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CO-REPRESENTATIONS OF HOPF–VON NEUMANN ALGEBRAS ON OPERATOR SPACES OTHER THAN COLUMN HILBERT SPACE

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Abstract

Recently, Daws introduced a notion of co-representation of abelian Hopf–von Neumann algebras on general reflexive Banach spaces. In this note, we show that this notion cannot be extended beyond subhomogeneous Hopf–von Neumann algebras. The key is our observation that, for a von Neumann algebra $\mathfrak{M}$ and a reflexive operator space E, the normal spatial tensor product $\mathfrak{M} \bar{\otimes} \mathcal{CB}(E)$ is a Banach algebra if and only if $\mathfrak{M}$ is subhomogeneous or E is completely isomorphic to column Hilbert space.

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1. Introduction

If $\mathfrak{A}$ is a Banach algebra, then its dual space $\mathfrak{A}^*$ is a Banach $\mathfrak{A}$-bimodule through

$$\langle x, a \cdot \phi \rangle := \langle xa, \phi \rangle \quad \text{and} \quad \langle x, \phi \cdot a \rangle := \langle ax, \phi \rangle \quad (a, x \in \mathfrak{A}, \phi \in \mathfrak{A}^*).$$

A functional $\phi \in \mathfrak{A}^*$ is said to be *weakly almost periodic* if $\{a \cdot \phi : a \in \mathfrak{A}, \|a\| \leq 1\}$ is relatively weakly compact in $\mathfrak{A}^*$. There appears to be some asymmetry in the definition of a weakly almost periodic functional, but thanks to Grothendieck's double limit criterion [7, Proposition 4], $\phi \in \mathfrak{A}^*$ is weakly almost periodic if and only if $\{\phi \cdot a : a \in \mathfrak{A}, \|a\| \leq 1\}$ is relatively weakly compact in $\mathfrak{A}^*$. The collection of all weakly almost periodic functionals on $\mathfrak{A}$ is a closed subspace of $\mathfrak{A}^*$, which we denote by $\mathrm{WAP}(\mathfrak{A})$.

Let G be a locally compact group, and let $\mathfrak{A}$ be its group algebra $L^1(G)$. In this case, it is not difficult to see that $\mathrm{WAP}(\mathfrak{A})$ is nothing more than $\mathrm{WAP}(G)$, the commutative C^*-algebra of all weakly almost periodic functions on G (see [3] for the definition and properties of $\mathrm{WAP}(G)$). Now, let $\mathfrak{A}$ be Eymard's Fourier algebra $A(G)$ with dual $\mathrm{VN}(G)$ (see [6]); in this case, we denote $\mathrm{WAP}(\mathfrak{A})$ by $\mathrm{WAP}(\hat{G})$. If G is abelian, then $\mathrm{WAP}(\hat{G})$ is indeed just the weakly almost periodic functions on the dual group $\hat{G}$

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and therefore, in particular, is a C^*-subalgebra of $L^\infty(\hat{G}) \cong \mathrm{VN}(G)$. With a little more effort, one can show that $\mathrm{WAP}(\hat{G})$ is a C^*-subalgebra of $\mathrm{VN}(G)$ whenever G has an abelian subgroup of finite index. For general G, however, it has been an open question for decades whether or not $\mathrm{WAP}(\hat{G})$ is always a C^*-subalgebra of $\mathrm{VN}(G)$.

2. Hopf–von Neumann algebras and co-representations

Recently, Daws showed in [4] that $\mathrm{WAP}(M(G))$ is a C^*-subalgebra of $\mathcal{C}_0(G)^{**}$ for any G, where $M(G)$ is the measure algebra of G. In fact, Daws proved a much more general result about abelian Hopf–von Neumann algebras.

DEFINITION 2.1. A *Hopf–von Neumann algebra* is a pair $(\mathfrak{M}, \Gamma)$, where $\mathfrak{M}$ is a von Neumann algebra, and Γ is a *co-multiplication*: a unital, injective, normal $*$-homomorphism $\Gamma : \mathfrak{M} \to \mathfrak{M} \bar{\otimes} \mathfrak{M}$ which is co-associative, that is,

$$(\mathrm{id} \otimes \Gamma) \circ \Gamma = (\Gamma \otimes \mathrm{id}) \circ \Gamma.$$

We call a Hopf–von Neumann algebra $(\mathfrak{M}, \Gamma)$ abelian, semidiscrete, and so on, if the underlying von Neumann algebra $\mathfrak{M}$ has the corresponding property.

EXAMPLE 2.2. Let G be a locally compact group.

(a) A co-multiplication

$$\Gamma : L^\infty(G) \to L^\infty(G) \bar{\otimes} L^\infty(G) \cong L^\infty(G \times G)$$

is given by

$$(\Gamma\phi)(x, y) := \phi(xy) \quad (\phi \in L^\infty(G), x, y \in G).$$

(Restricting Γ to $\mathcal{C}_0(G)$ and then taking section adjoints yields another co-multiplication $\tilde{\Gamma} : \mathcal{C}_0(G)^{**} \to \mathcal{C}_0(G)^{**} \bar{\otimes} \mathcal{C}_0(G)^{**}$.)

(b) Let $\lambda : G \to \mathcal{B}(L^2(G))$ be the left regular representation of G. Then a co-multiplication $\hat{\Gamma} : \mathrm{VN}(G) \to \mathrm{VN}(G) \bar{\otimes} \mathrm{VN}(G)$ is given by

$$\hat{\Gamma}(\lambda(x)) = \lambda(x) \otimes \lambda(x) \quad (x \in G).$$

Whenever $(\mathfrak{M}, \Gamma)$ is a Hopf–von Neumann algebra, the unique predual $\mathfrak{M}_*$ of $\mathfrak{M}$ becomes a Banach algebra via

$$\langle x, f * g\rangle := \langle \Gamma x, f \otimes g\rangle \quad (f, g \in \mathfrak{M}_*, x \in \mathfrak{M}).$$

EXAMPLE 2.3. If G is a locally compact group, then $*$ defined in this manner for $(L^\infty(G), \Gamma)$ is the usual convolution product on $L^1(G)$, whereas $*$ for $(\mathrm{VN}(G), \hat{\Gamma})$ is the pointwise product on $A(G)$.

Any von Neumann algebra $\mathfrak{M}$ is a (concrete) operator space, so that $\mathfrak{M}_*$ is an abstract operator space. (For background on operator space theory, we refer to [5], the notation of which we adopt.) If $(\mathfrak{M}, \Gamma)$ is a Hopf–von Neumann algebra, then Γ is a *complete* isometry. Consequently, $(\mathfrak{M}_*, *)$ is not only a Banach algebra, but a *completely contractive Banach algebra* [5, p. 308].

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The main result of [4] is as follows.

THEOREM 2.4. *Let* $(\mathfrak{M}, \Gamma)$ *be an abelian Hopf–von Neumann algebra. Then* $\mathrm{WAP}(\mathfrak{M}_*)$ *is a* C^**-subalgebra of* $\mathfrak{M}$.

At the heart of Daws's proof is the notion of a co-representation of a Hopf–von Neumann algebra. Usually one considers co-representation on Hilbert spaces.

DEFINITION 2.5. Let $(\mathfrak{M}, \Gamma)$ be a Hopf–von Neumann algebra. A *co-representation* of $(\mathfrak{M}, \Gamma)$ on a Hilbert space $\mathfrak{H}$ is an operator $U \in \mathfrak{M} \bar{\otimes} \mathcal{B}(\mathfrak{H})$ such that

$$(\Gamma \otimes \mathrm{id})(U) = U_{1,3} U_{2,3}. \tag{$*$}$$

Here, $U_{1,3}$ and $U_{2,3}$ are what is called *leg notation*: if $\mathfrak{M}$ acts on a Hilbert space $\mathfrak{K}$, then $U_{1,3}$ is the linear operator on the Hilbert space tensor product $\mathfrak{K} \otimes_2 \mathfrak{K} \otimes_2 \mathfrak{H}$ that acts as U on the first and the third factor and as the identity on the second one; $U_{2,3}$ is defined similarly.

Commonly, co-representations are also required to be unitary, but we will not need that property.

By [5, Corollary 7.1.5 and Theorem 7.2.4], we have the completely isometric identifications

$$\mathfrak{M} \bar{\otimes} \mathcal{B}(\mathfrak{H}) = (\mathfrak{M}_* \hat{\otimes} \mathcal{B}(\mathfrak{H})_*)^* = \mathcal{CB}(\mathfrak{M}_*, \mathcal{B}(\mathfrak{H})).$$

Given an operator $U \in \mathfrak{M} \bar{\otimes} \mathcal{B}(\mathfrak{H})$, the corresponding map in $\mathcal{CB}(\mathfrak{M}_*, \mathcal{B}(\mathfrak{H}))$ is

$$\mathfrak{M}_* \to \mathcal{B}(\mathfrak{H}), \quad f \mapsto (f \otimes \mathrm{id})(U), \tag{$**$}$$

and ($*$) is equivalent to ($**$) being a multiplicative map from $(\mathfrak{M}_*, *)$ into $\mathcal{B}(\mathfrak{H})$. The advantage of looking at elements of $\mathfrak{M} \bar{\otimes} \mathcal{B}(\mathfrak{H})$ instead of $\mathcal{CB}(\mathfrak{M}_*, \mathcal{B}(\mathfrak{H}))$ is that $\mathfrak{M} \bar{\otimes} \mathcal{B}(\mathfrak{H})$ is again a von Neumann algebra, so that it makes sense to multiply its elements.

Let $(\mathfrak{M}, \Gamma)$ be an abelian Hopf–von Neumann algebra. Then $\mathfrak{M}$ is of the form $L^\infty(X)$ for some measure space X. The proof of Theorem 2.4 in [4] has three main ingredients.

(1) Elements of $\mathrm{WAP}(L^1(X))$ arise as coefficients of representations of $(L^1(X), *)$ on reflexive Banach spaces.

(2) For a reflexive Banach space E, the weak* closure of $L^\infty(X) \otimes \mathcal{B}(E)$ in $\mathcal{B}(L^2(X, E))$, denoted by $L^\infty(X) \bar{\otimes} \mathcal{B}(E)$, can be identified with $\mathcal{B}(L^1(X), \mathcal{B}(E))$ [4, Proposition 3.2]. This identification then allows co-representations of $(L^\infty(X), \Gamma)$ to be defined on E by analogy with Definition 2.5.

(3) The product in $L^\infty(X) \bar{\otimes} \mathcal{B}(E)$ corresponds to the product in $\mathrm{WAP}(L^1(X))$.

Is it possible to adapt this approach to more general Hopf–von Neumann algebras?

3. Co-representations on operator spaces

In [4], Daws uses the symbol $L^\infty(X) \bar{\otimes} \mathcal{B}(E)$ for the closure of $L^\infty(X) \otimes \mathcal{B}(E)$ in $\mathcal{B}(L^2(X, E))$. In operator space theory, the symbol $\bar{\otimes}$ is usually reserved for the

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normal spatial tensor product of dual operator spaces [5, p. 134]. Both usages are consistent: for a reflexive Banach space E, we have the isometric identifications

$$\begin{aligned}\mathcal{B}(L^1(X), \mathcal{B}(E)) &= \mathcal{CB}(L^1(X), \mathcal{CB}(\max E)) \\ &= (L^1(X) \hat{\otimes} (\max E \hat{\otimes} \min E^*))^* \\ &= L^\infty(X) \bar{\otimes} \mathcal{CB}(\max E).\end{aligned}$$

From [4, Proposition 3.2], we thus conclude that the product on $L^\infty(X) \otimes \mathcal{B}(E)$ extends to $L^\infty(X) \bar{\otimes} \mathcal{CB}(\max E)$, turning it into a Banach algebra. More generally, $L^\infty(X) \bar{\otimes} \mathcal{CB}(E)$ is a Banach algebra for any reflexive, homogeneous operator space E, that is, satisfying $\mathcal{CB}(E) = \mathcal{B}(E)$ with identical norms.

Let $\mathfrak{M}$ be a semidiscrete von Neumann algebra, and let E be a reflexive operator space. Then we have the completely isometric identifications

$$\mathcal{CB}(\mathfrak{M}_*, \mathcal{CB}(E)) = (\mathfrak{M}_* \hat{\otimes} (E \hat{\otimes} E^*))^* = \mathfrak{M} \bar{\otimes} \mathcal{CB}(E).$$

(We need the semidiscreteness of $\mathfrak{M}$ for the second equality: without it, we would not get $\mathfrak{M} \bar{\otimes} \mathcal{CB}(E)$, but the *normal Fubini tensor product* $\mathfrak{M} \bar{\otimes}_{\mathcal{F}} \mathcal{CB}(E)$; see [8].) This suggests that it might be possible to define a notion of co-representation of semidiscrete Hopf–von Neumann algebras on general reflexive operator spaces. Just to meaningfully state the right-hand side ($*$) for some $U \in \mathfrak{M} \bar{\otimes} \mathcal{CB}(E)$, we need $\mathfrak{M} \bar{\otimes} \mathfrak{M} \bar{\otimes} \mathcal{CB}(E)$ to be multiplicatively closed, and to adapt the proof of Theorem 2.4 to general (semidiscrete) Hopf–von Neumann algebras, we need $\mathfrak{M} \bar{\otimes} \mathcal{CB}(E)$ also to be multiplicatively closed. It all comes down to the question whether or not $\mathfrak{M} \bar{\otimes} \mathcal{CB}(E)$ is a Banach algebra for a (semidiscrete) von Neumann algebra and a reflexive operator space E.

For abelian $(\mathfrak{M}, \Gamma)$, this is indeed the case, and it is not difficult to extend [4, Proposition 3.2] to a general operator space setting.

PROPOSITION 3.1. *Let $L^\infty(X)$ be an abelian von Neumann algebra, and let E be a reflexive operator space. Then the closure of $L^\infty(X) \otimes \mathcal{CB}(E)$ in $\mathcal{CB}(L^2(X, E))$ is isometrically isomorphic to $\mathcal{CB}(L^1(X), \mathcal{CB}(E))$. In particular, the product of $L^\infty(X) \otimes \mathcal{CB}(E)$ extends to $L^\infty(X) \bar{\otimes} \mathcal{CB}(E)$, turning it into a Banach algebra.*

Here, the operator space structure on $L^2(X, E)$ is that obtained through complex interpolation between $L^\infty(X) \check{\otimes} E$ and $L^1(X) \hat{\otimes} E$ as described in [9].

We can even go a little beyond the abelian framework. If $\mathfrak{M}$ is a subhomogeneous von Neumann algebra, it is of the form

$$\mathfrak{M} \cong M_{n_1}(\mathfrak{M}_1) \oplus \cdots \oplus M_{n_k}(\mathfrak{M}_k)$$

with $n_1, \ldots, n_k \in \mathbb{N}$ and abelian von Neumann algebras $\mathfrak{M}_1, \ldots, \mathfrak{M}_k$. This yields the following corollary.

COROLLARY 3.2. *Let $\mathfrak{M}$ be a subhomogeneous von Neumann algebra, and let E be a reflexive operator space. Then the product of $\mathfrak{M} \otimes \mathcal{CB}(E)$ extends to $\mathfrak{M} \bar{\otimes} \mathcal{CB}(E)$,*

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turning it into a Banach algebra (with bounded, but not necessarily contractive multiplication).

So, if $(\mathfrak{M}, \Gamma)$ is a subhomogeneous Hopf–von Neumann algebra, we can meaningfully speak of its co-representations on reflexive operator spaces. But what if we go beyond subhomogeneous von Neumann algebras? For certain operator spaces, this is no problem. Let $\mathfrak{H}$ be a Hilbert space, and let $\mathfrak{H}_c$ be column Hilbert space [5, Section 3.4]. Then $\mathcal{CB}(\mathfrak{H}_c) = \mathcal{B}(\mathfrak{H})$ as operator spaces [5, Theorem 3.4.1], so that

$$\mathfrak{M} \bar{\otimes} \mathcal{CB}(\mathfrak{H}_c) = \mathfrak{M} \bar{\otimes} \mathcal{B}(\mathfrak{H})$$

is a von Neumann algebra, and co-representations on $\mathfrak{H}_c$ are nothing more than co-representations on $\mathfrak{H}$ in the sense of Definition 2.5.

As we shall see, this is about as far as we can get, and we state the following theorem.

THEOREM 3.3. *Let* $(\mathfrak{M}, \Gamma)$ *be a Hopf–von Neumann algebra, and let E be a reflexive operator space. Then the following are equivalent.*

(i) *The product of* $\mathfrak{M} \otimes \mathcal{CB}(E)$ *extends to* $\mathfrak{M} \bar{\otimes} \mathcal{CB}(E)$*, turning it into a Banach algebra with bounded, but not necessarily contractive multiplication.*
(ii) *The product of* $\mathfrak{M} \otimes \mathcal{CB}(E)$ *extends to* $\mathfrak{M} \check{\otimes} \mathcal{CB}(E)$*, turning it into a Banach algebra with bounded, but not necessarily contractive multiplication.*
(iii) $\mathfrak{M}$ *is subhomogeneous or E is completely isomorphic to* $\mathfrak{H}_c$ *for some Hilbert space* $\mathfrak{H}$*.*

For the proof, we require a lemma.

LEMMA 3.4. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be completely contractive Banach algebras such that* $\mathfrak{A}$ *contains the full matrix algebra* M_n *as a subalgebra for each* $n \in \mathbb{N}$*, and suppose that the product of* $\mathfrak{A} \otimes \mathfrak{B}$ *extends to* $\mathfrak{A} \check{\otimes} \mathfrak{B}$*, turning it into a Banach algebra with bounded, but not necessarily contractive multiplication. Then* $\mathfrak{B}$ *is completely isomorphic to an operator algebra.*

Here, an operator algebra is a closed, but not necessarily self-adjoint subalgebra of $\mathcal{B}(\mathfrak{H})$ for some Hilbert space $\mathfrak{H}$.

PROOF. Let $C \geq 1$ be a bound for the multiplication in $\mathfrak{A} \check{\otimes} \mathfrak{B}$, and note that, for $n \in \mathbb{N}$, we have a complete isometry $M_n(\mathfrak{B}) \cong M_n \check{\otimes} \mathfrak{B}$ [5, Corollary 8.1.3]. Since M_n is a subalgebra of $\mathfrak{A}$, this means that formal matrix multiplication from $M_n(\mathfrak{B}) \times M_n(\mathfrak{B})$ to $M_n(\mathfrak{B})$ is bounded by C for each n. By the definition of the *Haagerup tensor product* (see [5, Ch. 9]) $\mathfrak{B} \otimes^h \mathfrak{B}$, this means that multiplication $m : \mathfrak{B} \otimes \mathfrak{B} \to \mathfrak{B}$ extends to a completely bounded map $m : \mathfrak{B} \otimes^h \mathfrak{B} \to \mathfrak{B}$. Hence, $\mathfrak{B}$ is completely isomorphic to an operator algebra by [2, Theorem 5.2.1]. □

PROOF OF THEOREM 3.3. We have previously seen that (iii) $\Longrightarrow$ (i) holds, and (i) $\Longrightarrow$ (ii) is obvious.

To prove (ii) $\Longrightarrow$ (iii), assume that $\mathfrak{M}$ is *not* subhomogeneous. Then the structure theory of von Neumann algebras yields that $\mathfrak{M}$ contains M_n as a subalgebra for each n.

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By the lemma, $\mathcal{CB}(E)$ is thus completely isomorphic to an operator algebra. By [1, Theorem 3.4], this means that E is completely isomorphic to $\mathfrak{H}_c$ for some Hilbert space $\mathfrak{H}$. □

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