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Bounded-Size Rules: The Barely Subcritical Regime

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Bounded-size rules (BSRs) are dynamic random graph processes which incorporate limited choice along with randomness in the evolution of the system. Typically one starts with the empty graph and at each stage two edges are chosen uniformly at random. One of the two edges is then placed into the system according to a decision rule based on the sizes of the components containing the four vertices. For bounded-size rules, all components of size greater than some fixed $K \geqslant 1$ are accorded the same treatment. Writing $\mathbf{BSR}(t)$ for the state of the system with $\lfloor nt/2 \rfloor$ edges, Spencer and Wormald [26] proved that for such rules, there exists a (rule-dependent) critical time t_c such that when $t < t_c$ the size of the largest component is of order $\log n$, while for $t > t_c$, the size of the largest component is of order n. In this work we obtain upper bounds (that hold with high probability) of order $n^{2\gamma} \log^4 n$, on the size of the largest component, at time instants $t_n = t_c - n^{-\gamma}$, where $\gamma \in (0, 1/4)$. This result for the barely subcritical regime forms a key ingredient in the study undertaken in [4], of the asymptotic dynamic behaviour of the process describing the vector of component sizes and associated complexity of the components for such random graph models in the critical scaling window. The proof uses a coupling of BSR processes with a certain family of inhomogeneous random graphs with vertices in the type space $\mathbb{R}_+ \times \mathcal{D}([0,\infty) : \mathbb{N}_0)$, where $\mathcal{D}([0,\infty) : \mathbb{N}_0)$ is the Skorokhod D-space of functions that are right continuous and have left limits, with values in the space of non-negative integers $\mathbb{N}_0$, equipped with the usual Skorokhod topology. The coupling construction also gives an alternative characterization (from the usual explosion time of the susceptibility function) of the critical time t_c for the emergence of the giant component in terms of the operator norm of integral operators on certain L^2 spaces.

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1. Introduction

The classical Erdős–Rényi random graph can be thought of as a dynamic random graph process on the vertex set $[n] := \{1, 2, \ldots, n\}$, where one starts with the empty graph $\mathbf{0}_n$ (the graph with n vertices and no edges) and at each discrete time step chooses an edge uniformly at random and places it in the configuration. Let $\mathbf{ER}^{(n)}(t)$ denote the state of the graph obtained after $\lfloor nt/2 \rfloor$ steps. For any graph $\mathbf{G}$, let $\mathcal{C}_i(\mathbf{G})$ denote the ith largest component and $|\mathcal{C}_i(\mathbf{G})|$ its size (number of vertices). Classical results [12, 11, 7] say that for fixed $t < 1$, the size of the largest component $|\mathcal{C}_1(\mathbf{ER}^{(n)}(t))|$ is $O(\log n)$, while for $t > 1$, $|\mathcal{C}_1(\mathbf{ER}^{(n)}(t))| \sim f(t)n$. Here $f(t) > 0$ is the survival probability of an associated supercritical branching process. For $t > 1$, the size of the second largest component $|\mathcal{C}_2(\mathbf{ER}^{(n)}(t))| = O(\log n)$. The largest component is often referred to as the giant component.

There have been several works aimed at understanding the nature and emergence of this giant as t transitions from below to above $t_c = 1$ [2, 17]. In recent years, motivated by a question of Achlioptas, there has been significant interest in investigating more general dynamical random graph models. The driving theme has been to understand the role of limited choice along with randomness in the evolution of the network, in particular the time and nature of the emergence of the giant component. The simplest such model that has been rigorously analysed, referred to as the Bohman–Frieze process, can be described as follows. Start with the empty graph at time $t = 0$. At each discrete time step, choose two edges e_1, e_2 uniformly at random amongst all pairs of ordered edges. Place edge $e_1 = (v_1, v_2)$ if both end points v_1, v_2 are isolated vertices (components of size one); otherwise use edge e_2.

Despite this conceptually simple modification of the standard Erdős–Rényi random graph process, a rigorous understanding of this process turns out to be non-trivial. Write $\mathbf{BF}^{(n)}(t)$ for the state of the system when we have placed $\lfloor nt/2 \rfloor$ edges. Bohman and Frieze [5, 6] showed that there exists a time $t > 1$ and $\varepsilon \in (0, 1)$ such that the size of the largest component $|\mathcal{C}_1(\mathbf{BF}^{(n)}(t))| = o(n^{\varepsilon})$. Thus this simple modification delays the time of emergence of a giant component.

Spencer and Wormald [26] substantially refined and extended these results to the context of all *bounded-size rules* (BSRs), which we now describe.

The bounded-size rule process $\{\mathbf{BSR}^{(n)}(t)\}_{t \geqslant 0}$. Fix $K \geqslant 0$; this will be a parameter in the construction of the process. Bounded-size rules treat all components of size greater than K in an identical fashion. Let $\Omega_K = \{1, 2, \ldots, K, \varpi\}$. Conceptually ϖ represents components of size greater than K. Given a graph $\mathbf{G}$ and a vertex $v \in \mathbf{G}$, write $\mathcal{C}_v(\mathbf{G})$ for the component that contains v. Define

$$c_{\mathbf{G}}(v) := \begin{cases} |\mathcal{C}_v(\mathbf{G})| & \text{if } |\mathcal{C}_v(\mathbf{G})| \leqslant K, \\ \varpi & \text{if } |\mathcal{C}_v(\mathbf{G})| > K. \end{cases} \tag{1.1}$$

For a quadruple of (not necessarily distinct) vertices v_1, v_2, v_3, v_4, write $\vec{v}$ for the ordered quadruple $\vec{v} = (v_1, v_2, v_3, v_4)$. Let $c_{\mathbf{G}}(\vec{v}) = (c_{\mathbf{G}}(v_1), c_{\mathbf{G}}(v_2), c_{\mathbf{G}}(v_3), c_{\mathbf{G}}(v_4))$. Fix $F \subseteq \Omega_K^4$. The set F will be another parameter in the construction of the process. The F-bounded-size rule(F-BSR) is defined as follows.

(a) At time $k = 0$ start with the empty graph $\mathbf{BSR}_0^{(n)} := \mathbf{0}_n$ on $[n]$ vertices.

(b) For $k \geqslant 0$, having constructed the graph $\mathbf{BSR}_k^{(n)}$, construct $\mathbf{BSR}_{k+1}^{(n)}$ as follows. Choose four vertices $\vec{v} = (v_1, v_2, v_3, v_4)$ uniformly at random amongst all n^4 possible quadruples and let $c_k(\vec{v}) = c_{\mathbf{BSR}_k}(\vec{v})$. If $c_k(\vec{v}) \in F$ then

$$\mathbf{BSR}_{k+1}^{(n)} = \mathbf{BSR}_k^{(n)} \cup (v_1, v_2)$$

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else

$$\mathbf{BSR}^{(n)}_{k+1} = \mathbf{BSR}^{(n)}_k \cup (v_3, v_4).$$

Mathematically it is more convenient to work with a formulation in which edges are added at Poissonian time instants rather than at fixed discrete times. More precisely, we will consider the following continuous-time version of the bounded-size rule process $\{\mathbf{BSR}^{(n)}(t)\}_{t\geqslant 0}$. For every quadruple of vertices $\vec{v} = (v_1, v_2, v_3, v_4) \in [n]^4$, let $\mathcal{P}_{\vec{v}}$ be a Poisson process with rate $1/2n^3$, independent between quadruples. Note that this implies that the rate of creation of edges is $n^4 \times 1/2n^3 = n/2$. Thus we have sped up time by a factor $n/2$ as in the above discrete-time construction. Start with $\mathbf{BSR}^{(n)}(0) = \mathbf{0}_n$. For any $t \geqslant 0$ at which there is a point in $\mathcal{P}_{\vec{v}}$ for a quadruple $\vec{v} \in [n]^4$, define

$$\mathbf{BSR}^{(n)}(t) = \begin{cases} \mathbf{BSR}^{(n)}(t-) \cup (v_1, v_2) & \text{if } c_{t-}(\vec{v}) \in F, \\ \mathbf{BSR}^{(n)}(t-) \cup (v_3, v_4) & \text{otherwise,} \end{cases} \tag{1.2}$$

where $c_{t-}(\vec{v}) = c_{\mathbf{BSR}(t-)}(\vec{v})$.

Two examples of such processes are the Erdős–Rényi process (where $K = 0$, $\Omega_K = \{\varpi\}$ and $F = \{(\varpi, \varpi, \varpi, \varpi)\}$) and the Bohman–Frieze process (where $K = 1$, $\Omega_K = \{1, \varpi\}$ and $F = \{(1, 1, j_3, j_4) : j_3, j_4 \in \Omega_K\}$). Spencer and Wormald [26] showed that every bounded-size rule exhibits a phase transition similar to the Erdős–Rényi random graph process. More precisely, write $\mathcal{C}_i^{(n)}(t)$ for the ith largest component in $\mathbf{BSR}^{(n)}(t)$, and $|\mathcal{C}_i^{(n)}(t)|$ for the size of this component. Define the *susceptibility* functions

$$\mathcal{S}_k(t) = \sum_{i=1}^{\infty} |\mathcal{C}_i^{(n)}(t)|^k, \quad \text{for } k = 1, 2, \dots. \tag{1.3}$$

Then [26] proves the following result.

Theorem 1.1 (Theorem 1.1 of [26]). *Fix $F \in \Omega_K^4$. Then for the random graph process associated with the F-BSR, there exists a deterministic monotonically increasing function $s_2(t)$ and a critical time t_c such that $\lim_{t\uparrow t_c} s_2(t) = \infty$ and*

$$\frac{\mathcal{S}_2(t)}{n} \xrightarrow{\mathbb{P}} s_2(t) \quad \text{as } n \to \infty, \quad \text{for all } t \in [0, t_c).$$

For fixed $t < t_c$, $|\mathcal{C}_1^{(n)}(t)| = O(\log n)$, while for $t > t_c$, $|\mathcal{C}_1^{(n)}(t)| = \Theta_P(n)$.

Here we use o, O, Θ in the usual manner. Given a sequence of random variables $\{\xi_n\}_{n\geqslant 1}$ and a function $f(n)$, we say that $\xi_n = O(f)$ if there is a constant C such that $\xi_n \leqslant Cf(n)$ with high probability (w.h.p.), and we say that $\xi_n = \Omega(f)$ if there is a constant C such that $\xi_n \geqslant Cf(n)$ w.h.p. Say that $\xi_n = \Theta(f)$ if $\xi_n = O(f)$ *and* $\xi_n = \Omega(f)$. In addition, we say that $\xi_n = o(f)$ if $\xi_n/f(n) \xrightarrow{\mathbb{P}} 0$.

Thus, as t transitions from less than t_c to greater than t_c, the size of the largest component jumps from size $O(\log n)$ to a giant component $\Theta(n)$. The aim of this work is to study the barely subcritical regime, *i.e.*, to analyse the behaviour of the size of the largest component at times $t = t_c - \varepsilon_n$ where $\varepsilon_n \to 0$. The main result is as follows.

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Theorem 1.2 (barely subcritical regime). *Fix $F \subset \Omega_K^4$ and $\gamma \in (0, 1/4)$. Then there exists $B \in (0, \infty)$ such that*

$$\mathbb{P}\left\{|\mathcal{C}_1^{(n)}(t)| \leqslant B\frac{(\log n)^4}{(t_c - t)^2},\ \forall t \leqslant t_c - n^{-\gamma}\right\} \to 1,$$

as $n \to \infty$.

As another consequence of our proofs, we obtain an alternative characterization of the critical time for a bounded-size rule given in Theorem 1.3 below. Let $\mathcal{X} = [0, \infty) \times \mathcal{D}([0, \infty) : \mathbb{N}_0)$, where $\mathcal{D}([0, \infty) : \mathbb{N}_0)$ is the Skorokhod D-space of functions that are right continuous and have left limits with values in the space of non-negative integers, equipped with the usual Skorokhod topology. Given a finite measure μ on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ and a measurable map $\kappa : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ satisfying $\int_{\mathcal{X}\times\mathcal{X}} \kappa^2(\mathbf{x}, \mathbf{y})\mu(d\mathbf{x})\mu(d\mathbf{y}) < \infty$, define the integral operator $\mathcal{K} : L^2(\mathcal{X}, \mu) \to L^2(\mathcal{X}, \mu)$ as

$$\mathcal{K}f(\mathbf{x}) = \int_{\mathcal{X}} \kappa(\mathbf{x}, \mathbf{y})f(\mathbf{y})\mu(d\mathbf{y}), \quad f \in L^2(\mathcal{X}, \mu), \quad \mathbf{x} \in \mathcal{X}.$$

We refer to κ as a kernel on $\mathcal{X} \times \mathcal{X}$ and $\mathcal{K}$ as the integral operator associated with (κ, μ). We will show the following result.

Theorem 1.3 (characterization of the critical time). *Fix $F \subset \Omega_K^4$. Then there exists a collection of F-dependent kernels $\{\kappa_t\}_{t\geqslant 0}$ on $\mathcal{X} \times \mathcal{X}$ and finite measures $\{\mu_t\}_{t\geqslant 0}$ on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ such that the integral operators $\mathcal{K}_t$ associated with (κ_t, μ_t), $t > 0$, have the property that the operator norms $\rho(t) = \|\mathcal{K}_t\|$ are continuous and strictly increasing in t. Furthermore, t_c is the unique time instant such that $\rho(t_c) = 1$.*

See Section 4.3 for a precise definition of κ_t and μ_t. We postpone the discussion of the connection between the integral operators in Theorem 1.3 and the BSR processes to Section 2.

1.1. Organization of the paper

The paper is organized as follows. In Section 2 we give a discussion of the main result. Section 3 collects some notation used in this work. In Section 4 we introduce and analyse certain inhomogeneous random graph processes associated with the BSR process. Finally, in Section 5 we complete the proofs of Theorems 1.2 and 1.3.

2. Discussion

We now give some background, open problems and general discussion of the results in this work.

2.1. Subcritical and supercritical random graphs

There has been considerable interest in understanding various properties of random graph models in the barely subcritical and supercritical regime. See, for example, [22], [14] and [15] for various results on complex network models such as the configuration model in the

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subcritical regime, [10] and [19] for structural properties of such graphs including mixing times of the nearly supercritical Erdős–Rényi random graph, [13] for an analysis of the Hamming cube near the critical regime, and [8] for an extensive analysis of a general class of the inhomogeneous random graphs. In recent years there has been significant effort in understanding a special (non-bounded-size) rule called the product rule [1], where one uses the edge that minimizes the product of the components at the end points. Simulations in [1] suggest that the nature of the emergence of the giant component is different from that observed for rules such as the Erdős–Rényi or Bohman–Frieze process. Conceptually such rules tend to be harder to analyse, since one needs to keep track not just of vertices in components up to size K for $K < \infty$ but for all K. There has been recent progress in understanding such rules [23]. The subcritical regime for such processes has been studied by Riordan and Warnke [24], who showed that there exists a critical time t_c such that, for $t > t_c$, the susceptibility function defined as in (1.3) satisfies $\mathcal{S}_2(t)/n \xrightarrow{\mathbb{P}} \infty$ as $n \to \infty$. Furthermore, there exist functions $\{f_k(\cdot)\}_{k\geqslant 1}$ such that, for $t < t_c$, the proportion of vertices in components of size k remains closely concentrated about $f_k(t)$ as $n \to \infty$.

For bounded-size rules, the only known results in the barely subcritical regime are in the context of the Bohman–Frieze process in [18] and [3] (see also [16], where scaling exponents for susceptibility functions in the Bohman–Frieze process were derived). Kang, Perkins and Spencer [18] showed that for fixed ε, the largest component in the Bohman–Frieze process at time $t_c - \varepsilon$ satisfies $|\mathcal{C}_1(\mathbf{BF}^{(n)}(t_c - \varepsilon))| = \Theta(\log n/\varepsilon^2)$. Bhamidi, Budhiraja and Wang [3] obtained upper bounds when $\varepsilon = \varepsilon_n$, where $\varepsilon_n \to 0$, and showed a result analogous to Theorem 1.2 using the special structure of certain differential equations associated with the Bohman–Frieze process.

2.2. Optimal scaling of the largest component in the subcritical regime

Note that for a fixed $\gamma \in (0, 1/4)$, at time $t_n = t_c - n^{-\gamma}$, Theorem 1.2 gives an upper bound of $Bn^{2\gamma}(\log n)^4$. One would expect, as suggested by Kang, Perkins and Spencer [18] for the special case of the Bohman–Frieze process, that $|\mathcal{C}_1(\mathbf{BF}^{(n)}(t_n))| = \Theta(n^{2\gamma} \log n)$. It would be interesting to see if the results in the paper can be refined to prove this result for general bounded-size rules. In fact, Sen [25], using the connection of the bounded-size rules to inhomogeneous random graphs established in this paper and in [3], proves a version of Theorem 1.2 with the improved bound $\log n/(t_c - t)^2$ for the Bohman–Frieze model for $t \leqslant t_c - n^{-\gamma}$ for $\gamma \leqslant 1/3$. Extending this to all bounded-size rules appears to be a challenging problem. The contribution of this paper is twofold.

(i) We establish bounds on the maximal component which are near-optimal up to $(\log n)^3$ factors all the way to $t = t_c - n^{-\gamma}$. This bound is good enough for one of our main goals, which is to study the limit behaviour in the critical scaling window (see Section 2.3).

(ii) We give a new characterization of the critical time as the first time when the operator norm of the kernel controlling the associated inhomogeneous random graph is one. In our future work we plan to use this characterization to study the supercritical regime as well.

2.3. Relevance to the critical regime

Theorem 1.2 plays a central role in the study of the asymptotics for bounded-size rules in the critical window. More precisely, in [4] we establish the critical scaling window for all bounded-size rules and show that all bounded-size rules lie in the same universality class as the Erdős–Rényi random graph process, in the sense that there is a $t_c > 0$ such that at time $t_c + \lambda/n^{1/3}$ for fixed $\lambda \in \mathbb{R}$, the sizes of maximal components $\mathcal{C}_n^{(i)}(t_c + \lambda/n^{1/3})$ scale like $n^{2/3}$ for any fixed $i \geqslant 1$, and furthermore the merging dynamics of these maximal components can be described by Aldous's multiplicative coalescent. In fact, in [4] we study the joint asymptotic behaviour of sizes and surplus of maximal components, and prove finite-dimensional convergence for multiple time instants $\{t_c + \lambda_j/n^{1/3}, j = 1, \ldots, m\}$, $m \geqslant 1$. Theorem 1.2 plays a crucial role in this analysis as follows. Direct proof of the result by exploring the graph in a breadth-first manner turns out to be infeasible since, unlike the Erdős–Rényi case, one does not have enough independence to prove scaling limits for the associated breadth-first walks. The strategy is to study the entire dynamics of the BSR processes in three steps.

(i) *Barely subcritical regime.* Study the behaviour of $|\mathcal{C}_1^{(n)}(t)|$, $\mathcal{S}_2(t)$ and $\mathcal{S}_3(t)$ at time $t = t_c - n^{-\gamma}$ for some fixed $\gamma \in (0, 1/3)$, and check that these quantities satisfy the regularity conditions for component sizes in Proposition 4 of [2]. To treat the asymptotics of surplus, additional regularity conditions need to be verified (see Theorem 5.1 and the proof of Lemma 7.2 in [4]).

(ii) *Modification of the BSR process.* Consider two 'modified processes' (upper bound and lower bound processes, $\mathbf{BSR}^+$ and $\mathbf{BSR}^-$) which start at time $t_c - n^{\gamma}$ with the same configuration of component sizes as the original process **BSR**, but then evolve as Erdős–Rényi random graph processes for $t \in [t_c - n^{-\gamma}, t_c + \lambda n^{-1/3}]$ and furthermore with high probability $\mathbf{BSR}^-(t) \subseteq \mathbf{BSR}(t) \subseteq \mathbf{BSR}^+(t)$ for all t in this interval. Using Proposition 4 of [2], the regularity conditions of the initial state as established in Step 1 and additional estimates, one can then prove the desired limit theorem for both of the 'modified processes'.

(iii) *Control errors.* By controlling errors between the original process and the two modified processes, we then show that the original process has the same limit behaviour as these two in the critical window.

The upper bound in Theorem 1.2 for $|\mathcal{C}_1^{(n)}(t)|$ in the subcritical regime plays a pivotal role in step (i) of the above program as well as in studying the susceptibility functions $\mathcal{S}_2(t)$ and $\mathcal{S}_3(t)$ at the entrance boundary of the critical scaling window. We refer the interested reader to [4] for details.

2.4. Connection to the discrete-time system

We use the continuous-time construction given in terms of Poisson processes, as opposed to the discrete-time construction, for mathematical convenience (see, *e.g.*, the various martingale estimates in Section 4). It is easy to show that in the continuous-time construction, by time $t_n = t_c - n^{-\gamma}$, the number of edges in the system is of order $nt_c - n^{1-\gamma} + O(\sqrt{n})$. Using this and the monotonicity of the process it is easy to check that Theorem 1.2 holds for the discrete-time version as well.

3. Notation

We collect some notation used throughout the rest of the paper. All unspecified limits are taken as $n \to \infty$. We use $\xrightarrow{\mathbb{P}}$ and $\xrightarrow{d}$ to denote convergence in probability and in distribution respectively. Given a sequence of events $\{E_n\}_{n\geqslant 1}$, we say that E_n occurs with high probability (w.h.p.) if $\mathbb{P}\{E_n\} \to 1$.

For a set S and a function $\mathbf{g} : S \to \mathbb{R}^k$, we write

$$\|\mathbf{g}\|_\infty = \sum_{i=1}^{k} \sup_{s\in S} |g_i(s)|,$$

where $\mathbf{g} = (g_1, \ldots, g_k)$. For a Polish space S, we let $\mathrm{BM}(S)$ denote the space of bounded measurable functions on S (equipped with the Borel sigma-field $\mathcal{B}(S)$). For a finite set S, $|S|$ denotes the number of elements in the set. $\mathbb{N}_0$ is the set of non-negative integers. For ease of notation, we shall often suppress the dependence on n and write, for example, $\mathbf{BSR}(t) = \mathbf{BSR}^{(n)}(t)$. Recall the Poisson processes $\mathcal{P}_{\vec{v}}$ used to construct $\mathbf{BSR}(\cdot)$ in the Introduction. Let $\{\mathcal{F}_t\}_{t\geqslant 0}$ be the associated filtration: $\mathcal{F}_t = \sigma\{\mathcal{P}_{\vec{v}}(s) : s \leqslant t, \vec{v} \in [n]^4\}$. We shall often deal with $\{\mathcal{F}_t\}$-semi-martingales $\{J(t)\}_{t\geqslant 0}$ of the form

$$dJ(t) := \alpha(t)\, dt + dM(t), \tag{3.1}$$

where M is a $\{\mathcal{F}_t\}$ local martingale. We shall denote $\alpha = \mathbf{d}(J)$ and $M = \mathbf{M}(J)$. For a local martingale $M(t)$, we shall write $\langle M, M\rangle(t)$ for the predictable quadratic variation process, namely the predictable process of bounded variation such that $M(t)^2 - \langle M, M\rangle(t)$ is a local martingale.

4. Inhomogeneous random graphs

Fix $K \geqslant 0$ and a general bounded-size rule $F \subseteq \Omega_K^4$ and recall that $\{\mathbf{BSR}(t)\}_{t\geqslant 0}$ denotes the continuous-time bounded-size rule process starting with the empty graph at $t = 0$. Note that the case $K = 0$ corresponds to the Erdős–Rényi random graph process, for which results such as Theorem 1.2 are well known. Thus, henceforth we shall assume $K \geqslant 1$. We begin in Section 4.1 by analysing the proportion of vertices in components of size i for $i \leqslant K$. As shown in [26], these converge to a set of deterministic functions which can be characterized as the unique solution of a set of differential equations. We will need precise rates of convergence for these proportions, which we establish in Lemma 4.2. We then study the evolution of components of size larger than K in Section 4.2. Finally, we relate the evolution of these components to an inhomogeneous random graph (IRG) model in Section 4.3.

4.1. Density of vertices in components of size bounded by K

Recall from (1.1) that $c_t(v) = c_{\mathbf{BSR}(t)}(v)$, for $v \in [n]$. For $t \geqslant 0$ and $i \in \Omega_K$, define

$$X_i(t) = |\{v \in [n] : c_t(v) = i\}| \quad \text{and} \quad \bar{x}_i(t) = X_i(t)/n. \tag{4.1}$$

Following [26], the first step in analysing bounded-size rules is understanding the evolution of $\bar{x}_i(\cdot)$ as functions of time as $n \to \infty$. Although [26] proves the convergence of $\bar{x}_i(t)$ as

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$n \to \infty$, we give a self-contained proof of this convergence with precise rates of convergence that will be needed in the proof of Theorem 1.2. Our notation follows [26]. Note that the BSR process changes values at the occurrence of points in the Poisson processes $\mathcal{P}_{\vec{v}}$, $\vec{v} \in [n]^4$. We call each such occurrence a 'round', and call a round *redundant* if the added edge in that round joins two vertices in the same component. Note that such rounds do not have any effect on component sizes or on the vector $\bar{\mathbf{x}}(t) = (\bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_K(t), \bar{x}_\varpi(t))$. We will in fact observe that such rounds are quite rare. We now describe the effect of non-redundant rounds on $\bar{\mathbf{x}}(\cdot)$. For $\vec{j} \in \Omega_K^4$ and $i \in \Omega_K$, write $\Delta(\vec{j}; i)$ for the change $\Delta X_i(t) := X_i(t) - X_i(t-)$ at an occurrence time t if the chosen quadruple $\vec{v} \in [n]^4$ satisfies $c_{t-}(\vec{v}) = \vec{j}$ and the round is not redundant. It is easy to check (see Section 2.1 of [26]) that when $\vec{j} = (j_1, j_2, j_3, j_4) \in F$,

$$\Delta(\vec{j}; i) = i \cdot (\mathbb{1}_{\{j_1+j_2=i\}} - \mathbb{1}_{\{j_1=i\}} - \mathbb{1}_{\{j_2=i\}}), \quad \text{for } 1 \leqslant i \leqslant K,$$
$$\Delta(\vec{j}; \varpi) = \mathbb{1}_{\{j_1+j_2=\varpi\}}(j_1 \mathbb{1}_{\{j_1 \leqslant K\}} + j_2 \mathbb{1}_{\{j_2 \leqslant K\}}),$$

with the convention $j_1 + j_2 = \varpi$ when the sum of j_1, j_2 is greater than K, and $j_1 + \varpi = \varpi + j_1 = \varpi$ for all $j_1 \in \Omega_K$. For $\vec{j} = (j_1, j_2, j_3, j_4) \in F^c$ one uses the second edge $\{v_3, v_4\}$ and the expressions for $\Delta(\vec{j}; i)$ are identical to the above, with (j_3, j_4) replacing (j_1, j_2). Note that the corresponding change in the density $\bar{x}_i(t) = X_i(t)/n$ is given by $\Delta(\vec{j}; i)/n$. For $\vec{j} \in \Omega_K^4$ and $t > 0$, write

$$\mathcal{Q}(t; \vec{j}) := \{\vec{v} \in [n]^4 : c_t(\vec{v}) = \vec{j}\}.$$

Then, for any set $E \subset \Omega_K^4$, the set $\{\vec{v} \in [n]^4 : c_t(\vec{v}) \in E\}$ has the partition

$$\{\vec{v} \in [n]^4 : c_t(\vec{v}) \in E\} = \cup_{\vec{j} \in E} \mathcal{Q}(t; \vec{j}).$$

Since each quadruple $\vec{v} \in [n]^4$ is selected according to the Poisson process $\mathcal{P}_{\vec{v}}$ with rate $1/2n^3$, the above description of the jumps of $X_i(\cdot)$ leads to a semi-martingale decomposition of $\bar{x}_i$ of the form (3.1) with

$$\begin{aligned}
\mathbf{d}(\bar{x}_i)(t) &= \sum_{\vec{v}: c_t(\vec{v}) \in F} \frac{\Delta(\vec{j}; i)}{2n^4} \mathbb{1}\{\mathcal{C}_{v_1}(t) \neq \mathcal{C}_{v_2}(t)\} + \sum_{\vec{v}: c_t(\vec{v}) \in F^c} \frac{\Delta(\vec{j}; i)}{2n^4} \mathbb{1}\{\mathcal{C}_{v_3}(t) \neq \mathcal{C}_{v_4}(t)\} \\
&= \sum_{\vec{v}: c_t(\vec{v}) \in F} \frac{\Delta(\vec{j}; i)}{2n^4} + \sum_{\vec{v}: c_t(\vec{v}) \in F^c} \frac{\Delta(\vec{j}; i)}{2n^4} - \sum_{\vec{v}: c_t(\vec{v}) \in F} \frac{\Delta(\vec{j}; i)}{2n^4} \mathbb{1}\{\mathcal{C}_{v_1}(t) = \mathcal{C}_{v_2}(t)\} \\
&\quad - \sum_{\vec{v}: c_t(\vec{v}) \in F^c} \frac{\Delta(\vec{j}; i)}{2n^4} \mathbb{1}\{\mathcal{C}_{v_3}(t) = \mathcal{C}_{v_4}(t)\},
\end{aligned} \tag{4.2}$$

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where $\mathcal{C}_v(t) := \mathcal{C}_v(\mathbf{BSR}(t))$ denotes the component containing v in $\mathbf{BSR}(t)$. Note that

$$\sum_{\vec{v}:c_t(\vec{v})\in F} \frac{\Delta(\vec{j};i)}{2n^4} = \sum_{\vec{j}\in F}\sum_{\vec{v}\in\mathcal{Q}(t;\vec{j})} \frac{\Delta(\vec{j};i)}{2n^4}$$
$$= \frac{1}{2n^4}\sum_{\vec{j}\in F}\Delta(\vec{j};i)\sum_{\vec{v}\in\mathcal{Q}(t;\vec{j})} 1$$
$$= \frac{1}{2}\sum_{\vec{j}\in F}\Delta(\vec{j};i)\bar{x}_{j_1}\bar{x}_{j_2}\bar{x}_{j_3}\bar{x}_{j_4}.$$

One can treat the term

$$\sum_{\vec{v}:c_t(\vec{v})\in F} \frac{\Delta(\vec{j};i)}{2n^4}$$

similarly. For $i \in \Omega_K$, define the functions $F_i : [0,1]^{K+1} \to \mathbb{R}$ mapping the vector

$$\mathbf{x} = (x_1, x_2, \ldots, x_K, x_\varpi) \in \mathbb{R}^{K+1}$$

to

$$F_i^x(\mathbf{x}) = \frac{1}{2}\sum_{\vec{j}\in F}\Delta(\vec{j};i)x_{j_1}x_{j_2}x_{j_3}x_{j_4} + \frac{1}{2}\sum_{\vec{j}\in F^c}\Delta(\vec{j};i)x_{j_1}x_{j_2}x_{j_3}x_{j_4}. \tag{4.3}$$

By (4.2) we have

$$|\mathbf{d}(\bar{x}_i)(t) - F_i^x(\bar{\mathbf{x}}(t))|$$
$$\leqslant \sum_{\vec{v}:c_t(\vec{v})\in F} \frac{\Delta(\vec{j};i)}{2n^4}\mathbb{1}\{\mathcal{C}_{v_1}(t) = \mathcal{C}_{v_2}(t)\} + \sum_{\vec{v}:c_t(\vec{v})\in F^c} \frac{\Delta(\vec{j};i)}{2n^4}\mathbb{1}\{\mathcal{C}_{v_3}(t) = \mathcal{C}_{v_4}(t)\}. \tag{4.4}$$

Note that if $\vec{j} \in F$ and $j_1 = j_2 = \varpi$, then $\Delta(\vec{j};i) = 0$ for all $i \in \Omega_K$, and thus, for the first term above,

$$\sum_{\vec{v}:c_t(\vec{v})\in F}\Delta(\vec{j};i)\mathbb{1}\{\mathcal{C}_{v_1}(t) = \mathcal{C}_{v_2}(t)\} \leqslant \sum_{\vec{v}:c_t(\vec{v})\in F}\Delta(\vec{j};i)\mathbb{1}\{\mathcal{C}_{v_1}(t) = \mathcal{C}_{v_2}(t), |\mathcal{C}_{v_1}(t)| \leqslant K\}$$
$$\leqslant 2K \cdot Kn^3,$$

where the last expression uses the fact that $|\Delta(\vec{j};i)| \leqslant 2K$ and a counting argument. By applying a similar argument to the $\sum_{\vec{v}:c_t(\vec{v})\in F^c}$ term, from (4.4) we have

$$|\mathbf{d}(\bar{x}_i)(t) - F_i^x(\bar{\mathbf{x}}(t))| \leqslant \frac{1}{2n^4}(2K^2n^3 + 2K^2n^3) = \frac{2K^2}{n}. \tag{4.5}$$

Note that $\bar{x}_1(0) = 1$, while $\bar{x}_i(0) = 0$ for other $i \in \Omega_K$. From equation (4.5) one is led to the following system of differential equations for $\mathbf{x}(t) := (x_j(t) : j \in \Omega_K)$:

$$x_i'(t) = F_i^x(\mathbf{x}(t)), \quad i \in \Omega_K, \quad t \geqslant 0, \quad \mathbf{x}(0) = (1, 0, \ldots, 0), \tag{4.6}$$

These equations were studied in [26] and the following result was established.

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Theorem 4.1 (Theorem 2.1 of [26]). *Equation* (4.6) *has a unique solution. For all* $i \in \Omega_K$ *and* $t > 0$, $x_i(t) > 0$. *Furthermore,* $\sum_{i\in\Omega_K} x_i(t) = 1$ *and* $\lim_{t\to\infty} x_\varpi(t) = 1$.

The paper [26] also showed that the functions $\bar{x}_i(t) \xrightarrow{\mathbb{P}} x_i(t)$ for each fixed $t \geqslant 0$. We will need precise rates of convergence for our proofs, for which we establish the following result.

Lemma 4.2. *Fix* $\delta \in (0, 1/2)$ *and* $T > 0$. *There exist* $C_1, C_2 \in (0, \infty)$ *such that, for all* n,

$$\mathbb{P}\left(\sup_{i\in\Omega_K}\sup_{s\in[0,T]} |\bar{x}_i(t) - x_i(t)| > n^{-\delta}\right) < C_1 \exp\left(-C_2 n^{1-2\delta}\right).$$

Proof. Note that $F_i^x(\cdot)$ is a Lipschitz function. Indeed, for $\mathbf{x}, \tilde{\mathbf{x}} \in [0, 1]^{K+1}$,

$$|F_i^x(\mathbf{x}) - F_i^x(\tilde{\mathbf{x}})| \leqslant 4K(K+1)^4 \sum_{i\in\Omega_K} |x_i - \tilde{x}_i| \leqslant 4K(K+1)^5 \sup_{i\in\Omega_K} |x_i - \tilde{x}_i|.$$

Write $D(t) := \sup_{i\in\Omega_K} |\bar{x}_i(t) - x_i(t)|$ and $M_i(t) := \mathbf{M}(\bar{x}_i)(t)$. Using (4.5), we get for all $i \in \Omega_K$ and $t \in [0, T]$

$$\begin{aligned}|\bar{x}_i(t) - x_i(t)| &\leqslant \int_0^t |F_i^x(\bar{\mathbf{x}}(s)) - F_i^x(\mathbf{x}(s))|ds + T \cdot \frac{2K^2}{n} + |M_i(t)| \\ &\leqslant 4K(K+1)^5 \int_0^t D(s)\, ds + T \cdot \frac{2K^2}{n} + |M_i(t)|.\end{aligned}$$

Taking $\sup_{i\in\Omega_K}$ on both sides and using Gronwall's lemma, we have

$$\sup_{t\in[0,T]} D(t) \leqslant \left(\sup_{i\in\Omega_K}\sup_{t\in[0,T]} |M_i(t)| + \frac{2TK^2}{n}\right) e^{4K(K+1)^5 T}.$$

Thus, for a suitable $d_1 \in (0, \infty)$,

$$\mathbb{P}\left\{\sup_{t\in[0,T]} D(t) > n^{-\delta}\right\} \leqslant \sum_{i\in\Omega_K} \mathbb{P}\left\{\sup_{t\in[0,T]} |M_i(t)| > d_1 n^{-\delta}\right\}. \tag{4.7}$$

To complete the proof we will use exponential tail bounds for martingales. From Theorem 5 in Section 4.13 of [20] we have that, for a square-integrable martingale M with $M(0) = 0$, $|\Delta M(t)| \leqslant c$ for all t, and $\langle M, M\rangle(T) \leqslant Q$, a.s., for some $c, Q \in (0, \infty)$,

$$\mathbb{P}\left\{\sup_{0\leqslant t\leqslant T} |M(t)| > \alpha\right\} \leqslant 2\exp\left\{-\sup_{\lambda>0}\left[\alpha\lambda - Q\psi(\lambda)\right]\right\}, \quad \text{for all } \alpha > 0,$$

where

$$\psi(\lambda) = \frac{e^{\lambda c} - 1 - \lambda c}{c^2}.$$

Optimizing over λ, we get the bound

$$\mathbb{P}\left\{\sup_{0\leqslant t\leqslant T} |M(t)| > \alpha\right\} \leqslant 2\exp\left\{-\frac{\alpha}{c}\log\left(1 + \frac{\alpha c}{Q}\right) + \left[\frac{\alpha}{c} - \frac{Q}{c^2}\log\left(1 + \frac{\alpha c}{Q}\right)\right]\right\}. \tag{4.8}$$

In our context, note that for any $i \in \Omega_K$, $|\Delta M_i(t)| = |\Delta \bar{x}_i(t)| \leqslant 2K/n$. Also, the total rate of jumps is bounded by $n^4 \cdot 1/2n^3$. Thus, for all $i \in \Omega_K$, the quadratic variation process

$$\langle M_i, M_i\rangle(T) \leqslant \int_0^T \left(\frac{2K}{n}\right)^2 \times \frac{n^4}{2n^3} dt = \frac{2K^2T}{n}.$$

Taking $\alpha = d_1 n^{-\delta}, Q = 2K^2T/n$ and $c = 2K/n$ in (4.8) completes the proof. □

4.2. Evolution of components of size larger than K

Let $\mathbf{BSR}^*(t)$ denote the subgraph of $\mathbf{BSR}(t)$ consisting of components of size greater than K. In this section we will focus on the dynamics and evolution of $\mathbf{BSR}^*(t)$. Note that $\mathbf{BSR}^*(0) = \emptyset$, *i.e.*, a graph with no vertices or edges. As time progresses, three distinct types of events affect the evolution $\mathbf{BSR}^*(t)$.

(1) *Immigration.* This occurs when two components of size $\leqslant K$ merge into a single component of size $> K$. We view the resulting component as a new immigrant into $\mathbf{BSR}^*(t)$. Note that the first component to appear in $\mathbf{BSR}^*(t)$ is an immigrant.

(2) *Attachments.* This occurs when a component of size $\leqslant K$ gets linked to a component of size larger than K. The former component enters $\mathbf{BSR}^*(t)$ via attaching itself to a component of size larger than K.

(3) *Edge formation.* Two distinct components of size larger than K merge into a single component via formation of an edge between these components. In this case, the vertex set of $\mathbf{BSR}^*(t)$ remains unchanged.

We now introduce some functions that describe the rate of occurrence for each of the three types of events. For $i_1, i_2 \in \Omega_K$ and $(i_1, i_2) \neq (\varpi, \varpi)$, let $n \cdot R_{i_1,i_2}(t)$ denote the rate at which two *distinct* components of size i_1, i_2 merge; for $(i_1, i_2) = (\varpi, \varpi)$, let $n \cdot R_{\varpi,\varpi}(t)$ denote the rate at which an edge is added to two vertices in components of size greater than K (these vertices may possibly be in the same component).

Thus, when $i_1 \neq i_2$, the rate $n \cdot R_{i_1,i_2}(t)$ is precisely

$$nR_{i_1,i_2}(t) =$$
$$(2n^3)^{-1}\left[\sum_{\substack{\vec{j}\in F\\ \{j_1,j_2\}=\{i_1,i_2\}}} X_{j_1}(t)X_{j_2}(t)X_{j_3}(t)X_{j_4}(t) + \sum_{\substack{\vec{j}\in F^c\\ \{j_3,j_4\}=\{i_1,i_2\}}} X_{j_1}(t)X_{j_2}(t)X_{j_3}(t)X_{j_4}(t)\right].$$

For $i_1, i_2 \in \Omega_K$, define $F^x_{i_1,i_2} : [0,1]^{K+1} \to \mathbb{R}$ as

$$F^x_{i_1,i_2}(\mathbf{x}) := \frac{1}{2}\Bigg(\sum_{\vec{j}\in F:\{j_1,j_2\}=\{i_1,i_2\}} x_{j_1}x_{j_2}x_{j_3}x_{j_4} + \sum_{\vec{j}\in F^c:\{j_3,j_4\}=\{i_1,i_2\}} x_{j_1}x_{j_2}x_{j_3}x_{j_4}\Bigg). \tag{4.9}$$

Thus when $i_1, i_2 \in \Omega_K$ and $i_1 \neq i_2$, we have $R_{i_1,i_2}(t) = F^x_{i_1,i_2}(\bar{\mathbf{x}}(t))$. The case $i_1 = i_2$ is more subtle due to the possibly redundant rounds that link vertices in the same component. When $i_1 = i_2 = \varpi$, by definition, we still have $R_{\varpi,\varpi}(t) = F^x_{\varpi,\varpi}(\bar{\mathbf{x}}(t))$. When $i_1 = i_2 = i \leqslant K$ the rate of redundant rounds can be bounded by $1/2n^3 \cdot Kn^3 \cdot 2 = K$, from which it follows that

$$|R_{i,i}(t) - F^x_{i,i}(\bar{\mathbf{x}}(t))| \leqslant \frac{K}{n}.$$

We now give expressions for the rates for the three types of events that govern the evolution of $\mathbf{BSR}^*(t)$. The convention followed for the rest of this section is that for $i_1, i_2 \in \Omega_K$, $i_1 + i_2 = \varpi$ when the sum of is greater than K, and $\varpi + i_1 = i_1 + \varpi = \varpi$ for all $i_1 \in \Omega_K$.

I. Immigrating vertices. For $1 \leqslant i \leqslant K$, write $n \cdot a_i^*(t)$ for the rate at which components of size $K + i$ immigrate into $\mathbf{BSR}^*(t)$ at time t. Using the above expressions for the rate of merger of components of various sizes, we have $a_i^*(t) = \sum_{1 \leqslant i_i, i_2 \leqslant K : i_1 + i_2 = K + i} R_{i_1, i_2}(t)$ and thus

$$\left| a_i^*(t) - \sum_{1 \leqslant i_i, i_2 \leqslant K : i_1 + i_2 = K + i} F_{i_1, i_2}^x(\bar{\mathbf{x}}(t)) \right| \leqslant \frac{K}{n}. \tag{4.10}$$

As before, the error is due to redundant rounds which can only occur for $i_1 = i_2 = (K+i)/2$ (and when $(K+i)/2$ is an integer). Now define functions $F_i^a : [0,1]^{K+1} \to \mathbb{R}_+$, and $a_i(\cdot) : [0, \infty) \to [0, \infty)$ by

$$F_i^a(\mathbf{x}) := \sum_{\substack{1 \leqslant i_i, i_2 \leqslant K, \\ i_1 + i_2 = K + i}} F_{i_1, i_2}^x(\mathbf{x}), \quad a_i(t) := F_i^a(\mathbf{x}(t)), \tag{4.11}$$

where $\mathbf{x}(t)$ is as in (4.6). Then (4.10) says that

$$\sup_{t \in [0, \infty)} |a_i^*(t) - F_i^a(\bar{\mathbf{x}}(t))| \leqslant K/n. \tag{4.12}$$

Note that for any $\delta < 1$, the error term in (4.12) is $o(n^{-\delta})$. Using this observation along with the Lipschitz property of F_{i_1, i_2}^x, we have from Lemma 4.2 that for any fixed $T > 0$ and $\delta < 1/2$,

$$\mathbb{P}\left(\sup_{1 \leqslant i \leqslant K} \sup_{s \in [0, T]} |a_i^*(t) - a_i(t)| > n^{-\delta} \right) \leqslant C_1 \exp(-C_2 n^{1 - 2\delta}). \tag{4.13}$$

The constants C_1, C_2 here may be different from those in Lemma 4.2, but for notational ease we use the same symbols.

II. Attachments. Fix $1 \leqslant i \leqslant K$ and a vertex v contained in a component in $\mathbf{BSR}^*(t)$. For $i \leqslant K$, let $c_i^*(t)$ denote the rate at which a component of size i attaches itself to the component of v through an edge connecting the former component to v. This rate can be calculated as follows. First note that the total rate of merger between a component of size i and a component in $\mathbf{BSR}^*(t)$ is $nR_{i,\varpi}^x(\bar{\mathbf{x}}(t)) = nF_{i,\varpi}^x(\bar{\mathbf{x}}(t))$. Since there are $X_\varpi(t)$ vertices in $\mathbf{BSR}^*(t)$, each of which has the same probability of being the vertex through which this attachment event happens, the rate at which a component of size i attaches to v is given by $nF_{i,\varpi}^x(\bar{\mathbf{x}}(t))/X_\varpi(t) = F_{i,\varpi}^x(\bar{\mathbf{x}}(t))/\bar{x}_\varpi(t)$. Since x_ϖ is a factor of $F_{i,\varpi}^x(\mathbf{x})$, $c_i^*(t)$ is a polynomial in $\bar{\mathbf{x}}(t)$. Define the functions $F_i^c : [0,1]^{K+1} \to \mathbb{R}$ and $c_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ as

$$F_i^c(\mathbf{x}) := F_{i,\varpi}^x(\mathbf{x})/x_\varpi, \quad c_i(t) := F_i^c(\mathbf{x}(t)). \tag{4.14}$$

Then $c_i^*(t) = F_i^c(\bar{\mathbf{x}}(t))$. Once again using Lemma 4.2, we get for any $\delta < 1/2$ and $T > 0$,

$$\mathbb{P}\left(\sup_{1 \leqslant i \leqslant K} \sup_{s \in [0, T]} |c_i^*(t) - c_i(t)| > n^{-\delta} \right) \leqslant C_1 \exp(-C_2 n^{1 - 2\delta}). \tag{4.15}$$

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III. Edge formation. Let $b^*(t)/n$ denote the rate of creation of an edge between two specified vertices $\{v_1, v_2\}$ in $\mathbf{BSR}^*(t)$. Note that the total rate of creation of an edge in $\mathbf{BSR}^*(t)$ is $nR_{\varpi,\varpi}(t) = nF^x_{\varpi,\varpi}(\bar{\mathbf{x}}(t))$. Since such an edge is equally likely to be between any of the $X^2_\varpi(t)$ pairs of vertices in $\mathbf{BSR}^*(t)$, we have that $b^*(t)/n = nF^x_{\varpi,\varpi}(\bar{\mathbf{x}}(t))/X^2_\varpi(t)$ and thus $b^*(t) = F^x_{\varpi,\varpi}(\bar{\mathbf{x}}(t))/x^2_\varpi(t)$. Define $F^b : [0,1]^{K+1} \to \mathbb{R}$ and $b(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ as

$$F^b(\mathbf{x}) := F^x_{\varpi,\varpi}(\mathbf{x})/x^2_\varpi \quad \text{and} \quad b(t) := F^b(\mathbf{x}(t)). \tag{4.16}$$

Once more it is clear that $F^b(\mathbf{x})$ is a polynomial and furthermore $b^*(t) = F^b(\bar{\mathbf{x}}(t))$, so by Lemma 4.2, for any $\delta < 1/2$ and $T > 0$,

$$\mathbb{P}\left(\sup_{s\in[0,T]} |b^*(t) - b(t)| > n^{-\delta}\right) \leqslant C_1 \exp(-C_2 n^{1-2\delta}). \tag{4.17}$$

Write $\mathbf{a}(t) := \{a_i(t)\}_{1\leqslant i\leqslant K}$ and $\mathbf{c}(t) := \{c_i(t)\}_{1\leqslant i\leqslant K}$. We refer to $(\mathbf{a}, b, \mathbf{c})$ as rate functions. In the proposition below we collect some properties of these rate functions. These properties are easy consequences of Theorem 4.1.

Proposition 4.3.

(a) *For all* $1 \leqslant i \leqslant K$ *and* $t > 0$, $b(t), a_i(t), c_i(t) > 0$.

(b) *We have*

$$\|\mathbf{a}\|_\infty := \sup_{t\geqslant 0} \sum_{i=1}^K a_i(t) \leqslant 1/2, \quad \|\mathbf{c}\|_\infty := \sup_{t\geqslant 0} \sum_{i=1}^K c_i(t) \leqslant 1/2, \quad \|b\|_\infty := \sup_{t\geqslant 0} b(t) \leqslant 1/2.$$

(c) $\lim_{t\to\infty} b(t) = 1/2$.

□

Proof. Part (a) follows from Theorem 4.1 and the definition of the functions. For (b) observe that

$$\sum_{i=1}^K a_i(t) = \sum_{i=1}^K F^a_i(\mathbf{x}(t)) \leqslant \frac{1}{2} \sum_{\vec{j}\in\Omega_K} x_{j_1}(t)x_{j_2}(t)x_{j_3}(t)x_{j_4}(t) = \frac{1}{2}\left[\sum_{i\in\Omega_K} x_i(t)\right]^4 = \frac{1}{2}.$$

Statements on $\|c\|_\infty, \|b\|_\infty$ follow similarly.

For (c), note that $F^x_{\varpi,\varpi}(\mathbf{x}) \geqslant x^4_\varpi/2$, since when all the four vertices selected are from components of size greater than K, two components of size greater than K will surely be linked. From Theorem 4.1, $\lim_{t\to\infty} x_\varpi(t) = 1$, and thus $\limsup_{t\to\infty} b(t) \geqslant x^2_\varpi(t)/2$. The result now follows on combining this with (b). □

4.3. Connection to inhomogeneous random graphs

In this section we describe the inhomogeneous random graph (IRG) models that have been studied extensively in [8], and then approximate $\mathbf{BSR}^*(t)$ by a special class of such models. We will in fact use a variation of the models in [8] which uses a suitable weight function to measure the volume of a component. We begin by defining the basic ingredients in this model. Let $\mathcal{X}$ be a Polish space, equipped with the Borel σ-field $\mathcal{B}(\mathcal{X})$. We shall sometimes refer to this as the *type space*. Let μ be a non-atomic finite measure on

$\mathcal{X}$ which we shall call the *type measure* on $\mathcal{X}$. A *kernel* will be a symmetric non-negative product measurable function $\kappa : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ and a *weight function* $\phi : \mathcal{X} \to \mathbb{R}$ will be a non-negative measurable function on $\mathcal{X}$. We call such a quadruple $\{\mathcal{X}, \mu, \kappa, \phi\}$ a *basic structure*.

The inhomogeneous random graph with weight function (IRG). Associated with a basic structure $\{\mathcal{X}, \mu, \kappa, \phi\}$, the IRG model $\mathbf{RG}^{(n)}(\mathcal{X}, \mu, \kappa, \phi)$ is a random graph described as follows.

(a) *Vertices.* The vertex set $\mathcal{V}$ of this random graph is a Poisson point process on the space $\mathcal{X}$ with intensity measure $n\mu$.
(b) *Edges.* An edge is added between vertices $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ with probability $1 \wedge (\kappa(\mathbf{x}, \mathbf{y}))/n$, independent across different pairs. This defines the random graph.
(c) *Volume.* The volume of a component $\mathcal{C}$ of $\mathbf{RG}^{(n)}(\mathcal{X}, \mu, \kappa, \phi)$ is defined as

$$\mathrm{vol}_\phi(\mathcal{C}) := \sum_{x \in \mathcal{C}} \phi(x). \tag{4.18}$$

For the rest of this section we take

$$\mathcal{X} := [0, \infty) \times \mathcal{W} \text{ where } \mathcal{W} := \mathcal{D}([0, \infty) : \mathbb{N}_0). \tag{4.19}$$

We first describe how, for each $t > 0$, $\mathbf{BSR}^*(t)$ can be identified with a random graph, denoted by $\Gamma(t)$, with vertex set in $\mathcal{X}$. Recall the three types of events governing the evolution of $\mathbf{BSR}^*(t)$, described in Section 4.2. Each component in $\mathbf{BSR}^*(t)$ contains at least one group of $K + i$ vertices, $i = 1, \ldots, K$, which appeared at some time instant $s \leqslant t$ in $\mathbf{BSR}^*(\cdot)$, as an immigrant. We denote the induced subgraph consisting of all immigrants of $\mathbf{BSR}^*(t)$ by $\mathrm{Imm}(t)$. Each component of $\mathrm{Imm}(t)$ corresponds to one immigrant. For a component $\mathcal{C} \subset \mathrm{Imm}(t)$, we let $\tau_{\mathcal{C}} \in (0, t]$ denote the instant this immigrant appears. For each such $\mathcal{C} \subset \mathrm{Imm}(t)$, we associate a path in $\mathcal{D}([0, \infty) : \mathbb{N}_0)$, denoted by $w_{\mathcal{C}}$, such that

$$w_{\mathcal{C}}(s) := \begin{cases} 0 & \text{when } s < \tau_{\mathcal{C}}, \\ |\mathcal{A}_{\mathcal{C}}(s)| & \text{when } \tau_{\mathcal{C}} \leqslant s \leqslant t, \\ w_{\mathcal{C}}(t) & \text{when } s > t, \end{cases}$$

where $\mathcal{A}_{\mathcal{C}}(s)$ is a graph defined as follows. Let $\mathrm{Att}(s)$ be the subgraph of $\mathbf{BSR}^*(s)$ obtained by deleting all the edges formed in the 'edge formation' stage. Thus $\mathrm{Att}(s)$ consists of only immigrants and attachments. Then $\mathcal{A}_{\mathcal{C}}(s)$ is defined to be the component of $\mathrm{Att}(s)$ such that $\mathcal{C} \subset \mathcal{A}_{\mathcal{C}}(s)$. Note that for $\mathcal{C}, \mathcal{C}' \subset \mathrm{Imm}(t)$ and $\mathcal{C} \neq \mathcal{C}'$, we must have $\mathcal{A}_{\mathcal{C}}(t) \neq \mathcal{A}_{\mathcal{C}'}(t)$.

Then $\{(\tau_{\mathcal{C}}, w_{\mathcal{C}}) : \mathcal{C} \subset \mathrm{Imm}(t)\}$ is a point process on $\mathcal{X}$. Now we define the random graph, $\Gamma(t)$, as follows. Let $\{(\tau_{\mathcal{C}}, w_{\mathcal{C}}) : \mathcal{C} \subset \mathrm{Imm}(t)\}$ be the vertex set. We form edges between any two vertices $(\tau_{\mathcal{C}}, w_{\mathcal{C}})$, $(\tau'_{\mathcal{C}}, w'_{\mathcal{C}})$ in $\Gamma(t)$ if the two subgraphs $\mathcal{A}_{\mathcal{C}}(t)$ and $\mathcal{A}_{\mathcal{C}'}(t)$ are directly linked by some edge in $\mathbf{BSR}^*(t)$.

The sizes of components in $\mathbf{BSR}^*(t)$ are equivalent to the *volumes* of components in $\Gamma(t)$, where the volume of a component is defined next. For $t > 0$, define the weight function $\phi_t : \mathcal{X} \to [0, \infty)$ as

$$\phi_t(\mathbf{x}) = \phi_t(s, w) = w(t), \quad \mathbf{x} = (s, w) \in \mathcal{X}. \tag{4.20}$$

Note that by construction there is a one-to-one correspondence between the components in $\mathbf{BSR}^*(t)$ and the components in $\Gamma(t)$. For a component $\mathcal{C}_0$ in $\mathbf{BSR}^*(t)$, let $I_{\mathcal{C}_0}$ denote the corresponding component in $\Gamma(t)$. Then we have

$$|\mathcal{C}_0| = \mathrm{vol}_{\phi_t}(I_{\mathcal{C}_0}). \tag{4.21}$$

Then the sizes of components in $\mathbf{BSR}^*(t)$ are equivalent to the volumes of components in $\Gamma(t)$.

We will now describe inhomogeneous random graph models that approximate $\Gamma(t)$ (and hence $\mathbf{BSR}^*(t)$). Given a set of non-negative continuous bounded functions $\boldsymbol{\alpha} = \{\alpha_i\}_{1\leqslant K}$, β and $\boldsymbol{\gamma} = \{\gamma_i\}_{1\leqslant i\leqslant K}$ on $[0,\infty)$, we construct, for each $t > 0$, type measures $\mu_t(\boldsymbol{\alpha},\beta,\boldsymbol{\gamma})$ and kernels $\kappa_t(\boldsymbol{\alpha},\beta,\boldsymbol{\gamma})$ on $\mathcal{X}$ as follows. For $i = 1,\ldots,K$ and $s \in [0,\infty)$, let $\tilde{\nu}_{s,i}$ denote the probability law on $\mathcal{D}([s,\infty) : \mathbb{N}_0)$ of the Markov process $\{\tilde{w}(r)\}_{r\in[s,\infty)}$ with infinitesimal generator

$$(\mathcal{A}(u)f)(k) = \sum_{j=1}^{K} k\gamma_j(u)(f(k+j) - f(k)), \quad f \in \mathrm{BM}(\mathbb{N}_0), \tag{4.22}$$

and initial condition $\tilde{w}(s) = K + i$. In words, this is a pure jump Markov process which starts at time s at state $K + i$ and then, at any time instant $u > s$, has jumps of size j at rate $\gamma_j(u)$. Let $\nu_{s,i}$ denote the probability law on $\mathcal{D}([0,\infty) : \mathbb{N}_0)$ of the stochastic process $\{w(r)\}_{r\in[0,\infty)}$, defined as

$$w(r) = \tilde{w}(r) \quad \text{for } r \geqslant s\,, \quad w(r) = 0 \quad \text{otherwise.} \tag{4.23}$$

Now define the finite measure $\mu_t(\boldsymbol{\alpha},\beta,\boldsymbol{\gamma}) \equiv \mu_t$ as

$$\int_{\mathcal{X}} f(\mathbf{x})\, d\mu_t(\mathbf{x}) = \sum_{i=1}^{K} \int_0^t \alpha_i(u) \left(\int_{\mathcal{W}} f(u,w)\nu_{u,i}(dw) \right) du, \quad f \in \mathrm{BM}(\mathcal{X}). \tag{4.24}$$

Next, define the kernel $\kappa_t(\boldsymbol{\alpha},\beta,\boldsymbol{\gamma}) \equiv \kappa_t$ on $\mathcal{X}\times\mathcal{X}$ as

$$\kappa_t(\mathbf{x},\mathbf{y}) = \kappa_t((s,w),(r,\tilde{w})) = \int_0^t w(u)\tilde{w}(u)\beta(u)\, du, \quad \mathbf{x} = (s,w), \mathbf{y} = (r,\tilde{w}) \in \mathcal{X}. \tag{4.25}$$

With the above choice of μ_t, κ_t and with weight function ϕ_t as in (4.20), we now construct the random graph $\mathbf{RG}^{(n)}(\mathcal{X},\mu_t,\kappa_t,\phi_t)$, which we denote by $\mathbf{RG}^{(n)}(\boldsymbol{\alpha},\beta,\boldsymbol{\gamma})(t)$. We will refer to the set of functions $(\boldsymbol{\alpha},\beta,\boldsymbol{\gamma})$ as above, as *rate functions*. These rate functions will typically arise as small perturbations of the functions $(\mathbf{a},b,\mathbf{c})$; thus in view of Proposition 4.3(b) it will suffice to consider $(\boldsymbol{\alpha},\beta,\boldsymbol{\gamma})$ such that $\max\{\|\boldsymbol{\alpha}\|_\infty,\|\beta\|_\infty,\|\boldsymbol{\gamma}\|_\infty\} < 1$. Throughout this work we will assume that all rate functions (and their perturbations) satisfy this bound.

The following key result says that for large n, $\Gamma(t)$ is suitably close to $\mathbf{RG}(\mathbf{a},b,\mathbf{c})(t)$, where $(\mathbf{a},b,\mathbf{c})$ are the rate functions introduced below (4.17). In order to state the result precisely, we extend the notion of a 'subgraph' to the setting with type space $\mathcal{X}$ and weight function ϕ. For $i = 1,2$, consider graphs $\mathbf{G}_i$, with finite vertex set $\mathcal{V}_i \subset \mathcal{X}$ and edge set $\mathcal{E}_i$. We say $\mathbf{G}_1$ is a subgraph of $\mathbf{G}_2$, and write $\mathbf{G}_1 \subset \mathbf{G}_2$ if there exists a one-to-one mapping $\Psi : \mathcal{V}_1 \to \mathcal{V}_2$ such that

(i) $\phi(\mathbf{x}) \leqslant \phi(\Psi(\mathbf{x}))$, for all $\mathbf{x} \in \mathcal{V}_1$,

(ii) $\{\mathbf{x}_1,\mathbf{x}_2\} \in \mathcal{E}_1$ implies $\{\Psi(\mathbf{x}_1),\Psi(\mathbf{x}_2)\} \in \mathcal{E}_2$.

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Lemma 4.4. *Fix $\delta \in (0,1/2)$ and let $\varepsilon_n = n^{-\delta}$, $n \geqslant 1$. For $t > 0$, define the set of functions*

$$\mathbf{a}^-(t) := \{(a_i(t) - \varepsilon_n) \vee 0\}_{1 \leqslant i \leqslant K}, \quad \mathbf{a}^+(t) := \{a_i(t) + \varepsilon_n\}_{1 \leqslant i \leqslant K}$$

and similarly $\mathbf{c}^-(t), \mathbf{c}^+(t)$ and $b^-(t), b^+(t)$. Define the inhomogeneous random graphs (IRG) with the above rate functions by

$$\mathbf{RG}^-(t) := \mathbf{RG}(\mathbf{a}^-, b^-, \mathbf{c}^-)(t), \quad \mathbf{RG}^+(t) := \mathbf{RG}(\mathbf{a}^+, b^+, \mathbf{c}^+)(t).$$

Then, for every $T > 0$ there exist $C_3, C_4 \in (0,\infty)$, such that, for all $t \in [0,T]$, there is a coupling of $\mathbf{RG}^-(t)$, $\mathbf{RG}^+(t)$ and $\Gamma(t)$ such that

$$\mathbb{P}\{\mathbf{RG}^-(t) \subset \Gamma(t) \subset \mathbf{RG}^+(t)\} > 1 - C_3 \exp\{-C_4 n^{1-2\delta}\}.$$

Proof. The coupling between the three graphs is done in a manner such that $\Gamma(t)$ is obtained by a suitable thinning of vertices and edges in $\mathbf{RG}^+(t)$ and $\mathbf{RG}^-(t)$ is obtained by a thinning of $\Gamma(t)$. We will only provide details of the first thinning step. We first construct the vertex sets $\mathcal{V}^+$ and $\mathcal{V}^*$ in $\mathbf{RG}^+(t)$ and $\Gamma(t)$ respectively.

Let $\mathcal{V}^+$ be a Poisson point process on $\mathcal{X}$ with intensity $n\mu_t^+$, where $\mu_t^+ := \mu_t(\mathbf{a}^+, b^+, \mathbf{c}^+)$. For a fixed realization of $\mathcal{V}^+$, let $(x_1^+, \ldots, x_N^+)$ denote the points in $\mathcal{V}^+$, with $x_i^+ = (s_i^+, w_i^+)$ and $0 < s_1^+ < s_2^+ \cdots s_N^+ < t$. Write $\mathbf{w}^+ = (w_1^+, \ldots, w_N^+)$. We now construct vertices in the corresponding realization of $\Gamma(t)$ (denoted by $\{x_1, \ldots, x_{N_0}\}$), along with the realizations of $\bar{x}_i(s)$, $i \in \Omega_K$, $0 \leqslant s \leqslant t$, which then defines $(a_j^*(s), b^*(s), c_j^*(s))$, $0 \leqslant s \leqslant t$, $j = 1, \ldots, K$, as functions of $\bar{\mathbf{x}}(s) = (\bar{x}_i(s))_{i \in \Omega_K}$ as in Section 4.2. For that, we will construct functions $w_j : [0,t] \to \mathbb{N}_0$, $1 \leqslant j \leqslant N$ and $\bar{x}_i : [0,t] \to [0,1]$, $i \in \Omega_K$. We will only describe the construction of $w_j, \bar{x}_i$ until the first time instant $s \in (0,t]$, when the property

$$a_j^*(s) \leqslant a_j^+(s), \quad b^*(s) \leqslant b^+(s), \quad c_j^*(s) \leqslant c_j^+(s) \quad \text{for all } 1 \leqslant k \leqslant K \tag{4.26}$$

is violated. Let σ denote the first time that (4.26) is violated, with σ taken to be t if the property holds for all $s \in [0,t]$. Subsequent to that time instant the construction can be done in any fashion that yields the correct probability law for $\Gamma(t)$. For simplicity, we assume henceforth that $\sigma = t$. After obtaining the functions $w_j, \bar{x}_i$, we set $x_i^* = (\tau_i^*, w_i^*)$, where τ_i^* is the first jump instant of w_i (taken to be $+\infty$ if there are no jumps) and $w_i^* \in \mathcal{D}([0,\infty) : \mathbb{N}_0)$ is defined as $w_i^*(s) = w_i(s)1_{[0,t]}(s) + w_i(t)1_{(t,\infty)}(s)$. The vertex set $\mathcal{V}^*$ is then defined as

$$\mathcal{V}^* = \{x_1, \ldots, x_{N_0}\} = \{x_i^* : \tau_i^* < t, i = 1, \ldots, N\}.$$

We now give the construction of $\mathbf{w}(s) = (w_1(s), \ldots, w_N(s))$ and $\bar{\mathbf{x}}(s)$ for $s \leqslant t$. Let $\{t_i\}_{i=1}^M$ denote $0 = t_0 < t_1 < t_2 < \cdots < t_M < t$, the collection of all time instants of jumps of $\{w_i^+\}_{i=1}^N$ before time t. Let i_k denote the coordinate of $\mathbf{w}^+$ that has a jump at time t_k, and denote the corresponding jump size by j_k. We set $\mathbf{w}(0) = 0$, $\bar{x}_i(0) = 0$ for $i \neq 1$ and $\bar{x}_1(0) = 1$. The construction proceeds recursively over the time intervals $(t_{k-1}, t_k]$, $k = 1, \ldots, M+1$, where $t_{M+1} = t$. Suppose that $(\mathbf{w}(s), \bar{\mathbf{x}}(s))$ have been defined for $s \in [0, t_{k-1}]$, for some $k \geqslant 1$. We now define these functions over the interval $(t_{k-1}, t_k]$.

Step 1: $s \in (t_{k-1}, t_k)$. Set $\mathbf{w}(s) = \mathbf{w}(t_{k-1})$. The values of $\bar{\mathbf{x}}(s)$ over the interval will be given as a realization of a jump process, for which jumps at time instant s occur at rates $n \cdot R_{i_1,i_2}(s)$, $i_1, i_2 \in \{1, \ldots, K\}$, $i_1 + i_2 \leqslant K$, where the function $R_{i_1,i_2}(s)$, given as a function of $\bar{\mathbf{x}}(s)$ is defined as in Section 4.2. A jump at time instant s, corresponding to the pair (i_1, i_2) as above, changes the values of $\bar{\mathbf{x}}$ to

$$\bar{x}_{i_1}(s) = \bar{x}_{i_1}(s-) - \frac{i_1}{n}, \quad \bar{x}_{i_2}(s) = \bar{x}_{i_2}(s-) - \frac{i_2}{n}, \quad \bar{x}_{i_1+i_2}(s) = \bar{x}_{i_1+i_2}(s-) + \frac{i_1 + i_2}{n}$$

if $i_1 \neq i_2$, and

$$\bar{x}_{i_1}(s) = \bar{x}_{i_1}(s-) - \frac{2i_1}{n}, \quad \bar{x}_{i_1+i_2}(s) = \bar{x}_{2i_1}(s-) + \frac{2i_1}{n}$$

if $i_1 = i_2$. The remaining $\bar{x}_i$ are unchanged. The values of $a_i^*(s)$, $b^*(s)$, $c_i^*(s)$ are determined accordingly.

Step 2: $s = t_k$. Recall that $w_{i_k}^+(t_k) - w_{i_k}^+(t_k-) = j_k$. We define $w_i(t_k) = w_i(t_k-)$ for all $i \neq i_k$. The values of $w_{i_k}(t_k)$ and $\bar{\mathbf{x}}(t_k)$ are determined as follows.

Case 1: $w_{i_k}^+(t_k-) = 0$. In this case $K + 1 \leqslant j_k \leqslant 2K$ and t_k is the first jump of $w_{i_k}^+$. Define for $1 \leqslant l \leqslant K$, $Q_k(l) := \sum_{i=1}^{l} R_{i,j_k-i}(t_k-)$, where by definition $R_{i,i'} = 0$ if $i' > K$. Note that $Q_k(K) = a_{j_k}^*(t_k-)$. We set $Q_k(0) = 0$. The values of $w_{i_k}(t_k)$ and $\bar{\mathbf{x}}(t_k)$ are now determined according to the realization of an independent uniform $[0, 1]$ random variable u_k as follows.

- If $u_k > Q_k(K)/a_{j_k}(t_k-)$, define $(w_{i_k}(t_k), \bar{\mathbf{x}}(t_k)) = (w_{i_k}(t_k-), \bar{\mathbf{x}}(t_k-))$.
- Otherwise, suppose $1 \leqslant l_k \leqslant K$ is such that

$$Q_k(l_k - 1) < u_k \leqslant Q_k(l_k).$$

Then define $w_{i_k}(t_k) = j_k$, $\bar{x}_{\varpi}(t_k) = \bar{x}_{\varpi}(t_k-) + j_k/n$ and

$$\bar{x}_{l_k}(t_k) = \bar{x}_{l_k}(t_k-) - \frac{l_k}{n}, \quad \bar{x}_{j_k-l_k}(t_k) = \bar{x}_{j_k-l_k}(t_k-) - \frac{j_k - l_k}{n}, \quad \text{if } l_k \neq j_k - l_k,$$
$$\bar{x}_{l_k}(t_k) = \bar{x}_{l_k}(t_k-) - \frac{2l_k}{n}, \quad \text{if } l_k = j_k - l_k.$$

The values of all other x_i processes at t_k stay the same as their values at t_k-.

Case 2: $w_{i_k}^+(t_k-) \neq 0$. In this case $1 \leqslant j_k \leqslant K$. Once again, the values of $w_{i_k}(t_k)$ and $\bar{\mathbf{x}}(t_k)$ are determined according to the realization of an independent uniform $[0, 1]$ random variable u_k as follows.

- If

$$u_k > \frac{w_{i_k}(t_k-) c_{j_k}^*(t_k-)}{w_{i_k}^+(t_k-) c_{j_k}^+(t_k-)},$$

define $(w_{i_k}(t_k), \bar{\mathbf{x}}(t_k)) = (w_{i_k}(t_k-), \bar{\mathbf{x}}(t_k-))$.
- Otherwise,

$$w_{i_k}(t_k) = w_{i_k}(t_k-) + j_k, \quad \bar{x}_{j_k}(t_k) = \bar{x}_{j_k}(t_k-) - \frac{j_k}{n}, \quad \bar{x}_{\varpi}(t_k) = \bar{x}_{\varpi}(t_k-) + \frac{j_k}{n},$$

and the value of all other x_i processes stay the same as their value at t_k-.

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This completes the construction of $(\mathbf{w}(s), \bar{\mathbf{x}}(s))$ for $s \in (t_{k-1}, t_k]$ and thus by this recursive procedure and our earlier discussion we obtain the vertex set

$$\mathcal{V}^* = \{x_1, \ldots, x_{N_0}\} = \{x_i^* : \tau_i^* < t, i = 1, \ldots, N\},$$

which will be used to construct $\Gamma(t)$.

Having constructed vertex sets $\mathcal{V}^+$ and $\mathcal{V}^*$, we now construct edges. For this we take realizations of independent uniform $[0,1]$ random variables $\{u_{i,j}\}_{1 \leqslant i < j < \infty}$ and construct edge between vertices x_i^+ and x_j^+ in $\mathcal{V}^+$ if

$$u_{i,j} \leqslant \frac{1}{n} \int_0^t b^+(s) w_i^+(s) w_j^+(s)\, ds.$$

This completes the construction of $\mathbf{RG}^+(t)$. Finally, construct an edge between x_i^* and x_j^* if both vertices are in $\mathcal{V}^*$ and

$$u_{i,j} \leqslant 1 - \exp\left\{ -\frac{1}{n} \int_0^t b^*(s) w_i(s) w_j(s)\, ds \right\}.$$

This completes the construction of $\Gamma(t)$. By construction, $\Gamma(t) \subset \mathbf{RG}(\mathbf{a}^+, b^+, \mathbf{c}^+)(t)$ on the set $\sigma = t$. Also, from (4.13), (4.17) and (4.15) it follows that $\mathbb{P}(\sigma < t) \leqslant C_3 \exp\{-C_4 n^{1-2\delta}\}$ for suitable constants C_3, C_4. The result follows. □

The following is an immediate corollary of Lemma 4.4.

Corollary 4.5. *Fix $T > 0$. Then, for $C_3, C_4 \in (0, \infty)$ and $t \in [0, T]$, a coupling of $\mathbf{RG}^-(t)$, $\mathbf{RG}^+(t)$ and $\Gamma(t)$ as in Lemma 4.4,*

$$\mathbb{P}\{\mathrm{vol}_{\phi_t}(\mathcal{I}_1^{\mathbf{RG}^-}(t)) \leqslant \mathrm{vol}_{\phi_t}(\mathcal{I}_1^{\Gamma}(t)) \leqslant \mathrm{vol}_{\phi_t}(\mathcal{I}_1^{\mathbf{RG}^+}(t))\} \geqslant 1 - C_3 \exp(-C_4 n^{1-2\delta}), \tag{4.27}$$

where $\mathcal{I}_1^{\Gamma}(t)$ denotes the component in $\Gamma(t)$ with the largest volume with respect to the weight function ϕ_t, and $\mathcal{I}_1^{\mathbf{RG}^-}(t)$, $\mathcal{I}_1^{\mathbf{RG}^+}(t)$ are defined similarly.

5. Proof of the main results

In this section we will complete the proof of Theorems 1.2 and 1.3. The proof of Theorem 1.3 is given in Section 5.4, while the proof of Theorem 1.2 is given in Section 5.5. Recall that Lemma 4.4 says that $\mathbf{BSR}^*$ can be approximated by $\mathbf{RG}(\mathbf{a}, b, \mathbf{c})$. Sections 5.2 and 5.3 analyse properties of integral operators associated with $\mathbf{RG}(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ for a general family of rate functions $(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$. We begin in Section 5.1 by presenting some results about an IRG model $\mathbf{RG}^{(n)}(\mathcal{X}, \mu, \kappa, \phi)$ on a general type space $\mathcal{X}$.

5.1. Preliminary lemmas

In this section we collect some results about the general IRG model $\mathbf{RG}^{(n)}(\mathcal{X}, \mu, \kappa, \phi)$. Let $\mathcal{K}$ be the integral operator associated with (κ, μ), as defined in the Introduction. Recall that the operator norm of $\mathcal{K}$, denoted as $\|\mathcal{K}\|$, is defined as

$$\|\mathcal{K}\| = \sup_{f \in L^2(\mathcal{X}, \mu), f \neq 0} \frac{\|\mathcal{K} f\|_2}{\|f\|_2}, \tag{5.1}$$

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where, for $f \in L^2(\mathcal{X}, \mu)$,

$$\|f\|_2 = \left(\int_{\mathcal{X}} |f(\mathbf{x})|^2 \mu(d\mathbf{x}) \right)^{1/2}.$$

Lemma 5.1. *Fix $(\mathcal{X}, \mu, \kappa, \phi)$. Denote the vertex set of $\mathbf{RG}^{(n)}(\mathcal{X}, \mu, \kappa, \phi) \equiv \mathbf{RG}^{(n)}$ by $\mathcal{P}_n$, which is a rate $n\mu$ Poisson point process on $\mathcal{X}$. Let $\mathcal{K}$ be the integral operator associated with (κ, μ). Suppose that $\|\mathcal{K}\| < 1$ and let $\Delta = 1 - \|\mathcal{K}\|$. Let $\mathcal{I}_1^{\mathbf{RG}}$ denote the component in $\mathbf{RG}^{(n)}$ with the largest volume (with respect to the weight function ϕ). Then the following hold.*

(i) *If $\|\phi\|_\infty < \infty$ and $\|\kappa\|_\infty < \infty$, then, for all $m \in \mathbb{N}$ and $D \in (0, \infty)$,*

$$\mathbb{P}\{\mathrm{vol}_\phi(\mathcal{I}_1^{\mathbf{RG}}) > m\} \leqslant 2nD \exp\{-C\Delta^2 m\} + \mathbb{P}(|\mathcal{P}_n| \geqslant nD), \tag{5.2}$$

where $C = (8\|\phi\|_\infty(1 + 3\|\kappa\|_\infty \mu(\mathcal{X})))^{-1}$.

(ii) *For $n \geqslant 1$, let $\Lambda_n \in \mathcal{B}(\mathcal{X})$ be such that*

$$g(n) := 8 \sup_{\mathbf{x} \in \Lambda_n} |\phi(\mathbf{x})| \left(1 + 3\mu(\mathcal{X}) \sup_{(\mathbf{x},\mathbf{y}) \in \Lambda_n \times \Lambda_n} |\kappa(\mathbf{x}, \mathbf{y})| \right) < \infty.$$

Then, for all $m \in \mathbb{N}$,

$$\mathbb{P}\{\mathrm{vol}_\phi(\mathcal{I}_1^{\mathbf{RG}}) > m\} \leqslant n\mu(\Lambda_n^c) + 2nD \exp(-\Delta^2 m/g(n)) + \mathbb{P}(|\mathcal{P}_n| \geqslant nD).$$

Proof. Part (i) was proved in [3] (see Lemmas 6.12 and 6.13 therein). We now prove (ii). Consider the truncated version of $\mathbf{RG}^{(n)}$ constructed using the basic structure $\{\mathcal{X}, \bar{\mu}, \bar{\kappa}, \bar{\phi}\}$, where $\bar{\mu} := \mu|_{\Lambda_n}$ (*i.e.*, the restriction of μ to Λ_n), $\bar{\kappa}(\mathbf{x}, \mathbf{y}) = \kappa(\mathbf{x}, \mathbf{y}) \mathbb{1}_{\Lambda_n}(\mathbf{x}) \mathbb{1}_{\Lambda_n}(\mathbf{y})$ and $\bar{\phi}(\mathbf{x}) = \phi(\mathbf{x}) \mathbb{1}_{\Lambda_n}(\mathbf{x})$. Note that $\|\bar{\kappa}\|_\infty < \infty$ and $\|\bar{\phi}\|_\infty < \infty$. Let $\bar{\mathcal{K}}$ denote the integral operator associated with $(\bar{\kappa}, \bar{\mu})$. Clearly $\|\bar{\mathcal{K}}\| \leqslant \|\mathcal{K}\|$ and thus $\bar{\Delta} = 1 - \|\bar{\mathcal{K}}\| \geqslant \Delta$. Consider the natural coupling between the truncated and original model by using the vertex set $\bar{\mathcal{P}}_n := \mathcal{P}_n \cap \Lambda_n$. Write $\bar{\mathcal{I}}_1^{\mathbf{RG}}$ for the component with the largest volume in the truncated model. Then we have

$$\begin{aligned}
\mathbb{P}\{\mathrm{vol}_\phi(\mathcal{I}_1^{\mathbf{RG}}) > m\} &\leqslant \mathbb{P}\{\mathcal{P}_n \cap \Lambda_n^c \neq \emptyset\} + \mathbb{P}\{\mathcal{P}_n \subset \Lambda_n, \mathrm{vol}_\phi(\mathcal{I}_1^{\mathbf{RG}}) > m\} \\
&= \mathbb{P}\{\mathcal{P}_n \cap \Lambda_n^c \neq \emptyset\} + \mathbb{P}\{\mathrm{vol}_\phi(\bar{\mathcal{I}}_1^{\mathbf{RG}}) > m\} \\
&\leqslant (1 - \exp\{-n\mu(\Lambda_n^c)\}) + 2nD \exp\{-\Delta^2 m/g(n)\} + \mathbb{P}(|\mathcal{P}_n| \geqslant nD),
\end{aligned}$$

where the last inequality follows from part (i) and the fact that $\Delta \leqslant \bar{\Delta}$. □

For the proof of the following elementary lemma we refer the reader to Lemma 6.5 in [3].

Lemma 5.2. *Let κ, κ' be kernels on a common finite measure space $(\mathcal{X}, \mu)$, with the associated integral operators $\mathcal{K}, \mathcal{K}'$ respectively. Then:*

(a) $\|\mathcal{K}\| \leqslant \|\kappa\|_{2,\mu} := \left(\int_{\mathcal{X} \times \mathcal{X}} \kappa^2(\mathbf{x}, \mathbf{y}) \mu(d\mathbf{x}) \mu(d\mathbf{y}) \right)^{1/2}$,

(b) *if $\kappa \leqslant \kappa'$, then $\|\mathcal{K}\| \leqslant \|\mathcal{K}'\|$,*

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(c) $\|\mathcal{K} - \mathcal{K}'\| \leqslant \|\bar{\mathcal{K}}\|$, *where* $\bar{\mathcal{K}}$ *is the integral operator associated with* $(|\kappa - \kappa'|, \mu)$.

For the proof of the following lemma we refer the reader to Lemma 6.18 of [3].

Lemma 5.3. *Let* $\tilde{\mu}, \mu$ *be two finite measures on the space* $\mathcal{X}$. *Assume* $\tilde{\mu} \ll \mu$ *and let* $g = d\tilde{\mu}/d\mu$ *be the Radon–Nikodym derivative. Let* $\tilde{\kappa}$ *be a kernel on* $\mathcal{X} \times \mathcal{X}$, *and define* κ *as*

$$\kappa(\mathbf{x}, \mathbf{y}) := \sqrt{g(\mathbf{x})g(\mathbf{y})}\tilde{\kappa}(\mathbf{x}, \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathcal{X}.$$

Let $\mathcal{K}$ (*resp.* $\tilde{\mathcal{K}}$) *denote the integral operator on* $L^2(\mathcal{X}, \mu)$ (*resp.* $L^2(\mathcal{X}, \tilde{\mu})$) *associated with* (κ, μ) (*resp.* $(\tilde{\kappa}, \tilde{\mu})$). $\|\mathcal{K}\|_{L^2(\mu)} = \|\tilde{\mathcal{K}}\|_{L^2(\tilde{\mu})}$, *where* $\|\mathcal{K}\|_{L^2(\mu)}$ (*resp.* $\|\tilde{\mathcal{K}}\|_{L^2(\tilde{\mu})}$) *is the norm of the operator* $\mathcal{K}$ (*resp.* $\tilde{\mathcal{K}}$) *on* $L^2(\mu)$ (*resp.* $L^2(\tilde{\mu})$).

We end this section with a lemma drawing a connection between the Yule process and the pure jump Markov processes with distribution $\nu_{s,i}$ that arose in the construction of the inhomogeneous random graphs $\mathbf{RG}(\boldsymbol{\alpha}, \beta, \gamma)$: see (4.23). Fix $j \geqslant 1$ and recall that a rate one Yule process starting at time $t = 0$ with j individuals is a pure birth Markov process $Y(t)$ with $Y(0) = j$ and the rate of going from state i to $i+1$ given by $\lambda(i) := i$. Also recall from (4.19) that $\mathcal{W}$ denotes the Skorokhod space $\mathcal{W} := \mathcal{D}([0, \infty) : \mathbb{N}_0)$.

Lemma 5.4. *Fix* $1 \leqslant i \leqslant K$ *and* $s \geqslant 0$ *and rate functions* $\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}$. *Let* $\{w(t)\}_{t \geqslant 0}$ *be a pure jump Markov process with law* $\nu_{s,i} := \nu_{s,i}(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ *as in* (4.23).

(i) *The process* $w^*(t) := w(t/K\|\boldsymbol{\gamma}\|_\infty)/K$ *can be stochastically dominated by a Yule process* $Y(\cdot)$ *starting with two particles* (*i.e.,* $Y(0) = 2$).

(ii) *Fix* $t > 0$, $s \in [0, t]$ *and* $1 \leqslant i \leqslant K$. *Then we have*

$$\int_{\mathcal{W}} [w(t)]^2 \nu_{s,i}(dw) \leqslant 6K^2 e^{2tK\|\boldsymbol{\gamma}\|_\infty},$$

and for any $A > 0$ *we have*

$$\nu_{s,i}(w(t) > A) \leqslant 2(1 - e^{-tK\|\boldsymbol{\gamma}\|_\infty})^{A/2K}.$$

Proof. **(i)** Note that under the law $\nu_{s,i}$, the process w satisfies $w(u) = 0$ for $u < s$ and $w(s) = K + i \leqslant 2K$. Further, for times $t > s$, by (4.22), the jumps of the w can be bounded as $\Delta w(t) := w(t) - w(t-) \leqslant K$ at rate at most $w(t)\|\boldsymbol{\gamma}\|_\infty$. The process $w^*(\cdot)$ is obtained by rescaling time and space for the process $w(\cdot)$. This is once again a pure jump Markov process with jump sizes $\Delta w^*(t) \leqslant 1$ which happen at rate at most one. Further, $w^*(0) \leqslant 2$. This immediately implies that this process is stochastically dominated by a Yule process with $Y(0) = 2$. This completes the proof.

(ii) We will use the result in part (i). Note that a Yule process starting with two individuals at time $t = 0$ has the same distribution as the sum of two independent Yule processes $\{Y_1(t)\}_{t \geqslant 0}$ and $\{Y_2(t)\}_{t \geqslant 0}$ with $Y_1(0) = Y_2(0) = 1$. Now fix $t > 0$, $s \leqslant t$ and $1 \leqslant i \leqslant K$. Let $w(\cdot)$ have distribution $\nu_{s,i}$. From (i) we have

$$w(t) \leqslant_d K \cdot (Y_1(tK\|\boldsymbol{\gamma}\|_\infty) + Y_2(tK\|\boldsymbol{\gamma}\|_\infty)). \tag{5.3}$$

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For simplicity write $X_1 = Y_1(tK\|\gamma\|_\infty)$ and $X_2 = Y_2(tK\|\gamma\|_\infty)$. Well-known results about Yule processes [21, Chapter 2] say that the random variables X_1 and X_2 have a geometric distribution with $p := e^{-tK\|\gamma\|_\infty}$. The first bound in (ii) follows from the geometric distribution and the fact

$$\int_{\mathcal{W}} [w(t)]^2 \nu_{s,i}(dw) \leqslant K^2 \mathbb{E}[(X_1 + X_2)^2].$$

The second bound in (ii) follows from

$$\nu_{s,i}(\{w(t) > A\}) \leqslant 2\mathbb{P}\{X_1 > A/2K\}.$$

This completes the proof. □

5.2. Some perturbation estimates for $\mathbf{RG}(\mathbf{a}, b, \mathbf{c})$

Recall that Lemma 4.4 coupled the evolution of $\Gamma(t)$ (equivalently $\mathbf{BSR}^*(t)$) with two inhomogeneous random graphs $\mathbf{RG}(\mathbf{a}^+, b, \mathbf{c}^+)(t)$ and $\mathbf{RG}(\mathbf{a}^-, b, \mathbf{c}^-)(t)$, which can be considered as perturbations of $\mathbf{RG}(\mathbf{a}, b, \mathbf{c})(t)$. The aim of this section is to understand the effect of such perturbations on the associated operator norms. Throughout this section $\mathcal{X}$ and ϕ_t are as in (4.19) and (4.20), respectively. Given the basic structure $\{\mathcal{X}, \mu_t, \kappa_t, \phi_t\}$, $t > 0$, associated with rate functions $(\boldsymbol{\alpha}, \beta, \gamma)$, we denote the norm of the integral operator $\mathcal{K}_t$ associated with (κ_t, ϕ_t) as $\rho_t(\boldsymbol{\alpha}, \beta, \gamma)$.

The following proposition, which is the main result of this section, studies the effect of perturbations of $(\boldsymbol{\alpha}, \beta, \gamma)$ on this norm. For a K-dimensional vector $\mathbf{v} = (v_1, \ldots, v_K)$ and a scalar θ, $\mathbf{v} + \theta$ and $(\mathbf{v} + \theta)^+$ denote the vectors

$$(v_1 + \theta, \ldots, v_K + \theta) \quad \text{and} \quad ((v_1 + \theta)^+, \ldots, (v_K + \theta)^+),$$

respectively.

Proposition 5.5. *For $\varepsilon > 0$, let*

$$\rho_t^+ = \rho_t(\boldsymbol{\alpha} + \varepsilon, \beta + \varepsilon, \gamma + \varepsilon) \quad \textit{and} \quad \rho_t^- = \rho_t((\boldsymbol{\alpha} - \varepsilon)^+, (\beta - \varepsilon)^+, (\gamma - \varepsilon)^+),$$

where $(\boldsymbol{\alpha}, \beta, \gamma)$ are rate functions. Assume that

$$\max\{\|\boldsymbol{\alpha} + \varepsilon\|_\infty, \|\beta + \varepsilon\|_\infty, \|\gamma + \varepsilon\|_\infty\} < 1.$$

For every $T > 0$, there is a $C_5 \in (0, \infty)$ such that, for all $\varepsilon > 0$ and $t \in [0, T]$,

$$\max\{|\rho_t - \rho_t^+|, |\rho_t - \rho_t^-|\} \leqslant C_5 \sqrt{\varepsilon} \cdot (-\log \varepsilon)^2.$$

□

The proof of Proposition 5.5 relies on Lemmas 5.6–5.10 below, and is given at the end of the section. We analyse the effect of perturbation of β, $\boldsymbol{\alpha}$ and γ separately in Lemmas 5.6, 5.8 and 5.10, respectively.

Lemma 5.6 (perturbations of β). *Let $(\boldsymbol{\alpha}, \beta, \gamma)$ be rate functions and β^ε be be a non-negative function on $[0, \infty)$ with $\sup_{0 \leqslant s < \infty} |\beta^\varepsilon(s) - \beta(s)| \leqslant \varepsilon$. Then*

$$|\rho_t(\boldsymbol{\alpha}, \beta, \gamma) - \rho_t(\boldsymbol{\alpha}, \beta^\varepsilon, \gamma)| \leqslant C\varepsilon,$$

where $C = 6t^2K^3\|\boldsymbol{\alpha}\|_\infty e^{2t\|\gamma\|_\infty}$.

Proof. Let (μ_t, κ_t) be the type measure and kernel associated with $(\boldsymbol{\alpha}, \beta, \gamma)$. Note that a perturbation in β only affects the kernel κ_t and not μ_t. Recall the representation of μ_t in terms of probability measures $\{\nu_{u,i}, u \in [0,t], i = 1, \ldots, K\}$. From Lemma 5.4(ii),

$$\int_{\mathcal{W}} [w(t)]^2 \nu_{u,i}(dw) \leqslant 6K^2 e^{2tK\|\gamma\|_\infty}, \quad \text{for all } u \in [0,t], i = 1, \ldots, K. \tag{5.4}$$

Let κ_t^ε denote the kernel obtained by replacing β by β^ε in (4.25). Since $\|\beta - \beta^\varepsilon\|_\infty < \varepsilon$, we have from (4.25) that

$$|\kappa_t(\mathbf{x}, \mathbf{y}) - \kappa_t^\varepsilon(\mathbf{x}, \mathbf{y})| \leqslant \varepsilon \int_0^t w(u)\tilde{w}(u)\, du \leqslant \varepsilon t w(t)\tilde{w}(t),$$

for $\mu_t \otimes \mu_t$ a.e. $(\mathbf{x}, \mathbf{y}) = ((s,w),(r,\tilde{w}))$.

By Lemma 5.2(a,c), we now have

$$\begin{aligned}
|\rho_t(\boldsymbol{\alpha}, \beta, \gamma) - \rho_t(\boldsymbol{\alpha}, \beta^\varepsilon, \gamma)| &\leqslant \left(\int_{\mathcal{X}\times\mathcal{X}} |\kappa_t(\mathbf{x}, \mathbf{y}) - \kappa_t^\varepsilon(\mathbf{x}, \mathbf{y})|^2 d\mu_t(\mathbf{x})\, d\mu_t(\mathbf{y}) \right)^{1/2} \\
&\leqslant \left(\int_{\mathcal{X}\times\mathcal{X}} (\varepsilon t w(t)\tilde{w}(t))^2 d\mu_t(\mathbf{x})\, d\mu_t(\mathbf{y}) \right)^{1/2} \\
&= \varepsilon t \sum_{i=1}^K \int_0^t \alpha_i(s) \left[\int_{\mathcal{W}} [w(t)]^2 \nu_{s,i}(dw) \right] ds \\
&\leqslant \varepsilon t \cdot t\|\boldsymbol{\alpha}\|_\infty \cdot K \cdot 6K^2 e^{2tK\|\gamma\|_\infty},
\end{aligned}$$

where the last inequality follows from (5.4). The result follows. □

When $\boldsymbol{\alpha}$ or γ is perturbed, the underlying measure μ_t changes as well and thus one needs to analyse the corresponding Radon–Nikodym derivatives. This is done in the following two lemmas. We let $\{\mathcal{G}_s\}_{0\leqslant s<\infty}$ denote the canonical filtration on $\mathcal{D}([0,\infty) : \mathbb{N}_0)$. In the following we use the convention that $0/0 = 1$.

Lemma 5.7. *Fix* $\varepsilon > 0$ *and let* $(\boldsymbol{\alpha}, \beta, \gamma)$, $(\tilde{\boldsymbol{\alpha}}, \tilde{\beta}, \tilde{\gamma})$ *be two sets of rate functions such that for all* $1 \leqslant i \leqslant K$ *and* $s \geqslant 0$,

$$\alpha_i(s) - \varepsilon \leqslant \tilde{\alpha}_i(s) \leqslant \alpha_i(s), \quad \text{and} \quad \gamma_i(s) - \varepsilon \leqslant \tilde{\gamma}_i(s) \leqslant \gamma_i(s).$$

Fix $t \geqslant 0$ *and let* μ_t *and* $\tilde{\mu}_t$ *be the corresponding type measures on* $\mathcal{X}$. *For* $(s,w) \in \mathcal{X}$ *and* $j \geqslant 1$, *let* τ_j^s *be the time of the jth jump of* $w(\cdot)$ *after time* s (μ_t *a.s.*). *Further, write* $\Delta(u) = w(u) - w(u-)$ *for* $u \geqslant 0$. *Then there exists* $\varepsilon_0 > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_0)$, $\tilde{\mu} \ll \mu$ *and*

$$\begin{aligned}
\frac{d\tilde{\mu}_t}{d\mu_t}(s,w) = \frac{\tilde{\alpha}_{\Delta(s)-K}(s)}{\alpha_{\Delta(s)-K}(s)} &\times \Pi_{j:\tau_j^s \leqslant t} \frac{\tilde{\gamma}_{\Delta(\tau_j^s)}(\tau_j^s)}{\gamma_{\Delta(\tau_j^s)}(\tau_j)} \\
&\times \exp\left\{ -\int_s^t w(u) \left[\sum_{i=1}^K \tilde{\gamma}_i(u) - \sum_{i=1}^K \gamma_i(u) \right] du \right\}.
\end{aligned}$$

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Proof. For $i = 1, \ldots, K$, define finite measures μ_t^i, $\tilde{\mu}_t^i$ on $\mathcal{X}$ as

$$\mu_t^i(du\,dw) = \alpha_i(u)\nu_{u,i}(dw)1_{[0,t]}(u)\,du, \quad \tilde{\mu}_t^i(du\,dw) = \tilde{\alpha}_i(u)\tilde{\nu}_{u,i}(dw)1_{[0,t]}(u)\,du,$$

where $\nu_{u,i}$ is defined above (4.24) and $\tilde{\nu}_{u,i}$ is defined similarly on replacing γ_i with $\tilde{\gamma}_i$. We will show that

$$\text{for all } 1 \leqslant k \leqslant K \text{ and } s \in [0, T], \quad \tilde{\nu}_{s,k} \ll \nu_{s,k} \quad \text{and} \quad \frac{d\tilde{\nu}_{s,k}}{d\nu_{s,k}}(w) = L_s^t(w), \tag{5.5}$$

where

$$L_s^t := \Pi_{j\geqslant 1}\left(\frac{\tilde{\gamma}_{\Delta(\tau_j^s)}(\tau_j^s)}{\gamma_{\Delta(\tau_j^s)}(\tau_j^s)}\mathbb{1}_{\{\tau_j^s\leqslant t\}}\right) \times \exp\left\{-\int_s^t w(u)\left[\sum_{i=1}^K \tilde{\gamma}_i(u) - \sum_{i=1}^K \gamma_i(u)\right]du\right\}. \tag{5.6}$$

The lemma is an immediate consequence of (5.5) on observing that μ_t^i and μ_t^j are mutually singular when $i \neq j$, and the relation

$$\mu_t = \sum_{i=1}^K \mu_t^i, \quad \tilde{\mu}_t = \sum_{i=1}^K \tilde{\mu}_t^i.$$

We now show (5.5). From the construction of $\nu_{s,k}$, it follows that there are counting processes $\{N_i(u)\}_{u\in[s,t]}$, $i = 1, \ldots, K$, on $\mathcal{W}$ such that

$$w(u) = w(s) + \sum_{i=1}^K iN_i(u), \quad \text{for } u \in [s,t], \ a.s.\ \nu_{s,k} \tag{5.7}$$

and

$$\begin{aligned} &M_i(u) := N_i(u) - \int_s^u w(r)\gamma_i(r)\,dr \quad \text{under } \nu_{s,k} \\ &\text{is a } \{\mathcal{G}_u\}_{u\in[s,t]} \text{ local martingale for } u \in [s,t]. \end{aligned} \tag{5.8}$$

From standard results it follows that L_s^t is a local martingale and super-martingale (see Theorem VI.T2 of [9]). In order to show (5.5), it suffices to show that $\{L_s^u\}_{u\in[s,t]}$ is a martingale. By a change of variable formula it follows that (see, *e.g.*, Theorem A4.T4 of [9])

$$L_s^v = 1 + \sum_{i=1}^K \int_s^v L_s^{u-} \cdot \left(\frac{\tilde{\gamma}_i(u)}{\gamma_i(u)} - 1\right) dM_i(u), \quad v \in [s,t]. \tag{5.9}$$

In order to show that L_s^t is a martingale, it then suffices, in view of (5.8), to show that (see, *e.g.*, Theorem II.T8 in [9]), for all $1 \leqslant i \leqslant K$,

$$\int_{\mathcal{W}}\left[\int_s^t L_s^u \cdot |\tilde{\gamma}_i(u) - \gamma_i(u)|w(u)\,du\right]d\nu_{s,k}(w) < \infty.$$

Finally note that $L_s^u \leqslant e^{\varepsilon t w(t)}$. Using Lemma 5.4(i) and standard estimates for Yule processes, it follows that for ε sufficiently small,

$$\sup_{s\in[0,t]}\sup_{1\leqslant k\leqslant K}\int_{\mathcal{W}} w(t)e^{\varepsilon t w(t)}d\nu_{s,k}(w) < \infty.$$

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The result follows. □

We will now use the above lemma to study the effect of perturbations in $\boldsymbol{\alpha}$ on $\rho_t(\boldsymbol{\alpha}, \beta, \gamma)$.

Lemma 5.8 (perturbations of $\boldsymbol{\alpha}$). *Fix $\varepsilon > 0$. Let $(\boldsymbol{\alpha}, \beta, \gamma)$ be rate functions and let $\boldsymbol{\alpha}^\varepsilon = (\alpha_1^\varepsilon, \ldots, \alpha_K^\varepsilon)$, where α_i^ε are continuous non-negative bounded functions on $[0, \infty)$ such that, for all $1 \leqslant i \leqslant K$ and $s \in [0, T]$,*

$$\alpha_i(s) - \varepsilon \leqslant \alpha_i^\varepsilon(s) \leqslant \alpha_i(s).$$

Then, for every $t > 0$,

$$|\rho_t(\boldsymbol{\alpha}, \beta, \gamma) - \rho_t(\boldsymbol{\alpha}^\varepsilon, \beta, \gamma)| \leqslant C\sqrt{\varepsilon},$$

where $C = t\|\beta\|_\infty \cdot 6K^2 e^{2tK\|\gamma\|_\infty} \cdot 4tK\sqrt{\|\boldsymbol{\alpha}\|_\infty}$.

Proof. Let (μ_t, κ_t) be the type measure and kernel associated with $(\boldsymbol{\alpha}, \beta, \gamma)$. Also, let μ_t^ε be the type measure associated with $(\boldsymbol{\alpha}^\varepsilon, \beta, \gamma)$. By Lemma 5.7,

$$g(s, w) := \frac{d\mu_t^\varepsilon}{d\mu_t}(s, w) = \frac{\alpha^\varepsilon_{\Delta(s)-K}(s)}{\alpha_{\Delta(s)-K}(s)} \quad \text{for} \quad (s, w) \in [0, t] \times \mathcal{W}.$$

Using Lemma 5.2(a,c), Lemma 5.3, and the fact that $|\kappa_t(\mathbf{x}, \mathbf{y})| \leqslant t\|\beta\|_\infty w(t)\tilde{w}(t)$, $\mu_t \otimes \mu_t$ a.e. $(\mathbf{x}, \mathbf{y}) = ((s, w), (\tilde{s}, \tilde{w}))$, we have

$$\begin{aligned}
&|\rho_t(\boldsymbol{\alpha}, \beta, \gamma) - \rho_t(\boldsymbol{\alpha}^\varepsilon, \beta, \gamma)| \\
&\quad \leqslant \left(\int_{\mathcal{X}\times\mathcal{X}} |\sqrt{g(\mathbf{x})g(\mathbf{y})} - 1|^2 |\kappa_t(\mathbf{x}, \mathbf{y})|^2 d\mu_t(\mathbf{x})\, d\mu_t(\mathbf{y}) \right)^{1/2} \\
&\quad \leqslant t\|\beta\|_\infty \left(\int_{\mathcal{X}\times\mathcal{X}} |\sqrt{g(\mathbf{x})g(\mathbf{y})} - 1|^2 w^2(t)\tilde{w}^2(t)\, d\mu_t(\mathbf{x})\, d\mu_t(\mathbf{y}) \right)^{1/2} \\
&\quad \leqslant t\|\beta\|_\infty d_1 \left(\sum_{i,j=1}^K \int_{[0,t]^2} \left(\sqrt{\frac{\alpha_i^\varepsilon(s)\alpha_j^\varepsilon(u)}{\alpha_i(s)\alpha_j(u)}} - 1 \right)^2 \alpha_i(s)\alpha_j(u)\, ds\, du \right)^{1/2}, \qquad (5.10)
\end{aligned}$$

where

$$d_1 = \sup_{s\in[0,T]} \sup_{1\leqslant i\leqslant K} \int_{\mathcal{W}} |w(t)|^2 \nu_{s,i}(dw) \leqslant 6K^2 e^{2tK\|\gamma\|_\infty},$$

and the last inequality follows from (5.4). In order to bound (5.10), note that

$$\begin{aligned}
&\left| \sqrt{\alpha_i(s)\alpha_j(u)} - \sqrt{\alpha_i^\varepsilon(s)\alpha_j^\varepsilon(u)} \right| \\
&\quad = \left| \sqrt{\alpha_i(s)} \left(\sqrt{\alpha_j(u)} - \sqrt{\alpha_j^\varepsilon(u)} \right) + \left(\sqrt{\alpha_i(s)} - \sqrt{\alpha_i^\varepsilon(s)} \right) \sqrt{\alpha_j^\varepsilon(u)} \right| \\
&\quad \leqslant 2\sqrt{\varepsilon} \left(\sqrt{\alpha_i(s)} + \sqrt{\alpha_j(u)} \right).
\end{aligned}$$

Plugging the above bound in (5.10) gives the desired result. □

We will now analyse the effect of perturbations in γ on $\rho_t(\boldsymbol{\alpha}, \beta, \gamma)$. We need the following preliminary truncation lemma.

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Lemma 5.9. *For every $T > 0$, there exist $C_6, C_7, A_0 \in (0, \infty)$ such that, for any $t \in [0, T]$ and rate functions $(\boldsymbol{\alpha}, \beta, \gamma)$, the following holds. Let μ_t, κ_t be the type measure and kernel associated with $(\boldsymbol{\alpha}, \beta, \gamma)$. For $A \in (0, \infty)$, define the kernel $\kappa_{A,t}$ as*

$$\kappa_{A,t}(\mathbf{x}, \mathbf{y}) = \kappa_t(\mathbf{x}, \mathbf{y}) \mathbb{1}_{\{w(t) \leqslant A, \tilde{w}(t) \leqslant A\}}, \quad \text{where } \mathbf{x} = (s, w), \mathbf{y} = (r, \tilde{w}). \tag{5.11}$$

Then, for all $A > A_0$,

$$\rho(\kappa_t) - C_6 e^{-C_7 A} \leqslant \rho(\kappa_{A,t}) \leqslant \rho(\kappa_t),$$

where $\rho(\kappa_t)$ (resp. $\rho(\kappa_{A,t})$) denotes the norm of the operator associated with (κ_t, μ_t) (resp. $(\kappa_{A,t}, \mu_t)$).

Proof. The upper bound in the lemma is immediate from Lemma 5.2(b). We now consider the lower bound. For the rest of the proof, we suppress the dependence of $\kappa_t, \kappa_{A,t}, \mu_t$ on t. Note that, from Lemma 5.2(a,c),

$$\begin{aligned}
\rho(\kappa) - \rho(\kappa_A) &\leqslant \left(\int_{\mathcal{X} \times \mathcal{X}} (\kappa(\mathbf{x}, \mathbf{y}) - \kappa_A(\mathbf{x}, \mathbf{y}))^2 d\mu(\mathbf{x})\, d\mu(\mathbf{y}) \right)^{1/2} \\
&\leqslant 2 \left(\int_{\mathcal{X} \times \mathcal{X}} (t \|\beta\|_\infty w(t) \tilde{w}(t) \mathbb{1}\{\tilde{w}(t) > A\})^2 d\mu(\mathbf{x})\, d\mu(\mathbf{y}) \right)^{1/2} \\
&\leqslant 2t \|\beta\|_\infty \left(d_1 \sum_{i=1}^{K} \sum_{j=1}^{K} \int_{[0,t] \times [0,t]} \alpha_i(s) \alpha_j(u)\, ds\, du \right)^{1/2} \\
&\leqslant 2t \|\beta\|_\infty \cdot t \|\boldsymbol{\alpha}\|_\infty \cdot \sqrt{d_1},
\end{aligned} \tag{5.12}$$

where

$$d_1 = \int_{\mathcal{W}} [w(t)]^2 \nu_{s,i}(dw) \int_{\mathcal{W}} [w(t)]^2 \mathbb{1}_{\{w(t) > A\}} \nu_{s,i}(dw). \tag{5.13}$$

By (5.3), $w(t) \leqslant_d K(X_1 + X_2)$, where X_1, X_2 are independent and identically distributed with geometric $p = e^{-tK\|\gamma\|_\infty}$ distribution:

$$\begin{aligned}
&\int_{\mathcal{W}} [w(t)]^2 \mathbb{1}_{\{w(t) > A\}} \nu_{s,i}(dw) \\
&\quad \leqslant K^2 \mathbb{E}\big[(X_1 + X_2)^2 \mathbb{1}_{\{X_1 + X_2 > A/K\}}\big] \\
&\quad = K^2 \mathbb{E}\big[(X_1 + X_2)^2 (\mathbb{1}_{\{X_1 + X_2 > C, X_1 \geqslant X_2\}} + \mathbb{1}_{\{X_1 + X_2 > C, X_1 < X_2\}})\big] \\
&\quad \leqslant 4K^2 \mathbb{E}\big[X_1^2 \mathbb{1}_{\{X_1 > A/2K\}} + X_2^2 \mathbb{1}_{\{X_2 > A/2K\}}\big],
\end{aligned}$$

The above quantity can be bounded by

$$d_2 (1 - e^{-2TK\|\gamma\|_\infty})^{A/2K} \leqslant d_2 \exp\left\{ -\frac{e^{-2TK\|\gamma\|_\infty}}{K} A \right\}$$

for some constant d_2. The result now follows on using the above bound and (5.4) in (5.13) and (5.12). □

Lemma 5.10 (perturbations of γ). *For every $T > 0$, there exists $C_8 \in (0, \infty)$ and $\varepsilon_0 \in (0, 1)$ such that, for all $t \in [0, T]$ and rate functions $(\boldsymbol{\alpha}, \beta, \gamma)$, the following holds. Suppose $\varepsilon \in (0, \varepsilon_0)$*

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and $\boldsymbol{\gamma}^\varepsilon = (\gamma_1^\varepsilon, \ldots, \gamma_K^\varepsilon)$*, where* γ_i^ε *are continuous, non-negative maps on* $[0, T]$ *such that, for all* $1 \leqslant i \leqslant K$*,*

$$\gamma_i(s) - \varepsilon \leqslant \gamma_i^\varepsilon(s) \leqslant \gamma_i(s), \quad \text{for all } s \in [0, T].$$

Then

$$|\rho_t(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}) - \rho_t(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}^\varepsilon)| \leqslant C_8 \sqrt{\varepsilon} \cdot (-\log \varepsilon)^2.$$

Proof. Let (μ_t, κ_t) (resp. $(\mu_t^\varepsilon, \kappa_t^\varepsilon)$) be the type measure and kernel associated with $(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ (resp. $(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}^\varepsilon)$). By Lemma 5.7, for $(s, w) \in [0, t] \times \mathcal{W}$,

$$\frac{d\mu_t^\varepsilon}{d\mu_t}(s, w) = \Pi_{j \geqslant 1} \left(\frac{\gamma^\varepsilon_{\Delta(\tau_j^s)}(\tau_j^s)}{\gamma_{\Delta(\tau_j^s)}(\tau_j^s)} \mathbb{1}_{\{\tau_j^s \leqslant t\}} \right) \times \exp\left\{ - \int_s^t w(u) \left[\sum_{i=1}^K \gamma_i^\varepsilon(u) - \sum_{i=1}^K \gamma_i(u) \right] du \right\}.$$

Denote the right side as $L_s^t(w)$. Then, as in the proof of Lemma 5.7, it follows that $\{L_s^u(w)\}_{u \in [s,t]}$ is a $\{\mathcal{G}_u\}_{u \in [s,t]}$ martingale under $\nu_{s,k}$ for every $k = 1, \ldots, K$. Fix $A \in (A_0, \infty)$, where A_0 is as in Lemma 5.9, and let $\kappa_{A,t}$ be defined by (5.11). Similarly define $\kappa_{A,t}^\varepsilon$ by replacing κ_t with κ_t^ε in (5.11). Denote the operator norm of the integral operators associated with $(\kappa_{A,t}, \mu_t)$ and $(\kappa_{A,t}^\varepsilon, \mu_t^\varepsilon)$ by $\rho_{A,t}(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma})$ and $\rho_{A,t}(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}^\varepsilon)$, respectively. Then, by Lemmas 5.3 and 5.2(a,c),

$$\begin{aligned}
&|\rho_{A,t}(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}) - \rho_{A,t}(\boldsymbol{\alpha}, \beta, \boldsymbol{\gamma}^\varepsilon)| \\
&\quad \leqslant \left(\int_{\mathcal{X} \times \mathcal{X}} \left| \sqrt{\frac{d\mu_t^\varepsilon}{d\mu_t}(s, w) \frac{d\mu_t^\varepsilon}{d\mu_t}(u, \tilde{w})} - 1 \right|^2 (\kappa_{A,t}(\mathbf{x}, \mathbf{y}))^2 d\mu_t(\mathbf{x})\, d\mu_t(\mathbf{y}) \right)^{1/2} \\
&\quad \leqslant tA^2 \|\beta\|_\infty \qquad (5.14) \\
&\qquad \times \left(\sum_{i=1}^K \sum_{j=1}^K \int_{[0,t] \times [0,t]} \alpha_i(s) \alpha_j(u) \int_{\mathcal{W} \times \mathcal{W}} |\sqrt{L_s^t(w) L_u^t(w)} - 1|^2 \nu_{s,i}(dw) \nu_{u,j}(d\tilde{w}) \right)^{1/2}.
\end{aligned}$$

Next, using the martingale property of L_s^t, we have

$$\begin{aligned}
&\int_{\mathcal{W} \times \mathcal{W}} |\sqrt{L_s^t(w) L_u^t(w)} - 1|^2 \nu_{s,i}(dw) \nu_{u,j}(d\tilde{w}) \\
&\quad = 2 - 2 \int_{\mathcal{W}} \sqrt{L_s^t(w)} \nu_{s,i}(dw) \int_{\mathcal{W}} \sqrt{L_u^t(w)} \nu_{u,j}(dw) \\
&\quad \leqslant 4 - 2 \int_{\mathcal{W}} \sqrt{L_s^t(w)} \nu_{s,i}(dw) - 2 \int_{\mathcal{W}} \sqrt{L_u^t(w)} \nu_{u,j}(dw), \qquad (5.15)
\end{aligned}$$

where the inequality on the last line follows on observing that from Jensen's inequality the two integrals on the second line are bounded by 1 and using the elementary inequality $a_1 + a_2 \leqslant a_1 a_2 + 1$, for $a_1, a_2 \in [0, 1]$. We will now estimate the two integrals on the last line of (5.15) by using the martingale properties of $\{L_s^u\}_{u \in [s,t]}$ and the representations (5.7) and (5.9) in the proof of Lemma 5.7. For the rest of the proof we write L_s^u as $L_s(u)$. By an application of Ito's formula, we have that for every $k = 1, \ldots, K$, $\nu_{s,k}$ a.s.

$$\begin{aligned}
\sqrt{L_s(t)} - 1 - \sum_{i=1}^{K} \int_s^t \frac{\sqrt{L_s(u-)}}{2} \left(\frac{\gamma_i^\varepsilon(u)}{\gamma_i(u)} - 1 \right) dM_i(u) \\
= \sum_{s<u\leqslant t} \left(\sqrt{L_s(u)} - \sqrt{L_s(u-)} \right) - \sum_{i=1}^{K} \int_s^t \frac{\sqrt{L_s(u-)}}{2} \left(\frac{\gamma_i^\varepsilon(u)}{\gamma_i(u)} - 1 \right) dN_i(u) \\
= \sum_{i=1}^{K} \int_s^t \sqrt{L_s(u-)} \left(\sqrt{\frac{\gamma_i^\varepsilon(u)}{\gamma_i(u)}} - 1 \right) dN_i(u) - \sum_{i=1}^{K} \int_s^t \frac{\sqrt{L_s(u-)}}{2} \left(\frac{\gamma_i^\varepsilon(u)}{\gamma_i(u)} - 1 \right) dN_i(u) \\
= -\frac{1}{2} \sum_{i=1}^{K} \int_s^t \sqrt{L_s(u-)} \left(\sqrt{\frac{\gamma_i^\varepsilon(u)}{\gamma_i(u)}} - 1 \right)^2 dN_i(u), \qquad (5.16)
\end{aligned}$$

where the second equality follows on observing that, for $u \in (s, t]$,

$$\sqrt{L_s(u)} = \sum_{i=1}^{K} \sqrt{L_s(u-)} \sqrt{\frac{\gamma_i^\varepsilon(u)}{\gamma_i(u)}} \Delta N_i(u).$$

As in the proof of Lemma 5.7, we can check that, for all i, k,

$$\int_{\mathcal{W}} \left[\int_s^t \sqrt{L_s(u)} \cdot |\gamma_i^\varepsilon(u) - \gamma_i(u)| w(u)\, du \right] d\nu_{s,k}(w) < \infty,$$

and consequently the last term on the left side of (5.16) is a martingale. Denoting the expectation operator corresponding to the probability measure $\nu_{s,k}$ on $\mathcal{W}$ by $\mathbb{E}_{s,k}$, we have

$$\begin{aligned}
1 - \mathbb{E}_{s,k}[\sqrt{L_s(t)}] &= \frac{1}{2} \sum_{i=1}^{K} \mathbb{E}_{s,k} \left[\int_s^t \sqrt{L_s(u-)} \left(\sqrt{\frac{\gamma_i^\varepsilon(u)}{\gamma_i(u)}} - 1 \right)^2 dN_i(u) \right] \\
&= \frac{1}{2} \sum_{i=1}^{K} \mathbb{E}_{s,k} \left[\int_s^t \sqrt{L_s(u)} \left(\sqrt{\frac{\gamma_i^\varepsilon(u)}{\gamma_i(u)}} - 1 \right)^2 w(u) \gamma_i(u)\, du \right] \\
&= \frac{1}{2} \sum_{i=1}^{K} \int_s^t \mathbb{E}_{s,k} \left[\sqrt{L_s(u)} w(u) \left(\sqrt{\gamma_i^\varepsilon(u)} - \sqrt{\gamma_i(u)} \right)^2 \right] du \\
&\leqslant \frac{1}{2} \int_s^t K\varepsilon \cdot \mathbb{E}_{s,k} \left[\sqrt{L_s(u)} w(u) \right] du \\
&\leqslant \frac{K\varepsilon}{2} \int_s^t \left(\mathbb{E}_{s,k}[L_s(u)] \mathbb{E}_{s,k}[w^2(u)] \right)^{1/2} du \\
&\leqslant \frac{K\varepsilon}{2} \cdot t \cdot (6K^2 e^{2TK\|\gamma\|_\infty})^{1/2},
\end{aligned}$$

where the last inequality follows from (5.4). Using the above bound in (5.14), we now have

$$|\rho_{A,t}(\boldsymbol{\alpha}, \beta, \gamma) - \rho_{A,t}(\boldsymbol{\alpha}, \beta, \gamma^\varepsilon)| \leqslant t A^2 \|\beta\|_\infty \cdot t \|\boldsymbol{\alpha}\|_\infty \cdot \left[2K\varepsilon t (6K^2 e^{2TK\|\gamma\|_\infty})^{1/2} \right]^{1/2}.$$

Finally, by Lemma 5.9 we have

$$\begin{aligned}
|\rho_t(\boldsymbol{\alpha}, \beta, \gamma) - \rho_t(\boldsymbol{\alpha}, \beta, \gamma^\varepsilon)| &\leqslant |\rho_{A,t}(\boldsymbol{\alpha}, \beta, \gamma) - \rho_{A,t}(\boldsymbol{\alpha}, \beta, \gamma^\varepsilon)| + 2C_6 e^{-C_7 A} \\
&< d_1 A^2 \varepsilon^{1/2} + 2C_6 e^{-C_7 A},
\end{aligned}$$

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where

$$d_1 = tA^2\|\beta\|_\infty \cdot t\|\boldsymbol{\alpha}\|_\infty \cdot \left[2Kt(6K^2 e^{2TK\|\gamma\|_\infty})^{1/2}\right]^{1/2}.$$

The result now follows on taking $A = -\log \varepsilon$ in the above display and taking ε_0 sufficiently small (in particular such that $-\log(\varepsilon_0) > A_0$). □

Now we combine all the above ingredients to complete the proof of Proposition 5.5.

Proof of Proposition 5.5. Using Lemmas 5.10, 5.6 and 5.8, we get

$$\begin{aligned}|\rho_t^+ - \rho_t| &\leqslant |\rho_t(\boldsymbol{\alpha}+\varepsilon, \beta+\varepsilon, \gamma+\varepsilon) - \rho_t(\boldsymbol{\alpha}+\varepsilon, \beta+\varepsilon, \gamma)| \\ &\quad + |\rho_t(\boldsymbol{\alpha}+\varepsilon, \beta+\varepsilon, \gamma) - \rho_t(\boldsymbol{\alpha}+\varepsilon, \beta, \gamma)| + |\rho_t(\boldsymbol{\alpha}+\varepsilon, \beta, \gamma) - \rho_t(\boldsymbol{\alpha}, \beta, \gamma)| \\ &\leqslant C_8\varepsilon^{1/2}(-\log\varepsilon)^2 + d_1\varepsilon + d_2\varepsilon^{1/2},\end{aligned}$$

where $d_1 = 6T^2K^3e^{2TK}$ and $d_2 = 24K^3T^2e^{2TK}$. A similar bound holds for $|\rho_t^- - \rho_t|$. The result follows. □

5.3. Effect of time perturbation on ρ_t

Throughout this section we fix rate functions $(\boldsymbol{\alpha}, \beta, \gamma)$. The aim of this section is to understand the evolution of the operator norm $\rho_t(\boldsymbol{\alpha}, \beta, \gamma)$ as t changes. The main result of the section is Proposition 5.11, which studies continuity and differentiability properties of the function $\rho(t) := \rho_t(\boldsymbol{\alpha}, \beta, \gamma)$, $t \geqslant 0$.

Proposition 5.11. *Suppose that $\beta(t) > 0$ for $t > 0$ and $\liminf_{t\to\infty} \beta(t) > 0$. Then:*

(i) *ρ is a continuous strictly increasing function on $\mathbb{R}_+$ with $\rho(0) = 0$ and $\lim_{t\to\infty} \rho(t) = \infty$,*
(ii) *there is a unique value $t_c' = t_c'(\boldsymbol{\alpha}, \beta, \gamma)$ such that $\rho(t_c') = 1$.*

□

The proof of the proposition relies on the following lemma and is given after the proof of the lemma.

Lemma 5.12. *Let $0 < t_1 \leqslant t_2 < \infty$. Then*

$$|t_2 - t_1| \cdot \frac{\inf_{t_1\leqslant u\leqslant t_2} \beta(u)}{t_1\|\beta\|_\infty} \cdot \rho(t_1) \leqslant \rho(t_2) - \rho(t_1) \leqslant |t_2 - t_1| \cdot 6t_2K^2\|\beta\|_\infty\|\boldsymbol{\alpha}\|_\infty e^{2t_2K\|\gamma\|_\infty}.$$

Proof. Letting $\mu := \mu_{t_2}$, we have

$$\begin{aligned}|\rho(t_2) - \rho(t_1)| &\leqslant \left(\int_{\mathcal{X}\times\mathcal{X}} (\kappa_{t_2}(\mathbf{x}, \mathbf{y}) - \kappa_{t_1}(\mathbf{x}, \mathbf{y}))^2 d\mu(\mathbf{x})\, d\mu(\mathbf{y})\right)^{1/2} \\ &\leqslant \left(\int_{\mathcal{X}\times\mathcal{X}} (\|\beta\|_\infty w(t_2)\tilde{w}(t_2)|t_2 - t_1|)^2 d\mu(\mathbf{x})\, d\mu(\mathbf{y})\right)^{1/2} \\ &\leqslant |t_2 - t_1| \cdot \|\beta\|_\infty \cdot t_2\|\boldsymbol{\alpha}\|_\infty \cdot 6K^2e^{2t_2K\|\gamma\|_\infty},\end{aligned}$$

where the last inequality once again follows from (5.4). This proves the upper bound.

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Next note that, for $\mu \otimes \mu$ a.e. $(\mathbf{x}, \mathbf{y})$ such that $\kappa_{t_1}(\mathbf{x}, \mathbf{y}) \neq 0$, we have

$$\begin{aligned}\frac{\kappa_{t_2}(\mathbf{x}, \mathbf{y})}{\kappa_{t_1}(\mathbf{x}, \mathbf{y})} &= 1 + \frac{\int_{t_1}^{t_2} w(u)\tilde{w}(u)\beta(u)\, du}{\int_0^{t_1} w(u)\tilde{w}(u)\beta(u)\, du} \\ &\geqslant 1 + \frac{w(t_1)\tilde{w}(t_1)\inf_{t_1 \leqslant u \leqslant t_2} \beta(u) \cdot (t_2 - t_1)}{w(t_1)\tilde{w}(t_1)\|\beta\|_\infty t_1}.\end{aligned}$$

Thus

$$\kappa_{t_2}(\mathbf{x}, \mathbf{y}) \geqslant \left(1 + |t_2 - t_1| \cdot \frac{\inf_{t_1 \leqslant u \leqslant t_2} \beta(u)}{t_1 \|\beta\|_\infty}\right) \kappa_{t_1}(\mathbf{x}, \mathbf{y}),$$

which from Lemma 5.2(b) implies

$$\rho(t_2) - \rho(t_1) \geqslant |t_2 - t_1| \cdot \frac{\inf_{t_1 \leqslant u \leqslant t_2} \beta(u)}{t_1 \|\beta\|_\infty} \cdot \rho(t_1).$$

This completes the proof of the lower bound. □

Proof of Proposition 5.11. Since $\kappa_0 = 0$, $\rho(0) = 0$ is immediate. Further, Lemma 5.12 shows that ρ is continuous and strictly increasing. Finally, since $\inf_{t \to \infty} \beta(t) > 0$, there exists $\delta > 0$ and a $t^* \in (0, \infty)$ such that for all $t \geqslant t^*$, $\beta(t) \geqslant \delta$. From Lemma 5.12 we then have, for $t \geqslant t^*$,

$$\rho(t) - \rho(t^*) \geqslant \frac{(t - t^*)\delta}{t^* \|\beta\|_\infty}.$$

This proves that $\rho(t) \to \infty$ as $t \to \infty$ and completes the proof of (i). Part (ii) is immediate from (i). □

5.4. Operator norm of $\mathbf{RG}(\mathbf{a}, b, \mathbf{c})$ and critical time of BSR

In this section we will prove Theorem 1.3. Recall that by Lemma 4.4, for any fixed time t, $\mathbf{BSR}^*(t)$ (more precisely, $\Gamma(t)$) can be approximated by perturbations of $\mathbf{RG}(\mathbf{a}, b, \mathbf{c})(t)$. To estimate the volume of the largest component in $\mathbf{RG}(\mathbf{a}, b, \mathbf{c})(t)$ we will use Lemma 5.1. In order to identify suitable Λ_n as in part (ii) of the lemma, we start with the following lemma.

Lemma 5.13. *Let $(\boldsymbol{\alpha}, \beta, \gamma)$ be rate functions and let μ_t be the associated type measure. Fix $T > 0$. Define $\Lambda \in \mathcal{B}(\mathcal{X})$ as $\Lambda = \{(s, w) \in \mathcal{X} : w(T) \leqslant l\}$ for $l \in \mathbb{R}_+$. Then, for every $l \in \mathbb{R}_+$,*

$$\mu_t(\Lambda^c) < 2T\|\boldsymbol{\alpha}\|_\infty \cdot \exp\left(-l\frac{e^{-TK\|\gamma\|_\infty}}{2K}\right).$$

Proof. Note that

$$\mu_t(\Lambda^c) = \sum_{i=1}^{K} \int_0^t \alpha_i(u) \nu_{u,i}(\Lambda^c) \leqslant \|\boldsymbol{\alpha}\|_\infty T \sup_{u \in [0,T]} \sup_{1 \leqslant i \leqslant K} \nu_{u,i}(\Lambda^c). \tag{5.17}$$

By (5.3),

$$\nu_{u,i}(\Lambda^c) = \nu_{u,i}(\{w : w(T) \geqslant l\}) \leqslant \mathbb{P}(X_1 + X_2 \geqslant l/K) \leqslant 2(1 - e^{-TK\|\gamma\|_\infty})^{l/2K}.$$

where X_i are i.i.d. with $\mathrm{Geom}(e^{-T\|\gamma\|_\infty})$ distribution. Using this estimate in (5.17), we have

$$\mu_t(\Lambda^c) \leqslant \|\boldsymbol{\alpha}\|_\infty T \cdot 2(1 - e^{-TK\|\gamma\|_\infty})^{l/2K}.$$

The result follows. □

We will now use the above lemma along with Lemma 5.1 to estimate the largest component in $\mathbf{RG}^{(n)}(\boldsymbol{\alpha}, \beta, \gamma)(t)$. Recall the notation $\rho_t(\boldsymbol{\alpha}, \beta, \gamma)$ from Section 5.2.

Lemma 5.14. *Let $(\boldsymbol{\alpha}, \beta, \gamma)$ be rate functions and let $\mathcal{I}_1^{\mathbf{RG}}(t)$ denote the component with the largest volume, with respect to the weight function ϕ_t, in $\mathbf{RG}^{(n)}(t) := \mathbf{RG}^{(n)}(\boldsymbol{\alpha}, \beta, \gamma)(t)$. Then, for every $t > 0$ such that $\rho_t(\boldsymbol{\alpha}, \beta, \gamma) < 1$, there exists $A \in (0, \infty)$ such that*

$$\mathbb{P}(\mathrm{vol}_{\phi_t}(\mathcal{I}_1^{\mathbf{RG}}(t)) > A \log^4 n) \to 0, \quad \text{as } n \to \infty.$$

Proof. We will use Lemma 5.1(ii). Define

$$\Lambda_n := \{(s, w) \in \mathcal{X} : w(t) < B \log n\},$$

where B will be chosen appropriately later in the proof. Now consider the function $g(n)$ in Lemma 5.1(ii) with Λ_n defined as above and (μ, ϕ, κ) there replaced by $(\mu_t, \phi_t, \kappa_t)$, where (μ_t, κ_t) is the type measure and kernel associated with $(\boldsymbol{\alpha}, \beta, \gamma)$. Note that

$$\kappa_t(\mathbf{x}, \mathbf{y}) = \int_0^t \beta(u) w(u) \tilde{w}(u)\, du \leqslant t\|\beta\|_\infty w(t)\tilde{w}(t),$$

and therefore

$$g(n) \leqslant 8B \log n (1 + 3\mu_t(\mathcal{X}) \cdot t\|\beta\|_\infty B^2 \log^2 n). \tag{5.18}$$

Writing $m_n = A \log^4 n$, the bound in Lemma 5.1(ii) then gives

$$\mathbb{P}(\mathrm{vol}_{\phi_t}(\mathcal{I}_1^{\mathbf{RG}}(t)) > m_n) \leqslant n\mu_t(\Lambda_n^c) + 2n\mu_t(\mathcal{X}) \exp\big(-\Delta^2 A \log^4 n / g(n)\big), \tag{5.19}$$

where $\Delta = 1 - \rho_t(\boldsymbol{\alpha}, \beta, \gamma) > 0$. Using Lemma 5.13 with $l = B \log n$ gives

$$n\mu_t(\Lambda_n^c) \leqslant nt\|\boldsymbol{\alpha}\|_\infty \cdot n^{-Be^{-T\|\gamma\|_\infty}/2K} = o(1) \tag{5.20}$$

for $B > 2Ke^{T\|\gamma\|_\infty}$. Now fix $B > e^{T\|\gamma\|_\infty/2K}$, and choose A large such that

$$n\mu_t(\mathcal{X}) \exp\big(-\Delta^2 A \log^4 n / g(n)\big) \to 0$$

as $n \to \infty$. The result follows. □

Proof of Theorem 1.3. For $t \geqslant 0$, let (μ_t, κ_t) be the type measure and the kernel associated with rate functions $(\mathbf{a}, b, \mathbf{c})$. We will prove Theorem 1.3 with this choice of (μ_t, κ_t). From Proposition 5.11 we have that $\rho(t) = \rho_t(\mathbf{a}, b, \mathbf{c})$ is continuous and strictly increasing in t and there is a unique $t'_c \in (0, \infty)$ such that $\rho(t'_c) = 1$. It now suffices to show that (a) for $t < t'_c$, $|\mathcal{C}_1(t)|$ (the size of the largest component in $\mathbf{BSR}^*(t)$) is $O(\log^4 n)$, and (b) for $t > t'_c$, $|\mathcal{C}_1(t)| = \Omega(n)$.

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First consider (a). Fix $t < t_c'$. For $\delta > 0$, define rate functions

$$(\mathbf{a}^+, b^+, \mathbf{c}^+) = (\mathbf{a} + \delta, b + \delta, \mathbf{c} + \delta).$$

Since $\rho(t) < 1$, by Proposition 5.5, we can choose δ sufficiently small that $\rho_t(\mathbf{a}^+, b^+, \mathbf{c}^+) < 1$. Let $\mathcal{I}_1^{\mathbf{RG}^+}(t)$ denote the component of the largest volume in $\mathbf{RG}^+(t) := \mathbf{RG}(\mathbf{a}^+, b^+, \mathbf{c}^+)(t)$. From Lemma 5.14 there exists $A \in (0, \infty)$ such that

$$\mathbb{P}(\mathrm{vol}_{\phi_t}(\mathcal{I}_1^{\mathbf{RG}^+}(t)) > A \log^4 n) \to 0, \quad \text{as } n \to \infty.$$

Combining this result with Corollary 4.5, we see that

$$\mathbb{P}(\mathrm{vol}_{\phi_t}(\mathcal{I}_1^{\Gamma}(t)) > A \log^4 n) \to 0, \quad \text{as } n \to \infty,$$

where $\mathcal{I}_1^{\Gamma}(t)$ is the component with the largest volume in $\Gamma(t)$. Part (a) is now immediate from the one-to-one correspondence between the components in $\Gamma(t)$ and $\mathbf{BSR}^*(t)$ (see (4.21)).

We now consider (b). Fix $t > t_c'$. Then $\rho(t) > 1$. From Proposition 5.5 we can find $\delta > 0$ such that $\rho_t(\mathbf{a}^-, b^-, \mathbf{c}^-) > 1$, where $(\mathbf{a}^-, b^-, \mathbf{c}^-) = ((\mathbf{a} - \delta)^+, (b - \delta)^+, (\mathbf{c} - \delta)^+)$. Let $\mathcal{C}_1^{\mathbf{RG}^-}(t)$ be the component in $\mathbf{RG}^-(t) := \mathbf{RG}^{(n)}(\mathbf{a}^-, b^-, \mathbf{c}^-)(t)$ with the largest number of vertices. By Theorem 3.1 of [8], $|\mathcal{C}_1^{\mathbf{RG}^-}(t)| = \Theta(n)$. Since $\mathrm{vol}_{\phi_t}(\mathcal{C}_1^{\mathbf{RG}^-}(t)) \geqslant |\mathcal{C}_1^{\mathbf{RG}^-}(t)|$, we have

$$\mathrm{vol}_{\phi_t}(\mathcal{I}_1^{\mathbf{RG}^-}(t)) = \Omega(n),$$

where $\mathcal{I}_1^{\mathbf{RG}^-}(t)$ is the component with the largest volume in $\mathbf{RG}^-(t)$. Finally, in view of Corollary 4.5, we have the same result with $\mathcal{I}_1^{\mathbf{RG}^-}(t)$ replaced by $\mathcal{I}_1^{\Gamma}(t)$ and the result follows once more from the one-to-one correspondence between the components in $\Gamma(t)$ and $\mathbf{BSR}^*(t)$. □

5.5. Barely subcritical regime for bounded-size rules

In this section we complete the proof of Theorem 1.2. Throughout this section we fix $\gamma \in (0, 1/4)$ and let $t_n = t_c - n^{-\gamma}$. The main ingredient in the proof is the following proposition.

Proposition 5.15. *There exist $\bar{B}, \bar{C}, \bar{N} \in (0, \infty)$ such that, for all $n \geqslant \bar{N}$ and all $0 \leqslant t \leqslant t_n$,*

$$\mathbb{P}\{|\mathcal{C}_1^{(n)}(t)| \geqslant \bar{m}(n,t)\} \leqslant \frac{\bar{C}}{n^2}, \quad \text{where } \bar{m}(n,t) = \frac{\bar{B}(\log n)^4}{(t_c - t)^2}.$$

□

Let us first prove Theorem 1.2 assuming the above proposition.

Proof of Theorem 1.2. Write $\tau = \inf\{t \geqslant 0 : |\mathcal{C}_1^{(n)}(t)| \geqslant m(n,t)\}$, where

$$m(n,t) = \frac{2\bar{B}(\log n)^4}{(t_c - t)^2}.$$

Then

$$\mathbb{P}\{|\mathcal{C}_1^{(n)}(t)| \geqslant m(n,t) \text{ for some } t \leqslant t_n\} = \mathbb{P}\{\tau \leqslant t_n\}. \tag{5.21}$$

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Note that

$$\{\tau = t\} \subset \bigcup_{v,v'\in[n],v\neq v'} E^{v,v'}, \tag{5.22}$$

where, denoting the component in **BSR**(t) that contains the vertex $v \in [n]$ by $\mathcal{C}_v^{(n)}(t)$ and its size by $|\mathcal{C}_v^{(n)}(t)|$,

$$E^{v,v'} = \{\max\{|\mathcal{C}_v^{(n)}(t-)|, |\mathcal{C}_{v'}^{(n)}(t-)|\} < m(n,t); \mathcal{C}_v^{(n)}(t-) \neq \mathcal{C}_{v'}^{(n)}(t-)\}$$
$$\bigcap \{|\mathcal{C}_v^{(n)}(t-)| + |\mathcal{C}_{v'}^{(n)}(t-)| \geqslant m(n,t)\} \bigcap \{\mathcal{C}_v^{(n)}(t) = \mathcal{C}_{v'}^{(n)}(t)\}. \tag{5.23}$$

Note that

$$\mathbb{P}\{|\mathcal{C}_v^{(n)}(t)| + |\mathcal{C}_{v'}^{(n)}(t)| \geqslant m(n,t)\} \leqslant 2\mathbb{P}\{|\mathcal{C}_1^{(n)}(t)| \geqslant m(n,t)/2\} \tag{5.24}$$

and, on the set, $\{\max\{|\mathcal{C}_v^{(n)}(t)|, |\mathcal{C}_{v'}^{(n)}(t)|\} < m(n,t)\}$, the rate at which $\mathcal{C}_v^{(n)}(t)$ and $\mathcal{C}_{v'}^{(n)}(t)$ merge can be bounded by

$$\frac{1}{2n^3} \cdot 4|\mathcal{C}_v^{(n)}(t)||\mathcal{C}_{v'}^{(n)}(t)|n^2 \leqslant \frac{2m^2(n,t)}{n}.$$

Combining this observation with (5.22) and (5.24), we have

$$\begin{aligned}\mathbb{P}\{\tau \leqslant t_n\} &\leqslant \sum_{v,v'\in[n],v\neq v'} \int_0^{t_n} \mathbb{P}\{|\mathcal{C}_v^{(n)}(t)| + |\mathcal{C}_{v'}^{(n)}(t)| \geqslant m(n,t)\} \cdot \frac{2m^2(n,t)}{n} dt \\ &\leqslant 2n^2 \int_0^{t_n} \mathbb{P}\{|\mathcal{C}_1^{(n)}(t)| \geqslant m(n,t)/2\} \cdot \frac{2m^2(n,t)}{n} dt \\ &\leqslant 4nt_c \sup_{t\leqslant t_n}\{m^2(n,t)\mathbb{P}\{|\mathcal{C}_1^{(n)}(t)| \geqslant \bar{m}(n,t)\}\} \\ &= O(n \cdot n^{4\gamma}(\log n)^8 \cdot n^{-2}) = O(n^{-1+4\gamma}(\log n)^8) = o(1),\end{aligned}$$

where the last line follows from Proposition 5.15 and the fact that $\gamma < 1/4$. Using the above estimate in (5.21), we have the result. □

We will need the following lemma in the proof of Proposition 5.15.

Lemma 5.16. *Let*

$$(\mathbf{a}^+, b^+, \mathbf{c}^+) = (\mathbf{a} + \delta_n, b + \delta_n, \mathbf{c} + \delta_n),$$

where $\delta_n = n^{-2\gamma_0}$ *and* $\gamma_0 \in (\gamma, 1/4)$. *Let* $\rho_t^{(n),+} = \rho_t(\mathbf{a}^+, b^+, \mathbf{c}^+)$. *Then there exists* $C_9, N_0 \in (0,\infty)$ *such that, for all* $n \geqslant N_0$,

$$\rho_t^{(n),+} < 1 - C_9(t_c - t) \quad \textit{for all } 0 \leqslant t \leqslant t_n.$$

Proof of Lemma 5.16. From Proposition 5.5, there is a $d_1 \in (0,\infty)$ such that

$$\rho_t^{(n),+} \leqslant \rho_t(\mathbf{a}, b, \mathbf{c}) + d_1 n^{-\gamma_0} \log^2 n, \quad \text{for all } t \leqslant t_c.$$

By Lemma 5.12 and since $\rho_{t_c}(\mathbf{a}, b, \mathbf{c}) = 1$, there exists $d_2 \in (0,\infty)$ such that

$$\rho_t(\mathbf{a}, b, \mathbf{c}) \leqslant 1 - d_2(t_c - t), \quad \text{for all } t \leqslant t_n.$$

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Thus, since $\gamma < \gamma_0$, we have for some $N_0 > 0$

$$\rho_t^{(n),+} \leqslant 1 - d_2(t_c - t) + d_1 n^{-\gamma_0}(\log n)^2 < 1 - \frac{d_2}{2}(t_c - t),$$

for all $n \geqslant N_0$ and $0 \leqslant t \leqslant t_c - n^{-\gamma}$. The result follows. □

Proof of Proposition 5.15. Recall the rate functions $(\mathbf{a}, b, \mathbf{c})$ introduced in Section 4.2. Choose $\gamma_0 \in (\gamma, 1/4)$ and let $(\mathbf{a}^+, b^+, \mathbf{c}^+)$ be as in Lemma 5.16. Fix $t < t_n$ and consider the random graph $\mathbf{RG}^{(n)}(\mathbf{a}^+, b^+, \mathbf{c}^+)(t)$. From Lemma 4.4, we can couple $\Gamma(t)$ and $\mathbf{RG}^{(n)}(\mathbf{a}^+, b^+, \mathbf{c}^+)(t)$ such that

$$\mathbb{P}(\Gamma(t) \subseteq \mathbf{RG}^{(n)}(\mathbf{a}^+, b^+, \mathbf{c}^+)(t)) \geqslant 1 - C_3 \exp(-C_4 n^{1-4\gamma_0}), \quad \text{for all } t \in [0, T].$$

Recalling the one-to-one correspondence between components in $\mathbf{BSR}^*(t)$ and $\Gamma(t)$, and (4.21), for any $m \geqslant 1$ we have

$$\mathbb{P}\{|\mathcal{C}_1^{(n)}(t)| > m\} \leqslant \mathbb{P}\{\mathrm{vol}_{\phi_t}(\mathcal{I}_1^{\mathbf{RG}^+}(t)) \geqslant m\} + C_3 \exp\{-C_4 n^{1-4\gamma_0}\}, \tag{5.25}$$

where $\mathcal{I}_1^{\mathbf{RG}^+}(t)$ is the component of largest volume in $\mathbf{RG}^{(n)}(\mathbf{a}^+, b^+, \mathbf{c}^+)(t)$. From Lemma 5.16, there is an $N_0 > 0$ such that $\Delta_t^{(n),+} = 1 - \rho_t(\mathbf{a}^+, b^+, \mathbf{c}^+)$ satisfies

$$\Delta_t^{(n),+} \geqslant C_9(t_c - t), \quad \text{for all } t \leqslant t_n,\ n \geqslant N_0. \tag{5.26}$$

Using Lemma 5.1 and arguing as in equation (5.19), for all $t \in [0, T]$ and all $m \geqslant 1$ we have

$$\mathbb{P}\{\mathrm{vol}_{\phi_t}(\mathcal{I}_1^{\mathbf{RG}^+}(t)) \geqslant m\} \leqslant n d_1 \exp\{-(\Delta_t^{(n),+})^2 m/(d_2 \log^3 n)\} + d_3 n^{-2}, \tag{5.27}$$

where d_1, d_2, d_3 are suitable constants. Using (5.26) in (5.27), we get

$$\mathbb{P}\{\mathrm{vol}_{\phi_t}(\mathcal{I}_1^{\mathbf{RG}^+}(t)) \geqslant m\} \leqslant n d_1 \exp\{-d_4(t_c - t)^2 m/\log^3 n\} + d_3 n^{-2}.$$

The result now follows on substituting

$$m = m(n, t) = \frac{\bar{B}(\log n)^4}{(t_c - t)^2},$$

with $\bar{B} > 3/d_4$, in the above inequality. □

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