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Multiple correlation sequences not approximable by nilsequences

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Abstract. We show that there is a measure-preserving system $(X, \mathscr{B}, \mu, T)$ together with functions $F_0, F_1, F_2 \in L^\infty(\mu)$ such that the correlation sequence $C_{F_0,F_1,F_2}(n) = \int_X F_0 \cdot T^n F_1 \cdot T^{2n} F_2 \, d\mu$ is not an approximate integral combination of 2-step nilsequences.

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1. *Introduction*

Let $(X, \mathscr{B}, \mu, T)$ be a measure-preserving system and let $F_0, F_1, \ldots, F_k \in L^\infty(\mu)$. Motivated in large part by applications in combinatorics and, in particular, to questions about arithmetic progressions, there has been much interest in *multiple correlation sequences*

$$C_{F_0,\ldots,F_k}(n) := \int_X F_0 \cdot T^n F_1 \cdots T^{kn} F_k \, d\mu.$$

In fact, much more general types of correlation sequences in which the powers $T, T^2, \ldots, T^k$ appearing here are replaced by measure-preserving maps $T_1, \ldots, T_k$ have been studied, but here we restrict attention to this special form.

In the case $k = 1$, there is a very satisfactory spectral theory of such sequences and indeed one has

$$C_{F_0,F_1}(n) = \int_0^1 e^{-2\pi int} \, d\sigma(t) \tag{1.1}$$

for some complex Borel measure σ of bounded total variation. This follows from the Herglotz theorem on positive-definite sequences (which applies directly in the case $F_0 = \overline{F_1}$) and a depolarization identity.

It is natural to ask to what extent this generalizes to $k \geqslant 2$. In the words of Frantzikinakis [7]:

> 'Finding a formula analogous to (1.1), with the multiple correlation sequences in place of the single correlation sequences, is a problem of fundamental importance which has been in the mind of experts for several years. A satisfactory solution is going to give us new insights and significantly improve our ability to deal with multiple ergodic averages.'

A result of Bergelson *et al* [2] describes the structure of multiple correlation sequences up to an error in ℓ^1 or ℓ^2. To state their result, we need to recall the notion of a nilsequence.

Definition 1.1. (Nilsequence) Let $k \geqslant 1$ be an integer. A k-step nilsequence is a sequence $(\phi(g^n x_0))_{n\in\mathbb{Z}}$. Here, $\phi : G \to \mathbb{C}$ is a continuous function satisfying the automorphy (essentially equivalently, ϕ is a function on the *nilmanifold* G/Γ) condition $\phi(x\gamma) = \phi(x)$ for all $x \in G$ and all $\gamma \in \Gamma$, where G is a simply connected k-step nilpotent Lie group with discrete and cocompact subgroup Γ, and g, x_0 are fixed elements of G.

A careful discussion of this notion may be found in many places, for instance [2]. The following result is [2, Theorem 1.9].

THEOREM 1.2. *Suppose that* $(X, \mathscr{B}, \mu, T)$ *is a measure-preserving system and that* $F_0, F_1, \ldots, F_k \in L^\infty(\mu)$. *Suppose that* $\|F_i\|_\infty \leqslant 1$. *Then we have a decomposition*

$$C_{F_0,F_1,\ldots,F_k}(n) = a(n) + b(n),$$

where $a(n)$ *is a uniform limit of k-step nilsequences with* $\|a\|_\infty \leqslant 1$, *and b is small in the sense that*

$$\lim_{|I|\to\infty} \frac{1}{|I|} \sum_{n\in I} |b(n)| = 0$$

as I ranges over all subintervals of $\mathbb{N}$.

For applications involving the behaviour of correlation sequences at a sparse sequence of n, the error term here is too big. Frantzikinakis [7, Problem 1] has suggested, in the context of seeking a generalization of (1.1), that a variant of Theorem 1.1 should hold with an ℓ^∞ error term. Note that in (1.1), we have not just one nilsequence $(e^{2\pi i n t})_{n\in\mathbb{N}}$, but an integral combination of (1-step) nilsequences. Frantzikinakis's formulation generalizes this concept to higher-step nilsequences.

Definition 1.3. An integral combination of k-step nilsequences is a sequence of the form

$$a(n) = \int_M a_m(n)\, d\sigma(m).$$

Here, M is a compact metric space, σ is a complex Borel measure of bounded variation and the a_m are k-step nilsequences, where the map $m \mapsto a_m(n)$ is measurable for each n.

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Our main theorem states that, even in the case $k = 2$, one cannot hope for a version of Theorem 1.1 in which the error b is small in ℓ^∞, even if one allows a to be an integral combination of nilsequences.

THEOREM 1.4. *There is a measure-preserving system $(X, \mathscr{B}, \mu, T)$, functions $F_0, F_1, F_2 \in L^\infty(\mu)$ and an $\varepsilon > 0$ such that the correlation sequence*

$$C_{F_0,F_1,F_2}(n) := \int_X F_0 \cdot T^n F_1 \cdot T^{2n} F_2 \, d\mu$$

cannot be written as $a(n) + b(n)$, where $\|b\|_\infty \leqslant \varepsilon$ and a is an integral combination of 2-step nilsequences.

This theorem casts some serious doubt on the existence of a formula generalizing (1.1).

Theorem 1.2 does not provide a negative answer to [7, Problem 1], because Frantzikinakis allows the automorphic functions ϕ in the definition of a nilsequence to be merely Riemann-integrable, rather than continuous. He calls these *generalized* nilsequences. An explanation of why our construction does not allow one to establish an analogue of Theorem 1.2 for generalized nilsequences is given in Appendix A. Note, however, that the Riemann-integrable functions ϕ appearing in Appendix A are very singular and we certainly do not expect that the corresponding generalized nilsequences have any important role to play in the theory.

One reason for considering Riemann-integrable functions rather than just continuous ones is that there is a somewhat natural and well-studied class of nilsequences in which ϕ is not continuous, namely the bracket polynomial phases [3]. In this case, the corresponding ϕ have only mild discontinuities, and our argument adapts easily to show that Theorem 1.2 remains true even if one allows a to be an integral combination of this more general class of nilsequences. We sketch the argument at the end of §3.

A key motivation for Frantzikinakis in formulating [7, Problem 1] was that it provides a potential route to understanding Szemerédi's theorem with common difference in a sparse random set, a problem for which our current understanding is extremely incomplete for progressions of length 3 or longer (see [4] for recent progress). Whilst Theorem 1.2 seems to rule this out as a viable strategy, our example unfortunately does not give any new information about Szemerédi's theorem with common differences from a random set, which remains a tantalizing open problem.

Notation. Our notation is standard. We occasionally write $\mathbb{E}_{x \in A}$ for $1/|A| \sum_{x \in A}$, where A is a finite set. We write $[N] = \{1, 2, \ldots, N\}$ as usual, and sometimes we write $[0, N-1] = \{0, 1, 2, \ldots, N-1\}$. For real t, we write $e(t) = e^{2\pi i t}$.

2. *Outline of the argument*

Our argument is part deterministic and part random. It is random in the sense that we do not explicitly construct a system $(X, \mathscr{B}, \mu, T)$ and functions F_0, F_1, F_2 for which the correlation sequence $C_{F_0,F_1,F_2}(n)$ is not approximable by an integral combination of nilsequences, but rather we show there are too many possibilities for the correlation functions $C_{F_0,F_1,F_2}(n)$ for this to be so.

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To do this, we first explicitly construct a certain infinite sequence $\mathscr{S} \subset \mathbb{N}$ whose growth is slower than exponential in the sense that

$$\lim_{N\to\infty} \frac{|\mathscr{S}[N]|}{\log N} = \infty, \tag{2.1}$$

where $\mathscr{S}[N] := \#\{n \in \mathscr{S} : n \leqslant N\}$.

We show that for any choice of function $\eta : \mathscr{S} \to \{1, -\frac{1}{3}\}$ there is a system $(X, \mathscr{B}, \mu, T)$ and functions F_0, F_1, F_2 such that $C_{F_0,F_1,F_2}(n) = \eta(n)$ for $n \in \mathscr{S}$.

For a random choice of η, such a function will almost surely not be approximable by an integral combination of nilsequences. We give the details of this deduction, which uses nothing about $\mathscr{S}$ other than the growth property (2.1), in Proposition 3.1.

The heart of the argument, then, is the construction of the system $(X, \mathscr{B}, \mu, T)$ and the functions F_0, F_1, F_2, given $\eta : \mathscr{S} \to \{1, -\frac{1}{3}\}$. This is assembled from a sequence of finitary examples, via a (well-known) variant of Furstenberg's correspondence principle, and here the specific nature of $\mathscr{S}$ is critical.

The idea behind the construction of these finitary examples ultimately comes from coding theory, and in particular a construction of Yekhanin [9]. We will only need the most basic form of these ideas; for instance, we can replace all the finite-field theory in Yekhanin's work with the simple observation that the function $\psi : \mathbb{Z} \to \{-1, 1\}$ defined by $\psi(0) = 1$, $\psi(1) = \psi(2) = -1$, and periodic mod 3 has the property that

$$\psi(x)\psi(x+d)\psi(x+2d) = \begin{cases} \psi(x), & d \equiv 0 \pmod 3, \\ 1, & d \not\equiv 0 \pmod 3. \end{cases}$$

The idea of using Yekhanin's construction to give interesting examples in the additive combinatorics of higher-order correlations first arose in the finite field setting, in a joint work of the first author and Labib [5]. Those ideas have inspired the present work.

Remark. The sequence $\mathscr{S}$ we construct has $|\mathscr{S}[N]| \gg_\varepsilon (\log N)^{2-\varepsilon}$; we do not know or have any reasonable guess as to what might be the best possible growth rate for sequences with the desired properties.

3. *Entropy and nilsequences*

PROPOSITION 3.1. *Let $\mathscr{S}$ be an increasing sequence of natural numbers such that*

$$\lim_{N\to\infty} \frac{|\mathscr{S}[N]|}{\log N} = \infty. \tag{3.1}$$

Then there is a function $\eta : \mathscr{S} \to \{1, -\frac{1}{3}\}$ such that

$$\lim_{N\to\infty} \frac{1}{|\mathscr{S}[N]|} \sum_{n\in\mathscr{S}[N]} \eta(s)a(s) = 0 \tag{3.2}$$

for all nilsequences a.

Proof. The space of C^∞-functions on G/Γ is dense in the space of continuous functions; to approximate a continuous function by a smooth function, average with respect to a smooth kernel supported near the identity on G. It therefore suffices to verify (3.2) for

$a(n) = \phi(g^n x)$ with $\phi \in C^\infty(G/\Gamma)$. Now we use the fact that there is a map

$$\text{Complexity} : \{\text{smooth nilsequences}\} \to (0, \infty)$$

and a function $M : (0, \infty) \times (0, 1) \to (0, \infty)$ such that the set

$$\{a : \text{Complexity}(a) \leqslant C\}$$

can be covered by $N^{M(C,\varepsilon)}$ balls of radius ε in $\ell^\infty[N]$.

Results of this type were first observed by Frantzikinakis [6, Proposition 6.2], and in fact Proposition 3.1 and its proof are very closely related to [6, Theorem 1.4]. A discussion which gives what we need here is in the appendix of Altman [1] (note that $(g^n x_0)_{n\in\mathbb{Z}}$ is a particular example of a polynomial sequence as considered by Altman).

We pick the values of $\eta(n)$ at random, choosing $\eta(n) = -\frac{1}{3}$ with probability $\frac{3}{4}$ and $\eta(n) = 1$ with probability $\frac{1}{4}$, these choices being independent for different values of $n \in \mathscr{S}$. Then $\mathbb{E}\eta(n) = 0$. By well-known large deviation estimates (Hoeffding's inequality), for any fixed 1-bounded function b and for any distinct $n_1, \ldots, n_m$,

$$\mathbb{P}\bigg(\bigg|\sum_{i=1}^{m} \eta(n_i)b(n_i)\bigg| \geqslant t\bigg) \ll e^{-ct^2/m}, \tag{3.3}$$

where $c > 0$ is absolute.

Let $\omega : \mathbb{N} \to (0, \infty)$ be some function tending to infinity, to be specified later.

For each N, let E_N be the following event: for all 1-bounded nilsequences a of complexity $\leqslant \omega(N)$,

$$\bigg|\sum_{n\in\mathscr{S}[N]} \eta(n)a(n)\bigg| \leqslant \frac{1}{\omega(N)}|\mathscr{S}[N]|. \tag{3.4}$$

We estimate $\mathbb{P}(E_N)$ as follows. Pick some collection $\{a_1, \ldots, a_J\}$, $J \leqslant N^{M(\omega(N),1/2\omega(N))}$ of functions such that, for every 1-bounded nilsequence a of complexity at most $\omega(N)$, there is some a_i with $\|a - a_i\|_{\ell^\infty[N]} \leqslant 1/2\omega(N)$. Note that we do not need to assume that the a_i are nilsequences (though this could be arranged if desired) and they are automatically 2-bounded.

If we are not in E_N, there is some a_i such that

$$\bigg|\sum_{n\in\mathscr{S}[N]} \eta(n)a_i(n)\bigg| \geqslant \frac{1}{2\omega(N)}|\mathscr{S}[N]|. \tag{3.5}$$

By (3.3), the probability of (3.5) happening, for some fixed i, is bounded above by $e^{-c'|\mathscr{S}[N]|/\omega(N)^2}$ for some $c' > 0$. Summing over i, it follows that

$$\mathbb{P}(\neg E_N) \leqslant N^{M(\omega(N),1/2\omega(N))} e^{-c'|\mathscr{S}[N]|/\omega(N)^2}.$$

Choose ω (with $\omega(N) \to \infty$) so that

$$\frac{|\mathscr{S}[N]|}{\log N} > \frac{\omega(N)^2}{c'}(10 + M(\omega(N), 1/2\omega(N)))$$

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for N sufficiently large. (Here, of course, we have used the assumption on $\mathscr{S}$.) This then means that

$$\mathbb{P}(\neg E_N) \leqslant N^{-10}$$

for large N. In particular, $\sum_N \mathbb{P}(\neg E_N) < \infty$ which, by Borel–Cantelli, implies that almost surely only finitely many of the $\neg E_N$ occur. In particular, there is some particular choice of η such that (3.4) holds for all sufficiently large N. Because every nilsequence has finite complexity, this implies the result. □

Remark. There is, of course, nothing special about $\{1, -\frac{1}{3}\}$; any set containing both positive and negative numbers would do.

To conclude this section, let us quickly sketch how one could extend Proposition 3.1 to include the case where $a()$ is a bracket polynomial or a product of such (and hence not a nilsequence with a *continuous* automorphic function ϕ). Write $\chi_{\alpha,\beta}(n) := e(\alpha n\lfloor \beta n\rfloor)$. The key point is that the set of functions $\chi_{\alpha,\beta}(n)$, like the set of nilsequences of fixed complexity, has polynomially bounded covering numbers in $\ell^\infty[N]$.

To see why this is so, first note that $\chi_{\alpha,\beta}$ depends only on $\alpha (\mathrm{mod}\ 1)$, so we may assume $0 \leqslant \alpha < 1$. Next, replacing β by $\beta + k$ for $k \in \mathbb{Z}$ has the effect of multiplying by a quadratic phase $e(\gamma n^2)$ (where $\gamma = \alpha k$). However, the set of all quadratic phases $e(\gamma n^2)$ is covered by $\ll_\varepsilon N^2$ balls of radius ε in $\ell^\infty[N]$, because we may assume $0 \leqslant \gamma < 1$ and changing γ by ε/N^2 only changes $e(\gamma n^2)$ by $O(\varepsilon)$, uniformly for $n \leqslant N$.

It therefore suffices to show that the covering numbers of the set $\Xi := \{\chi_{\alpha,\beta} : 0 \leqslant \alpha, \beta < 1\}$ are polynomially bounded in $\ell^\infty[N]$. Now, restricted to $n \leqslant N$, there are only polynomially many functions $\lfloor \beta n\rfloor$ as β ranges in $[0, 1)$. Indeed, the map $\beta \mapsto (\lfloor \beta n\rfloor)_{n\leqslant N}$ is only discontinuous at the points where $\beta n \in \mathbb{Z}$ for some $n \leqslant N$, of which there are no more than N^2 with $0 \leqslant \beta < 1$. Thus, $\chi_{\alpha,\beta} = \chi_{\alpha,\beta'}$, with β' varying in a set of size N^2. Changing α by ε/N^2 only changes $\chi_{\alpha,\beta}(n)$ by $O(\varepsilon)$, uniformly for $n \leqslant N$. Therefore, the covering number of Ξ in $\ell^\infty[N]$ is $\ll_\varepsilon N^4$.

It follows immediately that, for fixed C, the set of functions of type $e(\sum_{i=1}^k \alpha_i n[\beta_i n])$, where $k \leqslant C$, is covered by $N^{M(C,\varepsilon)}$ balls of radius ε in $\ell^\infty[N]$. One could include various types of 1-step nilsequence or bracket polynomial and obtain a similar result.

Bounds on covering numbers were all we needed to know about nilsequences, and the rest of the argument goes over verbatim.

4. *The heart of the construction*

Define $\psi : \mathbb{Z} \to \{-1, 1\}$ to be the function with $\psi(0) = 1$, $\psi(1) = \psi(2) = -1$, and periodic mod 3. The crucial property of this function we use is the following, which is easily checked:

$$\psi(x)\psi(x+d)\psi(x+2d) = \psi(x) \tag{4.1}$$

if $d \equiv 0 \pmod 3$, and 1 if $d \not\equiv 0 \pmod 3$.

Fix, once and for all, a sequence $M_1 < M_2 < \cdots$ of positive integers such that:

(1) each M_i is a multiple of 3;

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(2) $\lim_{k\to\infty} k^{-2}\sum_{i=1}^{k}\log M_i = 0$;
(3) $\prod_{i=1}^{\infty}(1-(3/M_i)) = \gamma > 0$.

For instance, one could take $M_i = 3i^2$.

Define

$$\Omega_k := \{(x_1, x_2, \ldots) : 0 \leqslant x_i < M_i, x_{k+1} = x_{k+2} = \cdots = 0\}.$$

Later on we will need the technical variant

$$\tilde{\Omega}_k := \{(x_1, x_2, \ldots) : 0 \leqslant x_i < M_i - 3, x_{k+1} = x_{k+2} = \cdots = 0\}.$$

Define also Σ_k to consist of all sequences $(x_1, x_2, \ldots)$ with precisely two non-zero entries x_a, x_b, both of which equal 1, and with $x_{k+1} = x_{k+2} = \cdots = 0$. Write

$$\Omega := \bigcup_k \Omega_k, \quad \tilde{\Omega} := \bigcup_k \tilde{\Omega}_k, \quad \Sigma := \bigcup_k \Sigma_k.$$

We have a bijective map

$$\beta : \Omega \to \mathbb{Z}_{\geqslant 0}$$

defined by

$$\beta(x_1, x_2, \ldots) = x_1 + M_1 x_2 + M_1 M_2 x_3 + \cdots .$$

Let $\mathscr{S} = \beta(\Sigma)$. Thus, $\mathscr{S}$ consists of the sums of two distinct elements of the sequence $\{1, M_1, M_1M_2, M_1M_2M_3, \ldots\}$. We claim that $\mathscr{S}$ satisfies the hypothesis (3.1) of Lemma 3.1, that is, $\lim_{N\to\infty} |\mathscr{S}[N]|/\log N = \infty$.

To see this, let k be maximal so that $M_1 \cdots M_k \leqslant N/2$. Then $|\mathscr{S}[N]| \geqslant \binom{k}{2}$, whilst $\log(N/2) \leqslant \sum_{i=1}^{k+1}\log M_i$. Therefore, it is enough that

$$\lim_{k\to\infty} k^{-2}\sum_{i=1}^{k+1}\log M_i = 0,$$

which follows immediately from assumption 2 above.

We now apply Proposition 3.1 to get a function $\eta : \mathscr{S} \to \{1, -\frac{1}{3}\}$ satisfying (3.2). Define

$$\Sigma_k^+ := \{x \in \Sigma_k : \eta(\beta(x)) = 1\} \quad \text{and} \quad \Sigma_k^- := \left\{x \in \Sigma_k : \eta(\beta(x)) = -\tfrac{1}{3}\right\}.$$

Thus, $\Sigma_k = \Sigma_k^- \cup \Sigma_k^+$.

We introduce one more piece of notation. If $z \in \Sigma_k$ and if $x \in \Omega_k$, then we write

$$\sigma_z(x) := \sum_{i\in[k]:z_i=0} x_i.$$

Now we come to the crucial definition. Let $k \in \mathbb{N}$. For $x \in \Omega_k$ define

$$f_k(\beta(x)) = \prod_{z\in\Sigma_-^k} \psi(\sigma_z(x)). \tag{4.2}$$

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Note that $\beta(\Omega_k) = [0, N_k - 1]$, where

$$N_k := M_1 \cdots M_k, \tag{4.3}$$

and so f_k is a well-defined function on $[0, N_k - 1]$, taking values in $\{-1, 1\}$.

Remark. As mentioned previously, the idea of making this definition ultimately comes from coding theory (see [5] for further discussion). The important feature is that averages of $f_k(n) f_k(n + d) f_k(n + 2d)$ over n simplify substantially by using (4.1); we give the details of this in (4.5).

Define also the technical variant

$$\tilde{f}_k(\beta(x)) := 1_{x \in \tilde{\Omega}_k} f_k(\beta(x)). \tag{4.4}$$

Thus, $\tilde{f}_k$ is defined on $[0, N_k - 1]$ and takes values in $\{-1, 0, 1\}$. Extend both f_k and $\tilde{f}_k$ to functions on all of $\mathbb{Z}_{\geqslant 0}$ by defining $f_k(n) = \tilde{f}_k(n) = 0$ for $n \geqslant N_k$.

The following lemma is the heart of the argument. Here, recall that $\gamma > 0$ is just a positive constant (appearing in point 3 of the list of properties satisfied by the M_i).

LEMMA 4.1. *For $d \in \mathbb{Z}_{\geqslant 0}$, write*

$$S_k(d) := \frac{1}{N_k} \sum_{n \in [0, N_k - 1]} \tilde{f}_k(n) f_k(n + d) f_k(n + 2d).$$

Then for $d \in \mathscr{S}$ we have $\lim_{k \to \infty} S_k(d) = \gamma \eta(d)$.

Proof. Let $d \in \mathscr{S} = \beta(\Sigma)$. For k large enough, $d \in \beta(\Sigma_k)$, and we assume this is so in what follows.

From the definition of $\tilde{f}_k$, we see that the sum over n ranges over $n = \beta(x)$, $x \in \tilde{\Omega}_k$. Now for n of this form and for $d = \beta(y)$, $y \in \Sigma_k$, we have $x + y, x + 2y \in \Omega_k$ and, moreover,

$$\beta(x + y) = \beta(x) + \beta(y) = n + d,$$
$$\beta(x + 2y) = \beta(x) + 2\beta(y) = n + 2d.$$

Note that this 'lack of carries' was precisely the reason for defining the set $\tilde{\Omega}_k$. It follows that

$$S_k(d) = \mathbb{E}_{x \in \Omega_k} \tilde{f}_k(\beta(x)) f_k(\beta(x + y)) f_k(\beta(x + 2y)),$$

for $d = \beta(y)$, $y \in \Sigma_k$. Substituting the definitions (4.2), (4.4) of f_k, $\tilde{f}_k$ (and noting that σ_z is linear), we see that

$$S_k(d) = \mathbb{E}_{x \in \Omega_k} 1_{x \in \tilde{\Omega}_k} \prod_{z \in \Sigma_k^-} \psi(\sigma_z(x)) \psi(\sigma_z(x) + \sigma_z(y)) \psi(\sigma_z(x) + 2\sigma_z(y)).$$

From (4.1) it follows that

$$S_k(d) = \mathbb{E}_{x \in \Omega_k} 1_{x \in \tilde{\Omega}_k} \prod_{z \in \Sigma_k^- : \sigma_z(y) \equiv 0 \pmod 3} \psi(\sigma_z(x)).$$

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Now both y and z here are vectors with only two non-zero entries and so $\sigma_z(y)$ takes only the values 0, 1, 2 with $\sigma_z(y) = 0$ if and only if $y = z$. Therefore,

$$S_k(d) = \begin{cases} \mathbb{E}_{x\in\Omega_k} 1_{x\in\tilde{\Omega}_k} \psi(\sigma_y(x)) & \text{if } y \in \Sigma_k^-, \\ \mathbb{E}_{x\in\Omega_k} 1_{x\in\tilde{\Omega}_k} & \text{if } y \in \Sigma_k^+. \end{cases} \tag{4.5}$$

The second expression is

$$\mathbb{E}_{x\in\Omega_k} 1_{x\in\tilde{\Omega}_k} = \frac{|\tilde{\Omega}_k|}{|\Omega_k|} = \prod_{i=1}^{k} \left(1 - \frac{3}{M_i}\right) \to \gamma$$

as $k \to \infty$. The first expression in (4.5) may be written explicitly as

$$\frac{|\tilde{\Omega}_k|}{|\Omega_k|} \mathbb{E}_{x\in\tilde{\Omega}_k} \psi(x_1 + \cdots + \hat{x}_i + \cdots + \hat{x}_j + \cdots + x_k), \tag{4.6}$$

where y has non-zero coordinates at i, j and the hat means that $\hat{x}_i$ does not appear in the sum. Note, however, that $\tilde{\Omega}_k$ is a box with sidelengths $M_i - 3$, each of which is a multiple of 3. Therefore $x_1 + \cdots + \hat{x}_i + \cdots + \hat{x}_j + \cdots + x_k$ is uniformly distributed mod 3, as x ranges uniformly over $\tilde{\Omega}_k$, and the average in (4.6) is

$$\frac{|\tilde{\Omega}_k|}{|\Omega_k|} \cdot \left(-\frac{1}{3}\right) = -\frac{1}{3} \prod_{i=1}^{k} \left(1 - \frac{3}{M_i}\right) \to -\frac{\gamma}{3}.$$

This completes the proof. □

5. *Putting everything together*

Our final task is to build a measure-preserving system from the functions constructed in the last section. For this we need a slight variant of the usual Furstenberg correspondence principle, proven in a very similar way. An essentially equivalent statement may be found, for instance, in [8, Proposition 3.3].

PROPOSITION 5.1. *Let $A \subset \mathbb{R}$ be a finite set. Suppose that for each $k \in \mathbb{N}$ we have functions $f_{0,k}, \cdots, f_{r,k} : \mathbb{Z}_{\geqslant 0} \to A$, and that $(N_k)_{k=1}^{\infty}$ is an increasing sequence of positive integers. Then there is a measure-preserving system $(X, \mathscr{B}, \mu, T)$ and functions $F_0, F_1, \ldots, F_r \in L^\infty(\mu)$ such that the following is true: if $(d_1, \ldots, d_r)$ is a tuple of distinct positive integers such that*

$$S(d_1, \ldots, d_r) := \lim_{k\to\infty} \frac{1}{N_k} \sum_{n\in[0,N_k-1]} f_{0,k}(n) f_{1,k}(n + d_1) \cdots f_{r,k}(n + d_r)$$

exists, then

$$S(d_1, \ldots, d_r) = \int_X F_0 \cdot T^{d_1} F_1 \cdot T^{d_2} F_2 \cdots T^{d_r} F_r d\mu.$$

We apply this with the functions constructed in the last section, taking $r = 2$, $f_{0,k} := \tilde{f}_k$, $f_{1,k} = f_{2,k} = f_k$ and $N_k = M_1 \cdots M_k$ as before.

By Proposition 5.1 and Lemma 4.1, there is a measure-preserving system $(X, \mathscr{B}, \mu, T)$ together with functions $F_0, F_1, F_2 \in L^\infty(\mu)$ such that, writing $C_{F_0,F_1,F_2}(d) := \int_X F_0 \cdot$

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$T^d F_1 \cdot T^{2d} F_2 \, d\mu$, we have

$$C_{F_0,F_1,F_2}(d) = \eta(d) \quad \text{for } d \in \mathscr{S}. \tag{5.1}$$

(Note it is clearly possible to scale the F_i to remove γ factor appearing in Lemma 4.1.) We claim that it is impossible to write

$$C_{F_0,F_1,F_2}(n) = a(n) + b(n)$$

with a an integral combination of 2-step nilsequences and $\|b\|_\infty \leqslant 1/100$. Suppose that this were possible. Then, from (5.1) and the fact that η takes values in $\{1, -\frac{1}{3}\}$, we would have $(a(d) + b(d))\eta(d) \in \{\frac{1}{9}, 1\}$ for all $d \in \mathscr{S}$. However, $|b(d)\eta(d)| \leqslant 1/100$ and, therefore,

$$\Re(a(d)\eta(d)) \geqslant \frac{1}{9} - \frac{1}{100} > \frac{1}{10} \tag{5.2}$$

for all $d \in \mathscr{S}$.

Suppose that

$$a(n) = \int_M a_m(n) \, d\sigma(m).$$

Here, M is a compact metric space, σ is a complex Borel measure of bounded variation and the a_m are nilsequences, with the map $m \mapsto a_m(n)$ being in $L^\infty(\sigma)$.

Then (5.2) implies that

$$\bigg| \frac{1}{|\mathscr{S}[N]|} \sum_{n \in \mathscr{S}[N]} a(n)\eta(n) \bigg| \geqslant \frac{1}{10}.$$

On the other hand, we have

$$\bigg| \frac{1}{|\mathscr{S}[N]|} \sum_{n \in \mathscr{S}[N]} a(n)\eta(n) \bigg| \leqslant \int_M \bigg| \frac{1}{|\mathscr{S}[N]|} \sum_{n \in \mathscr{S}[N]} a_m(n)\eta(n) \bigg| \, d|\sigma|.$$

However, by the choice of η (Lemma 3.1) we have

$$\lim_{N\to\infty} \frac{1}{|\mathscr{S}[N]|} \sum_{n \in \mathscr{S}[N]} a_m(n)\eta(n) = 0$$

for all m. Therefore, by the dominated convergence theorem,

$$\lim_{N\to\infty} \int_M \bigg| \frac{1}{|\mathscr{S}[N]|} \sum_{n \in \mathscr{S}[N]} a_m(n)\eta(n) \bigg| \, d|\sigma| = 0.$$

Putting these statements together gives a contradiction, and this completes the proof of Theorem 1.2.

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A. *Appendix. Generalized nilsequences*

In this appendix, we explain why our example does not seem to give a negative solution to [7, Problem 1]. That is, we explain why our example (or similar ones) do not seem to be able to rule out the possibility that $C_{F_0,F_1,F_2}(n)$ is an approximate integral combination of *generalized* 2-step nilsequences, in which the automorphic function ϕ is allowed to be merely Riemann-integrable. In fact, our examples agree with 1-step generalized nilsequences on the crucial set $\mathscr{S}$.

Recall that $\mathscr{S} = \mathscr{A} \hat{+} \mathscr{A}$, where

$$\mathscr{A} = \{N_0, N_1, N_2, \ldots\} \quad \text{and} \quad N_i := \prod_{j \leqslant i} M_j$$

(thus, $N_0 = 1$, $N_1 = M_1$, $N_2 = M_1 M_2$ and so on). Here, $\mathscr{A} \hat{+} \mathscr{A}$ means the restricted sumset of $\mathscr{A}$ with itself, that is, the set of sums of two distinct elements of $\mathscr{A}$.

PROPOSITION A.1. *There is $\theta \in \mathbb{R}/\mathbb{Z}$ such that the following is true. Let $\eta : \mathscr{S} \to [-1, 1]$ be any function. Then there is a Riemann-integrable function $\phi : \mathbb{R}/\mathbb{Z} \to [-1, 1]$ such that $\phi(\theta n) = \eta(n)$ for all $n \in \mathscr{S}$.*

Proof. Set $\theta := \sum_{i=1}^{\infty} 1/N_i$. Because $M_1 < M_2 < \cdots$, we certainly have $M_j \geqslant j$. As a consequence, the usual proof that e is irrational may be adapted easily to show that θ is irrational: if $\theta = p/q$, then $\alpha := (M_1 \cdots M_q p)/q \in (1/q)\mathbb{Z}$, but, on the other hand, the fractional part of α satisfies

$$0 < \{\alpha\} = \frac{1}{M_{q+1}} + \frac{1}{M_{q+1}M_{q+2}} + \cdots \leqslant \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \cdots < \frac{1}{q}.$$

Now define $\phi : \mathbb{R}/\mathbb{Z} \to [-1, 1]$ as follows: $\phi(\theta n) = \eta(n)$ for all $n \in \mathscr{S}$, and $\phi(x) = 0$ if $x \notin \theta\mathscr{S}$. Because θ is irrational, this is a well-defined function.

We claim that it is Riemann-integrable, with integral zero. It is enough to show that for every $\varepsilon > 0$, there is some finite collection of intervals, of total length $< \varepsilon$, whose union covers $\theta\mathscr{S}$.

Note that for every j we have

$$\|\theta N_j\|_{\mathbb{R}/\mathbb{Z}} = \frac{1}{M_{j+1}} + \frac{1}{M_{j+1}M_{j+2}} + \cdots < \frac{1}{M_{j+1} - 1}. \tag{A.1}$$

Moreover, condition 3 in the definition of the M_j implies that

$$\limsup_{j \to \infty} \frac{M_j}{j} = \infty. \tag{A.2}$$

In particular, we may choose k so that $1/(M_{k+1} - 1) < \varepsilon/10k$, and by (A.1) it follows that

$$\|\theta N_j\|_{\mathbb{R}/\mathbb{Z}} < \frac{\varepsilon}{10k} \quad \text{for } j \geqslant k.$$

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It follows that

$$\theta\mathscr{A} \subseteq \{\theta N_0, \ldots, \theta N_{k-1}\} \cup I,$$

where $I = (-\varepsilon/10k, \varepsilon/10k) \subseteq \mathbb{R}/\mathbb{Z}$. Therefore,

$$\theta\mathscr{S} \subseteq \theta\mathscr{A} + \theta\mathscr{A} \subseteq \bigcup_{i,j<k} \{\theta(N_i + N_j)\} \cup \bigcup_{i<k} (\theta N_i + I) \cup (I + I),$$

which makes it clear that $\theta\mathscr{S}$ is contained in a finite union of intervals of length less than ε. □

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