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A Liouville comparison principle for solutions of semilinear elliptic partial differential inequalities

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This work is devoted to the study of a Liouville-type comparison principle for entire weak solutions of semilinear elliptic partial differential inequalities of the form

$$\mathcal{L}u + |u|^{q-1}u \leqslant \mathcal{L}v + |v|^{q-1}v,$$

where $q > 0$ is a given real number and $\mathcal{L}$ is a linear (possibly non-uniformly) elliptic partial differential operator of second order in divergence form given by the relation

$$\mathcal{L} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ij}(x)\frac{\partial}{\partial x_j}\right].$$

We assume that $n \geqslant 2$, that the coefficients $a_{ij}(x)$, $i, j = 1, \dots, n$, are measurable bounded functions on $\mathbb{R}^n$ such that $a_{ij}(x) = a_{ji}(x)$ and that the corresponding quadratic form is non-negative. The results obtained in this work were announced by the author in 2005.

1. Introduction and definitions

This work is devoted to the study of a Liouville-type comparison principle for entire weak solutions of semilinear elliptic partial differential inequalities of the form

$$\mathcal{L}u + |u|^{q-1}u \leqslant \mathcal{L}v + |v|^{q-1}v, \tag{1.1}$$

where $q > 0$ is a given real number and $\mathcal{L}$ is a linear (possibly non-uniformly) elliptic partial differential operator of second order in divergence form given by the relation

$$\mathcal{L} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[a_{ij}(x)\frac{\partial}{\partial x_j}\right]. \tag{1.2}$$

Henceforth, we assume that $n \geqslant 2$ is a natural number, the coefficients $a_{ij}(x)$, $i, j = 1, \dots, n$, are measurable bounded functions on $\mathbb{R}^n$ such that $a_{ij}(x) = a_{ji}(x)$, and the corresponding quadratic form is non-negative, i.e.

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant 0 \tag{1.3}$$

for all $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ at almost all $x \in \mathbb{R}^n$.

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Remark 1.1. It is important to note that if u and v satisfy, respectively, the inequalities

$$-\mathcal{L}u \geqslant |u|^{q-1}u \tag{1.4}$$

and

$$-\mathcal{L}v \leqslant |v|^{q-1}v, \tag{1.5}$$

then the pair (u, v) satisfies inequality (1.1). Thus, all the results obtained in this work for solutions of (1.1) are valid for the corresponding solutions of the system (1.4), (1.5).

Remark 1.2. Our main goal here is to study solutions of inequality (1.1) with $\mathcal{L} = \Delta$, the Laplacian operator, and to obtain new sharp results in this case. Our secondary goal is to show that the approach used can be applied to study solutions of inequality (1.1) with $\mathcal{L}$ in a wide class of linear partial differential operators and also to obtain new results which are 'sharp over' a function space directly associated with the operator $\mathcal{L}$.

Remark 1.3. The results obtained, which evidently have a comparison principle character, we term Liouville-type comparison principles, since, in a particular case when $v = 0$, they become Liouville-type theorems for solutions of inequality (1.4). That it is possible to consider and compare arbitrary solutions (u, v) of inequality (1.1), not just solutions of a special form such as $(u, 0)$, is the main difference between the results in theorems 2.1–2.3, 2.5–2.7, 2.12 and their analogues in the literature.

Note that a Liouville theorem for solutions of linear uniformly elliptic second-order partial differential equations on $\mathbb{R}^n$, $n > 2$, was first obtained in [2] under some continuity assumptions on the coefficients of the equations and in [11] without continuity assumptions on the coefficients of the equations. In the case of linear uniformly elliptic second-order partial differential equations on $\mathbb{R}^2$, a Liouville theorem is a direct consequence of a Harnack inequality first obtained in [10]. A Liouville theorem for solutions of semilinear uniformly elliptic second-order partial differential equations on $\mathbb{R}^n$, $n > 2$, was first obtained in [1], and a Liouville theorem for solutions of semilinear (possibly non-uniformly) elliptic second-order partial differential inequalities of the form (1.4) on $\mathbb{R}^n$, $n \geqslant 2$, was, seemingly, first obtained in [6] (see also [7]).

Remark 1.4. The results obtained in this work were announced in [9]; they complete similar results on solutions of quasilinear elliptic partial differential inequalities which were announced in [8]. To prove these results we further develop the approach that was proposed for solving similar problems in wide classes of partial differential equations and inequalities in [6].

Remark 1.5. Note that the approach used in this paper can be applied, for example, in the context of similar problems with sub-Laplacians on stratified Lie groups, currently a research area of great and growing interest.

DEFINITION 1.6. Let $q > 0$, let $\hat{q} = \max\{1, q\}$ and let D be an arbitrary bounded domain in $\mathbb{R}^n$. By $W^{\mathcal{L},q}(D)$ we denote the completion of the function space $C^\infty(D)$ with respect to the norm

$$\|v\|_{W^{\mathcal{L},q}(D)} = \left[\int_D \sum_{i,j=1}^n a_{ij} v_{x_i} v_{x_j} \,\mathrm{d}x\right]^{1/2} + \left[\int_D |v|^{\hat{q}} \,\mathrm{d}x\right]^{1/\hat{q}}. \tag{1.6}$$

DEFINITION 1.7. Let $q > 0$. A function v belongs to the function space $W^{\mathcal{L},q}_{\mathrm{loc}}(\mathbb{R}^n)$ if $v \in W^{\mathcal{L},q}(D)$ for every bounded domain D in $\mathbb{R}^n$.

DEFINITION 1.8. Let $q > 0$. By an entire weak solution of inequality (1.1) we mean a pair (u, v) of functions $u(x)$ and $v(x)$ which are measurable on $\mathbb{R}^n$, belong to the function space $W^{\mathcal{L},q}_{\mathrm{loc}}(\mathbb{R}^n)$ and satisfy the integral inequality

$$\int_{\mathbb{R}^n} \left[\sum_{i,j=1}^n a_{ij} u_{x_i} \varphi_{x_j} - |u|^{q-1} u\varphi\right] \mathrm{d}x \geqslant \int_{\mathbb{R}^n} \left[\sum_{i,j=1}^n a_{ij} v_{x_i} \varphi_{x_j} - |v|^{q-1} v\varphi\right] \mathrm{d}x \tag{1.7}$$

for every non-negative function $\varphi \in C^\infty(\mathbb{R}^n)$ with compact support.

Analogous definitions for solutions of inequalities (1.4) and (1.5), which are the special cases of inequality (1.1) for $v = 0$ and $u = 0$, respectively, can immediately be obtained from definition 1.8.

DEFINITION 1.9. Let $q > 0$. By an entire weak solution of inequality (1.4) (respectively, (1.5)) we mean a function $w(x)$ measurable on $\mathbb{R}^n$ which belongs to the function space $W^{\mathcal{L},q}_{\mathrm{loc}}(\mathbb{R}^n)$ and satisfies the integral inequality

$$\int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij} w_{x_i} \varphi_{x_j} \,\mathrm{d}x - \int_{\mathbb{R}^n} |w|^{q-1} w\varphi \,\mathrm{d}x \geqslant 0 \quad (\text{respectively, } \leqslant 0) \tag{1.8}$$

for every non-negative function $\varphi \in C^\infty(\mathbb{R}^n)$ with compact support.

REMARK 1.10. We understand formulae (1.7) and (1.8) in the sense discussed, for example, in [5, 12].

2. Results

THEOREM 2.1. *Let $n = 2$, let $q > 0$ and let (u, v) be an entire weak solution of inequality (1.1) on $\mathbb{R}^n$ such that $u(x) \geqslant v(x)$. Then $u(x) = v(x)$ on $\mathbb{R}^n$.*

THEOREM 2.2. *Let $n > 2$, $1 < q \leqslant n/(n-2)$, and let (u, v) be an entire weak solution of inequality (1.1) on $\mathbb{R}^n$ such that $u(x) \geqslant v(x)$. Then $u(x) = v(x)$ on $\mathbb{R}^n$.*

THEOREM 2.3. *Let $n > 2$ and $q > n/(n-2)$. Then there exists no entire weak solution (u, v) of inequality (1.1) on $\mathbb{R}^n$ such that $u(x) \geqslant v(x)$ and the relation*

$$\limsup_{R\to+\infty} R^{-n+2(q-\nu)/(q-1)} \int_{|x|<R} (u(x) - v(x))^{q-\nu} \,\mathrm{d}x = +\infty \tag{2.1}$$

holds with any given $\nu \in (0, 1)$.

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Example 2.4. To illustrate the sharpness of theorem 2.3 we note that, for $n > 2$, $q > n/(n-2)$, and a suitable constant $c > 0$, the pair (u, v) of functions

$$u(x) = c(1 + |x|^2)^{-1/(q-1)} \quad \text{and} \quad v(x) = 0 \tag{2.2}$$

is an entire weak solution of inequality (1.1) on $\mathbb{R}^n$ with $\mathcal{L} = \Delta$ such that, for any given $\nu \in (0, 1)$, the relation

$$\limsup_{R\to+\infty} R^{-n+2(q-\nu)/(q-1)} \int_{|x|<R} (u(x) - v(x))^{q-\nu} \, \mathrm{d}x = C_1, \tag{2.3}$$

with C_1 a certain positive constant, holds.

The following two statements are simple special cases of theorem 2.3.

Theorem 2.5. *Let $n > 2$ and $q > n/(n-2)$. Then, for any given constants $c > 0$ and $\nu > 0$, there exists no entire weak solution (u, v) of inequality (1.1) on $\mathbb{R}^n$ such that $u(x) \geqslant v(x) + c(1 + |x|^2)^{-1/(q-1)+\nu}$.*

Theorem 2.6. *Let $n > 2$ and $q > n/(n-2)$. Then, for any given constant $c > 0$, there exists no entire weak solution (u, v) of inequality (1.1) on $\mathbb{R}^n$ such that $u(x) \geqslant v(x) + c$.*

Theorem 2.7. *Let $n > 2$ and $0 < q < 1$ or $n > 2$ and $q > n/(n-2)$. Then there exists no entire weak solution (u, v) of inequality (1.1) on $\mathbb{R}^n$ such that $u(x) \geqslant v(x)$ and the relation*

$$\limsup_{R\to+\infty} R^{2-n} \int_{\{|x|<R\}\cap\{x\in\mathbb{R}^n : u(x)\neq v(x)\}} (|u|^{q-1}u - |v|^{q-1}v)(u - v)^{-1} \, \mathrm{d}x = +\infty \tag{2.4}$$

holds.

Example 2.8. To illustrate the sharpness of theorem 2.7 in the case when $n > 2$ and $0 < q < 1$, we note that, for $n > 2$, $0 < q < 1$, $0 < \mu < (n-2)/n$, and a suitable constant $c > 0$, the pair (u, v) of functions

$$u(x) = c(1 + |x|^2)^{1/(1-q)} + (1 + |x|^2)^{-\mu}, \tag{2.5}$$

$$v(x) = c(1 + |x|^2)^{1/(1-q)} \tag{2.6}$$

is an entire weak solution of inequality (1.1) on $\mathbb{R}^n$ with $\mathcal{L} = \Delta$ such that $u(x) \geqslant v(x)$ and the relation

$$\limsup_{R\to+\infty} R^{2-n} \int_{\{|x|<R\}\cap\{x\in\mathbb{R}^n : u(x)\neq v(x)\}} (|u|^{q-1}u - |v|^{q-1}v)(u - v)^{-1} \, \mathrm{d}x = C_2, \tag{2.7}$$

with C_2 a certain positive constant, holds.

Example 2.9. We also note that for $n > 2$, $0 < q < 1$, $0 < \mu < (n-2)/n$, $\lambda > 1/(1-q)$, and a suitable constant $c > 0$, the pair (u, v) of functions

$$u(x) = c(1 + |x|^2)^{\lambda} + (1 + |x|^2)^{-\mu}, \tag{2.8}$$

$$v(x) = c(1 + |x|^2)^{\lambda} \tag{2.9}$$

is an entire weak solution of inequality (1.1) on $\mathbb{R}^n$ with $\mathcal{L} = \Delta$ such that $u(x) \geqslant v(x)$ and the relation

$$\limsup_{R\to+\infty} R^{2-n} \int_{\{|x|<R\}\cap\{x\in\mathbb{R}^n : u(x)\neq v(x)\}} (|u|^{q-1}u - |v|^{q-1}v)(u-v)^{-1}\,\mathrm{d}x = 0 \quad (2.10)$$

holds.

Example 2.10. To illustrate the importance of condition (2.4) in theorem 2.7, we note that, for $n > 2$, $0 < q < 1$, $0 < \mu < (n-2)/n$, $\lambda \geqslant (1+\mu)/(1-q)$, a suitable constant $c > 0$, and any given constant $\kappa > 0$, the pair (u, v) of functions

$$u(x) = c(1+|x|^2)^\lambda + \kappa + (1+|x|^2)^{-\mu}, \quad (2.11)$$

$$v(x) = c(1+|x|^2)^\lambda \quad (2.12)$$

is an entire weak solution of inequality (1.1) on $\mathbb{R}^n$ with $\mathcal{L} = \Delta$ such that $u(x) \geqslant v(x) + \kappa$ and the relation

$$\limsup_{R\to+\infty} R^{2-n} \int_{\{|x|<R\}\cap\{x\in\mathbb{R}^n : u(x)\neq v(x)\}} (|u|^{q-1}u - |v|^{q-1}v)(u-v)^{-1}\,\mathrm{d}x = 0 \quad (2.13)$$

holds.

Example 2.11. To illustrate the sharpness of theorem 2.7 in the case when $n > 2$ and $q > n/(n-2)$, we note that, for $n > 2$, $q > n/(n-2)$, and a suitable constant $c > 0$, the pair (u, v) of functions (2.2) is an entire weak solution of inequality (1.1) on $\mathbb{R}^n$ with $\mathcal{L} = \Delta$ such that the relation

$$\limsup_{R\to+\infty} R^{2-n} \int_{\{|x|<R\}\cap\{x\in\mathbb{R}^n : u(x)\neq v(x)\}} (|u|^{q-1}u - |v|^{q-1}v)(u-v)^{-1}\,\mathrm{d}x = C_3, \quad (2.14)$$

with C_3 a certain positive constant, holds.

Theorem 2.12. *Let $n > 2$ and let $q = 1$. Then there exists no entire weak solution (u, v) of inequality (1.1) on $\mathbb{R}^n$ such that $u(x) > v(x)$.*

3. Proofs

In what follows, a 'smooth' function is a C^∞-function, and $B(R)$ is an open ball on $\mathbb{R}^n$ centred at the origin with radius $r > 0$.

Proof of theorem 2.1. Let $n = 2$, let $q > 0$ and let (u, v) be an entire weak solution of inequality (1.1) on $\mathbb{R}^n$ such that $u(x) \geqslant v(x)$. It then follows from (1.7) that the inequality

$$\int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij}(u-v)_{x_i}\varphi_{x_j}\,\mathrm{d}x \geqslant \int_{\mathbb{R}^n} (|u|^{q-1}u - |v|^{q-1}v)\varphi\,\mathrm{d}x \quad (3.1)$$

holds for every non-negative function $\varphi \in C^\infty(\mathbb{R}^n)$ with compact support. Let R and ε be arbitrary positive numbers, and let $\zeta : \mathbb{R}^n \to [0, 1]$ be a smooth function which equals 1 on $\overline{B(R/2)}$ and 0 outside $B(R)$. Without loss of generality, we substitute $\varphi(x) = (w(x)+\varepsilon)^{-\nu}\zeta^s(x)$ as a test function in inequality (3.1), where $w(x) = u(x) - v(x)$, $\nu > 1$ and $s \geqslant 2$ are real numbers. Integrating by parts in (3.1) we have

$$-\nu \int_{B(R)} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w+\varepsilon)^{-\nu-1}\zeta^s \,\mathrm{d}x + s\int_{B(R)} \sum_{i,j=1}^{n} a_{ij} w_{x_i} \zeta_{x_j} (w+\varepsilon)^{-\nu}\zeta^{s-1}\,\mathrm{d}x \equiv I_1 + I_2 \geqslant \int_{B(R)} (|u|^{q-1}u - |v|^{q-1}v)(w+\varepsilon)^{-\nu}\zeta^s\,\mathrm{d}x. \qquad (3.2)$$

Furthermore, using the obvious inequality

$$|I_2| \leqslant \int_{B(R)} s\biggl(\sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j}\biggr)^{1/2}\biggl(\sum_{i,j=1}^{n} a_{ij}\zeta_{x_i}\zeta_{x_j}\biggr)^{1/2}(w+\varepsilon)^{-\nu}\zeta^{s-1}\,\mathrm{d}x \qquad (3.3)$$

and estimating the integrand on the right-hand side of (3.3) by Young's inequality

$$AB \leqslant \rho A^{\beta/(\beta-1)} + \rho^{1-\beta}B^\beta \qquad (3.4)$$

with $\rho = \frac{1}{2}\nu$ and $\beta = 2$, we arrive at

$$|I_2| \leqslant \tfrac{1}{2}\nu \int_{B(R)} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j}(w+\varepsilon)^{-\nu-1}\zeta^s\,\mathrm{d}x + \int_{B(R)} c_1 \sum_{i,j=1}^{n} a_{ij}\zeta_{x_i}\zeta_{x_j}(w+\varepsilon)^{1-\nu}\zeta^{s-2}\,\mathrm{d}x. \qquad (3.5)$$

Henceforth, we use the symbols $c_i, i = 1, 2, \dots,$ to denote constants depending possibly on n, q, s, ν, and the coefficients of the operator $\mathcal{L}$ but not on R and ε. Now, from (3.2) and (3.5) we obtain the inequality

$$\int_{B(R)} c_1 \sum_{i,j=1}^{n} a_{ij}\zeta_{x_i}\zeta_{x_j}(w+\varepsilon)^{1-\nu}\zeta^{s-2}\,\mathrm{d}x \geqslant \int_{B(R)} (|u|^{q-1}u - |v|^{q-1}v)(w+\varepsilon)^{-\nu}\zeta^s\,\mathrm{d}x + \tfrac{1}{2}\nu\int_{B(R)} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j}(w+\varepsilon)^{-\nu-1}\zeta^s\,\mathrm{d}x. \qquad (3.6)$$

Since $\nu > 1$, the function $w(x)$ is non-negative, and the first integral on the right-hand side of (3.6) is also non-negative, from (3.6), we easily obtain the inequality

$$c_2\varepsilon^{1-\nu}\int_{B(R)} \sum_{i,j=1}^{n} a_{ij}\zeta_{x_i}\zeta_{x_j}\,\mathrm{d}x \geqslant \int_{B(R)} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j}(w+\varepsilon)^{-\nu-1}\zeta^s\,\mathrm{d}x. \qquad (3.7)$$

Now, without loss of generality, for any $R > 0$, in (3.7) we choose the function $\zeta(x)$ in the form $\zeta(x) = \psi(|x|/R)$, where $\psi : [0, \infty) \to [0, 1]$ is a smooth function which equals 1 on $[0, \frac{1}{2}]$ and 0 on $[1, \infty)$ and such that the inequality

$$|\nabla\zeta| \leqslant c_3 R^{-1} \tag{3.8}$$

holds. Then, owing to the boundedness of the coefficients of the operator $\mathcal{L}$, there exists a constant, M, such that the inequality

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leqslant M|\xi|^2 \tag{3.9}$$

holds for all $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ at almost all $x \in \mathbb{R}^n$ and, therefore, relations (3.7)–(3.9) yield the inequality

$$c_4\varepsilon^{1-\nu}R^{n-2} \geqslant \int_{B(R)} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w+\varepsilon)^{-\nu-1}\,\mathrm{d}x. \tag{3.10}$$

Since $n = 2$, the integral on the right-hand side of (3.10) is bounded for all $R > 0$. Hence, owing to the monotonicity, the relation

$$\int_{B(R_k)\setminus B(R_k/2)} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w+\varepsilon)^{-\nu-1}\,\mathrm{d}x \to 0 \tag{3.11}$$

holds for all sequences $R_k \to \infty$. On the other hand, from (3.2), (3.3) we arrive at

$$s\int_{B(R)} \biggl(\sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j}\biggr)^{1/2} \biggl(\sum_{i,j=1}^{n} a_{ij} \zeta_{x_i}\zeta_{x_j}\biggr)^{1/2} (w+\varepsilon)^{-\nu}\zeta^{s-1}\,\mathrm{d}x \\ \geqslant \int_{B(r)} (|u|^{q-1}u - |v|^{q-1}v)(w+\varepsilon)^{-\nu}\,\mathrm{d}x. \tag{3.12}$$

Estimating the integrand on the left-hand side of (3.12) by Hölder's inequality, we obtain

$$c_5\biggl(\int_{B(R)\setminus B(R/2)} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w+\varepsilon)^{-\nu-1}\,\mathrm{d}x\biggr)^{1/2} \\ \times \biggl(\int_{B(R)} \sum_{i,j=1}^{n} a_{ij} \zeta_{x_i}\zeta_{x_j} (w+\varepsilon)^{1-\nu}\,\mathrm{d}x\biggr)^{1/2} \\ \geqslant \int_{B(R/2)} (|u|^{q-1}u - |v|^{q-1}v)(w+\varepsilon)^{-\nu}\,\mathrm{d}x. \tag{3.13}$$

Now, as above, for any $R > 0$, in (3.13) we choose the function $\zeta(x)$ in the form $\zeta(x) = \psi(|x|/R)$, where $\psi : [0, \infty) \to [0, 1]$ is a smooth function which equals 1 on $[0, \frac{1}{2}]$ and 0 on $[1, \infty)$ and such that the inequality (3.8) holds. Then, since $\nu > 1$

and $w(x)$ is non-negative, (3.8), (3.9) and (3.13) yield the inequality

$$c_6\varepsilon^{(1-\nu)/2}R^{(n-2)/2}\biggl(\int_{B(R)\setminus B(R/2)}\sum_{i,j=1}^{n}a_{ij}w_{x_i}w_{x_j}(w+\varepsilon)^{-\nu-1}\,\mathrm{d}x\biggr)^{1/2} \geqslant \int_{B(R/2)}(|u|^{q-1}u-|v|^{q-1}v)(w+\varepsilon)^{-\nu}\,\mathrm{d}x. \quad (3.14)$$

It then follows directly from (3.11) and (3.14) that for $n=2$ the relation

$$\int_{B(R_k/2)}(|u|^{q-1}u-|v|^{q-1}v)(w+\varepsilon)^{-\nu}\,\mathrm{d}x\to 0 \quad (3.15)$$

holds as $R_k\to\infty$. This in turn easily yields, from (3.15), that $u(x)=v(x)$ on $\mathbb{R}^n$. □

Proof of theorem 2.2. Let $n>2$, $1<q\leqslant n/(n-2)$, and (u,v) be an entire weak solution of inequality (1.1) on $\mathbb{R}^n$ such that $u(x)\geqslant v(x)$. By the inequality

$$(|u|^{q-1}u-|v|^{q-1}v)(u-v)\geqslant c_1|u-v|^{q+1}, \quad (3.16)$$

which holds with any $q\geqslant 1$ and a certain positive constant c_1 depending only on q, from (1.7) we obtain the inequality

$$\int_{\mathbb{R}^n}\sum_{i,j=1}^{n}a_{ij}(u-v)_{x_i}\varphi_{x_j}\,\mathrm{d}x\geqslant c_1\int_{\mathbb{R}^n}(u-v)^q\varphi\,\mathrm{d}x, \quad (3.17)$$

which holds for every non-negative function $\varphi\in C^\infty(\mathbb{R}^n)$ with compact support; actually, $c_1=2^{1-q}$ (see, for example, [3]), but we do not need the precise value of this constant. Let R and ε be arbitrary positive numbers, and let $\zeta:\mathbb{R}^n\to[0,1]$ be a smooth function which equals 1 on $\overline{B(R/2)}$ and 0 outside $B(R)$. Without loss of generality, we substitute $\varphi(x)=(w(x)+\varepsilon)^{-\nu}\zeta^s(x)$ as a test function in (3.17), where $w(x)=u(x)-v(x)$, $s\geqslant 2$ and $\nu\in(0,1)\cap(0,q-1)$. Integrating by parts in (3.17) we obtain

$$-\nu\int_{B(R)}\sum_{i,j=1}^{n}a_{ij}w_{x_i}w_{x_j}(w+\varepsilon)^{-\nu-1}\zeta^s\,\mathrm{d}x + s\int_{B(R)}\sum_{i,j=1}^{n}a_{ij}w_{x_i}\zeta_{x_j}(w+\varepsilon)^{-\nu}\zeta^{s-1}\,\mathrm{d}x \equiv I_1+I_2\geqslant c_1\int_{B(R)}w^q(w+\varepsilon)^{-\nu}\zeta^s\,\mathrm{d}x. \quad (3.18)$$

As above, using the obvious inequality

$$|I_2|\leqslant\int_{B(R)}s\biggl(\sum_{i,j=1}^{n}a_{ij}w_{x_i}w_{x_j}\biggr)^{1/2}\biggl(\sum_{i,j=1}^{n}a_{ij}\zeta_{x_i}\zeta_{x_j}\biggr)^{1/2}(w+\varepsilon)^{-\nu}\zeta^{s-1}\,\mathrm{d}x \quad (3.19)$$

and estimating the integrand on the right-hand side of (3.19) by Young's inequality (3.4) with $\rho = \frac{1}{2}\nu$ and $\beta = 2$, we arrive at

$$|I_2| \leqslant \tfrac{1}{2}\nu \int_{B(R)} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w+\varepsilon)^{-\nu-1} \zeta^s \,\mathrm{d}x + \int_{B(R)} c_2 \sum_{i,j=1}^{n} a_{ij} \zeta_{x_i} \zeta_{x_j} (w+\varepsilon)^{1-\nu} \zeta^{s-2} \,\mathrm{d}x. \quad (3.20)$$

Also, as above, we use the symbols c_i, $i = 1, 2, \dots,$ to denote constants depending possibly on n, q, s, ν, and the coefficients of the operator $\mathcal{L}$ but not on R and ε. Furthermore, from (3.18) and (3.20) we have

$$\int_{B(R)} c_2 \sum_{i,j=1}^{n} a_{ij} \zeta_{x_i} \zeta_{x_j} (w+\varepsilon)^{1-\nu} \zeta^{s-2} \,\mathrm{d}x \geqslant c_1 \int_{B(R)} w^q (w+\varepsilon)^{-\nu} \zeta^s \,\mathrm{d}x + \tfrac{1}{2}\nu \int_{B(R)} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w+\varepsilon)^{-\nu-1} \zeta^s \,\mathrm{d}x. \quad (3.21)$$

Estimating the integrand on the left-hand side of (3.21) by Young's inequality (3.4) with

$$\rho = \tfrac{1}{2}c_1 \quad \text{and} \quad \beta = (q-\nu)/(q-1),$$

we obtain

$$\tfrac{1}{2}c_1 \int_{B(R)\setminus B(R/2)} (w+\varepsilon)^{q-\nu} \zeta^s \,\mathrm{d}x + c_3 \int_{B(R)} \biggl(\sum_{i,j=1}^{n} a_{ij} \zeta_{x_i} \zeta_{x_j} \biggr)^{(q-\nu)/(q-1)} \zeta^{(s-2)(q-\nu)/(q-1)} \,\mathrm{d}x \geqslant c_1 \int_{B(R)} w^q (w+\varepsilon)^{-\nu} \zeta^s \,\mathrm{d}x + \tfrac{1}{2}\nu \int_{B(R)} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w+\varepsilon)^{-\nu-1} \zeta^s \,\mathrm{d}x. \quad (3.22)$$

Now, using (3.22), we estimate the integral

$$\int_{B(R)} w^q \zeta^s \,\mathrm{d}x.$$

To this end, we substitute $\varphi(x) = \zeta^s(x)$ in (3.17) and arrive at

$$c_1 \int_{B(R)} w^q \zeta^s \,\mathrm{d}x \leqslant s \int_{B(R)} \sum_{i,j=1}^{n} a_{ij} w_{x_i} \zeta_{x_j} \zeta^{s-1} \,\mathrm{d}x. \quad (3.23)$$

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Estimating the right-hand side of (3.23) by using Hölder's inequality, we have

$$c_1 \int_{B(R)} w^q \zeta^s \,\mathrm{d}x \leqslant s\biggl(\int_{B(R)} \sum_{i,j=1}^{n} a_{ij}\zeta_{x_i}\zeta_{x_j}(w+\varepsilon)^{\nu+1}\zeta^{s-2}\,\mathrm{d}x \biggr)^{1/2} \times \biggl(\int_{B(R)} \sum_{i,j=1}^{n} a_{ij}w_{x_i}w_{x_j}(w+\varepsilon)^{-\nu-1}\zeta^{s}\,\mathrm{d}x \biggr)^{1/2} \tag{3.24}$$

which holds with any $\nu \in (0,1)\cap(0,q-1)$ and $\varepsilon > 0$. Since the inequality

$$\int_{B(R)} \sum_{i,j=1}^{n} a_{ij}\zeta_{x_i}\zeta_{x_j}(w+\varepsilon)^{\nu+1}\zeta^{s-2}\,\mathrm{d}x \leqslant \biggl(\int_{B(R)\setminus B(R/2)} (w+\varepsilon)^{d(1+\nu)}\zeta^{s}\,\mathrm{d}x \biggr)^{1/d} \times \biggl(\int_{B(R)} \biggl(\sum_{i,j=1}^{n} a_{ij}\zeta_{x_i}\zeta_{x_j} \biggr)^{d/(d-1)} \zeta^{s-2d/(d-1)}\,\mathrm{d}x \biggr)^{(d-1)/d} \tag{3.25}$$

formally holds with any $d > 1$, then, by choosing for any sufficiently small ν from the interval $(0,1)\cap(0,q-1)$ the parameter $d = q/(1+\nu)$ such that $d(1+\nu) = q$, from (3.24) and (3.25) we obtain the inequality

$$c_1 \int_{B(R)} w^q \zeta^s \,\mathrm{d}x \leqslant s\biggl(\int_{B(R)} \biggl(\sum_{i,j=1}^{n} a_{ij}\zeta_{x_i}\zeta_{x_j} \biggr)^{d/(d-1)} \zeta^{s-2d/(d-1)}\,\mathrm{d}x \biggr)^{(d-1)/2d} \times \biggl(\int_{B(R)\setminus B(R/2)} (w+\varepsilon)^{q}\zeta^{s}\,\mathrm{d}x \biggr)^{1/2d} \times \biggl(\int_{B(R)} \sum_{i,j=1}^{n} a_{ij}w_{x_i}w_{x_j}(w+\varepsilon)^{-\nu-1}\zeta^{s}\,\mathrm{d}x \biggr)^{1/2}. \tag{3.26}$$

Now estimating the last term on the right-hand side of (3.26) by using inequality (3.22), we have

$$c_1 \int_{B(R)} w^q \zeta^s \,\mathrm{d}x \leqslant s\biggl(\int_{B(R)} \biggl(\sum_{i,j=1}^{n} a_{ij}\zeta_{x_i}\zeta_{x_j} \biggr)^{d/(d-1)} \zeta^{s-2d/(d-1)}\,\mathrm{d}x \biggr)^{(d-1)/2d} \times \biggl(\int_{B(R)\setminus B(R/2)} (w+\varepsilon)^{q}\zeta^{s}\,\mathrm{d}x \biggr)^{1/2d} \times \biggl(c_4 \int_{B(R)} \biggl(\sum_{i,j=1}^{n} a_{ij}\zeta_{x_i}\zeta_{x_j} \biggr)^{(q-\nu)/(q-1)} \zeta^{(s-2)(q-\nu)/(q-1)}\,\mathrm{d}x + \frac{c_1}{\nu} \int_{B(R)\setminus B(R/2)} (w+\varepsilon)^{q-\nu}\zeta^{s}\,\mathrm{d}x - \frac{2c_1}{\nu} \int_{B(R)} w^q (w+\varepsilon)^{-\nu}\zeta^{s}\,\mathrm{d}x \biggr)^{1/2}. \tag{3.27}$$

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In (3.27), passing to the limit as $\varepsilon \to 0$ as justified by Lebesgue's theorem (see, for example, [4, p. 303]), we arrive at

$$\begin{aligned}\int_{B(R)} w^q \zeta^s \,\mathrm{d}x \leqslant c_5 \biggl(\int_{B(R)} \biggl(\sum_{i,j=1}^{n} a_{ij} \zeta_{x_i} \zeta_{x_j} \biggr)^{d/(d-1)} \zeta^{s-2d/(d-1)} \,\mathrm{d}x \biggr)^{(d-1)/2d} \\ \times \biggl(\int_{B(R)\setminus B(R/2)} w^q \zeta^s \,\mathrm{d}x \biggr)^{1/2d} \\ \times \biggl(\int_{B(R)} \biggl(\sum_{i,j=1}^{n} a_{ij} \zeta_{x_i} \zeta_{x_j} \biggr)^{(q-\nu)/(q-1)} \zeta^{(s-2)(q-\nu)/(q-1)} \,\mathrm{d}x \biggr)^{1/2},\end{aligned} \tag{3.28}$$

which, in turn, yields, for sufficiently large s, the inequality

$$\begin{aligned}\biggl(\int_{B(R/2)} w^q \,\mathrm{d}x \biggr)^{(2d-1)/2d} \leqslant c_5 \biggl(\int_{B(R)} \biggl(\sum_{i,j=1}^{n} a_{ij} \zeta_{x_i} \zeta_{x_j} \biggr)^{d/(d-1)} \mathrm{d}x \biggr)^{(d-1)/2d} \\ \times \biggl(\int_{B(R)} \biggl(\sum_{i,j=1}^{n} a_{ij} \zeta_{x_i} \zeta_{x_j} \biggr)^{(q-\nu)/(q-1)} \mathrm{d}x \biggr)^{1/2}.\end{aligned} \tag{3.29}$$

Now, without loss of generality, for any $R > 0$, in (3.29) we choose the function $\zeta(x)$ in the form $\zeta(x) = \psi(|x|/R)$, where $\psi : [0, \infty) \to [0, 1]$ is a smooth function which equals 1 on $[0, \frac{1}{2}]$ and 0 on $[1, \infty)$ and such that the inequality

$$|\nabla \zeta| \leqslant c_6 R^{-1} \tag{3.30}$$

holds. Then, owing to the boundedness of the coefficients of the operator $\mathcal{L}$, (3.29) and (3.30) yield the inequality

$$\biggl(\int_{B(R)} w^q \,\mathrm{d}x \biggr)^{(2d-1)/2d} \leqslant c_7 R^p, \tag{3.31}$$

which holds with

$$p = \frac{n - p_1}{p_1} + \frac{n - p_2}{2}, \tag{3.32}$$

where

$$p_1 = \frac{2d}{d-1} \quad \text{and} \quad p_2 = \frac{2(q-\nu)}{q-1}. \tag{3.33}$$

It is easy to calculate that

$$p = \frac{(n-2)(2q-1-\nu)}{2q(q-1)} \biggl(q - \frac{n}{n-2} \biggr). \tag{3.34}$$

It then follows from (3.34) that, for $1 < q < n/(n-2)$ and any $\nu \in (0,1) \cap (0, q-1)$, the parameter p is negative. Therefore, in this case, (3.31) yields that

$$\int_{\mathbb{R}^n} w^q \,\mathrm{d}x = 0$$

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and, therefore, that $u(x)=v(x)$ on $\mathbb{R}^n$. Also, it follows from (3.31) and (3.34) that if $n>2$ and $q=n/(n-2)$, then the integral

$$\int_{\mathbb{R}^n} w^q \,\mathrm{d}x$$

is bounded. Hence, owing to the monotonicity, the relation

$$\int_{B(R_k)\setminus B(R_k/2)} w^q \,\mathrm{d}x \to 0 \tag{3.35}$$

holds for all sequences $R_k\to\infty$. On the other hand, from (3.28), for sufficiently large s, owing to the boundedness of the coefficients of the operator $\mathcal{L}$, we have

$$\begin{aligned}\int_{B(R/2)} w^q \,\mathrm{d}x \leqslant c_5 \biggl(\int_{B(R)\setminus B(R/2)} w^q \,\mathrm{d}x\biggr)^{1/2d} \\ \times \biggl(\int_{B(R)} |\nabla\zeta|^{2d/(d-1)} \,\mathrm{d}x\biggr)^{(d-1)/2d} \\ \times \biggl(\int_{B(R)} |\nabla\zeta|^{2(q-\nu)/(q-1)} \,\mathrm{d}x\biggr)^{1/2}.\end{aligned} \tag{3.36}$$

Now, by choosing in (3.36) the function $\zeta(x)$ of the form indicated above, from (3.30) and (3.36) we arrive at

$$\int_{B(R/2)} w^q \,\mathrm{d}x \leqslant c_8 R^p \biggl(\int_{B(R)\setminus B(R/2)} w^q \,\mathrm{d}x\biggr)^{1/2d}, \tag{3.37}$$

where p is given by (3.34). Furthermore, (3.34), (3.35) and (3.37) imply directly, for $n>2$ and $q=n/(n-2)$, the relation

$$\int_{B(R_k)} w^q \,\mathrm{d}x \to 0, \tag{3.38}$$

which holds as $R_k\to\infty$. This, in turn, again implies that

$$\int_{\mathbb{R}^n} w^q \,\mathrm{d}x = 0$$

and, therefore, that $u(x)=v(x)$ on $\mathbb{R}^n$. □

Proof of theorem 2.3. The proof is by contradiction. Let $n>2$ and $q>n/(n-2)$. Assume that there exists an entire weak solution (u,v) of inequality (1.1) on $\mathbb{R}^n$ such that $u(x)\geqslant v(x)$ and relation (2.1) holds with any given $\nu\in(0,1)$. From (1.7) and (3.16) we then have the inequality

$$\int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij}(u-v)_{x_i}\varphi_{x_j} \,\mathrm{d}x \geqslant c_1 \int_{\mathbb{R}^n} (u-v)^q \varphi \,\mathrm{d}x, \tag{3.39}$$

with c_1 the constant from (3.16), which holds for every non-negative function $\varphi\in C^\infty(\mathbb{R}^n)$ with compact support. Let R and ε be arbitrary positive numbers, and let $\zeta : \mathbb{R}^n \to [0,1]$ be a smooth function which equals 1 on $\overline{B(R/2)}$ and 0 outside $B(R)$. Without loss of generality, we substitute $\varphi(x) = (w(x)+\varepsilon)^{-\nu}\zeta^s(x)$ as a test function in inequality (3.39), where $w(x) = u(x) - v(x)$, $s \geqslant 2$ and $1 > \nu > 0$. By integrating by parts in (3.39) we arrive at

$$-\nu \int_{B(R)} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w+\varepsilon)^{-\nu-1}\zeta^s \,\mathrm{d}x + s\int_{B(R)} \sum_{i,j=1}^{n} a_{ij} w_{x_i}\zeta_{x_j}(w+\varepsilon)^{-\nu}\zeta^{s-1}\,\mathrm{d}x \equiv I_1 + I_2 \geqslant c_1 \int_{B(R/2)} w^q (w+\varepsilon)^{-\nu}\zeta^s \,\mathrm{d}x. \quad (3.40)$$

Since

$$|I_2| \leqslant \int_{B(R)} s\Big(\sum_{i,j=1}^{n} a_{ij}\zeta_{x_i}\zeta_{x_j}\Big)^{1/2}\Big(\sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j}\Big)^{1/2}(w+\varepsilon)^{-\nu}\zeta^{s-1}\,\mathrm{d}x, \quad (3.41)$$

estimating the integrand on the right-hand side of (3.41) by Young's inequality (3.4) with $\rho = \frac{1}{2}\nu$ and $\beta = 2$, we obtain the inequality

$$|I_2| \leqslant \tfrac{1}{2}\nu \int_{B(R)} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j}(w+\varepsilon)^{-\nu-1}\zeta^s\,\mathrm{d}x + c_2 \int_{B(R)} \sum_{i,j=1}^{n} a_{ij}\zeta_{x_i}\zeta_{x_j}(w+\varepsilon)^{1-\nu}\zeta^{s-2}\,\mathrm{d}x. \quad (3.42)$$

As above, we use the symbols c_i, $i = 1,2,\dots$, to denote constants depending possibly on n, q, s, ν, and the coefficients of the operator $\mathcal{L}$ but not on R and ε. Furthermore, from (3.40) and (3.42) we have

$$c_2 \int_{B(R)} \sum_{i,j=1}^{n} a_{ij}\zeta_{x_i}\zeta_{x_j}(w+\varepsilon)^{1-\nu}\zeta^{s-2}\,\mathrm{d}x \geqslant \int_{B(R)} w^q(w+\varepsilon)^{-\nu}\zeta^s\,\mathrm{d}x. \quad (3.43)$$

In (3.43), passing to the limit as $\varepsilon \to 0$, as justified by Lebesgue's theorem (see, for example, [4, p. 303]), we have

$$c_2 \int_{B(R)} \sum_{i,j=1}^{n} a_{ij}\zeta_{x_i}\zeta_{x_j} w^{1-\nu}\zeta^{s-2}\,\mathrm{d}x \geqslant \int_{B(R)} w^{q-\nu}\zeta^s\,\mathrm{d}x. \quad (3.44)$$

Since, by the hypotheses of theorem 2.3, $q > 1$, in (3.44) choose $s = 2(q-\nu)/(q-1)$ such that $(s-2)(q-\nu)/(1-\nu) = s$. Then, estimating the left-hand side of (3.44) by Hölder's inequality, we obtain

$$c_3\Big(\int_{B(R)}\Big(\sum_{i,j=1}^{n} a_{ij}\zeta_{x_i}\zeta_{x_j}\Big)^{(q-\nu)/(q-1)}\mathrm{d}x\Big)^{(q-1)/(q-\nu)} \times \Big(\int_{B(R)} w^{q-\nu}\zeta^s\,\mathrm{d}x\Big)^{(1-\nu)/(q-\nu)} \geqslant \int_{B(R)} w^{q-\nu}\zeta^s\,\mathrm{d}x. \quad (3.45)$$

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In turn, the inequality

$$c_3 \int_{B(R)} \biggl(\sum_{i,j=1}^{n} a_{ij} \zeta_{x_i} \zeta_{x_j} \biggr)^{(q-\nu)/(q-1)} \mathrm{d}x \geqslant \int_{B(R)} w^{q-\nu} \zeta^s \, \mathrm{d}x \tag{3.46}$$

follows directly from (3.45). As above, without loss of generality, for any $R > 0$, in (3.46) we choose the function $\zeta(x)$ in the form $\zeta(x) = \psi(|x|/R)$, where $\psi : [0, \infty) \to [0, 1]$ is a smooth function which equals 1 on $[0, \frac{1}{2}]$ and 0 on $[1, \infty)$ and such that the inequality

$$|\nabla \zeta| \leqslant c_4 R^{-1} \tag{3.47}$$

holds. Then, owing to the boundedness of the coefficients of the operator $\mathcal{L}$, (3.46) and (3.47) yield

$$c_5 R^{n-2(q-\nu)/(q-1)} \geqslant \int_{B(R)} w^{q-\nu} \, \mathrm{d}x. \tag{3.48}$$

From (3.48) we arrive at the inequality

$$\limsup_{R \to +\infty} R^{-n+2(q-\nu)/(q-1)} \int_{B(R)} (u(x) - v(x))^{q-\nu} \, \mathrm{d}x < +\infty, \tag{3.49}$$

which holds with any fixed $\nu \in (0, 1)$, and which contradicts the hypotheses of theorem 2.3; namely, it contradicts (2.1). Thus, we have a contradiction of our assumption. □

Proof of theorem 2.7. The proof is by contradiction. Let $n > 2$ and $q > 0$; namely, we prove this theorem for $n > 2$ and all $q > 0$. Assume there exists an entire weak solution (u, v) of inequality (1.1) on $\mathbb{R}^n$ such that $u(x) \geqslant v(x)$ and relation (2.4) holds. Then, from (1.7), we have the inequality

$$\int_{\mathbb{R}^n} \sum_{i,j=1}^{n} a_{ij} (u - v)_{x_i} \varphi_{x_j} \, \mathrm{d}x \geqslant \int_{\mathbb{R}^n} (|u|^{q-1} u - |v|^{q-1} v) \varphi \, \mathrm{d}x, \tag{3.50}$$

which holds for every non-negative function $\varphi \in C^\infty(\mathbb{R}^n)$ with compact support. Let R and ε be arbitrary positive numbers, and let $\zeta : \mathbb{R}^n \to [0, 1]$ be a smooth function which equals 1 on $\overline{B(R/2)}$ and 0 outside $B(R)$. Without loss of generality, we substitute $\varphi(x) = (w + \varepsilon)^{-1} \zeta^2(x)$ as a test function in (3.50), where $w(x) = u(x) - v(x)$. Integrating by parts in (3.50) we obtain

$$\begin{aligned} - \int_{B(R)} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} (w + \varepsilon)^{-2} \zeta^2 \, \mathrm{d}x + 2 \int_{B(R)} \sum_{i,j=1}^{n} a_{ij} w_{x_i} \zeta_{x_j} (w + \varepsilon)^{-1} \zeta \, \mathrm{d}x \\ \equiv I_1 + I_2 \geqslant \int_{B(R/2)} (|u|^{q-1} u - |v|^{q-1} v)(w + \varepsilon)^{-1} \, \mathrm{d}x. \end{aligned} \tag{3.51}$$

Since

$$|I_2| \leqslant \int_{B(R)} 2 \biggl(\sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} \biggr)^{1/2} \biggl(\sum_{i,j=1}^{n} a_{ij} \zeta_{x_i} \zeta_{x_j} \biggr)^{1/2} (w + \varepsilon)^{-1} \zeta \, \mathrm{d}x, \tag{3.52}$$

estimating the integrand on the right-hand side of (3.52) by Young's inequality (3.4) with $\rho = \frac{1}{2}$ and $\beta = 2$, we arrive at the inequality

$$|I_2| \leqslant \frac{1}{2}\int_{B(R)}\sum_{i,j=1}^{n} a_{ij}w_{x_i}w_{x_j}(w+\varepsilon)^{-2}\zeta^2\,\mathrm{d}x + c_1\int_{B(R)}\sum_{i,j=1}^{n} a_{ij}\zeta_{x_i}\zeta_{x_j}\,\mathrm{d}x. \quad (3.53)$$

As above, we use the symbols c_i, $i = 1, 2, \dots$, to denote constants depending possibly on n, q and the coefficients of the operator $\mathcal{L}$ but not on R and ε. In turn, (3.51) and (3.53) yield the inequality

$$c_1\int_{B(R)}\sum_{i,j=1}^{n} a_{ij}\zeta_{x_i}\zeta_{x_j}\,\mathrm{d}x \geqslant \int_{B(R/2)}(|u|^{q-1}u - |v|^{q-1}v)(w+\varepsilon)^{-1}\,\mathrm{d}x. \quad (3.54)$$

Furthermore, as above, without loss of generality, for any $R > 0$, in (3.54) we choose the function $\zeta(x)$ in the form $\zeta(x) = \psi(|x|/R)$, where $\psi : [0,\infty) \to [0,1]$ is a smooth function which equals 1 on $[0, \frac{1}{2}]$ and 0 on $[1,\infty)$ and such that the inequality

$$|\nabla\zeta| \leqslant c_2 R^{-1} \quad (3.55)$$

holds. Then, owing to the boundedness of the coefficients of the operator $\mathcal{L}$, from (3.54) and (3.55) we obtain the inequality

$$c_3 R^{n-2} \geqslant \int_{B(R/2)}(|u|^{q-1}u - |v|^{q-1}v)(w+\varepsilon)^{-1}\,\mathrm{d}x \quad (3.56)$$

and, therefore, the inequality

$$c_3 R^{n-2} \geqslant \int_{B(R/2)\cap\{x\in\mathbb{R}^n : u(x)\neq v(x)\}}(|u|^{q-1}u - |v|^{q-1}v)(w+\varepsilon)^{-1}\,\mathrm{d}x, \quad (3.57)$$

which hold for all $R > 0$. In (3.57), passing to the limit as $\varepsilon \to 0$ as justified by Levi's theorem (see, for example, [4, p. 305]), we arrive at

$$c_3 R^{n-2} \geqslant \int_{B(R/2)\cap\{x\in\mathbb{R}^n : u(x)\neq v(x)\}}(|u|^{q-1}u - |v|^{q-1}v)(u-v)^{-1}\,\mathrm{d}x. \quad (3.58)$$

Finally, from (3.58) we obtain the inequality

$$\limsup_{R\to+\infty} R^{2-n}\int_{B(R)\cap\{x\in\mathbb{R}^n : u(x)\neq v(x)\}}(|u|^{q-1}u - |v|^{q-1}v)(u-v)^{-1}\,\mathrm{d}x < +\infty, \quad (3.59)$$

which contradicts the hypotheses of theorem 2.7; namely, it contradicts (2.4). Thus, we have a contradiction of our assumption. □

Proof of theorem 2.12. The proof is by contradiction. Let $n > 2$ and $q = 1$. Assume that there exists an entire weak solution (u, v) of inequality (1.1) on $\mathbb{R}^n$ such that $u(x) > v(x)$. Then, from (1.7), we have the inequality

$$\int_{\mathbb{R}^n}\sum_{i,j=1}^{n} a_{ij}(u-v)_{x_i}\varphi_{x_j}\,\mathrm{d}x \geqslant \int_{\mathbb{R}^n}(u-v)\varphi\,\mathrm{d}x, \quad (3.60)$$

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which holds for every non-negative function $\varphi \in C^\infty(\mathbb{R}^n)$ with compact support. Let R be an arbitrary positive number, and let $\zeta : \mathbb{R}^n \to [0,1]$ be a smooth function which equals 1 on $\overline{B(R/2)}$ and 0 outside $B(R)$. Without loss of generality, we substitute $\varphi(x) = w^{-1}\zeta^2(x)$ as a test function in inequality (3.60), where $w(x) = u(x) - v(x)$. Integrating by parts in (3.60), we obtain

$$-\int_{B(R)} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} w^{-2} \zeta^2 \,\mathrm{d}x + 2\int_{B(R)} \sum_{i,j=1}^{n} a_{ij} w_{x_i} \zeta_{x_j} w^{-1} \zeta \,\mathrm{d}x \equiv I_1 + I_2 \geqslant \int_{B(R)} \zeta^2 \,\mathrm{d}x. \tag{3.61}$$

Since

$$|I_2| \leqslant \int_{B(R)} 2\biggl(\sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j}\biggr)^{1/2} \biggl(\sum_{i,j=1}^{n} a_{ij} \zeta_{x_i} \zeta_{x_j}\biggr)^{1/2} w^{-1}\zeta \,\mathrm{d}x, \tag{3.62}$$

by estimating the integrand on the right-hand side of (3.62) by Young's inequality (3.4) with $\rho = \frac{1}{2}$ and $\beta = 2$, we arrive at the inequality

$$|I_2| \leqslant \frac{1}{2} \int_{B(R)} \sum_{i,j=1}^{n} a_{ij} w_{x_i} w_{x_j} w^{-2} \zeta^2 \,\mathrm{d}x + c_1 \int_{B(R)} \sum_{i,j=1}^{n} a_{ij} \zeta_{x_i} \zeta_{x_j} \,\mathrm{d}x. \tag{3.63}$$

As above, we use the symbols c_i, $i = 1, 2, \dots$, to denote constants depending possibly on n, q and the coefficients of the operator $\mathcal{L}$ but not on R. Furthermore, (3.61) and (3.63) yield

$$c_1 \int_{B(R)} \sum_{i,j=1}^{n} a_{ij} \zeta_{x_i} \zeta_{x_j} \,\mathrm{d}x \geqslant \int_{B(R)} \zeta^2 \,\mathrm{d}x. \tag{3.64}$$

Now, without loss of generality, for any $R > 0$, we choose in (3.64) the function $\zeta(x)$ in the form $\zeta(x) = \psi(|x|/R)$, where $\psi : [0,\infty) \to [0,1]$ is a smooth function which equals 1 on $[0, \frac{1}{2}]$ and 0 on $[1,\infty)$ and such that the inequality

$$|\nabla\zeta| \leqslant c_2 R^{-1} \tag{3.65}$$

holds. Then, owing to the boundedness of the coefficients of the operator $\mathcal{L}$, from (3.64) and (3.65) we obtain the inequality

$$c_3 R^{n-2} \geqslant R^n, \tag{3.66}$$

which holds for all $R > 0$. Thus, we have a contradiction of our assumption. □

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