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Existence of positive radial solutions for a superlinear semipositone p-Laplacian problem on the exterior of a ball

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We prove the existence of positive radial solutions to a class of semipositone p-Laplacian problems on the exterior of a ball subject to Dirichlet and nonlinear boundary conditions. Using variational methods we prove the existence of a solution, and then use *a priori* estimates to prove the positivity of the solution.

Keywords: semipositone; p-Laplacian; nonlinear boundary conditions

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Secondary 35J20

1. Introduction

We study positive, radial solutions to equations of the form

$$\left.\begin{aligned} -\Delta_p u = \lambda K(|x|) f(u), &\quad x \in \Omega_e,\\ u = 0, &\quad |x| = r_0,\\ u \to 0, &\quad |x| \to \infty, \end{aligned}\right\} \tag{1.1}$$

and

$$\left.\begin{aligned} -\Delta_p u = \lambda K(|x|) f(u), &\quad x \in \Omega_e,\\ \frac{\partial u}{\partial \eta} + \tilde{c}(u) u = 0, &\quad |x| = r_0,\\ u \to 0, &\quad |x| \to \infty, \end{aligned}\right\} \tag{1.2}$$

where $\lambda > 0$ is a parameter,

$$\Delta_p w = \nabla \cdot (|\nabla w|^{p-2} \nabla w),$$

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$p > 1$ is the p-Laplacian, and $\Omega_e = \{x \in \mathbb{R}^N \mid |x| > r_0,\ r_0 > 0,\ N > p\}$. We assume that the reaction term $f\colon [0,\infty) \to \mathbb{R}$ is a non-decreasing, continuous function such that

(F1) there exist $A, B \in (0,\infty)$ and $q \in (p-1,\infty)$ such that $A(s^q - 1) \leqslant f(s) \leqslant B(s^q + 1)$ $\forall s \geqslant 0$ (which implies that f is p-superlinear at infinity);

(F2) $f(0) < 0$ (semipositone); and

(F3) there exists $\theta > p$ such that for s sufficiently large, $sf(s) > \theta F(s)$, where

$$F(s) = \int_0^s f(t)\,\mathrm{d}t.$$

The weight $K\colon [r_0,\infty) \to (0,\infty)$ is a continuous function such that

(K1) there exists $\mu \in (0,(N-p)/(p-1))$ so that $K(r) \leqslant 1/r^{N+\mu}$ for $r \gg 1$, and

(K2) $K(r)$ is decreasing on $[\tilde{R},\infty)$ for some $\tilde{R} \gg 1$.

When analysing (1.2), we further assume that $\tilde{c}\colon [0,\infty) \to (0,\infty)$ is continuous. Here $\partial u/\partial\eta$ is the outward normal derivative.

Applying the change of variables $\zeta = |x|$ and $t = (\zeta/r_0)^{(p-N)/(p-1)}$ transforms (1.1) and (1.2) to the boundary-value problems

$$\left.\begin{aligned} -(\phi_p(u'))' = \lambda h(t) f(u), \quad t \in (0,1),\\ u(0) = 0 = u(1), \end{aligned}\right\} \tag{D}$$

and

$$\left.\begin{aligned} -(\phi_p(u'))' = \lambda h(t) f(u), \quad t \in (0,1),\\ u(0) = 0,\\ \phi_p(u'(1)) + c(u(1))\phi_p(u(1)) = 0, \end{aligned}\right\} \tag{NL}$$

respectively, where $\phi_p(s) = |s|^{p-2}s$,

$$h(t) = \left(\frac{p-1}{N-p} r_0\right)^p t^{-p(N-1)/(N-p)} K(r_0 t^{(1-p)/(N-p)}),$$

and $c(s) = (r_0(p-1)/(N-p)\tilde{c}(s))^{p-1}$.

Conditions (K1) and (K2) imply that

$$h \in L^1(0,1) \cap C(0,1] \quad \text{and} \quad \hat{h} = \inf_{t\in(0,1]} h(t) > 0.$$

Note that if one assumes that $\mu \geqslant (N-p)/(p-1)$, then $h \in C[0,1]$ and is a simpler case to study. Here we allow $\mu < (N-p)/(p-1)$, which may result in h being singular at $t = 0$.

Now let

$$g(s) = \int_0^s c(t)\phi_p(t)\,\mathrm{d}t.$$

We will assume that $c(s)$ satisfies the growth condition:

(C1) for θ satisfying (F3), we have $c(s)s^p < \theta g(s)$ for s sufficiently large.

By a solution u to problem (D) (or (NL)), we mean a $u \in C^1[0,1]$ and $\phi_p(u') \in W^{1,1}(0,1)$ satisfying (D) (or (NL)).

We will establish the following results.

THEOREM 1.1. *Assume that (F1)–(F3) and (K1), (K2) hold. Then (D) has a positive solution for* $\lambda \approx 0$.

THEOREM 1.2. *Assume that (F1)–(F3), (K1), (K2) and (C1) hold. Then (NL) has a positive solution for* $\lambda \approx 0$.

In order to make use of variational techniques, we extend the functions f and c to all of $\mathbb{R}$ by setting $f(s) = f(0)$ and $c(s) = c(-s)$ for $s < 0$.

REMARK 1.3. Let $f(s) = s^q - 1$ and $c(s) = s^n + 1$, $n > 0$. Then, choosing $\theta = \frac{1}{2}(n+p+q+1)$, f satisfies (F1)–(F3) and c satisfies (C1).

REMARK 1.4. Given the extension of $f(s) = f(0)$, $s < 0$, (F1) implies that

$$f(s) \leqslant B(|s|^q + 1) \quad \forall s \in \mathbb{R}.$$

Furthermore, we note that (F1) implies that there exists constant $A_1 > 0$ such that

$$A_1(s^{q+1} - 1) \leqslant F(s) \quad \forall s \geqslant 0,$$

and constant $B_1 > 0$ such that

$$F(s) \leqslant B_1(|s|^{q+1} + 1) \quad \forall s \in \mathbb{R}.$$

Similarly, if f satisfies (F3), then there exists a constant $\tilde{\theta} > 0$ such that

$$sf(s) > \theta F(s) - \tilde{\theta} \quad \forall s \geqslant 0.$$

Finally, (C1) combined with the extension $c(s) = c(-s)$, $s < 0$, implies that there exists $\tilde{\theta}_1 \in \mathbb{R}$ such that $\tilde{\theta}_1 < \theta g(s) - c(s)|s|^p$ for all $s \in \mathbb{R}$ since g is an even function.

For a rich history of the study of existence results for the case of the Laplacian and the p-Laplacian operator with Dirichlet boundary conditions on bounded domains, see [2, 4–6, 8–11, 15–17]. In all of these works the authors studied equations of the form

$$\begin{aligned} -\Delta_p u &= \lambda f(u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^N$ (including the cases in which Ω is a ball or an annulus). Assuming that $f \in C[0,\infty)$, $f(0) < 0$ and f has p-superlinear growth at infinity, they discussed the existence of a positive solution for $\lambda \approx 0$. Recently, when $p = 2$, Dhanya *et al*. [13] proved the existence of a positive radial solution when Ω is the region exterior to a ball. Their study also included the case in which a nonlinear condition (as in (1.2)) was satisfied on the inner boundary (i.e. the boundary of the ball). The focus of this paper is to extend this result for all $p > 1$. In [13], the

Dhanya *et al*. used variational methods (the mountain pass theorem) combined with the properties of the Green function. In the p-Laplacian case (when $p \neq 2$), the help of a Green function is unavailable, which necessitates a deeper analysis. Extending recent ideas from [11] to the case of boundary-value problems with singular weights as well as to boundary-value problems with nonlinear boundary conditions, we establish our results in this paper.

In § 2 we recall the mountain pass theorem and an important property (the (S^+) property) of the p-Laplacian operator. In § 3 we prove theorem 1.1 and in § 4 we prove theorem 1.2.

2. Preliminaries

We will use the mountain pass theorem, as in [3], which is stated below.

Theorem 2.1 (mountain pass theorem). *Let X be a Banach space and let $J \in C^1(X;\mathbb{R})$ satisfy the following:*

(I) *(Palais–Smale condition) any sequence $\{u_n\} \subset X$ such that $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence,*

(II) $J(0) = 0$,

(III) *there exist $\alpha, R > 0$ such that $J(u) \geqslant \alpha\ \forall \|u\|_X = R$, and*

(IV) *there exists $v \in X$ such that $\|v\|_X > R$ and $J(v) < 0$.*

Furthermore, let

$$\Gamma := \{\gamma \in C([0,1];X) \colon \gamma(0) = 0,\ \gamma(1) = v\},$$

and

$$\hat{c} := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)).$$

Then $\hat{c}$ is a critical value of the functional J.

In order to apply the mountain pass theorem, we employ several Banach spaces, $W_0^{1,p}(0,1)$, $C[0,1]$, $C^1[0,1]$ and $L^s(0,1)$, each equipped with the usual norms, $\|\cdot\|_{1,p}$, $\|\cdot\|_\infty$, $\|\cdot\|_{C^1}$ and $\|\cdot\|_s$, respectively. We also recall that $W^{1,p}(0,1)$ is compactly embedded in $C[0,1]$, which implies the existence of a constant $k > 0$ such that $\|u\|_\infty \leqslant k\|u\|_{1,p}$ for every $u \in W_0^{1,p}(0,1)$ (see [1]).

Finally, we recall the concept of the (S^+) condition (see [7]). The proof of the following proposition can be found in [14].

Proposition 2.2 ((S^+) property). *Let $\Psi \colon W^{1,p}(0,1) \to [0,\infty)$ be defined by*

$$\Psi(u) = \frac{1}{p}\int_0^1 |u'|^p \,\mathrm{d}x.$$

Then Ψ' exists,

$$\langle \Psi'(u), v\rangle = \int_0^1 |u'|^{p-2}u'v' \,\mathrm{d}x,$$

and if $u_n \rightharpoonup u$ and $\limsup_{n\to\infty}\langle\Psi'(u_n), u_n - u\rangle \leqslant 0$, then $u_n \to u$ strongly in $W^{1,p}(0,1)$.

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3. Proof of theorem 1.1

Let $J\colon W_0^{1,p}(0,1) \to \mathbb{R}$ be defined by

$$J(u) = \frac{1}{p}\int_0^1 (u')^p \,\mathrm{d}x - \lambda \int_0^1 hF(u)\,\mathrm{d}x. \tag{3.1}$$

The second term in the definition of J is well defined since $W_0^{1,p}(0,1) \hookrightarrow C[0,1]$ and

$$\left|\lambda \int_0^1 hF(u)\,\mathrm{d}x\right| \leqslant \lambda \|h\|_1 \max_{-M_1\leqslant s\leqslant M_1} |F(s)|, \quad \text{where } M_1 = \|u\|_\infty.$$

Furthermore, the map J is continuously differentiable and

$$\langle J'(u), v\rangle = \int_0^1 |u'|^{p-2}u'v'\,\mathrm{d}x - \lambda\int_0^1 hf(u)v\,\mathrm{d}x \quad \forall v \in W_0^{1,p}(0,1).$$

Clearly, the first term of J' is well defined. The second term is well defined since $W_0^{1,p}(0,1) \hookrightarrow C[0,1]$ and the extended function $f \in C(\mathbb{R})$. Indeed, to show that J' is a continuous map, let us show that

$$L_u(v) := \int_0^1 hf(u)v\,\mathrm{d}x$$

is continuous for any $v \in W_0^{1,p}(0,1)$.

Let $\varepsilon > 0$ be given. Since the extended function f is continuous, there exists $\delta_1 > 0$ so that for every $t_1, t_2 \in \mathbb{R}$ such that $|t_2 - t_1| < \delta_1$, $|f(t_2) - f(t_1)| < \varepsilon/k\|h\|_1$. Choose $\delta = \delta_1/k$ so that when $\|u_1 - u_2\|_{1,p} < \delta$, we have $\|u_1 - u_2\|_\infty < \delta_1$. Then for any fixed $v \in W_0^{1,p}(0,1)$ with $\|v\|_{1,p} \leqslant 1$,

$$\begin{aligned}
|L_{u_1}(v) - L_{u_2}(v)| &= \left|\int_0^1 h(f(u_1) - f(u_2))v\,\mathrm{d}x\right| \\
&\leqslant \int_0^1 h|f(u_1) - f(u_2)|\|v\|_\infty\,\mathrm{d}x \\
&\leqslant k\int_0^1 h|f(u_1) - f(u_2)|\,\mathrm{d}x \\
&\leqslant k\int_0^1 h\frac{\varepsilon}{k\|h\|_1}\,\mathrm{d}x \\
&= \varepsilon
\end{aligned}$$

for all u_1, u_2 with $\|u_1 - u_2\|_{1,p} < \delta$. Hence,

$$\|L_{u_1} - L_{u_2}\| = \sup_{\|v\|_{1,p}\leqslant 1} \{|L_{u_1}(v) - L_{u_2}(v)|\} \leqslant \varepsilon.$$

Therefore, J is C^1.

We will first establish the existence of a solution for (D) using the mountain pass theorem and then prove that the solution thus obtained is positive.

LEMMA 3.1. *The critical point $u \in W_0^{1,p}(0,1)$ of (3.1) is a solution of (D).*

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Proof. If u is a critical point of (3.1), then

$$\int_0^1 \phi_p(u'(s))v'(s)\,\mathrm{d}s = \lambda \int_0^1 h(s)f(u(s))v(s)\,\mathrm{d}s \quad \forall v \in C_0^\infty[0,1].$$

Using integration by parts, we then have,

$$\int_0^1 (\phi_p(u'(s))' + \lambda h(s)f(u(s)))v(s)\,\mathrm{d}s = 0 \quad \forall v \in C_0^\infty[0,1].$$

Hence, $(\phi_p(u'(x)))' = -\lambda h(x)f(u(x))$ almost everywhere in (0,1). But since f is continuous, $u \in C[0,1]$ and $h \in C(0,1)$, so $(\phi_p(u'(x)))' = -\lambda h(x)f(u(x))$ holds for every $x \in (0,1)$. Furthermore, since $h \in L^1(0,1)$, f is continuous and $u \in C[0,1]$, we have that $(\phi_p(u'))' \in L^1(0,1)$, i.e. $\phi_p(u') \in W^{1,1}(0,1)$.

Let $x_0 \in (0,1)$ so that $u'(x_0) = 0$. Then

$$u'(x) = \phi_p^{-1}\left(-\lambda \int_{x_0}^x h(s)f(u(s))\,\mathrm{d}s\right).$$

For $x \in (0,1]$, h is continuous on $[x_0,x]$, and therefore, $-\lambda\int_{x_0}^x h(s)f(u(s))\,\mathrm{d}s$ is also continuous. Since ϕ_p^{-1} is also continuous, we find that u' is continuous.

For $x = 0$, we have that

$$\begin{aligned}\lim_{x\to 0^+} u'(x) &= \lim_{x\to 0^+} \phi_p^{-1}\left(-\lambda \int_{x_0}^x h(s)f(u(s))\,\mathrm{d}s\right)\\ &= \phi_p^{-1}\left(-\lambda \int_{x_0}^0 h(s)f(u(s))\,\mathrm{d}s\right)\end{aligned}$$

exists since ϕ_p^{-1} is a continuous function and $h \in L^1(0,1)$. Hence, $u \in C^1[0,1]$. □

3.1. Existence of a mountain pass solution

In the following theorem, we establish the existence of a mountain pass solution.

THEOREM 3.2. *Assume that (F1)–(F3) and (K1), (K2) hold. Then, for $\lambda \approx 0$, the hypotheses of the mountain pass theorem are satisfied, and there exists a solution u_λ to (D).*

In order to prove theorem 3.2, we first prove several lemmas. Throughout the calculations to follow, we let $r = 1/(q+1-p)$.

LEMMA 3.3. *The map J satisfies the Palais–Smale condition (see theorem 2.1(I)).*

Proof. First, we wish to show that any sequence $\{u_n\}$ satisfying the hypotheses of theorem 2.1(I) must be bounded. Assume to the contrary that $\{u_n\}$ is a sequence such that $J'(u_n) \to 0$, there exists some $M > 0$ such that $|J(u_n)| < M$ $\forall n \geqslant 1$, and $\|u_n\|_{1,p} \to \infty$. Then consider the quantity

$$\frac{\theta J(u_n) - \langle J'(u_n), u_n\rangle}{\|u_n\|_{1,p}},$$

where $\theta > p$ is chosen as in (F3). Taking the limit as $n \to \infty$, we see that

$$\lim_{n\to\infty} \frac{\theta J(u_n) - \langle J'(u_n), u_n\rangle}{\|u_n\|_{1,p}} = 0,$$

since $J(u_n)$ is bounded and $J'(u_n) \to 0$. Also, we can write

$$\begin{aligned}\theta J(u_n) - \langle J'(u_n), u_n\rangle = \left(\frac{\theta}{p} - 1\right) \int_0^1 (u_n')^p \, \mathrm{d}x \\ - \lambda \int_0^1 h(\theta F(u_n) - f(u_n)u_n) \, \mathrm{d}x.\end{aligned}$$

Note that when $u_n \geqslant 0$, $\theta F(u_n) - f(u_n)u_n \leqslant \tilde{\theta}$ and when $u_n < 0$,

$$\begin{aligned}\theta F(u_n) - f(u_n)u_n &= \theta u_n f(0) - f(0)u_n \\ &= (\theta - 1)f(0)u_n.\end{aligned}$$

Hence,

$$\begin{aligned}&\theta J(u_n) - \langle J'(u_n), u_n\rangle \\ &\quad\geqslant \left(\frac{\theta}{p} - 1\right) \int_0^1 (u_n')^p \, \mathrm{d}x - \lambda\tilde{\theta}\|h\|_1 - \lambda(\theta - 1)|f(0)|\|u_n\|_\infty\|h\|_1 \\ &\quad\geqslant \left(\frac{\theta}{p} - 1\right)\|u_n\|_{1,p}^p - \lambda\tilde{\theta}\|h\|_1 - \lambda k(\theta - 1)|f(0)|\|u_n\|_{1,p}\|h\|_1.\end{aligned}$$

But by dividing both sides through by $\|u_n\|_{1,p}$ and taking a limit as $n \to \infty$, we get a contradiction. Hence, $\{u_n\}$ is bounded in $W_0^{1,p}(0,1)$, and therefore there exists a subsequence, call it again $\{u_n\}$, that converges weakly in $W_0^{1,p}(0,1)$ and strongly in $C[0,1]$.

Since $u_n \to u$ strongly in $C[0,1]$, we have

$$\lim_{n\to\infty} \int_0^1 hf(u_n)(u_n - u) \, \mathrm{d}x \to 0.$$

Furthermore, since $\{u_n\}$ is a Palais–Smale sequence, $J'(u_n) \to 0$. Therefore, since $u_n - u$ is bounded in $W_0^{1,p}(0,1)$, we obtain

$$\lim_{n\to\infty} \langle J'(u_n), u_n - u\rangle \to 0.$$

Hence,

$$\langle J'(u_n), u_n - u\rangle + \lambda \int_0^1 hf(u_n)(u_n - u) \, \mathrm{d}x = \langle \Psi'(u_n), u_n - u\rangle \to 0.$$

Therefore, by the (S^+) property, $u_n \to u$ strongly in $W_0^{1,p}(0,1)$, and so J satisfies theorem 2.1(I). □

Lemma 3.4. *There exists $\bar{\lambda} > 0$ and $u \in W_0^{1,p}(0,1)$ such that if $\lambda \in (0, \bar{\lambda})$, then $J(u) < 0$.*

Proof. Let $v_1 \in W_0^{1,p}(0,1)$ be such that $\|v_1\|_{1,p} = 1$, $v_1(x) > 0 \; \forall x \in (0,1)$ (which implies that $v_1 \in L^{q+1}(0,1)$), and let $c_1 = (2/pA_1\hat{h}\|v_1\|_{q+1}^{q+1})^r$. Then, for $s = c_1\lambda^{-r}$,

$$\begin{aligned} J(sv_1) &= \frac{1}{p}\int_0^1 ((sv_1)')^p \,\mathrm{d}x - \lambda\int_0^1 hF(sv_1)\,\mathrm{d}x \\ &\leqslant \frac{s^p}{p} - \lambda A_1 \int_0^1 h(s^{q+1}v_1^{q+1} - 1)\,\mathrm{d}x \\ &\leqslant \frac{s^p}{p} - \lambda A_1 s^{q+1}\hat{h}\|v_1\|_{q+1}^{q+1} + \lambda A_1\|h\|_1 \\ &= c_1^p\left(\frac{\lambda^{-rp}}{p} - \lambda\hat{h}A_1c_1^{q+1-p}\lambda^{-r(q+1)}\|v_1\|_{q+1}^{q+1}\right) + \lambda A_1\|h\|_1. \end{aligned} \tag{3.2}$$

Now, substituting in our choice of c_1, we have

$$\begin{aligned} J(sv_1) &\leqslant c_1^p\left(\frac{\lambda^{-rp}}{p} - \frac{2}{p}\lambda^{1-r(q+1)}\right) + \lambda A_1\|h\|_1 \\ &= c_1^p\lambda^{-rp}\left(\frac{1}{p} - \frac{2}{p}\lambda^{1-r(q+1-p)}\right) + \lambda A_1\|h\|_1 \\ &= -c_1^p\lambda^{-rp}\frac{1}{p} + \lambda A_1\|h\|_1 \\ &= \lambda^{-rp}\left(\frac{-c_1^p}{p} + \lambda^{1+rp}A_1\|h\|_1\right). \end{aligned}$$

Hence, choosing $\bar{\lambda} < (p\|h\|_1 A_1 c_1^{-p})^{-1/(1+rp)}$, we see that for all $\lambda \in (0,\bar{\lambda})$ there exists s^* (for example, $s^* = c_1(\bar{\lambda}/2)^{-r}$) such that $J(u) < 0$ for $u = s^*v_1$. □

LEMMA 3.5. *There exist $\tau \in (0, c_1)$ and $\tilde{\lambda} > 0$ such that if $\|u\|_{1,p} = \tau\lambda^{-r}$, then $J(u) \geqslant c_2(\tau\lambda^{-r})^p$ for all $\lambda \in (0,\tilde{\lambda})$, where $c_2 = 1/4p$.*

Proof. Let $\|u\|_{1,p} = \tau\lambda^{-r}$, where $\tau > 0$ is to be chosen later. Then

$$\begin{aligned} J(u) &= \frac{(\tau\lambda^{-r})^p}{p} - \lambda\int_0^1 hF(u)\,\mathrm{d}x \\ &\geqslant \frac{(\tau\lambda^{-r})^p}{p} - \lambda B_1\int_0^1 h|u|^{q+1}\,\mathrm{d}x - \lambda B_1\|h\|_1 \\ &\geqslant \frac{(\tau\lambda^{-r})^p}{p} - \lambda B_1\|h\|_1\|u\|_\infty^{q+1} - \lambda B_1\|h\|_1 \\ &\geqslant \frac{(\tau\lambda^{-r})^p}{p} - \lambda k^{q+1}B_1\|h\|_1\|u\|_{1,p}^{q+1} - \lambda B_1\|h\|_1 \\ &= \frac{(\tau\lambda^{-r})^p}{p} - \lambda k^{q+1}B_1\|h\|_1(\tau\lambda^{-r})^{q+1} - \lambda B_1\|h\|_1 \\ &\geqslant \lambda^{-rp}\left(\frac{\tau^p}{2p} - \lambda^{1+rp}B_1\|h\|_1\right), \end{aligned}$$

where $\tau < \min\{(1/2pB_1\|h\|_1k^{q+1})^{1/r}, c_1\}$ has now been chosen. Taking

$$\tilde{\lambda} = \tau^{p/(1+rp)}(4pB_1\|h\|_1)^{-1/(1+rp)},$$

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we have

$$J(u) \geqslant c_2 \tau^p \lambda^{-rp}$$

for all $\lambda \in (0, \tilde{\lambda})$, which proves the claim. □

3.1.1. *Proof of theorem 3.2*

We have already established that $J \in C^1(W_0^{1,p}(0,1); \mathbb{R})$. Observe that $J(0) = 0$ and by lemmas 3.3–3.5, for $\lambda < \min\{\bar{\lambda}, \tilde{\lambda}\}$, we have satisfied hypotheses (I)–(IV) of the mountain pass theorem (where we note that the choice $\tau < c_1$ in lemma 3.5 is sufficient to ensure that $\|v\|_X > R$ in hypothesis (IV)). Hence, there exists a solution u_λ to (D).

REMARK 3.6. To show the simple existence of a mountain pass solution (not necessarily positive) to (D), we may choose $\|u\|_{1,p}$ sufficiently small and quickly get the desired result. However, this solution likely has negative values and therefore does not make sense in the context of problem (1.1) since $f(s)$ is only defined for $s \geqslant 0$.

3.2. Positivity of solution

Let u_λ be the mountain pass solution to (D), as in theorem 3.2. We first establish two *a priori* bounds on u_λ that are necessary for establishing positivity.

LEMMA 3.7. *Let u_λ be as in theorem 3.2. Then there exist an $M_0 > 0$ and $\hat{\lambda} > 0$ such that*

$$M_0 \lambda^{-r} \leqslant \|u_\lambda\|_\infty$$

for all $\lambda \in (0, \hat{\lambda})$.

Proof. Recall that

$$J(u_\lambda) \geqslant c_2 \tau^p \lambda^{-rp} \quad \text{for } \lambda \in (0, \tilde{\lambda}),$$
$$0 > \hat{F} := \inf_{s \in \mathbb{R}} F(s) > -\infty \quad \text{and} \quad f(s)s \leqslant B(|s|^{q+1} + |s|) \ \forall s \in \mathbb{R}.$$

Letting

$$\hat{\lambda} = \min\left\{ \left(\frac{(p-1)c_2 \tau^p}{p|\hat{F}| \|h\|_1} \right)^{1/(1+rp)}, (2B\|h\|_1 c_2^{-1} \tau^{-p})^{-1/(1+rp)}, \tilde{\lambda} \right\},$$

we have that

$$\begin{aligned} \lambda \int_0^1 h f(u_\lambda) u_\lambda \, \mathrm{d}x &= \int_0^1 |u_\lambda'|^p \, \mathrm{d}x \\ &= pJ(u_\lambda) + p\lambda \int_0^1 hF(u_\lambda) \, \mathrm{d}x \\ &\geqslant pc_2 \tau^p \lambda^{-rp} - p|\hat{F}| \|h\|_1 \lambda \\ &\geqslant c_2 \tau^p \lambda^{-rp} \end{aligned} \tag{3.3}$$

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for $\lambda \in (0, \hat{\lambda})$. We further note that

$$\begin{aligned} c_2\tau^p\lambda^{-rp} &\leqslant \lambda \int_0^1 hf(u_\lambda)u_\lambda \,\mathrm{d}x \\ &\leqslant B\lambda \int_0^1 h(|u_\lambda|^{q+1} + |u_\lambda|)\,\mathrm{d}x \\ &\leqslant B\lambda \int_0^1 h(\|u_\lambda\|_\infty^{q+1} + \|u_\lambda\|_\infty)\,\mathrm{d}x \\ &\leqslant B\lambda \|h\|_1(\|u_\lambda\|_\infty^{q+1} + \|u_\lambda\|_\infty), \end{aligned}$$

so that for $\lambda < \hat{\lambda} \leqslant (2B\|h\|_1 c_2^{-1}\tau^{-p})^{-1/(1+rp)}$, $\|u_\lambda\|_\infty \geqslant 1$. We also have that

$$\begin{aligned} \lambda \int_0^1 hf(u_\lambda)u_\lambda \,\mathrm{d}x &\leqslant B\lambda \int_0^1 h(|u_\lambda|^{q+1} + |u_\lambda|)\,\mathrm{d}x \\ &\leqslant B\lambda \int_0^1 h(\|u_\lambda\|_\infty^{q+1} + \|u_\lambda\|_\infty)\,\mathrm{d}x \\ &\leqslant 2B\lambda\|h\|_1\|u_\lambda\|_\infty^{q+1}, \end{aligned} \tag{3.4}$$

since $\|u_\lambda\|_\infty \geqslant 1$. We combine (3.3) and (3.4) and take $M_0 = (c_2\tau^p/2B\|h\|_1)^{1/(q+1)}$ to complete the proof. □

LEMMA 3.8. *Let u_λ be as in theorem 3.2. Then there exist $c_3 > 0$ and $\lambda^* > 0$ such that*

$$\|u_\lambda\|_{1,p}^p \leqslant c_3\lambda^{-rp}$$

for all $\lambda \in (0, \lambda^*)$.

Proof. Let $\Omega^+ = \{x \in [0,1] \mid u_\lambda(x) \geqslant 0\}$ and $\Omega^- = [0,1]\setminus\Omega^+$. Since u_λ is a critical point of J, and using remark 1.4,

$$\begin{aligned} \|u_\lambda\|_{1,p}^p &= pJ(u_\lambda) + p\lambda\int_{\Omega^-} hF(u_\lambda)\,\mathrm{d}x + p\lambda\int_0^1 hF(u_\lambda)\,\mathrm{d}x - p\lambda\int_{\Omega^-} hF(u_\lambda)\,\mathrm{d}x \\ &\leqslant pJ(u_\lambda) + p\lambda\int_{\Omega^-} hu_\lambda f(0)\,\mathrm{d}x + p\lambda\int_0^1 h\biggl(\frac{u_\lambda f(u_\lambda)}{\theta} + \frac{\tilde{\theta}}{\theta}\biggr)\,\mathrm{d}x \\ &\quad - p\lambda\int_{\Omega^-} h\biggl(\frac{u_\lambda f(0)}{\theta} + \frac{\tilde{\theta}}{\theta}\biggr)\,\mathrm{d}x \\ &= pJ(u_\lambda) + p\lambda\biggl(1 - \frac{1}{\theta}\biggr)\int_{\Omega^-} hu_\lambda f(0)\,\mathrm{d}x - p\lambda\int_{\Omega^-} h\frac{\tilde{\theta}}{\theta}\,\mathrm{d}x \\ &\quad + \frac{p\lambda}{\theta}\int_0^1 hu_\lambda f(u_\lambda)\,\mathrm{d}x + p\lambda\frac{\tilde{\theta}}{\theta}\|h\|_1 \\ &\leqslant pJ(u_\lambda) + p\lambda k|f(0)|\|h\|_1\|u_\lambda\|_{1,p} + \frac{p}{\theta}\|u_\lambda\|_{1,p}^p + p\lambda\frac{\tilde{\theta}}{\theta}\|h\|_1. \end{aligned} \tag{3.5}$$

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On the other hand, by the mountain pass characterization of u_λ,

$$
\begin{aligned}
J(u_\lambda) &\leqslant \max_{s\geqslant 0}\{J(sv_1)\} \\
&\leqslant \max_{s\geqslant 0}\left\{\frac{s^p}{p} - \lambda A_1 s^{q+1}\hat{h}\|v_1\|_{q+1}^{q+1} + \lambda A_1\|h\|_1\right\},
\end{aligned} \tag{3.6}
$$

as in (3.2). Let

$$
p(s) := \frac{s^p}{p} - \lambda A_1 s^{q+1}\hat{h}\|v_1\|_{q+1}^{q+1} + \lambda A_1\|h\|_1,
$$

so that by solving $p'(s) = 0$ we find that $p(s)$ is maximized when $s = \bar{K}\lambda^{-r}$, where

$$
\bar{K} = (A_1(q+1)\hat{h}\|v_1\|_{q+1}^{q+1})^{-r}.
$$

Hence, if $\lambda \leqslant 1$, then $\lambda^{-rp} \geqslant \lambda$, and therefore

$$
\begin{aligned}
pJ(u_\lambda) + p\lambda\frac{\tilde{\theta}}{\theta}\|h\|_1 &\leqslant \bar{K}^p\lambda^{-rp} - p\lambda A_1\hat{h}\bar{K}^{q+1}\lambda^{-r(q+1)}\|v_1\|_{q+1}^{q+1} + \lambda p\left(A_1 + \frac{\tilde{\theta}}{\theta}\right)\|h\|_1 \\
&\leqslant \bar{K}^p\lambda^{-rp} - pA_1\hat{h}\bar{K}^{q+1}\lambda^{-rp}\|v_1\|_{q+1}^{q+1} + \lambda^{-rp}p\left(A_1 + \frac{\tilde{\theta}}{\theta}\right)\|h\|_1 \\
&\leqslant \left(\bar{K}^p - pA_1\hat{h}\bar{K}^{q+1}\|v_1\|_{q+1}^{q+1} + p\left(A_1 + \frac{\tilde{\theta}}{\theta}\right)\|h\|_1\right)\lambda^{-rp} \\
&= \tilde{c}_3\lambda^{-rp},
\end{aligned} \tag{3.7}
$$

where

$$
\tilde{c}_3 = \bar{K}^p - pA_1\hat{h}\bar{K}^{q+1}\|v_1\|_{q+1}^{q+1} + p\left(A_1 + \frac{\tilde{\theta}}{\theta}\right)\|h\|_1.
$$

By lemma 3.7, if $\lambda < \min\{\hat{\lambda}, (k/M_0)^{-1/r}\}$, then

$$
\|u_\lambda\|_{1,p} \geqslant \frac{1}{k}\|u_\lambda\|_\infty \geqslant \frac{M_0}{k}\lambda^{-r} \geqslant 1.
$$

From (3.5) and (3.7), we have that

$$
a\|u_\lambda\|_{1,p}^p \leqslant b\lambda\|u_\lambda\|_{1,p} + \tilde{c}_3\lambda^{-rp}
$$

for $a = 1 - p/\theta > 0$ and $b = pk|f(0)|\|h\|_1 > 0$. Since $\|u_\lambda\|_{1,p} \geqslant 1$,

$$
a\|u_\lambda\|_{1,p}^p \leqslant b\lambda\|u_\lambda\|_{1,p}^p + \tilde{c}_3\lambda^{-rp}.
$$

Hence, if

$$
\lambda \leqslant \frac{a}{2b} = \frac{\theta - p}{2\theta pk|f(0)|\|h\|_1},
$$

then

$$
(a - b\lambda)\|u_\lambda\|_{1,p}^p \leqslant \tilde{c}_3\lambda^{-rp},
$$

which implies that

$$
\tfrac{1}{2}a\|u_\lambda\|_{1,p}^p \leqslant \tilde{c}_3\lambda^{-rp}.
$$

The lemma is proven taking $c_3 = 2\tilde{c}_3/a$ and

$$
\lambda^* = \min\left\{1, \hat{\lambda}, \left(\frac{k}{M_0}\right)^{-1/r}, \frac{\theta - p}{2\theta pk|f(0)|\|h\|_1}\right\}. \qquad \square
$$

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3.2.1. *Proof of theorem 1.1*

We prove the theorem by contradiction. Suppose that there exists a sequence $\{(\lambda_j, u_{\lambda_j})\}_{j=1}^{\infty} \subset (0,1) \times C^1[0,1]$ of mountain pass solutions to (D), as in theorem 3.2, such that $\lambda_j \to 0$ and $m(\{x \in (0,1) \mid u_{\lambda_j}(x) \leqslant 0\}) > 0$. Let $w_j = u_{\lambda_j}/\|u_{\lambda_j}\|_\infty$. Then we have

$$-(\phi_p(w_j'))' = \lambda_j h \frac{f(u_{\lambda_j})}{\|u_{\lambda_j}\|_\infty^{p-1}}.$$

By (F1) and lemmas 3.7 and 3.8,

$$\begin{aligned} |\lambda_j f(u_{\lambda_j})\|u_{\lambda_j}\|_\infty^{1-p}| &\leqslant \lambda_j B(\|u_{\lambda_j}\|_\infty^{q+1-p} + \|u_{\lambda_j}\|_\infty^{1-p}) \\ &\leqslant \lambda_j B(k^{1/r}\|u_{\lambda_j}\|_{1,p}^{1/r} + M_0^{1-p}\lambda^{-r(1-p)}) \\ &\leqslant \lambda_j B c_4(\lambda_j^{-1} + \lambda_j^{-r(1-p)}), \end{aligned} \tag{3.8}$$

where $c_4 = \max\{(c_3 k)^{1/r}, M_0^{1-p}\}$. Hence, we observe from (3.8) that

$$\begin{aligned} |\lambda_j f(u_{\lambda_j})\|u_{\lambda_j}\|_\infty^{1-p}| &\leqslant Bc_4 + Bc_4\lambda_j^{-r(1-p)+1} \\ &\leqslant Bc_4 + Bc_4\lambda_j^{q/(q+1-p)} \\ &\leqslant 2Bc_4 \end{aligned} \tag{3.9}$$

for λ_j sufficiently small. Hence, $\lambda_j f(u_{\lambda_j}(x))\|u_{\lambda_j}\|_\infty^{1-p}$ converges to a limit, $z_1(x)$, for every $x \in [0,1]$. Furthermore, since $\lambda_j\|u_{\lambda_j}\|_\infty^{1-p} \to 0$ as $j \to \infty$, and f is bounded from below,

$$\begin{aligned} z_1(x) &= \lim_{j\to\infty} \lambda_j f(u_{\lambda_j}(x))\|u_{\lambda_j}\|_\infty^{1-p} \\ &\geqslant \lim_{j\to\infty} -\lambda_j|f(0)|\|u_{\lambda_j}\|_\infty^{1-p} \\ &= 0. \end{aligned}$$

Therefore,

$$\lambda_j h(x) f(u_{\lambda_j}(x))\|u_{\lambda_j}\|_\infty^{1-p} \to h(x)z_1(x) =: z(x) \quad \forall x \in (0,1],$$

and $z(x) \geqslant 0$ $\forall x \in (0,1]$.

Let $x_j \in (0,1)$ be a maximum of $w_j(x)$. Then

$$\begin{aligned} \phi_p(w_j'(x)) &= \int_x^{x_j} -(\phi_p(w_j'(s)))' \, \mathrm{d}s \\ &= \int_x^{x_j} \lambda_j h(s) f(u_{\lambda_j}(s))\|u_{\lambda_j}\|_\infty^{1-p} \, \mathrm{d}s. \end{aligned}$$

By (3.9), this implies that $|\phi_p(w_j'(x))| \leqslant 2Bc_4\|h\|_1$ $\forall x \in [0,1]$, and therefore $|w_j'(x)| \leqslant (2Bc_4\|h\|_1)^{1/(p-1)}$ $\forall x \in [0,1]$. By the Arzelà–Ascoli theorem, this implies that there exists $w \in C[0,1]$ such that $w_j \to w$ in $C[0,1]$.

Meanwhile, again by (3.9), we have that $|\lambda_j h(x) f(u_{\lambda_j}(x))\|u_{\lambda_j}\|_\infty^{1-p}| \leqslant 2Bc_4 h(x)$ $\forall x \in (0,1]$. Since $h \in L^1(0,1)$, by the Lebesgue dominated convergence theorem,

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we may choose a subsequence u_{λ_j} with $x_j \to x_0$ such that

$$\int_x^{x_j} \lambda_j h(s) f(u_{\lambda_j}(s)) \|u_{\lambda_j}\|_\infty^{1-p}\, \mathrm{d}s \to \int_x^{x_0} h(s) z_1(s)\, \mathrm{d}s = \int_x^{x_0} z(s)\, \mathrm{d}s.$$

Hence,

$$\phi_p^{-1}\biggl(\int_x^{x_j} \lambda_j h(s) f(u_{\lambda_j}(s)) \|u_{\lambda_j}\|_\infty^{1-p}\, \mathrm{d}s\biggr) \to \phi_p^{-1}\biggl(\int_x^{x_0} z(s)\, \mathrm{d}s\biggr),$$

and therefore

$$\int_0^t \phi_p^{-1}\biggl(\int_x^{x_j} \lambda_j h(s) f\Bigl(u_{\lambda_j}(s)\Bigr) \|u_{\lambda_j}\|_\infty^{1-p}\, \mathrm{d}s\biggr)\, \mathrm{d}x \to \int_0^t \phi_p^{-1}\biggl(\int_x^{x_0} z(s)\, \mathrm{d}s\biggr)\, \mathrm{d}x.$$

Therefore, we see that $w_j(t) \to \int_0^t \phi_p^{-1}(\int_x^{x_0} z(s)\, \mathrm{d}s)\, \mathrm{d}x = w(t)$, and hence

$$\begin{aligned} w_j'(t) &= \phi_p^{-1}\biggl(\int_t^{x_j} \lambda_j h(s) f(u_{\lambda_j}(s)) \|u_{\lambda_j}\|_\infty^{1-p}\, \mathrm{d}s\biggr) \\ &\to \phi_p^{-1}\biggl(\int_t^{x_0} z(s)\, \mathrm{d}s\biggr) \\ &= w'(t) \end{aligned}$$

for all $t \in [0,1]$.

Hence, $-(\phi_p(w'))' = z \geqslant 0$ with $w(0) = 0 = w(1)$. Since $\|w_j\|_\infty = 1$, $w \not\equiv 0$. Hence, since w is concave, $w > 0$ in $(0,1)$, $w'(0) > 0$, and $w'(1) < 0$. Since $w_j \to w$ in $C^1[0,1]$, we have $w_j(x) > 0$ for all $x \in (0,1)$ for j sufficiently large. Hence, $u_{\lambda_j}(x) > 0$ for all $x \in (0,1)$ for j sufficiently large, which implies that $m(\{x \in (0,1); u_{\lambda_j}(x) \leqslant 0\}) = 0$ for all j sufficiently large, which is a contradiction. Hence, there exists some $\check{\lambda}$ such that (D) has a positive solution for all $\lambda \in (0, \check{\lambda})$.

Remark 3.9. Note that since we now have a positive solution to (D), by reversing the earlier change of variables, we have a positive radial solution to (1.1) on Ω_e.

Remark 3.10. By lemmas 3.7 and 3.8, we have

$$\|w_j\|_{1,p} \leqslant \frac{c_3}{M_0},$$

where we note that c_3 and M_0 are independent of λ, and therefore independent of j. In the case in which $h \in C[0,1]$ (that is, $\mu \geqslant (N-p)/(p-1)$), [12, proposition 3.7] implies that the sequence $\{w_j\}_{j=1}^\infty$ is uniformly bounded in $C_0^{1,\beta}[0,1]$ for some $\beta \in (0,1)$. We could then conclude that $w \in C^{1,\beta^*}[0,1]$ for some $\beta^* \in (0,\beta)$. This makes the proof simpler when $h \in C[0,1]$.

4. Proof of theorem 1.2

We begin by establishing the appropriate variational formulation of the problem. Let $W := \{u \in W^{1,p}(0,1) \mid u(0) = 0\}$. Then W is a Banach space with the induced norm of $W^{1,p}(0,1)$. Let E be defined on W as

$$E(u) = J(u) + g(u(1)), \tag{4.1}$$

where $J(u)$ is as before. Once again, the compact embedding of $W^{1,p}(0,1)$ into $C[0,1]$ implies that E is well defined.

Since we have already established that J is a C^1 functional, we need only show that $H(u) := g(u(1))$ is C^1. Fix $u \in W$ so that for any $v \in W$, $\langle H'(u), v\rangle = g'(u(1))v(1)$. It is clear that the function $g(s)$ as previously defined is continuously differentiable, and furthermore, since pointwise evaluation is a continuous operation, we may conclude that $H'(u)$ is a continuous functional on W. Hence, $E(u)$ is a C^1 functional as it is the sum of two C^1 functionals.

Proposition 4.1. *Let*

$$\|u\|_W = \left(\int_0^1 |u|^p \,\mathrm{d}x + \int_0^1 |u'|^p \,\mathrm{d}x\right)^{1/p}$$

be the norm induced on W as a subspace of $W^{1,p}(0,1)$, and let $\|u\| = (\int_0^1 |u'|^p \,\mathrm{d}x)^{1/p}$ on W. Then $\|\cdot\|$ is equivalent to $\|\cdot\|_W$ on W.

Proof. Let $u \in W$. Then clearly, $\|u\| \leqslant \|u\|_W$. Furthermore, applying Jensen's inequality, we have

$$\begin{aligned}\int_0^1 |u(x)|^p \,\mathrm{d}x &= \int_0^1 \left|\int_0^x u'(s)\,\mathrm{d}s\right|^p \mathrm{d}x \\ &\leqslant \int_0^1 \left(\int_0^x |u'(s)|\,\mathrm{d}s\right)^p \mathrm{d}x \\ &\leqslant \int_0^1 \left(\int_0^1 |u'(s)|\,\mathrm{d}s\right)^p \mathrm{d}x \\ &\leqslant \int_0^1 \int_0^1 |u'(s)|^p \,\mathrm{d}s\,\mathrm{d}x \\ &= \int_0^1 |u'(s)|^p \,\mathrm{d}s,\end{aligned}$$

which implies that

$$\|u\|_W = \left(\int_0^1 |u|^p \,\mathrm{d}x + \int_0^1 |u'|^p \,\mathrm{d}x\right)^{1/p} \leqslant \left(2\int_0^1 |u'|^p \,\mathrm{d}x\right)^{1/p} = 2^{1/p}\|u\|.$$

Hence, $\|\cdot\|$ is equivalent to $\|\cdot\|_W$ on W. □

By proposition 4.1, we may continue our analysis using

$$\|u\|_W = \left(\int_0^1 |u'|^p \,\mathrm{d}x\right)^{1/p}.$$

Lemma 4.2. *The critical point $u \in W$ of (4.1) is a solution of (NL).*

Proof. If u is a critical point of (4.1), then

$$\int_0^1 \phi_p(u'(s))v'(s)\,\mathrm{d}s + g'(u(1))v(1) = \lambda \int_0^1 h(s)f(u(s))v(s)\,\mathrm{d}s \quad \forall v \in C_0^\infty[0,1].$$

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Using integration by parts and the fact that $v(1) = 0$, we have that

$$\int_0^1 (\phi_p(u'(s))' + \lambda h(s) f(u(s)))v(s)\,\mathrm{d}s = 0 \quad \forall v \in C_0^\infty[0,1].$$

As in the proof of lemma 3.1, we have that $(\phi_p(u'(x)))' = -\lambda h(x) f(u(x))\ \forall x \in (0,1)$, $\phi_p(u') \in W^{1,1}(0,1)$, and $u \in C^1[0,1]$.

Clearly, $u(0) = 0$ since $u \in W$. Let $\tilde{C} = \{v \in C^\infty[0,1] \mid v(0) = 0\}$. Then since $\tilde{C} \subset W$ and u is a critical point of (4.1),

$$\int_0^1 \phi(u'(s))v'(s)\,\mathrm{d}s + g'(u(1))v(1) = \lambda \int_0^1 h(s) f(u(s))v(s)\,\mathrm{d}s \quad \forall v \in \tilde{C}.$$

Hence, using integration by parts,

$$\begin{aligned} \phi_p(u'(1))v(1) - \int_0^1 (\phi_p(u'(s)))'v(s)\,\mathrm{d}s + g'(u(1))v(1) \\ = \lambda \int_0^1 h(s) f(u(s))v(s)\,\mathrm{d}s \quad \forall v \in \tilde{C}, \end{aligned}$$

which implies that, for all $v \in \tilde{C}$,

$$\begin{aligned} (\phi_p(u'(1)) + c(u(1))\phi_p(u(1)))v(1) &= \phi_p(u'(1))v(1) + g'(u(1))v(1) \\ &= \int_0^1 ((\phi_p(u'(s)))' + \lambda h(s) f(u(s)))v(s)\,\mathrm{d}s \\ &= 0 \end{aligned}$$

since $(\phi_p(u'(x)))' + \lambda h(x) f(u(x)) = 0$ almost everywhere in $(0,1)$. Since $v(1)$ is arbitrary, we may conclude that $\phi_p(u'(1)) + c(u(1))\phi_p(u(1)) = 0$, and therefore the boundary conditions are satisfied. □

4.1. Existence of a mountain pass solution

Again, our goal will be to establish the existence of a mountain pass solution.

THEOREM 4.3. *Assume that (F1)–(F3), (K1), (K2) and (C1) hold. Then, for $\lambda \approx 0$, the hypotheses of the mountain pass theorem are satisfied and there exists a solution u_λ to (NL).*

We again establish several lemmas that will help to prove the theorem.

LEMMA 4.4. *The map E satisfies the Palais–Smale condition (see theorem 2.1(I)).*

Proof. As before, we first wish to show that any sequence $\{u_n\}$ satisfying the hypotheses of theorem 2.1(I) must be bounded. Assume to the contrary that $\{u_n\}$ is a sequence such that $E'(u_n) \to 0$, there exists some $M > 0$ such that $|E(u_n)| < M\ \forall n \geqslant 1$, and $\|u_n\|_W \to \infty$. Then, choosing $\theta > p$ satisfying (F3) and (C1), we note that

$$\lim_{n\to\infty} \frac{\theta E(u_n) - \langle E'(u_n), u_n\rangle}{\|u_n\|_W} = 0.$$

Also note that

$$\begin{aligned}\theta E(u_n) - \langle E'(u_n), u_n\rangle &= (\theta J(u_n) - \langle J'(u_n), u_n\rangle) + (\theta g(u_n(1)) - c(u_n(1))(u_n(1))^p) \\ &\geqslant \left(\frac{\theta}{p} - 1\right)\|u_n\|_W^p - \lambda\tilde{\theta}\|h\|_1 \\ &\quad - \lambda k(\theta - 1)|f(0)|\|u_n\|_W\|h\|_1 + \tilde{\theta}_1\end{aligned}$$

by combining the earlier estimate on $\theta J(u_n) - \langle J'(u_n), u_n\rangle$ with (C1). But this implies that

$$\begin{aligned}0 &= \lim_{n\to\infty} \frac{\theta E(u_n) - \langle E'(u_n), u_n\rangle}{\|u_n\|_W} \\ &\geqslant \lim_{n\to\infty} \frac{(\theta/p - 1)\|u_n\|_W^p - \lambda\tilde{\theta}\|h\|_1 - \lambda k(\theta - 1)|f(0)|\|u_n\|_W\|h\|_1 + \tilde{\theta}_1}{\|u_n\|_W} \\ &= \infty,\end{aligned}$$

which is a contradiction. Hence, $\{u_n\}$ is bounded in W and therefore contains a subsequence that converges weakly in W and strongly in $C[0,1]$.

Since $u_n \to u$ strongly in $C[0,1]$, we have

$$\lim_{n\to\infty} \int_0^1 hf(u_n)(u_n - u)\,\mathrm{d}x \to 0.$$

Furthermore, since $\{u_n\}$ is a Palais–Smale sequence, $E'(u_n) \to 0$. Therefore, since $u_n - u$ is bounded in W, we obtain

$$\lim_{n\to\infty} \langle E'(u_n), u_n - u\rangle \to 0.$$

Finally, we note that

$$c(u_n(1)) \cdot \phi_p(u_n(1)) \cdot (u_n(1) - u(1)) \to 0$$

since $u_n \to u$ strongly in $C[0,1]$ implies pointwise convergence and c, ϕ_p are both continuous functions. Hence,

$$\begin{aligned}\langle E'(u_n), u_n - u\rangle &+ \lambda \int_0^1 hf(u_n)(u_n - u)\,\mathrm{d}x \\ &- c(u_n(1)) \cdot \phi_p(u_n(1)) \cdot (u_n(1) - u(1)) = \langle \Psi'(u_n), u_n - u\rangle \\ &\to 0.\end{aligned}$$

Therefore, by the (S^+) property, $u_n \to u$ strongly in W, and so E satisfies theorem 2.1(I). □

The following two lemmas are analogous to lemmas 3.4 and 3.5 presented in the Dirichlet case, and rely heavily on the estimates there.

LEMMA 4.5. *Let u and $\bar{\lambda} > 0$ be as in lemma 3.4. Then, for $\lambda \in (0, \bar{\lambda})$, $E(u) < 0$.*

Proof. Choose $v_1 \in W_0^{1,p}(0,1) \subset W$ as in the proof of lemma 3.4. Then $E(sv_1) = J(sv_1) + g(sv_1(1)) = J(sv_1)$ since $v_1(1) = 0$ and $g(0) = 0$. The conclusion follows from lemma 3.4. □

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LEMMA 4.6. *Let* $\tau \in (0, c_1)$ *and let* $c_2, \tilde{\lambda} > 0$ *be as in lemma 3.5. Then if* $\|u\|_W = \tau\lambda^{-r}$, $E(u) \geqslant c_2(\tau\lambda^{-r})^p$ *for all* $\lambda \in (0, \tilde{\lambda})$.

Proof. Since $g(s) \geqslant 0$ $\forall s \in \mathbb{R}$, we have that $E(u) \geqslant J(u)$. From lemma 3.5, $J(u) \geqslant c_2(\tau\lambda^{-r})^p$ for all $\lambda \in (0, \tilde{\lambda})$. This completes the proof. □

4.1.1. *Proof of theorem 4.3*

Again, $E \in C^1(W_0^{1,p}(0,1), \mathbb{R})$, $E(0) = 0$ and, by lemmas 4.4–4.6, for $\lambda < \min\{\bar{\lambda}, \tilde{\lambda}\}$ we have satisfied hypotheses (I)–(IV) of the mountain pass theorem. Hence, there exists a solution u_λ to (NL).

4.2. Positivity of solution

To follow the same argument as in the proof of theorem 1.1, we need two lemmas, as before.

LEMMA 4.7. *Let* u_λ *be as in theorem 4.3. For* $M_0 > 0$ *and* $\hat{\lambda} > 0$ *as in lemma 3.7,*

$$M_0\lambda^{-r} \leqslant \|u_\lambda\|_\infty$$

for all $\lambda \in (0, \hat{\lambda})$.

Proof. Using the same notation as in the proof of lemma 3.7, since u_λ is a solution to (NL) we have that

$$\begin{aligned} \lambda \int_0^1 hf(u_\lambda)u_\lambda \, \mathrm{d}x &= \int_0^1 |u_\lambda'|^p \, \mathrm{d}x + c(u_\lambda(1))\phi_p(u_\lambda(1))u_\lambda(1) \\ &= pJ(u_\lambda) + p\lambda \int_0^1 hF(u_\lambda) \, \mathrm{d}x + c(u_\lambda(1))|u_\lambda(1)|^p \\ &\geqslant pc_2\lambda^{-rp} - p|\hat{F}|\|h\|_1\lambda \\ &\geqslant c_2\lambda^{-rp} \end{aligned} \tag{4.2}$$

for $\lambda \in (0, \hat{\lambda})$. The conclusion follows from the argument in the proof of lemma 3.7. □

LEMMA 4.8. *Let* u_λ *be as in theorem 4.3. There exist* $C_3 > 0$ *and* $\Lambda^* > 0$ *such that*

$$\|u_\lambda\|_W^p \leqslant C_3\lambda^{-rp}$$

for all $\lambda \in (0, \Lambda^*)$.

Proof. Since u_λ is a critical point of E and using remark 1.4,

$$\begin{aligned} \|u_\lambda\|_W^p &= pE(u_\lambda) + p\lambda \int_{\Omega^-} hF(u_\lambda) \, \mathrm{d}x + p\lambda \int_{\Omega^+} hF(u_\lambda) \, \mathrm{d}x - pg(u_\lambda(1)) \\ &\leqslant pE(u_\lambda) + p\lambda \int_{\Omega^-} hu_\lambda f(0) \, \mathrm{d}x + p\lambda \int_0^1 h\bigg(\frac{u_\lambda f(u_\lambda)}{\theta} + \frac{\tilde{\theta}}{\theta}\bigg) \, \mathrm{d}x \\ &\quad - p\lambda \int_{\Omega^-} h\bigg(\frac{u_\lambda f(0)}{\theta} + \frac{\tilde{\theta}}{\theta}\bigg) \, \mathrm{d}x - pg(u_\lambda(1)) \end{aligned}$$

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$$
\begin{aligned}
&= pE(u_\lambda) + p\lambda\left(1 - \frac{1}{\theta}\right)\int_{\Omega^-} hu_\lambda f(0)\,\mathrm{d}x - p\lambda\int_{\Omega^-} h\frac{\tilde{\theta}}{\theta}\,\mathrm{d}x \\
&\quad + \frac{p\lambda}{\theta}\int_0^1 hu_\lambda f(u_\lambda)\,\mathrm{d}x + p\lambda\frac{\tilde{\theta}}{\theta}\|h\|_1 - pg(u_\lambda(1)) \\
&\leqslant pE(u_\lambda) + p\lambda k|f(0)|\|h\|_1\|u_\lambda\|_W + \frac{p}{\theta}\|u_\lambda\|_W^p \\
&\quad + \frac{p}{\theta}c(u_\lambda(1))\phi_p(u_\lambda(1))u_\lambda(1) + p\lambda\frac{\tilde{\theta}}{\theta}\|h\|_1 - pg(u_\lambda(1)) \\
&= pE(u_\lambda) + p\lambda k|f(0)|\|h\|_1\|u_\lambda\|_W + \frac{p}{\theta}\|u\|_W^p + p\lambda\frac{\tilde{\theta}}{\theta}\|h\|_1 \\
&\quad + \frac{p}{\theta}(c(u_\lambda(1))|u_\lambda(1)|^p - \theta g(u_\lambda(1))) \\
&\leqslant pE(u_\lambda) + p\lambda k|f(0)|\|h\|_1\|u_\lambda\|_W + \frac{p}{\theta}\|u_\lambda\|_W^p + p\lambda\frac{\tilde{\theta}}{\theta}\|h\|_1 - p\frac{\tilde{\theta}_1}{\theta}. \qquad (4.3)
\end{aligned}
$$

Finally, if we choose $\lambda \leqslant (|\tilde{\theta}_1|/M_0)^{-1/rp}$, then $-\tilde{\theta}_1 \leqslant M_0\lambda^{-rp}$, so that

$$
\begin{aligned}
\|u_\lambda\|_W^p &\leqslant pE(u_\lambda) + p\lambda k|f(0)|\|h\|_1\|u_\lambda\|_W + \frac{p}{\theta}\|u_\lambda\|_W^p + p\lambda\frac{\tilde{\theta}}{\theta}\|h\|_1 - p\frac{\tilde{\theta}_1}{\theta} \\
&\leqslant pE(u_\lambda) + p\lambda k|f(0)|\|h\|_1\|u_\lambda\|_W + \frac{p}{\theta}\|u_\lambda\|_W^p + p\lambda\frac{\tilde{\theta}}{\theta}\|h\|_1 + p\frac{M_0\lambda^{-rp}}{\theta}.
\end{aligned}
\qquad (4.4)
$$

By the mountain pass characterization of u_λ,

$$
\begin{aligned}
E(u_\lambda) &\leqslant \max_{s\geqslant 0}\{E(sv_1)\} \\
&= \max_{s\geqslant 0}\{J(sv_1)\} \\
&\leqslant \max_{s\geqslant 0}\left\{\frac{s^p}{p} - \lambda A_1 s^{q+1}\hat{h}\|v_1\|_{q+1}^{q+1} + \lambda A_1\|h\|_1\right\},
\end{aligned}
\qquad (4.5)
$$

by (3.2).

Now, note that the inequality (4.5) is identical to the inequality (3.6), except that the functional J has now been replaced by the functional E. Hence, we may conclude from (4.5) that

$$
pE(u_\lambda) + p\lambda\frac{\tilde{\theta}}{\theta}\|h\|_1 \leqslant \tilde{c}_3\lambda^{-rp}, \qquad (4.6)
$$

where

$$
\tilde{c}_3 = \bar{K}^p - pA_1\hat{h}\bar{K}^{q+1}\|v_1\|_{q+1}^{q+1} + p\left(A_1 + \frac{\tilde{\theta}}{\theta}\right)\|h\|_1
$$

as in lemma 3.8.

Hence, following the proof of lemma 3.8, we may combine (4.3) and (4.6) to observe that

$$
a\|u_\lambda\|_W^p \leqslant b\lambda\|u_\lambda\|_W + \tilde{C}_3\lambda^{-rp}
$$

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for $a = 1 - p/\theta > 0$, $b = pk|f(0)|\|h\|_1 > 0$ and $\tilde{C}_3 = \tilde{c}_3 + pM_0/\theta$. Now, choosing $\lambda \leqslant a/2b$ and taking $C_3 = 2\tilde{C}_3/a$, we may follow the proof of lemma 3.8 to conclude that

$$\|u_\lambda\|_W^p \leqslant C_3 \lambda^{-rp}$$

for all $\lambda \in (0, \Lambda^*)$, where

$$\Lambda^* = \min\left\{1, \hat{\lambda}, \left(\frac{|\tilde{\theta}_1|}{M_0}\right)^{-1/rp}, \frac{\theta - p}{2\theta pk|f(0)|\|h\|_1}\right\}. \qquad \square$$

4.2.1. *Proof of theorem 1.2*

We again prove the theorem by contradiction. Suppose that there exists a sequence $\{(\lambda_j, u_{\lambda_j})\}_{j=1}^{\infty} \subset (0,1) \times C^1[0,1]$ of mountain pass solutions to (NL), as in theorem 4.3, such that $\lambda_j \to 0$ and $m(\{x \in (0,1) \mid u_{\lambda_j}(x) \leqslant 0\}) > 0$.

Let $w_j = u_{\lambda_j}/\|u_{\lambda_j}\|_\infty$. Then

$$\left.\begin{aligned} -(\phi_p(w_j'))' = \lambda h \frac{f(u_{\lambda_j})}{\|u_{\lambda_j}\|_\infty^{p-1}}, \quad x \in (0,1), \\ w_j(0) = 0, \\ \phi_p(w_j'(1)) + c(u_{\lambda_j}(1))\phi_p(w_j(1)) = 0, \end{aligned}\right\} \tag{4.7}$$

and, as in the proof of theorem 1.1, $w_j \to w$ strongly in $C^1[0,1]$ with w satisfying

$$\left.\begin{aligned} -(\phi_p(w'))' = z, \quad x \in (0,1), \\ w(0) = 0, \\ \phi_p(w'(1)) + c(L)\phi_p(w(1)) = 0, \end{aligned}\right\} \tag{4.8}$$

where $L = \lim_{j\to\infty} u_{\lambda_j}(1)$.

Since $\|w_j\|_\infty = 1$, $w \not\equiv 0$. Furthermore, since $z \geqslant 0$ and $c(L) > 0$, w is concave and satisfies the nonlinear boundary condition at $x = 1$ so that $w'(0) > 0$, $w'(1) < 0$, $w(1) > 0$, and $w > 0$ in $(0,1)$. The conclusion follows from the same argument as in the proof of theorem 1.1.

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