Ergod. Th. & Dynam. Sys., (2023), **43**, 3777–3795 © The Author(s), 2023. Published by Cambridge University Press.
doi:10.1017/etds.2022.120

The moduli space of polynomial maps and their fixed-point multipliers: II. Improvement to the algorithm and monic centered polynomials

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(*Received* 25 *October* 2021 *and accepted in revised form* 18 *December* 2022)

Abstract. We consider the family MC_d of monic centered polynomials of one complex variable with degree $d \geq 2$, and study the map $\widehat{\Phi}_d : \mathrm{MC}_d \to \widetilde{\Lambda}_d \subset \mathbb{C}^d/\mathfrak{S}_d$ which maps each $f \in \mathrm{MC}_d$ to its unordered collection of fixed-point multipliers. We give an explicit formula for counting the number of elements of each fiber $\widehat{\Phi}_d^{-1}(\bar{\lambda})$ for every $\bar{\lambda} \in \widetilde{\Lambda}_d$ except when the fiber $\widehat{\Phi}_d^{-1}(\bar{\lambda})$ contains polynomials having multiple fixed points. This formula is not a recursive one, and is a drastic improvement of our previous result [T. Sugiyama. The moduli space of polynomial maps and their fixed-point multipliers. *Adv. Math.* **322** (2017), 132–185] which gave a rather long algorithm with some induction processes.

Key words: complex dynamics, fixed-point multipliers, moduli space of polynomial maps, partition of integers, inclusion-exclusion formula
2020 Mathematics Subject Classification: 37F10 (Primary); 05A19, 14D20 (Secondary)

1. *Introduction*

This paper is a continuation of the author's previous work [**14**].

We first remind our setting from [**14**]. Let MP_d be the family of affine conjugacy classes of polynomial maps of one complex variable with degree $d \geq 2$, and $\mathbb{C}^d/\mathfrak{S}_d$ the set of unordered collections of d complex numbers, where $\mathfrak{S}_d$ denotes the dth symmetric group. We denote by Φ_d the map

$$\Phi_d : \mathrm{MP}_d \to \widetilde{\Lambda}_d \subset \mathbb{C}^d/\mathfrak{S}_d$$

which maps each $f \in \mathrm{MP}_d$ to its unordered collection of fixed-point multipliers. Here, fixed-point multipliers of $f \in \mathrm{MP}_d$ always satisfy a certain relation by the fixed point theorem for polynomial maps (see §12 in [11]), which implies that the image of Φ_d is contained in a certain hyperplane $\widetilde{\Lambda}_d$ in $\mathbb{C}^d/\mathfrak{S}_d$.

As mentioned in [14], it is well known that the map $\Phi_d : \mathrm{MP}_d \to \widetilde{\Lambda}_d$ is bijective for $d = 2$ and also for $d = 3$ (see [9]). For $d \geq 4$, Fujimura and Nishizawa have done some preliminary works in finding $\#(\Phi_d^{-1}(\bar{\lambda}))$ for $\bar{\lambda} \in \widetilde{\Lambda}_d$ in their series of papers such as [2, 3, 12]. Hereafter, $\#(X)$, or simply $\#X$, denotes the cardinality of a set X. Fujimura and Taniguchi [4] also constructed a compactification of MP_d, which gave us a strong geometric insight on the fiber structure of Φ_d. Other compactifications of MP_d were also constructed independently by Silverman [13] and by DeMarco and McMullen [1]. For rational maps and their periodic-point multipliers, McMullen [8] gave a general important result. In a special case of [8], there is a famous result by Milnor [10] for rational maps of degree two and their fixed-point multipliers. There is also a result by Hutz and Tepper [7] for rational maps of degree three and their periodic-point multipliers of period less than or equal to two. There are some other results [5, 6] concerning polynomial or rational maps and their periodic-point multipliers. (See [14] for more details.)

Following the results above, in [14], we succeeded in giving, for every $\bar{\lambda} = \{\lambda_1, \ldots, \lambda_d\} \in \widetilde{\Lambda}_d$, an algorithm for counting the number of elements of $\Phi_d^{-1}(\bar{\lambda})$ except when $\lambda_i = 1$ for some i. However, the algorithm was rather long and complicated. In this paper, we make a *drastic improvement* to its algorithm; we no longer need induction processes to find $\#(\Phi_d^{-1}(\bar{\lambda}))$ if we consider $\Phi_d^{-1}(\bar{\lambda})$ counted with multiplicity (see Theorem I). Moreover, if we consider the family MC_d of monic centered polynomials of degree d and the map $\widehat{\Phi}_d : \mathrm{MC}_d \to \widetilde{\Lambda}_d$, instead of MP_d and $\Phi_d : \mathrm{MP}_d \to \widetilde{\Lambda}_d$, we can always give an explicit expression of $\#(\widehat{\Phi}_d^{-1}(\bar{\lambda}))$ even when its multiplicity is ignored (see Theorem II and Corollary III). Here, $\widehat{\Phi}_d : \mathrm{MC}_d \to \widetilde{\Lambda}_d$ is defined to be the composite mapping of the natural projection $\mathrm{MC}_d \to \mathrm{MP}_d$ and Φ_d. Interestingly, the formula for finding $\#(\Phi_d^{-1}(\bar{\lambda}))$ in Theorem I has the form of the inclusion-exclusion formula.

There are five sections in this paper. In §§2 and 3, we shall review the results in [14] more precisely and state Theorems I, II, and Corollary III, which are the main results in this paper. Section 4 is devoted to the proof of Theorem I and §5 is devoted to the proof of Theorem II. The main part in this paper is the proof of Theorem I in §4, which consists of a good deal of combinatorial argument. Compared with the proof of Theorem I, the proof of Theorem II in §5 is relatively easy under the assumption of [14]. However, by combining Theorems I and II, we directly have Corollary III, which is, in some sense, a monumental achievement of our study.

2. *Main result 1*

In this section, we always consider $\Phi_d^{-1}(\bar{\lambda})$ *counted with multiplicity* and deal with improvements to the algorithm for finding $\#(\Phi_d^{-1}(\bar{\lambda}))$. We first fix our notation.

For $d \geq 2$, we put

$$\mathrm{Poly}_d := \{f \in \mathbb{C}[z] \mid \deg f = d\} \quad \text{and} \quad \mathrm{Aut}(\mathbb{C}) := \{\gamma(z) = az + b \mid a, b \in \mathbb{C},\ a \neq 0\}.$$

Since $\gamma \in \mathrm{Aut}(\mathbb{C})$ naturally acts on $f \in \mathrm{Poly}_d$ by $\gamma \cdot f := \gamma \circ f \circ \gamma^{-1}$, we can define its quotient $\mathrm{MP}_d := \mathrm{Poly}_d/\mathrm{Aut}(\mathbb{C})$, which we usually call the moduli space of polynomial maps of degree d. We put $\mathrm{Fix}(f) := \{z \in \mathbb{C} \mid f(z) = z\}$ for $f \in \mathrm{Poly}_d$, where $\mathrm{Fix}(f)$ is

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considered counted with multiplicity. Hence, we always have $\#(\mathrm{Fix}(f)) = d$. Since the unordered collection of fixed-point multipliers $(f'(\zeta))_{\zeta\in\mathrm{Fix}(f)}$ of $f \in \mathrm{Poly}_d$ is invariant under the action of $\mathrm{Aut}(\mathbb{C})$, we can naturally define the map $\Phi_d : \mathrm{MP}_d \to \mathbb{C}^d/\mathfrak{S}_d$ by $\Phi_d(f) := (f'(\zeta))_{\zeta\in\mathrm{Fix}(f)}$. Here, $\mathfrak{S}_d$ denotes the dth symmetric group which acts on $\mathbb{C}^d$ by the permutation of coordinates. Note that a fixed point $\zeta \in \mathrm{Fix}(f)$ is multiple if and only if $f'(\zeta) = 1$.

By the fixed point theorem for polynomial maps, we always have $\sum_{\zeta\in\mathrm{Fix}(f)} 1/(1 - f'(\zeta)) = 0$ for $f \in \mathrm{Poly}_d$ if f has no multiple fixed point. (See §12 in [11] or Proposition 1.1 in [14] for more details.) Hence, putting $\Lambda_d := \{(\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d \mid \sum_{i=1}^{d} \prod_{j\neq i}(1 - \lambda_j) = 0\}$ and $\widetilde{\Lambda}_d := \Lambda_d/\mathfrak{S}_d$, we have the inclusion relation $\Phi_d(\mathrm{MP}_d) \subseteq \widetilde{\Lambda}_d \subseteq \mathbb{C}^d/\mathfrak{S}_d$. We therefore have the map

$$\Phi_d : \mathrm{MP}_d \to \widetilde{\Lambda}_d$$

by $f \mapsto (f'(\zeta))_{\zeta\in\mathrm{Fix}(f)}$, which is the main object of our study.

In this paper, we again restrict our attention to the map Φ_d on the domain where polynomial maps have no multiple fixed points, that is, on the domains

$$V_d := \{(\lambda_1, \ldots, \lambda_d) \in \Lambda_d \mid \lambda_i \neq 1 \text{ for every } 1 \leq i \leq d\} \quad \text{and} \quad \widetilde{V}_d := V_d/\mathfrak{S}_d,$$

which are Zariski open subsets of Λ_d and $\widetilde{\Lambda}_d$, respectively. Here, note that we also have

$$V_d = \left\{(\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d \;\middle|\; \lambda_i \neq 1 \text{ for every } 1 \leq i \leq d, \sum_{i=1}^{d} \frac{1}{1-\lambda_i} = 0\right\}.$$

Throughout this paper, we always denote by $\bar{\lambda}$ the equivalence class of $\lambda \in \Lambda_d$ in $\widetilde{\Lambda}_d$, that is, $\bar{\lambda} = pr(\lambda)$, where $pr : \Lambda_d \to \widetilde{\Lambda}_d$ denotes the canonical projection. Hence, for $\lambda = (\lambda_1, \ldots, \lambda_d) \in \Lambda_d$, we sometimes express $\bar{\lambda} = \{\lambda_1, \ldots, \lambda_d\} \in \widetilde{\Lambda}_d$. We never denote by $\bar{\lambda}$ the complex conjugate of λ in this paper.

The objects defined in the following definition play a central roll in [14] and also in this paper.

Definition 2.1. For $\lambda = (\lambda_1, \ldots, \lambda_d) \in V_d$, we put

$$\mathfrak{I}(\lambda) := \left\{ \{I_1, \ldots, I_l\} \;\middle|\; \begin{array}{c} l \geq 2, \quad I_1 \amalg \cdots \amalg I_l = \{1, \ldots, d\}, \\ I_u \neq \emptyset \text{ for every } 1 \leq u \leq l, \\ \sum_{i\in I_u} 1/(1-\lambda_i) = 0 \text{ for every } 1 \leq u \leq l \end{array} \right\},$$

where $I_1 \amalg \cdots \amalg I_l$ denotes the disjoint union of $I_1, \ldots, I_l$. By definition, each element of $\mathfrak{I}(\lambda)$ is considered to be a partition of $\{1, \ldots, d\}$. The partial order $\prec$ in $\mathfrak{I}(\lambda)$ is defined by the refinement of partitions, namely, for $\mathbb{I}, \mathbb{I}' \in \mathfrak{I}(\lambda)$, the relation $\mathbb{I} \prec \mathbb{I}'$ holds if and only if $\mathbb{I}'$ is a refinement of $\mathbb{I}$ as partitions of $\{1, \ldots, d\}$.

For $\lambda \in V_d$ and for $I \in \mathbb{I} \in \mathfrak{I}(\lambda)$, we put $\lambda_I := (\lambda_i)_{i\in I}$.

In the above definition, note that the condition $I \in \mathbb{I} \in \mathfrak{I}(\lambda)$ for I is equivalent to the conditions $\emptyset \subsetneq I \subsetneq \{1, \ldots, d\}$ and $\sum_{i\in I} 1/(1-\lambda_i) = 0$. Hence, we always have $\lambda_I \in V_{\#I}$ for $\lambda \in V_d$ and $I \in \mathbb{I} \in \mathfrak{I}(\lambda)$ by definition. Also note that $\#I \geq 2$ holds for every $I \in \mathbb{I} \in \mathfrak{I}(\lambda)$.

The following object is also very important in this paper.

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Definition 2.2. For $\lambda \in V_d$, we put

$$\mathfrak{I}'(\lambda) := \mathfrak{I}(\lambda) \cup \{\{\{1, \ldots, d\}\}\}.$$

The partial order $\prec$ in $\mathfrak{I}(\lambda)$ is naturally extended to the partial order $\prec$ in $\mathfrak{I}'(\lambda)$.

By definition, $\mathfrak{I}'(\lambda)$ is obtained from $\mathfrak{I}(\lambda)$ by adding exactly one element $\mathbb{I}_0 := \{\{1, \ldots, d\}\}$. Here, $\mathbb{I}_0$ is the unique minimum element of $\mathfrak{I}'(\lambda)$ with respect to the partial order $\prec$. Moreover, $\mathbb{I}_0$ is considered to be a partition of $\{1, \ldots, d\}$ which, in practice, does not partition $\{1, \ldots, d\}$. We also have the equality

$$\mathfrak{I}'(\lambda) = \left\{ \{I_1, \ldots, I_l\} \;\middle|\; \begin{array}{c} l \geq 1,\, I_1 \amalg \cdots \amalg I_l = \{1, \ldots, d\}, \\ I_u \neq \emptyset \text{ for every } 1 \leq u \leq l, \\ \sum_{i \in I_u} 1/(1-\lambda_i) = 0 \text{ for every } 1 \leq u \leq l \end{array} \right\}.$$

We already have the following theorem by Main Theorem III and Remark 1.8 in [14] and by Theorem B and Proposition C in §6 in [14].

THEOREM 2.3. *We can define the non-negative integer $e_{\mathbb{I}}(\lambda)$ for each $d \geq 4$, $\lambda \in V_d$, and $\mathbb{I} \in \mathfrak{I}(\lambda)$, and can also define the non-negative integer $s_d(\lambda)$ for each $d \geq 2$ and $\lambda \in V_d$ inductively by the equalities*

$$s_d(\lambda) = (d-2)! - \sum_{\mathbb{I} \in \mathfrak{I}(\lambda)} \left(e_{\mathbb{I}}(\lambda) \cdot \prod_{k=d-\#\mathbb{I}+1}^{d-2} k \right) \tag{2.1}$$

for $d \geq 2$ and $\lambda \in V_d$, and

$$e_{\mathbb{I}}(\lambda) = \prod_{I \in \mathbb{I}} ((\#I - 1) \cdot s_{\#I}(\lambda_I)) \tag{2.2}$$

for $d \geq 4$, $\lambda \in V_d$, and $\mathbb{I} \in \mathfrak{I}(\lambda)$. Here, in the case $\#\mathbb{I} = 2$, we put $\prod_{k=d-\#\mathbb{I}+1}^{d-2} k = \prod_{k=d-1}^{d-2} k = 1$.

If we consider $\Phi_d^{-1}(\bar{\lambda})$ 'counted with multiplicity' for $d \geq 2$ and $\lambda \in V_d$, then we have

$$\#(\Phi_d^{-1}(\bar{\lambda})) = s_d(\lambda).$$

Remark 2.4. For $d = 2$ or 3, we always have $\mathfrak{I}(\lambda) = \emptyset$ for every $\lambda \in V_d$ by definition. Hence, by equation (2.1), we have $s_2(\lambda) = (2-2)! = 1$ for every $\lambda \in V_2$ and $s_3(\lambda) = (3-2)! = 1$ for every $\lambda \in V_3$. For $d \geq 4$, every $e_{\mathbb{I}}(\lambda)$ and $s_d(\lambda)$ are determined uniquely and can actually be found by equations (2.1) and (2.2) by induction on d, since $2 \leq \#I < d$ holds for $I \in \mathbb{I} \in \mathfrak{I}(\lambda)$ with $\lambda \in V_d$.

In the rest of this paper, we always assume that $e_{\mathbb{I}}(\lambda)$ and $s_d(\lambda)$ are the non-negative integers defined in Theorem 2.3.

We already made a minor improvement to the above algorithm by Main Theorem III in [14] and by Proposition D in §6 in [14], as in the following.

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THEOREM 2.5. *The non-negative integer $e_{\mathbb{I}}(\lambda)$ for $\lambda \in V_d$ and $\mathbb{I} \in \mathfrak{I}(\lambda)$ defined in Theorem 2.3 also satisfies the equality*

$$e_{\mathbb{I}}(\lambda) = \bigg(\prod_{I\in\mathbb{I}}(\#I-1)!\bigg) - \sum_{\substack{\mathbb{I}' \in \mathfrak{I}(\lambda) \\ \mathbb{I}' \succ \mathbb{I},\ \mathbb{I}' \neq \mathbb{I}}} \bigg(e_{\mathbb{I}'}(\lambda) \cdot \prod_{I\in\mathbb{I}} \bigg(\prod_{k=\#I-\chi_I(\mathbb{I}')+1}^{\#I-1} k\bigg)\bigg), \tag{2.3}$$

where we put $\chi_I(\mathbb{I}') := \#(\{I' \in \mathbb{I}' \mid I' \subseteq I\})$ for $\mathbb{I}' \succ \mathbb{I}$ and $I \in \mathbb{I}$. Here, in the case $\chi_I(\mathbb{I}') = 1$, we put $\prod_{k=\#I-\chi_I(\mathbb{I}')+1}^{\#I-1} k = \prod_{k=\#I}^{\#I-1} k = 1$.

Remark 2.6. By definition, we always have $\sum_{I\in\mathbb{I}} \chi_I(\mathbb{I}') = \#\mathbb{I}'$ for $\mathbb{I}' \succ \mathbb{I}$.

Remark 2.7. We can also find $s_d(\lambda)$ only by using equations (2.1) and (2.3). The algorithm using equations (2.1) and (2.3) is a little simpler than the algorithm in Theorem 2.3.

Remark 2.8. We present a rough outline of the proof of Theorem 2.5 in this remark, since the proof can be an easy exercise for the proof of Theorem I in this paper. (See 'Proof of Proposition D' on pp. 175–177 in [**14**] for details.) In the case where $d = \#I$ and $\lambda = \lambda_I$, equation (2.1) is equivalent to the following:

$$(\#I-1)! = (\#I-1)s_{\#I}(\lambda_I) + \sum_{\mathbb{I}\in\mathfrak{I}(\lambda_I)} \bigg(e_{\mathbb{I}}(\lambda_I) \cdot \prod_{k=\#I-\#\mathbb{I}+1}^{\#I-1} k\bigg). \tag{2.4}$$

Plugging equation (2.4) into $\prod_{I\in\mathbb{I}}(\#I-1)!$ and using equation (2.2) carefully, we have equation (2.3).

In this paper, we make a drastic improvement to the above algorithm as in the following.

THEOREM I. *The non-negative integer $s_d(\lambda)$ for $d \geq 2$ and $\lambda \in V_d$ defined in Theorem 2.3 is expressed in the form*

$$(d-1)s_d(\lambda) = \sum_{\mathbb{I}\in\mathfrak{I}'(\lambda)} \bigg(\{-(d-1)\}^{\#\mathbb{I}-1} \cdot \prod_{I\in\mathbb{I}}(\#I-1)!\bigg). \tag{2.5}$$

Hence, if we consider $\Phi_d^{-1}(\bar{\lambda})$ 'counted with multiplicity' for $d \geq 2$ and $\lambda \in V_d$, then we have

$$\#(\Phi_d^{-1}(\bar{\lambda})) = -\sum_{\mathbb{I}\in\mathfrak{I}'(\lambda)} \bigg(\{-(d-1)\}^{\#\mathbb{I}-2} \cdot \prod_{I\in\mathbb{I}}(\#I-1)!\bigg). \tag{2.6}$$

Theorem I is proved in §4.

Remark 2.9. By Theorem I, we no longer need induction processes to find $\#(\Phi_d^{-1}(\bar{\lambda}))$ if we consider $\Phi_d^{-1}(\bar{\lambda})$ counted with multiplicity. We only need to find $\mathfrak{I}'(\lambda)$ and to compute straightforward the right-hand side of equation (2.6).

However, there are some minor defects in the form of equation (2.6) comparing with equation (2.1). By equation (2.1), we can easily see the inequality $s_d(\lambda) \leq (d-2)!$; however, it cannot be easily seen by equation (2.6). The sum of the absolute value

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$\sum_{\mathbb{I}\in\mathfrak{I}'(\lambda)}((d-1)^{\#\mathbb{I}-2}\cdot\prod_{I\in\mathbb{I}}(\#I-1)!\,)$ in the right-hand side of equation (2.6) can be much greater than $(d-2)!$.

Remark 2.10. Each term in the right-hand side of equation (2.5) $\{-(d-1)\}^{\#\mathbb{I}-1}\cdot\prod_{I\in\mathbb{I}}(\#I-1)!$ is positive or negative, according to whether $\#\mathbb{I}$ is odd or even. Moreover, if $\mathbb{I}\in\mathfrak{I}'(\lambda)$ and $\mathbb{I}'\prec\mathbb{I}$, then we automatically have $\mathbb{I}'\in\mathfrak{I}'(\lambda)$. Hence, equation (2.5) is considered to be a kind of inclusion-exclusion formula.

Remark 2.11. Theorem I is derived from Theorem 2.3 with no extra information. Hence, the proof of Theorem I is self-contained and requires no prerequisites under the assumption of Theorem 2.3, whereas its proof is highly non-trivial. The proof consists of a good deal of combinatorial argument.

3. *Main result 2*

In this section, we proceed to the next step, in which we discuss the possibility of improving the algorithm for counting the number of *discrete* elements of $\Phi_d^{-1}(\bar{\lambda})$. Therefore, in this section, $\Phi_d^{-1}(\bar{\lambda})$ is *not* considered counted with multiplicity; $\Phi_d^{-1}(\bar{\lambda})$ is considered to be a set. In this setting, we have already obtained an algorithm for counting the number of discrete elements of $\Phi_d^{-1}(\bar{\lambda})$ by using $\{s_{d'}(\lambda')\mid 2\le d'\le d,\ \lambda'\in V_{d'}\}$ in the third and fourth steps in Main Theorem III in [**14**]. To review the result more precisely and to discuss further properties, we first fix our notation.

The following objects are important in this section.

Definition 3.1. For $\lambda=(\lambda_1,\ldots,\lambda_d)\in V_d$, we put

$$\mathfrak{K}(\lambda):=\left\{K\ \middle|\ \begin{array}{c}\emptyset\subsetneq K\subseteq\{1,\ldots,d\},\\ i,j\in K\Rightarrow\lambda_i=\lambda_j,\\ i\in K,\ j\in\{1,\ldots,d\}\setminus K\Longrightarrow\lambda_i\neq\lambda_j\end{array}\right\}.$$

Note that if we put $\mathfrak{K}(\lambda)=:\{K_1,\ldots,K_q\}$, then $K_1,\ldots,K_q$ are mutually disjoint, and the equality $K_1\amalg\cdots\amalg K_q=\{1,\ldots,d\}$ holds by definition; and hence $\mathfrak{K}(\lambda)$ is a partition of $\{1,\ldots,d\}$.

Definition 3.2. We denote the family of monic centered polynomials of degree d by

$$\mathrm{MC}_d:=\left\{f(z)=z^d+\sum_{k=0}^{d-2}a_kz^k\ \middle|\ a_k\in\mathbb{C}\text{ for }0\le k\le d-2\right\},$$

denote the composite mapping of $\mathrm{MC}_d\subset\mathrm{Poly}_d\twoheadrightarrow\mathrm{Poly}_d/\mathrm{Aut}(\mathbb{C})=\mathrm{MP}_d$ by $p:\mathrm{MC}_d\to\mathrm{MP}_d$, and also denote the composite mapping of $p:\mathrm{MC}_d\to\mathrm{MP}_d$ and $\Phi_d:\mathrm{MP}_d\to\widetilde{\Lambda}_d$ by $\widehat{\Phi}_d:\mathrm{MC}_d\to\widetilde{\Lambda}_d$, that is, $\widehat{\Phi}_d:=\Phi_d\circ p$.

In the above definition, the map p is surjective since every affine conjugacy class of polynomial maps contains monic centered polynomials. Moreover, two monic centered polynomials $f,g\in\mathrm{MC}_d$ are affinely conjugate if and only if there exists a $(d-1)$th radical root a of 1 such that the equality $g(z)=af(a^{-1}z)$ holds. Hence, the group $\{a\in\mathbb{C}\mid a^{d-1}=1\}\cong\mathbb{Z}/(d-1)\mathbb{Z}$ naturally acts on MC_d, and the induced mapping

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$\overline{p} : \mathrm{MC}_d/(\mathbb{Z}/(d-1)\mathbb{Z}) \to \mathrm{MP}_d$ is an isomorphism. Since $\mathrm{MC}_d \cong \mathbb{C}^{d-1}$, we also have $\mathrm{MP}_d \cong \mathbb{C}^{d-1}/(\mathbb{Z}/(d-1)\mathbb{Z})$. Here, the action of $\mathbb{Z}/(d-1)\mathbb{Z}$ on MC_d is *not* free for $d \geq 3$, and MP_d has the set of singular points $\mathrm{Sing}(\mathrm{MP}_d)$ for $d \geq 4$. Hence, in some sense, the map $p : \mathrm{MC}_d \to \mathrm{MP}_d$ can be considered to be a 'desingularization' of MP_d for $d \geq 4$.

We already have the following theorem by Remark 1.9 in [14].

THEOREM 3.3. *For $d \geq 2$ and $\lambda \in V_d$, we put $\mathfrak{K}(\lambda) =: \{K_1, \ldots, K_q\}$ and denote by g_w the greatest common divisor of $\#K_1, \ldots, \#K_{(w-1)}, (\#K_w) - 1, \#K_{(w+1)}, \ldots, \#K_q$ for each $1 \leq w \leq q$. If $g_w = 1$ holds for every $1 \leq w \leq q$, then we have*

$$\#(\Phi_d^{-1}(\bar{\lambda})) = \frac{s_d(\lambda)}{(\#K_1)! \cdots (\#K_q)!} = \frac{s_d(\lambda)}{\prod_{K \in \mathfrak{K}(\lambda)} (\#K)!}, \tag{3.1}$$

where $s_d(\lambda)$ is the non-negative integer defined in Theorem 2.3 and rewritten in Theorem I. Here, note that $\Phi_d^{-1}(\bar{\lambda})$ is not considered counted with multiplicity, and hence $\#(\Phi_d^{-1}(\bar{\lambda}))$ denotes the number of discrete elements of $\Phi_d^{-1}(\bar{\lambda})$.

In the case of $g_w \geq 2$ for some w, we also have an algorithm for finding $\#(\Phi_d^{-1}(\bar{\lambda}))$ in the third and fourth steps in Main Theorem III in [14]. However, it contains induction processes and is much more complicated than equation (3.1); and hence we omit to describe it again in this paper.

As we already mentioned in Remark 1.9 in [14], we find that for $d \geq 4$ and for $\lambda \in V_d$, the inequality $g_w \geq 2$ holds for some w only if $\bar{\lambda} \in \Phi_d(\mathrm{Sing}(\mathrm{MP}_d))$. Since MC_d is a 'desingularization' of MP_d, it is natural to expect that the map $\widehat{\Phi}_d = \Phi_d \circ p : \mathrm{MC}_d \to \widetilde{\Lambda}_d$ is simpler than the map $\Phi_d : \mathrm{MP}_d \to \widetilde{\Lambda}_d$ itself. In the following, we consider MC_d instead of MP_d, and also consider $\widehat{\Phi}_d : \mathrm{MC}_d \to \widetilde{\Lambda}_d$ instead of $\Phi_d : \mathrm{MP}_d \to \widetilde{\Lambda}_d$.

We now state the second main theorem in this paper.

THEOREM II. *For $d \geq 2$, $\lambda \in V_d$, and $\widehat{\Phi}_d : \mathrm{MC}_d \to \widetilde{\Lambda}_d$, we have*

$$\#(\widehat{\Phi}_d^{-1}(\bar{\lambda})) = \frac{(d-1)s_d(\lambda)}{\prod_{K \in \mathfrak{K}(\lambda)} (\#K)!}, \tag{3.2}$$

where $s_d(\lambda)$ is the non-negative integer defined in Theorem 2.3 and rewritten in Theorem I. Here, note that $\widehat{\Phi}_d^{-1}(\bar{\lambda})$ is not considered counted with multiplicity, and hence $\#(\widehat{\Phi}_d^{-1}(\bar{\lambda}))$ denotes the number of discrete elements of $\widehat{\Phi}_d^{-1}(\bar{\lambda})$.

Theorem II is proved in §5.

Remark 3.4. Theorem II holds for *every* $\lambda \in V_d$ with no exception, and has no induction process. Hence, we can say that the fiber structure of the map $\widehat{\Phi}_d : \mathrm{MC}_d \to \widetilde{\Lambda}_d$ is simpler than the fiber structure of the map $\Phi_d : \mathrm{MP}_d \to \widetilde{\Lambda}_d$, or moreover we can also say that the complexity of the map $\Phi_d : \mathrm{MP}_d \to \widetilde{\Lambda}_d$ is composed of the two complexities: one of them is the complexity of the map $\widehat{\Phi}_d : \mathrm{MC}_d \to \widetilde{\Lambda}_d$ and the other is the complexity of the map $p : \mathrm{MC}_d \to \mathrm{MP}_d$. Therefore, in some sense, consideration of the map $\widehat{\Phi}_d$ is more essential than that of the map Φ_d in the study of fixed-point multipliers for polynomial maps.

Remark 3.5. Theorem II is proved by a closer look at Propositions 4.3 and 9.1 in [14].

Combining Theorems I and II, we have the following.

COROLLARY III. *For $d \geq 2$, $\lambda \in V_d$, and $\widehat{\Phi}_d : \mathrm{MC}_d \to \widetilde{\Lambda}_d$, we have*

$$\#(\widehat{\Phi}_d^{-1}(\bar{\lambda})) = \frac{\sum_{\mathbb{I}\in\mathfrak{I}'(\lambda)}(\{-(d-1)\}^{\#\mathbb{I}-1} \cdot \prod_{I\in\mathbb{I}}(\#I-1)!\,)}{\prod_{K\in\mathfrak{K}(\lambda)}(\#K)!}.$$

4. *Proof of Theorem I*

In this section, we prove Theorem I. We assume $d \geq 2$ and $\lambda = (\lambda_1, \ldots, \lambda_d) \in V_d$, and denote by $\mathbb{I}_0 = \{\{1, \ldots, d\}\}$ the minimum element of $\mathfrak{I}'(\lambda)$, which are fixed throughout this section.

First we put

$$e_{\mathbb{I}_0}(\lambda) := (d-1)s_d(\lambda)$$

for $\mathbb{I}_0 = \{\{1, \ldots, d\}\} \in \mathfrak{I}'(\lambda)$. Then, equation (2.2) for $\mathbb{I} \in \mathfrak{I}(\lambda)$ is rewritten in the form

$$e_{\mathbb{I}}(\lambda) = \prod_{I\in\mathbb{I}} e_{\{I\}}(\lambda_I). \tag{4.1}$$

Here, $\{I\}$ denotes the minimum element of $\mathfrak{I}'(\lambda_I)$. Moreover, equation (2.1) is rewritten in the form

$$e_{\mathbb{I}_0}(\lambda) = (d-1)! - \sum_{\mathbb{I}\in\mathfrak{I}(\lambda)} \left(e_{\mathbb{I}}(\lambda) \cdot \prod_{k=d-\#\mathbb{I}+1}^{d-1} k\right), \tag{4.2}$$

which is also equivalent to the equality

$$(d-1)! = \sum_{\mathbb{I}\in\mathfrak{I}'(\lambda)} \left(e_{\mathbb{I}}(\lambda) \cdot \prod_{k=d-\#\mathbb{I}+1}^{d-1} k\right)$$

since for $\mathbb{I}_0 \in \mathfrak{I}'(\lambda)$, we have $e_{\mathbb{I}_0}(\lambda) \cdot \prod_{k=d-\#\mathbb{I}_0+1}^{d-1} k = e_{\mathbb{I}_0}(\lambda) \cdot \prod_{k=d}^{d-1} k = e_{\mathbb{I}_0}(\lambda)$. Equation (2.5), which we would like to prove in this section, is also rewritten in the form

$$e_{\mathbb{I}_0}(\lambda) = \sum_{\mathbb{I}\in\mathfrak{I}'(\lambda)} \left(\{-(d-1)\}^{\#\mathbb{I}-1} \cdot \prod_{I\in\mathbb{I}}(\#I-1)!\right). \tag{4.3$_d$}$$

Hence, to prove Theorem I, it suffices to derive equation $(4.3)_d$ from equations (4.1) and (4.2).

In the following, we show equation $(4.3)_d$ by induction on d.

For $d = 2$ or 3, we have $s_d(\lambda) = 1$ and $\mathfrak{I}'(\lambda) = \{\mathbb{I}_0\}$ for every $\lambda \in V_d$. Hence, for $\lambda \in V_d$, we always have

$$e_{\mathbb{I}_0}(\lambda) = (d-1)s_d(\lambda) = d-1$$

and also have

$$\begin{aligned}\sum_{\mathbb{I}\in\mathfrak{I}'(\lambda)} \left(\{-(d-1)\}^{\#\mathbb{I}-1} \cdot \prod_{I\in\mathbb{I}}(\#I-1)!\right) &= \{-(d-1)\}^{\#\mathbb{I}_0-1} \cdot \prod_{I\in\mathbb{I}_0}(\#I-1)! \\ &= \{-(d-1)\}^{1-1} \cdot (d-1)! = (d-1)!\,.\end{aligned}$$

Since $d-1 = (d-1)!$ for $d = 2$ or 3, we have equations $(4.3)_2$ and $(4.3)_3$.

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In the following, we assume $d \geq 4$ and show equation $(4.3)_d$ by the assumption of equations $(4.3)_2$, $(4.3)_3, \ldots, (4.3)_{d-1}$, (4.1), and (4.2).

For each $\mathbb{I} \in \mathfrak{I}(\lambda)$ with $\lambda \in V_d$, we put $\mathbb{I} =: \{I_1, \ldots, I_l\}$. Then, by using equations (4.1) and $(4.3)_{d'}$ for $2 \leq d' < d$, we have the following equalities:

$$
\begin{aligned}
e_{\mathbb{I}}(\lambda) &= \prod_{I \in \mathbb{I}} e_{\{I\}}(\lambda_I) = \prod_{u=1}^{l} e_{\{I_u\}}(\lambda_{I_u}) \\
&= \prod_{u=1}^{l} \Bigg(\sum_{\mathbb{I}'_u \in \mathfrak{I}'(\lambda_{I_u})} \Bigg[\{-(\#I_u - 1)\}^{\#\mathbb{I}'_u - 1} \cdot \prod_{I'_u \in \mathbb{I}'_u} (\#I'_u - 1)! \Bigg] \Bigg) \\
&= \sum_{\mathbb{I}'_1 \in \mathfrak{I}'(\lambda_{I_1})} \cdots \sum_{\mathbb{I}'_l \in \mathfrak{I}'(\lambda_{I_l})} \prod_{u=1}^{l} \Bigg[\{-(\#I_u - 1)\}^{\#\mathbb{I}'_u - 1} \cdot \prod_{I'_u \in \mathbb{I}'_u} (\#I'_u - 1)! \Bigg] \\
&= \sum_{\substack{\mathbb{I}' \in \mathfrak{I}(\lambda) \\ \mathbb{I}' \succ \mathbb{I}}} \Bigg[\Bigg(\prod_{I' \in \mathbb{I}'} (\#I' - 1)! \Bigg) \cdot \Bigg(\prod_{u=1}^{l} \{-(\#I_u - 1)\}^{\chi_{I_u}(\mathbb{I}') - 1} \Bigg) \Bigg]
\end{aligned}
\tag{4.4}
$$

since we have the equality

$$
\{\mathbb{I}'_1 \amalg \cdots \amalg \mathbb{I}'_l \mid \mathbb{I}'_1 \in \mathfrak{I}'(\lambda_{I_1}), \ldots, \mathbb{I}'_l \in \mathfrak{I}'(\lambda_{I_l})\} = \{\mathbb{I}' \in \mathfrak{I}(\lambda) \mid \mathbb{I}' \succ \mathbb{I}\}
$$

by definition. Here, since $\mathbb{I} \succ \mathbb{I}$ holds for $\mathbb{I} \in \mathfrak{I}(\lambda)$, we have $\mathbb{I} \in \{\mathbb{I}' \in \mathfrak{I}(\lambda) \mid \mathbb{I}' \succ \mathbb{I}\}$. Note that in equation (4.4), $\chi_{I_u}(\mathbb{I}') = \#(\{I' \in \mathbb{I}' \mid I' \subseteq I_u\})$ is the function defined in Theorem 2.5.

Substituting equation (4.4) into equation (4.2), we have

$$
\begin{aligned}
&e_{\mathbb{I}_0}(\lambda) \\
&= (d-1)! - \sum_{\mathbb{I} \in \mathfrak{I}(\lambda)} \Bigg\{ \sum_{\substack{\mathbb{I}' \in \mathfrak{I}(\lambda) \\ \mathbb{I}' \succ \mathbb{I}}} \Bigg(\prod_{I' \in \mathbb{I}'} (\#I' - 1)! \Bigg) \cdot \Bigg(\prod_{I \in \mathbb{I}} \{-(\#I - 1)\}^{\chi_I(\mathbb{I}') - 1} \Bigg) \Bigg\} \cdot \prod_{k=d-\#\mathbb{I}+1}^{d-1} k \\
&= (d-1)! - \sum_{\mathbb{I}' \in \mathfrak{I}(\lambda)} \Bigg\{ \prod_{I' \in \mathbb{I}'} (\#I' - 1)! \Bigg\} \cdot \Bigg\{ \sum_{\substack{\mathbb{I} \in \mathfrak{I}(\lambda) \\ \mathbb{I} \prec \mathbb{I}'}} \Bigg(\prod_{I \in \mathbb{I}} \{-(\#I - 1)\}^{\chi_I(\mathbb{I}') - 1} \Bigg) \cdot \prod_{k=d-\#\mathbb{I}+1}^{d-1} k \Bigg\}.
\end{aligned}
\tag{4.5}
$$

Here, equation $(4.3)_d$, which we would like to prove in this section, is equivalent to the equality

$$
e_{\mathbb{I}_0}(\lambda) = (d-1)! + \sum_{\mathbb{I} \in \mathfrak{I}(\lambda)} \Bigg(\{-(d-1)\}^{\#\mathbb{I} - 1} \cdot \prod_{I \in \mathbb{I}} (\#I - 1)! \Bigg),
$$

which is also equivalent to

$$
e_{\mathbb{I}_0}(\lambda) = (d-1)! + \sum_{\mathbb{I}' \in \mathfrak{I}(\lambda)} \Bigg[\Bigg\{ \prod_{I' \in \mathbb{I}'} (\#I' - 1)! \Bigg\} \cdot \{-(d-1)\}^{\#\mathbb{I}' - 1} \Bigg]. \tag{4.6}
$$

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Hence, comparing equations (4.5) and (4.6), we find that to prove equation $(4.3)_d$, we only need to show the following equality for each $\mathbb{I}' \in \mathfrak{I}(\lambda)$:

$$\{-(d-1)\}^{\#\mathbb{I}'-1} = -\sum_{\substack{\mathbb{I} \in \mathfrak{I}(\lambda) \\ \mathbb{I} \prec \mathbb{I}'}} \bigg(\prod_{I\in\mathbb{I}} \{-(\#I-1)\}^{\chi_I(\mathbb{I}')-1}\bigg) \cdot \prod_{k=d-\#\mathbb{I}+1}^{d-1} k. \tag{4.7}$$

Here, equation (4.7) is equivalent to the equality

$$\sum_{\substack{\mathbb{I} \in \mathfrak{I}'(\lambda) \\ \mathbb{I} \prec \mathbb{I}'}} \bigg(\prod_{k=d-\#\mathbb{I}+1}^{d-1} k\bigg) \cdot \bigg(\prod_{I\in\mathbb{I}} \{-(\#I-1)\}^{\chi_I(\mathbb{I}')-1}\bigg) = 0 \tag{4.8}$$

since for $\mathbb{I}_0 \in \mathfrak{I}'(\lambda)$ and $\mathbb{I}' \in \mathfrak{I}(\lambda)$, we have $\mathbb{I}_0 \prec \mathbb{I}'$ and

$$\bigg(\prod_{k=d-\#\mathbb{I}_0+1}^{d-1} k\bigg) \cdot \prod_{I\in\mathbb{I}_0} \{-(\#I-1)\}^{\chi_I(\mathbb{I}')-1} = \bigg(\prod_{k=d}^{d-1} k\bigg) \cdot \{-(d-1)\}^{\#\mathbb{I}'-1} = \{-(d-1)\}^{\#\mathbb{I}'-1}.$$

Hence, to prove Theorem I, we only need to show equation (4.8) for every $d \geq 4$, $\lambda \in V_d$, and $\mathbb{I}' \in \mathfrak{I}(\lambda)$. In the following, instead of expressing $\sum_{\mathbb{I}\in\mathfrak{I}'(\lambda),\, \mathbb{I}\prec\mathbb{I}'}$ for $\mathbb{I}' \in \mathfrak{I}(\lambda)$, we simply express $\sum_{\mathbb{I}\prec\mathbb{I}'}$, because if $\mathbb{I}$ is a partition of $\{1, \ldots, d\}$ and $\mathbb{I} \prec \mathbb{I}'$ for $\mathbb{I}' \in \mathfrak{I}(\lambda)$, then we automatically have $\mathbb{I} \in \mathfrak{I}'(\lambda)$.

To prove equation (4.8), we make use of the following.

Definition 4.1. For $\mathbb{I}' \in \mathfrak{I}(\lambda)$ with $\#\mathbb{I}' = l$ and for $k \in \mathbb{Z}$, we put

$$f_{l,k} := \sum_{\mathbb{I}\prec\mathbb{I}',\, \#\mathbb{I}=k} \prod_{I\in\mathbb{I}} \{-(\#I-1)\}^{\chi_I(\mathbb{I}')-1}.$$

Remark 4.2. For $\mathbb{I}' \in \mathfrak{I}(\lambda)$ with $\#\mathbb{I}' = l$ and for $\mathbb{I} \prec \mathbb{I}'$, we always have $1 \leq \#\mathbb{I} \leq l$. Hence, if $k \leq 0$ or $k \geq l+1$, then we have $f_{l,k} = 0$ by definition.

Example 4.3. Let us find $f_{l,l}$ and $f_{l,1}$ for $l \geq 2$ in this example.

Since $\{\mathbb{I} \mid \mathbb{I} \prec \mathbb{I}',\ \#\mathbb{I} = l\} = \{\mathbb{I}'\}$, we have

$$f_{l,l} = \prod_{I\in\mathbb{I}'} \{-(\#I-1)\}^{\chi_I(\mathbb{I}')-1} = \prod_{I\in\mathbb{I}'} \{-(\#I-1)\}^{1-1} = 1.$$

Let us consider $f_{l,1}$ next. Since $\{\mathbb{I} \mid \mathbb{I} \prec \mathbb{I}',\ \#\mathbb{I} = 1\} = \{\mathbb{I}_0\}$, we have

$$f_{l,1} = \prod_{I\in\mathbb{I}_0} \{-(\#I-1)\}^{\chi_I(\mathbb{I}')-1} = \{-(d-1)\}^{l-1}.$$

Example 4.4. Let us also find $f_{4,2}$ in this example. For $\mathbb{I}' \in \mathfrak{I}(\lambda)$ with $\#\mathbb{I}' = 4$, we can put $\mathbb{I}' = \{I_1, I_2, I_3, I_4\}$, and in this expression, we have $\{\mathbb{I} \mid \mathbb{I} \prec \mathbb{I}',\ \#\mathbb{I} = 2\} = \{\mathbb{I}_1, \ldots, \mathbb{I}_7\}$, where

$$\begin{aligned} &\mathbb{I}_1 = \{I_1,\ I_2 \amalg I_3 \amalg I_4\}, \quad \mathbb{I}_2 = \{I_2,\ I_1 \amalg I_3 \amalg I_4\}, \\ &\mathbb{I}_3 = \{I_3,\ I_1 \amalg I_2 \amalg I_4\}, \quad \mathbb{I}_4 = \{I_4,\ I_1 \amalg I_2 \amalg I_3\}, \\ &\mathbb{I}_5 = \{I_1 \amalg I_2,\ I_3 \amalg I_4\}, \quad \mathbb{I}_6 = \{I_1 \amalg I_3,\ I_2 \amalg I_4\}, \quad \text{and} \quad \mathbb{I}_7 = \{I_1 \amalg I_4,\ I_2 \amalg I_3\}. \end{aligned}$$

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We put $\#I_u =: i_u$ for $1 \leq u \leq 4$. Note that the equality $i_1 + i_2 + i_3 + i_4 = d$ holds. We have

$$\prod_{I \in \mathbb{I}_1} \{-(\#I - 1)\}^{\chi_I(\mathbb{I}')-1} = \{-(i_1 - 1)\}^{1-1} \cdot \{-(i_2 + i_3 + i_4 - 1)\}^{3-1},$$
$$\prod_{I \in \mathbb{I}_5} \{-(\#I - 1)\}^{\chi_I(\mathbb{I}')-1} = \{-(i_1 + i_2 - 1)\}^{2-1} \cdot \{-(i_3 + i_4 - 1)\}^{2-1},$$

for instance, which implies

$$\sum_{u=1}^{4} \prod_{I \in \mathbb{I}_u} \{-(\#I - 1)\}^{\chi_I(\mathbb{I}')-1} = \sum_{u=1}^{4} (i_1 + i_2 + i_3 + i_4 - i_u - 1)^2 = \sum_{u=1}^{4} (d - i_u - 1)^2$$
$$= 4(d-1)^2 - 2(d-1)d + \sum_{u=1}^{4} i_u^2,$$
$$\sum_{u=5}^{7} \prod_{I \in \mathbb{I}_u} \{-(\#I - 1)\}^{\chi_I(\mathbb{I}')-1} = (i_1 + i_2 - 1)(i_3 + i_4 - 1) + (i_1 + i_3 - 1)(i_2 + i_4 - 1)$$
$$+ (i_1 + i_4 - 1)(i_2 + i_3 - 1) = 2 \sum_{1 \leq u < v \leq 4} i_u i_v - 3d + 3.$$

Hence, we have

$$\begin{aligned} f_{4,2} &= \sum_{u=1}^{7} \prod_{I \in \mathbb{I}_u} \{-(\#I - 1)\}^{\chi_I(\mathbb{I}')-1} \\ &= 4(d-1)^2 - 2(d-1)d + \sum_{u=1}^{4} i_u^2 + 2 \sum_{1 \leq u < v \leq 4} i_u i_v - 3d + 3 \\ &= 2d^2 - 9d + 7 + \left(\sum_{u=1}^{4} i_u \right)^2 = 3d^2 - 9d + 7. \end{aligned}$$

Example 4.5. By a similar computation to Example 4.4, we have the following for $l \leq 5$:

$$\begin{array}{llll} f_{2,1} = -d + 1, & f_{3,1} = (d-1)^2, & f_{4,1} = \{-(d-1)\}^3, & f_{5,1} = \{-(d-1)\}^4, \\ f_{2,2} = 1, & f_{3,2} = -2d + 3, & f_{4,2} = 3d^2 - 9d + 7, & f_{5,2} = -4d^3 + 18d^2 - 28d + 15, \\ & f_{3,3} = 1, & f_{4,3} = -3d + 6, & f_{5,3} = 6d^2 - 24d + 25, \\ & & f_{4,4} = 1, & f_{5,4} = -4d + 10, \\ & & & f_{5,5} = 1. \end{array}$$

The following is the key proposition to prove equation (4.8).

PROPOSITION 4.6. *The number $f_{l,k}$ defined in Definition 4.1 is a function of l, k, and d, and does not depend on the choice of $\mathbb{I}' \in \mathfrak{I}(\lambda)$ with $\#\mathbb{I}' = l$. Moreover, for $l, k \in \mathbb{Z}$ with $l \geq 2$, we have the equality*

$$f_{l+1,k} = f_{l,k-1} - (d-k) f_{l,k}.$$

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PROPOSITION 4.7. *Admitting Proposition 4.6, we have equation (4.8) for every $d \geq 4$, $\lambda \in V_d$, and $\mathbb{I}' \in \mathfrak{I}(\lambda)$. Hence, Proposition 4.6 implies Theorem I.*

Proof of Proposition 4.7. If $\#\mathbb{I}' = 2$, then we can put $\mathbb{I}' = \{I_1, I_2\}$ and have $\{\mathbb{I} \mid \mathbb{I} \prec \mathbb{I}'\} = \{\mathbb{I}_0, \mathbb{I}'\}$. Hence, we have

$$\begin{aligned}
&\sum_{\mathbb{I} \prec \mathbb{I}'} \Bigg(\prod_{k=d-\#\mathbb{I}+1}^{d-1} k \Bigg) \cdot \Bigg(\prod_{I \in \mathbb{I}} \{-(\#I - 1)\}^{\chi_I(\mathbb{I}')-1} \Bigg) \\
&\quad = 1 \cdot \{-(d-1)\}^{2-1} + (d-1) \cdot \{-(\#I_1 - 1)\}^{1-1} \cdot \{-(\#I_2 - 1)\}^{1-1} \\
&\quad = -(d-1) + (d-1) = 0.
\end{aligned}$$

In the case where $\#\mathbb{I}' \geq 3$, we put $\#\mathbb{I}' =: l + 1$. Then we have $l \geq 2$ and have the following equalities by Proposition 4.6:

$$\begin{aligned}
&\sum_{\mathbb{I} \prec \mathbb{I}'} \Bigg(\prod_{k=d-\#\mathbb{I}+1}^{d-1} k \Bigg) \cdot \Bigg(\prod_{I \in \mathbb{I}} \{-(\#I - 1)\}^{\chi_I(\mathbb{I}')-1} \Bigg) \\
&\quad = \sum_{k=1}^{l+1} \Bigg(\prod_{k'=d-k+1}^{d-1} k' \Bigg) \cdot f_{l+1,k} \\
&\quad = \sum_{k=1}^{l+1} \Bigg(\prod_{k'=d-k+1}^{d-1} k' \Bigg) \cdot (f_{l,k-1} - (d-k) f_{l,k}) \\
&\quad = \sum_{k=1}^{l+1} \Bigg(\prod_{k'=d-k+1}^{d-1} k' \Bigg) \cdot f_{l,k-1} - \sum_{k=1}^{l+1} \Bigg(\prod_{k'=d-k+1}^{d-1} k' \Bigg) \cdot (d-k) f_{l,k} \\
&\quad = \sum_{k=0}^{l} \Bigg(\prod_{k'=d-k}^{d-1} k' \Bigg) \cdot f_{l,k} - \sum_{k=1}^{l+1} \Bigg(\prod_{k'=d-k}^{d-1} k' \Bigg) \cdot f_{l,k} \\
&\quad = \Bigg(\prod_{k'=d}^{d-1} k' \Bigg) \cdot f_{l,0} - \Bigg(\prod_{k'=d-(l+1)}^{d-1} k' \Bigg) \cdot f_{l,l+1} = 0,
\end{aligned}$$

which completes the proof of Proposition 4.7. □

In the rest of this section, we shall prove Proposition 4.6. We make use of the following polynomial to prove Proposition 4.6.

Definition 4.8. For $l, k \in \mathbb{Z}$ with $l \geq 2$, we define $\mathfrak{J}_l(k)$ as follows: if $k \leq 0$ or $k \geq l + 1$, then we put $\mathfrak{J}_l(k) = \emptyset$; if $1 \leq k \leq l$, then we put

$$\mathfrak{J}_l(k) := \left\{ \{J_1, \ldots, J_k\} \;\middle|\; \begin{array}{l} J_1 \amalg \cdots \amalg J_k = \{1, \ldots, l\}, \\ J_v \neq \emptyset \text{ for every } 1 \leq v \leq k \end{array} \right\},$$

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where $J_1 \amalg \cdots \amalg J_k$ denotes the disjoint union of $J_1, \ldots, J_k$. Moreover, for $l, k \in \mathbb{Z}$ with $l \geq 2$, we put

$$g_{l,k}(X_1, \ldots, X_l) := \sum_{\mathbb{J} \in \mathfrak{J}_l(k)} \prod_{J \in \mathbb{J}} \left\{ -\left(\sum_{u \in J} X_u - 1 \right) \right\}^{\#J-1}.$$

By definition, $\mathfrak{J}_l(k)$ is the set of all the partitions of $\{1, \ldots, l\}$ into k pieces. Note that the equality $g_{l,k}(X_1, \ldots, X_l) = 0$ trivially holds for $k \leq 0$ or $k \geq l+1$.

LEMMA 4.9. *For $\mathbb{I}' \in \mathfrak{I}(\lambda)$ with $\#\mathbb{I}' = l$ and for every $k \in \mathbb{Z}$, putting $\mathbb{I}' =: \{I_1, \ldots, I_l\}$ and $\#I_u =: i_u$ for $1 \leq u \leq l$, we have*

$$f_{l,k} = g_{l,k}(i_1, \ldots, i_l). \tag{4.9}$$

Proof. If $k \leq 0$ or $k \geq l+1$, then equation (4.9) trivially holds since both sides of equation (4.9) are equal to zero. In the following, we assume $1 \leq k \leq l$.

By definition, we have

$$f_{l,k} = \sum_{\mathbb{I} \prec \mathbb{I}',\, \#\mathbb{I}=k} \prod_{I \in \mathbb{I}} \{-(\#I - 1)\}^{\chi_I(\mathbb{I}')-1} = \sum_{\mathbb{I} \prec \mathbb{I}',\, \#\mathbb{I}=k} \prod_{I \in \mathbb{I}} \left\{ -\left(\sum_{1 \leq u \leq l,\ I_u \subset I} i_u - 1 \right) \right\}^{\chi_I(\mathbb{I}')-1}.$$

Hence, putting

$$\tilde{g}_{l,k}(X_1, \ldots, X_l) := \sum_{\mathbb{I} \prec \mathbb{I}',\, \#\mathbb{I}=k} \prod_{I \in \mathbb{I}} \left\{ -\left(\sum_{1 \leq u \leq l,\ I_u \subset I} X_u - 1 \right) \right\}^{\chi_I(\mathbb{I}')-1},$$

we obviously have $\tilde{g}_{l,k}(i_1, \ldots, i_l) = f_{l,k}$.

Here, we can make a bijection $\mathfrak{J}_l(k) \to \{\mathbb{I} \mid \mathbb{I} \prec \mathbb{I}',\ \#\mathbb{I} = k\}$ by

$$\mathbb{J} \mapsto \{\amalg_{u \in J} I_u \mid\ J \in \mathbb{J}\},$$

which implies that

$$\begin{aligned}
\tilde{g}_{l,k}(X_1, \ldots, X_l) &= \sum_{\mathbb{J} \in \mathfrak{J}_l(k)} \prod_{I \in \{\amalg_{u \in J} I_u \mid\ J \in \mathbb{J}\}} \left\{ -\left(\sum_{1 \leq u \leq l,\ I_u \subset I} X_u - 1 \right) \right\}^{\chi_I(\mathbb{I}')-1} \\
&= \sum_{\mathbb{J} \in \mathfrak{J}_l(k)} \prod_{J \in \mathbb{J}} \left\{ -\left(\sum_{1 \leq u \leq l,\ I_u \subset \amalg_{u' \in J} I_{u'}} X_u - 1 \right) \right\}^{\chi_{(\amalg_{u' \in J} I_{u'})}(\mathbb{I}')-1} \\
&= \sum_{\mathbb{J} \in \mathfrak{J}_l(k)} \prod_{J \in \mathbb{J}} \left\{ -\left(\sum_{u \in J} X_u - 1 \right) \right\}^{\#J-1} = g_{l,k}(X_1, \ldots, X_l).
\end{aligned}$$

Hence, we have equation (4.9). □

LEMMA 4.10. *The polynomial $g_{l,k}(X_1, \ldots, X_l)$ defined in Definition 4.8 is determined only by l and k, belongs to the polynomial ring $\mathbb{Z}[X_1, \ldots, X_l]$, and is symmetric in l variables $X_1, \ldots, X_l$. Moreover, the equality* $\deg g_{l,k} = l - k$ *holds for $l \geq 2$ and $1 \leq k \leq l$.*

Proof. The former two assertions are obvious by definition.

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The action of $\mathfrak{S}_l$ on $\{1, \ldots, l\}$ naturally induces the action of $\mathfrak{S}_l$ on $\mathfrak{J}_l(k)$ for each k, which implies that for every $\tau \in \mathfrak{S}_l$, we have $g_{l,k}(X_{\tau(1)}, \ldots, X_{\tau(l)}) = g_{l,k}(X_1, \ldots, X_l)$. Hence, $g_{l,k}(X_1, \ldots, X_l)$ is a symmetric polynomial in l variables $X_1, \ldots, X_l$.

Since $\sum_{J\in\mathbb{J}}(\#J - 1) = l - \#\mathbb{J} = l - k$ for every $\mathbb{J} \in \mathfrak{J}_l(k)$, we have $\deg g_{l,k} \leq l - k$. Moreover, for $\mathbb{J} \in \mathfrak{J}_l(k)$ with $1 \leq k \leq l$, the coefficient of each term of $\prod_{J\in\mathbb{J}}\{-(\sum_{u\in J} X_u - 1)\}^{\#J-1}$ with degree $l - k$ is positive or negative according to whether $l - k$ is even or odd. Hence, the terms with degree $l - k$ in $g_{l,k}(X_1, \ldots, X_l)$ are not canceled, which implies that the degree of $g_{l,k}(X_1, \ldots, X_l)$ is exactly equal to $l - k$ if $1 \leq k \leq l$. □

PROPOSITION 4.11. *For $l, k \in \mathbb{Z}$ with $l \geq 2$, we have*

$$g_{l+1,k}(X_1, \ldots, X_l, 0) = g_{l,k-1}(X_1, \ldots, X_l) - (X_1 + \cdots + X_l - k)g_{l,k}(X_1, \ldots, X_l).$$

Proof. First, we put

$$\mathfrak{J}^1_{l+1}(k) := \{\mathbb{J} \in \mathfrak{J}_{l+1}(k) \mid \{l+1\} \in \mathbb{J}\} \quad \text{and} \quad \mathfrak{J}^2_{l+1}(k) := \{\mathbb{J} \in \mathfrak{J}_{l+1}(k) \mid \{l+1\} \notin \mathbb{J}\}$$

for $l \geq 2$. Then we have $\mathfrak{J}^1_{l+1}(k) \amalg \mathfrak{J}^2_{l+1}(k) = \mathfrak{J}_{l+1}(k)$ for every k. Moreover, we have $\mathfrak{J}^1_{l+1}(k) = \emptyset$ for $k \leq 1$ or $k \geq l+2$, and $\mathfrak{J}^2_{l+1}(k) = \emptyset$ for $k \leq 0$ or $k \geq l+1$.

For $\mathbb{J} \in \mathfrak{J}^1_{l+1}(k)$, we can express $\mathbb{J} = \{J_1, \ldots, J_{k-1}, \{l+1\}\}$, where $J_1 \amalg \cdots \amalg J_{k-1} = \{1, \ldots, l\}$. Hence, we can make a bijection $\pi_1 : \mathfrak{J}^1_{l+1}(k) \to \mathfrak{J}_l(k-1)$ by $\mathbb{J} \mapsto \mathbb{J} \setminus \{\{l+1\}\}$. Moreover, for $J = \{l+1\} \in \mathbb{J} \in \mathfrak{J}^1_{l+1}(k)$, we have

$$\left\{-\left(\sum_{u\in J} X_u - 1\right)\right\}^{\#J-1} = \left\{-\left(X_{l+1} - 1\right)\right\}^{1-1} = 1.$$

Hence, we have

$$\begin{aligned}\sum_{\mathbb{J}\in\mathfrak{J}^1_{l+1}(k)} \prod_{J\in\mathbb{J}} \left\{-\left(\sum_{u\in J} X_u - 1\right)\right\}^{\#J-1} &= \sum_{\mathbb{J}\in\mathfrak{J}^1_{l+1}(k)} \prod_{J\in\pi_1(\mathbb{J})} \left\{-\left(\sum_{u\in J} X_u - 1\right)\right\}^{\#J-1} \\ &= \sum_{\mathbb{J}\in\mathfrak{J}_l(k-1)} \prod_{J\in\mathbb{J}} \left\{-\left(\sum_{u\in J} X_u - 1\right)\right\}^{\#J-1} \\ &= g_{l,k-1}(X_1, \ldots, X_l).\end{aligned} \tag{4.10}$$

For $\mathbb{J}' \in \mathfrak{J}^2_{l+1}(k)$, we can express $\mathbb{J}' = \{J_1, \ldots, J_k\}$ with $\{l+1\} \subsetneq J_k$, and in this expression, we have $\{J_1, \ldots, J_{k-1}, (J_k \setminus \{l+1\})\} \in \mathfrak{J}_l(k)$. Hence, we can make a surjection $\pi_2 : \mathfrak{J}^2_{l+1}(k) \to \mathfrak{J}_l(k)$ by $\mathbb{J}' \mapsto \{J \setminus \{l+1\} \mid J \in \mathbb{J}'\}$. For each $\mathbb{J} = \{J_1, \ldots, J_k\} \in \mathfrak{J}_l(k)$, its fiber $\pi_2^{-1}(\mathbb{J})$ consists of k elements, which are $\{J_v \mid 1 \leq v \leq k,\ v \neq v'\} \cup \{J_{v'} \amalg \{l+1\}\}$ for $1 \leq v' \leq k$. Hence, for each $\mathbb{J} = \{J_1, \ldots, J_k\} \in \mathfrak{J}_l(k)$, we have

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$$
\begin{aligned}
&\sum_{\mathbb{J}' \in \pi_2^{-1}(\mathbb{J})} \prod_{J \in \mathbb{J}'} \left\{ -\left(\sum_{u \in J} X_u - 1 \right) \right\}^{\#J-1} \Bigg|_{X_{l+1}=0} \\
&\quad = \sum_{v'=1}^{k} \Bigg[\left\{ -\left(\sum_{u \in J_{v'} \amalg \{l+1\}} X_u - 1 \right) \right\}^{\#(J_{v'} \amalg \{l+1\})-1} \\
&\qquad \times \prod_{1 \le v \le k,\ v \ne v'} \left\{ -\left(\sum_{u \in J_v} X_u - 1 \right) \right\}^{\#J_v - 1} \Bigg] \Bigg|_{X_{l+1}=0} \\
&\quad = \sum_{v'=1}^{k} \Bigg[\left\{ -\left(\sum_{u \in J_{v'}} X_u - 1 \right) \right\}^{\#J_{v'}} \cdot \prod_{1 \le v \le k,\ v \ne v'} \left\{ -\left(\sum_{u \in J_v} X_u - 1 \right) \right\}^{\#J_v - 1} \Bigg] \\
&\quad = \sum_{v'=1}^{k} \Bigg[\left\{ -\left(\sum_{u \in J_{v'}} X_u - 1 \right) \right\} \cdot \prod_{v=1}^{k} \left\{ -\left(\sum_{u \in J_v} X_u - 1 \right) \right\}^{\#J_v - 1} \Bigg] \\
&\quad = \Bigg[\sum_{v'=1}^{k} \left\{ -\left(\sum_{u \in J_{v'}} X_u - 1 \right) \right\} \Bigg] \cdot \prod_{v=1}^{k} \left\{ -\left(\sum_{u \in J_v} X_u - 1 \right) \right\}^{\#J_v - 1} \\
&\quad = -\left(\sum_{u=1}^{l} X_u - k \right) \cdot \prod_{J \in \mathbb{J}} \left\{ -\left(\sum_{u \in J} X_u - 1 \right) \right\}^{\#J-1}.
\end{aligned}
$$

We therefore have

$$
\begin{aligned}
&\sum_{\mathbb{J}' \in \mathfrak{J}_{l+1}^2(k)} \prod_{J \in \mathbb{J}'} \left\{ -\left(\sum_{u \in J} X_u - 1 \right) \right\}^{\#J-1} \Bigg|_{X_{l+1}=0} \\
&\quad = \sum_{\mathbb{J} \in \mathfrak{J}_l(k)} \sum_{\mathbb{J}' \in \pi_2^{-1}(\mathbb{J})} \prod_{J \in \mathbb{J}'} \left\{ -\left(\sum_{u \in J} X_u - 1 \right) \right\}^{\#J-1} \Bigg|_{X_{l+1}=0} \\
&\quad = \sum_{\mathbb{J} \in \mathfrak{J}_l(k)} \Bigg[-\left(\sum_{u=1}^{l} X_u - k \right) \cdot \prod_{J \in \mathbb{J}} \left\{ -\left(\sum_{u \in J} X_u - 1 \right) \right\}^{\#J-1} \Bigg] \\
&\quad = -\left(\sum_{u=1}^{l} X_u - k \right) \sum_{\mathbb{J} \in \mathfrak{J}_l(k)} \prod_{J \in \mathbb{J}} \left\{ -\left(\sum_{u \in J} X_u - 1 \right) \right\}^{\#J-1} \\
&\quad = -(X_1 + \cdots + X_l - k) g_{l,k}(X_1, \ldots, X_l).
\end{aligned} \tag{4.11}
$$

By equations (4.10) and (4.11), we have

$$
\begin{aligned}
g_{l+1,k}(X_1, \ldots, X_l, 0) &= \sum_{\mathbb{J} \in \mathfrak{J}_{l+1}(k)} \prod_{J \in \mathbb{J}} \left\{ -\left(\sum_{u \in J} X_u - 1 \right) \right\}^{\#J-1} \Bigg|_{X_{l+1}=0} \\
&= \sum_{\mathbb{J} \in \mathfrak{J}_{l+1}^1(k)} \prod_{J \in \mathbb{J}} \left\{ -\left(\sum_{u \in J} X_u - 1 \right) \right\}^{\#J-1}
\end{aligned}
$$

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$$+\sum_{\mathbb{J}'\in\mathfrak{J}^2_{l+1}(k)}\prod_{J\in\mathbb{J}'}\left\{-\left(\sum_{u\in J}X_u-1\right)\right\}^{\#J-1}\Bigg|_{X_{l+1}=0}$$
$$=g_{l,k-1}(X_1,\ldots,X_l)-(X_1+\cdots+X_l-k)g_{l,k}(X_1,\ldots,X_l),$$

which completes the proof of Proposition 4.11. □

LEMMA 4.12. *For every $l,k\in\mathbb{Z}$ with $l\geq 2$, there exists a polynomial $h_{l,k}(Y)\in\mathbb{Z}[Y]$ such that the equality*

$$g_{l,k}(X_1,\ldots,X_l)=h_{l,k}(X_1+\cdots+X_l) \tag{4.12}$$

holds. Moreover, for every $l,k\in\mathbb{Z}$ with $l\geq 2$, the equality

$$h_{l+1,k}(Y)=h_{l,k-1}(Y)-(Y-k)h_{l,k}(Y) \tag{4.13}$$

holds.

Proof. In the case where $l=2$, we have $g_{2,1}(X_1,X_2)=-(X_1+X_2-1)$ and $g_{2,2}(X_1,X_2)=1$ by a direct calculation. Hence, putting $h_{2,1}(Y)=-(Y-1)$, $h_{2,2}(Y)=1$, and $h_{2,k}(Y)=0$ for $k\neq 1,2$, we have $g_{2,k}(X_1,X_2)=h_{2,k}(X_1+X_2)$ for every $k\in\mathbb{Z}$.

For $l\geq 3$ and for every $k\in\mathbb{Z}$, we define the polynomials $h_{l,k}(Y)$ inductively by equation (4.13). Then we obviously have $h_{l,k}(Y)=0$ for $k\leq 0$ or $k\geq l+1$. Hence, equation (4.12) holds for $k\leq 0$ or $k\geq l+1$. In the following, we show equation (4.12) for $l\geq 3$ and $1\leq k\leq l$ by induction on l. Hence, we suppose equation (4.12) for every $k\in\mathbb{Z}$, and show the equality $g_{l+1,k}(X_1,\ldots,X_{l+1})=h_{l+1,k}(X_1+\cdots+X_{l+1})$ for $1\leq k\leq l+1$.

By the assumption and Proposition 4.11, we have

$$\begin{aligned}g_{l+1,k}(X_1,\ldots,X_l,0)&=g_{l,k-1}(X_1,\ldots,X_l)-(X_1+\cdots+X_l-k)g_{l,k}(X_1,\ldots,X_l)\\&=h_{l,k-1}(X_1+\cdots+X_l)-(X_1+\cdots+X_l-k)h_{l,k}(X_1+\cdots+X_l)\\&=h_{l+1,k}(X_1+\cdots+X_l).\end{aligned}$$

Hence, putting $P_{l+1,k}(X_1,\ldots,X_{l+1}):=g_{l+1,k}(X_1,\ldots,X_{l+1})-h_{l+1,k}(X_1+\cdots+X_{l+1})$, we have $P_{l+1,k}(X_1,\ldots,X_l,0)=0$. Moreover, by Lemma 4.10, the polynomial $P_{l+1,k}(X_1,\ldots,X_{l+1})$ is symmetric in $l+1$ variables $X_1,\ldots,X_{l+1}$.

We denote by $\sigma_{l+1,m}=\sigma_{l+1,m}(X_1,\ldots,X_{l+1})$ the elementary symmetric polynomial of degree m in $l+1$ variables $X_1,\ldots,X_{l+1}$. Since $P_{l+1,k}(X_1,\ldots,X_{l+1})$ is a symmetric polynomial with coefficients in $\mathbb{Z}$, we have $P_{l+1,k}(X_1,\ldots,X_{l+1})\in\mathbb{Z}[\sigma_{l+1,1},\ldots,\sigma_{l+1,l+1}]$. Moreover, since $\deg g_{l+1,k}=\deg h_{l+1,k}=l+1-k\leq l$, we have $\deg P_{l+1,k}\leq l$, which implies that $P_{l+1,k}(X_1,\ldots,X_{l+1})\in\mathbb{Z}[\sigma_{l+1,1},\ldots,\sigma_{l+1,l}]$.

Since $\sigma_{l+1,m}(X_1,\ldots,X_l,0)=\sigma_{l,m}(X_1,\ldots,X_l)$ for $1\leq m\leq l$, we have a ring isomorphism $\varphi:\mathbb{Z}[\sigma_{l+1,1},\ldots,\sigma_{l+1,l}]\to\mathbb{Z}[\sigma_{l,1},\ldots,\sigma_{l,l}]$ by substituting $X_{l+1}=0$, and under the map φ, we have $\varphi(P_{l+1,k})=P_{l+1,k}(X_1,\ldots,X_l,0)=0$. Hence, injectivity of φ implies $P_{l+1,k}(X_1,\ldots,X_{l+1})=0$. We therefore have $g_{l+1,k}(X_1,\ldots,X_{l+1})=h_{l+1,k}(X_1+\cdots+X_{l+1})$, which completes the proof of Lemma 4.12 by induction on l. □

Proof of Proposition 4.6. By Definition 4.1, $f_{l,k}$ is originally a function of $d \geq 4$, $\mathbb{I}' \in \mathfrak{I}(\lambda)$, and $k \in \mathbb{Z}$. However, putting $\#\mathbb{I}' = l$, $\mathbb{I}' =: \{I_1, \ldots, I_l\}$, and $\#I_u =: i_u$ for $1 \leq u \leq l$, we have by Lemmas 4.9 and 4.12 the equality

$$f_{l,k} = g_{l,k}(i_1, \ldots, i_l) = h_{l,k}(i_1 + \cdots + i_l) = h_{l,k}(d). \tag{4.14}$$

Hence, $f_{l,k}$ is in practice a function of l, k, and d since the polynomial $h_{l,k}(Y)$ depends only on l and k.

Moreover, by equation (4.14) and Lemma 4.12, we have

$$f_{l+1,k} = h_{l+1,k}(d) = h_{l,k-1}(d) - (d-k)h_{l,k}(d) = f_{l,k-1} - (d-k)f_{l,k}$$

for every $l, k \in \mathbb{Z}$ with $l \geq 2$, which completes the proof of Proposition 4.6. □

To summarize the above mentioned, we have completed the proof of Theorem I.

5. *Proof of Theorem II*

In this section, we prove Theorem II. Throughout this section, we always assume $\lambda = (\lambda_1, \ldots, \lambda_d) \in V_d$, and moreover assume that $s_d(\lambda)$ is the non-negative integer defined in Theorem 2.3.

First, we consider the case where $d = 2$. If $d = 2$, then the maps $p : \mathrm{MC}_2 \to \mathrm{MP}_2$ and $\Phi_2 : \mathrm{MP}_2 \to \widetilde{\Lambda}_2$ are bijective. Hence, we have $\#(\widehat{\Phi}_2^{-1}(\bar{\lambda})) = 1$ for every $\lambda \in V_2$. Regarding the right-hand side of equation (3.2), since $s_2(\lambda) = 1$ and $\mathfrak{K}(\lambda) = \{\{1\}, \{2\}\}$ for every $\lambda \in V_2$, we always have

$$\frac{(d-1)s_d(\lambda)}{\prod_{K \in \mathfrak{K}(\lambda)}(\#K)!} = \frac{(2-1)s_2(\lambda)}{1! \cdot 1!} = 1.$$

Hence, equation (3.2) holds for every $\lambda \in V_2$.

In the rest of this section, we consider the case $d \geq 3$. We denote by $\mathbb{P}^{d-1}$ the complex projective space of dimension $d-1$, and put

$$\Sigma_d(\lambda) := \left\{ (\zeta_1 : \cdots : \zeta_d) \in \mathbb{P}^{d-1} \;\middle|\; \begin{array}{c} \displaystyle\sum_{i=1}^{d} \zeta_i = 0 \\ \displaystyle\sum_{i=1}^{d} (1/(1-\lambda_i))\zeta_i^k = 0 \quad \text{for} \quad 1 \leq k \leq d-2 \\ \zeta_1, \ldots, \zeta_d \text{ are mutually distinct} \end{array} \right\}.$$

We already have the following proposition by Propositions 4.3 and 9.1 in [14].

PROPOSITION 5.1. *The equality $\#(\Sigma_d(\lambda)) = s_d(\lambda)$ holds. Moreover, we can define the surjection $\pi(\lambda) : \Sigma_d(\lambda) \to \Phi_d^{-1}(\bar{\lambda})$ by*

$$(\zeta_1 : \cdots : \zeta_d) \mapsto f(z) = z + \rho(z - \zeta_1) \cdots (z - \zeta_d),$$

where $-1/\rho = \sum_{i=1}^{d}(1/(1-\lambda_i))\zeta_i^{d-1}$.

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We put

$$\widetilde{\Sigma}_d(\lambda) := \left\{ (\zeta_1, \ldots, \zeta_d) \in \mathbb{C}^d \;\middle|\; \begin{array}{c} \sum_{i=1}^{d} \zeta_i = 0 \\ \sum_{i=1}^{d} (1/(1-\lambda_i))\zeta_i^k = \begin{cases} 0 & \text{for } 1 \le k \le d-2 \\ -1 & \text{for } k = d-1 \end{cases} \\ \zeta_1, \ldots, \zeta_d \text{ are mutually distinct} \end{array} \right\}.$$

Then the natural projection $\widetilde{\Sigma}_d(\lambda) \to \Sigma_d(\lambda)$ defined by $(\zeta_1, \ldots, \zeta_d) \mapsto (\zeta_1 : \cdots : \zeta_d)$ is a $(d-1)$-to-one map because for every $(\zeta_1 : \cdots : \zeta_d) \in \Sigma_d(\lambda)$, we have $\sum_{i=1}^{d}(1/(1-\lambda_i))\zeta_i^{d-1} \neq 0$ by Proposition 5.1. Hence, we have

$$\#(\widetilde{\Sigma}_d(\lambda)) = (d-1)\#(\Sigma_d(\lambda)) = (d-1)s_d(\lambda). \tag{5.1}$$

We consider next the relation between $\widetilde{\Sigma}_d(\lambda)$ and $\widehat{\Phi}_d^{-1}(\bar{\lambda})$. We can define the surjection $\widehat{\pi}(\lambda) : \widetilde{\Sigma}_d(\lambda) \to \widehat{\Phi}_d^{-1}(\bar{\lambda})$ by

$$(\zeta_1, \ldots, \zeta_d) \mapsto f(z) = z + (z-\zeta_1)\cdots(z-\zeta_d)$$

by lifting up the map $\pi(\lambda) : \Sigma_d(\lambda) \to \Phi_d^{-1}(\bar{\lambda})$ in Proposition 5.1. Here, since $d \geq 3$, every polynomial $f(z) = z + (z-\zeta_1)\cdots(z-\zeta_d)$ for $(\zeta_1, \ldots, \zeta_d) \in \widetilde{\Sigma}_d(\lambda)$ is monic and centered.

We put

$$\mathfrak{S}(\mathfrak{K}(\lambda)) := \{\sigma \in \mathfrak{S}_d \mid i \in K \in \mathfrak{K}(\lambda) \Longrightarrow \sigma(i) \in K\}.$$

Here, note that we also have $\mathfrak{S}(\mathfrak{K}(\lambda)) = \{\sigma \in \mathfrak{S}_d \mid \lambda_{\sigma(i)} = \lambda_i \text{ for every } 1 \le i \le d\}$. Moreover, $\mathfrak{S}(\mathfrak{K}(\lambda))$ is a subgroup of $\mathfrak{S}_d$ and is isomorphic to $\prod_{K \in \mathfrak{K}(\lambda)} \mathrm{Aut}(K) \cong \prod_{K \in \mathfrak{K}(\lambda)} \mathfrak{S}_{\#K}$.

The group $\mathfrak{S}(\mathfrak{K}(\lambda))$ naturally acts on $\widetilde{\Sigma}_d(\lambda)$ by the permutation of coordinates, and its action is free. Moreover, for $\zeta, \zeta' \in \widetilde{\Sigma}_d(\lambda)$, the equality $\widehat{\pi}(\lambda)(\zeta) = \widehat{\pi}(\lambda)(\zeta')$ holds if and only if the equality $\zeta' = \sigma \cdot \zeta$ holds for some $\sigma \in \mathfrak{S}(\mathfrak{K}(\lambda))$, which can be verified by a similar argument to the proof of Lemma 4.5(6) in [14]. We therefore have the bijection

$$\overline{\widehat{\pi}(\lambda)} : \widetilde{\Sigma}_d(\lambda)/\mathfrak{S}(\mathfrak{K}(\lambda)) \cong \widehat{\Phi}_d^{-1}(\bar{\lambda}),$$

which implies the equality

$$\#(\widehat{\Phi}_d^{-1}(\bar{\lambda})) = \frac{\#(\widetilde{\Sigma}_d(\lambda))}{\#(\mathfrak{S}(\mathfrak{K}(\lambda)))} = \frac{\#(\widetilde{\Sigma}_d(\lambda))}{\prod_{K \in \mathfrak{K}(\lambda)} (\#K)!}. \tag{5.2}$$

Combining equations (5.1) and (5.2), we have

$$\#(\widehat{\Phi}_d^{-1}(\bar{\lambda})) = \frac{(d-1)s_d(\lambda)}{\prod_{K \in \mathfrak{K}(\lambda)} (\#K)!},$$

which completes the proof of Theorem II.

https://doi.org/10.1017/etds.2022.120 Published online by Cambridge University Press

Acknowledgements. The author would like to express his thanks to Professor Hiroki Sumi for valuable advice on this paper. This work was supported by JSPS KAKENHI grant number JP19K14557.

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