation of the author T. Y. at Gebze Technical University. Part of the work was carried out when author Y. Z. was visiting Gebze Technical University. He gratefully acknowledges the hospitality from the host institute. ■

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