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OPTIMAL MAINTENANCE OF SEMI-MARKOV MISSIONS

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We analyze optimal replacement and repair problems of semi-Markov missions that are composed of phases with random sequence and durations. The mission process is the minimal semi-Markov process associated with a Markov renewal process. The system is a complex one consisting of non-identical components whose failure properties depend on the mission process. We prove some monotonicity properties for the optimal replacement policy and analyze the optimal repair problem under different cost structures.

1. INTRODUCTION

Many complex systems perform missions which are composed of different phases or stages. Deterioration of the components and configuration of the system change dramatically from phase to phase. Such systems are called phased-mission systems or mission-based systems in the literature where the sequence and the durations of phases can be deterministic or random. The most important property of these systems is that all stochastic and deterministic failure properties of the components depend on the phases of the mission. This creates stochastic dependence among the lifetimes of the components via the common mission process. In this paper, we analyze optimal replacement and repair problems for phased-mission systems assuming that the sequence of the phases and their durations are random. Our results are valid for any lifetime distribution which can be chosen arbitrarily for each component.

Phased-mission systems were introduced by Esary and Ziehms [8] and a vast literature has accrued since then. We refer the reader to Burdick et al. [3], Veatch [15], Kim and Park [9] and references cited in these papers for phased-mission systems with deterministic sequence of phases. There are also some papers on phased-mission systems with random sequence of phases such as Mura and Bondavalli [10], Mura and Bondavalli [11] and Bondavalli and Filippini [2].

The phased-mission system considered in this study is assumed to perform a mission with several phases whose sequence and durations are random. More specifically, we assume

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that the sequence of the successive phases follows a Markov chain and the phase durations are generally distributed. Some previous work and real-life applications are the main source of our motivation. In the prevailing phased-mission system literature, the phase durations are generally assumed to be deterministic, which may be realistic for some applications, such as aerospace applications where phases are preplanned on the ground (Mura and Bondavalli [11]). However, Alam and Al-Saggaf [1] state that random phase durations are more realistic in many systems including real-time control for air-craft and space vehicles in which different sets of computational tasks are executed during different phases of a control process. In NASA's Mars Exploration Mission, for example, the mission consists of many phases such as *Vehicle Launch*; *Cruise*; *Approach*; *Entry*; *Descent and Landing* to Mars; *Rover Egress*; and a number of *Surface Operations* that involve scientific data collection and transmission to the Earth. Some complex calculations and evaluations by the science and engineering teams determine what the rover actually does on the surface. The scientific investigations performed involve further phases with random sequence and durations. Moreover, systems may be affected by sources of randomness that are of an exogenous nature. For instance, the performance of the rovers depends on various atmospheric conditions which can be defined as different phases affecting the system. As another example, consider an online shopping system which performs a mission with four phases; namely, *Home*, *Search*, *Login*, and *Buy*. In phase *Home*, a user, visiting the Web page of the system, reads the advertisements on the home page. Then, the user can either search for a specific product or directly go to login page, where the system performs *Search* and *Login* phases, respectively. The user may complete the session either by entering credit card information while the system performs the *Buy* phase or by abandoning the system. It is clear that the mission performed by this online system has a dynamic structure and can be represented by using a Markovian model.

We will use the intrinsic aging model introduced by Çınlar and Özekici [6] who propose to construct an intrinsic clock which ticks differently in different environments or phases to measure the intrinsic age of the component. The environment is modelled by a semi-Markov jump process and the intrinsic age is represented by the cumulative hazard accumulated in time during the operation of the device in the randomly varying environment. Özekici [14] analyzed optimal replacement and repair problems for a single unit utilizing intrinsic aging concepts and showed that optimal replacement policy is a control limit policy. Moreover, some characterizations for the optimal repair policy under different cost structures are proposed in that study. In this study, we actually extend Özekici [14] to the multi-component case since we define the mission process as an environmental process which is not affected by the deterioration levels of the components. We refer the interested readers to Özekici [13], Cho and Parlar [4], Dekker, Wildeman and Van der Duyn Schouten [7], Wang [16], and Nicolai and Dekker [12] for related literature on optimal maintenance.

Our purpose in the present setting is to study the optimal replacement and repair problems of a multi-component phased-mission system. We assume that the sequence of the phases follows a Markov chain, the duration of each phase and the lifetime of each component are generally distributed. We aim to prove some monotonicity properties of the optimal policies under the usual assumptions requiring increasing failure rate (IFR) life distributions and reasonable cost structures.

This study contributes to the both phased-mission and optimal maintenance literatures. The prevailing studies in the phased-mission literature generally assume deterministic phase durations, which is not always reasonable as mentioned previously, or focus on specific system structures. However, we consider a multi-component phased-mission system with random durations, and our results are valid for any system structure. Our study is similar to the optimal maintenance models including an environmental process modulating the system failure parameters. In our setting, the environmental process is a Markov renewal process, and the system is a multi-component device with generally distributed lifetimes. To the best of our knowledge, this is the most general model analyzed in the optimal maintenance literature yet.

In Section 2, the mission process and the intrinsic aging model are described. We will analyze replacement and repair problems in Sections 3 and 4, respectively. The lengthy proofs of some results given in the paper are provided in the Appendix. Finally, throughout this paper, increasing means non-decreasing and decreasing means non-increasing.

2. THE MISSION PROCESS AND INTRINSIC AGING

Let X_n denote the nth phase of the mission and T_n denote the time at which the nth phase starts with $T_0 \equiv 0$. The main assumption is that the process $(X, T) = \{(X_n, T_n); n \geq 0\}$ is a Markov renewal process on the countable state space E with some semi-Markov kernel Q. The state space E is actually that of the process X and it is implicitly understood that the process T always takes values in $\mathbb{R}_+ = [0, +\infty)$ since they denote times at which certain events occur. We refer the reader to Çınlar [5] for a more rigorous and detailed treatment of Markov renewal processes and theory. The Markov renewal property states that

$$P\{X_{n+1} = j, T_{n+1} - T_n \leq t | X_0, \ldots, X_n; T_0, \ldots, T_n\} = P\{X_{n+1} = j, T_{n+1} - T_n \leq t | X_n\}, \quad \textbf{(1)}$$

where we suppose that the process is time-homogeneous with the semi-Markov kernel

$$Q(i, j, t) = P\{X_{n+1} = j, T_{n+1} - T_n \leq t | X_n = i\} \quad \textbf{(2)}$$

for any $i, j \in E$ and $t \in \mathbb{R}_+$. The mission process $Y = \{Y_t; t \geq 0\}$ is the minimal semi-Markov process associated with (X, T) so that Y_t is the stage or phase of the mission at time t. More precisely, $Y_t = X_n$ whenever $T_n \leq t < T_{n+1}$.

Throughout this paper, we consider a complex device with an arbitrary structure and m components. We let $S = \{1, \ldots, m\}$ denote the set of all components in the system and $L(k)$ denote the lifetime of component $k \in S$. The lifetime of the whole system is denoted by L. There is, of course, a relationship between the system and component failure times. For example, $L = \min\{L(k); k \in S\}$ for a series system and $L = \max\{L(k); k \in S\}$ for a parallel system.

Since all components perform the same phase at a given time, their lifetimes are dependent via their common mission process. The successive flights of an airplane, for instance, can be considered as a mission including simply five phases, namely take-off, smooth cruise, cruise in turbulence, landing, and idleness. It is well known that failure rates of airplane components, especially components of jet engine turbines, are much higher during take-off and landing than during cruise. Moreover, it is reasonable to assume that a turbulence increases the failure rates. For such an airplane, it is not realistic to assume that the lifetimes of the components constituting the airplane are independent. This is mainly because these components will operate in turbulence for the same amount of time, and they will experience the same number of take-offs and landings. The degradation rates of the components will increase and decrease simultaneously, and the corresponding transition times will be governed by the same mission performed by the airplane. Therefore, the component lifetimes are dependent via the common mission process. We further assume that the mission process is the only source of dependence among the component lifetimes, and they are otherwise independent during any given phase.

We consider a very general optimal maintenance problem. The system is composed of multiple non-identical components with generally distributed lifetimes. Furthermore, these general lifetime distributions are modulated by an external process which is also a very general stochastic process, namely a semi-Markov process. Since the memoryless property does not hold in this case, the mathematical analysis of such a complex model is quite hard. Consider a transition time at which a phase of the mission is completed and another phase starts. At this time point, to be able to compute the residual lifetime of the system probabilistically, the age of each component should be known. However, the real age of a component, which is equal to the time since the start of the mission, is not directly useful since different phases have different properties and, therefore, a same amount of time in different phases has different degradation effects on the components. That is why in addition to the real component ages, the previous phases and their durations should also be known at the transition time to analyze the residual lifetime of the system. It is clear that this issue complicates the mathematical analysis of the model. To attack this problem related to the systems working in random environments, Çınlar and Özekici [6] suggests the intrinsic aging model in which the intrinsic age of a component is defined as its cumulative hazard accumulated in time during the operation of the device. This definition implies that the authors design an intrinsic clock which ticks differently in different environments. This directly implies that the intrinsic clock associated with a component will tick differently in different phases of a mission since we define the mission process as an external environment process. One of the main advantages of using the intrinsic age of a component instead of the real age is that it is sufficient to know the intrinsic ages of the components to analyze the residual lifetime of the system at the transition times, that is, it is not necessary to store the previously performed phases. This is mainly due to the fact that the intrinsic clock already takes account of the effects of changing phases. The second, possibly more important, main advantage of the intrinsic aging model is that the intrinsic lifetime of any device is exponentially distributed with rate 1. This implies that it is possible to make use of the memoryless property if the intrinsic clock is used instead of real-time clock. Due to these advantages which are especially helpful in our setting, we use the intrinsic aging model in this study. The formal construction of the intrinsic aging model is given in the remainder of this section.

Let $H_k(i,t)$ be the cumulative hazard of component k at time t in phase i which is assumed to be continuously differentiable in t. Then, we have the well-known equality

$$P\{L(k) > t | Y = i\} = e^{-H_k(i,t)}$$

when the phase is fixed to be $\{Y = i\} = \{Y_t = i \text{ for all } t \geq 0\}$. Note that if $L(k)$ has a continuously differentiable distribution function in phase i, then $H_k(i,t)$ is continuously differentiable in t. The intrinsic age of component k at time t is defined as $H_k(i,t)$ provided that the system performs phase i throughout $[0,t]$. The intrinsic aging rate of component k during phase i is defined as

$$r_k(i,a) = \frac{d}{dt} H_k(i,t) \,|_{t=H_k^{-1}(i,a)} \tag{3}$$

at any age $a \in \mathbb{R}_+$ where $H_k^{-1}(i,a)$ is the time at which the intrinsic age of component k becomes a if the system performs phase i; or

$$H_k^{-1}(i,a) = \inf\{t \in \mathbb{R}_+; H_k(i,t) > a\}.$$

Since $H_k(i,t)$ is increasing in t, $H_k\left(i, H_k^{-1}(i,a)\right) = a$.

Let $A(k) = \{A_t(k); t \in \mathbb{R}_+\}$ denote the intrinsic age process of component k for all $k \in S$. We assume that the intrinsic age process satisfies

$$\frac{dA_t(k)}{dt} = r_k(Y_t, A_t(k))$$

for $0 \leq t < \min\{L(k), L\}$. Therefore, if the intrinsic age of component k at time s when phase i starts is $A_s(k) = a$, then after t units of time its age becomes

$$A_{s+t}(k) = h_k(i, a, t) = H_k\left(i, H_k^{-1}(i, a) + t\right). \tag{4}$$

Note that this definition requires that both component k and the system are functioning at time $s + t$. Let $B_0(k) = A_0(k)$ and define an embedded process $B(k) = \{B_n(k); n = 0, 1, \ldots\}$ recursively through

$$B_{n+1}(k) = h_k(X_n, B_n(k), T_{n+1} - T_n)$$

for $n \geq 0$. The intrinsic aging process $A = \{A_t(k); t \in \mathbb{R}_+, k \in S\}$ of the whole system consists of the aging processes of the components. Note that $A_t \in \mathcal{F} = \overline{\mathbb{R}}_+^m = [0, +\infty]^m$ for all $t \in \mathbb{R}_+$ and $\mathcal{F}$ is the state space of A. The intrinsic age process of component k can be constructed recursively by

$$A_{T_n+t}(k) = h_k(X_n, B_n(k), t)$$

provided that $t \leq T_{n+1} - T_n$ and both the component and the whole system are functioning at time $T_n + t$. As soon as component k fails at some time $L(k)$, we set the intrinsic age to $A_{L(k)+t}(k) = +\infty$ for all $t \in \mathbb{R}_+$. Clearly, $+\infty$ denotes the failure state. We extend the definition of h_k in Eq. (4) such that $h_k(i, +\infty, t) = +\infty$ since a failed component remains failed.

Following the construction in Çınlar and Özekici [6], it is clear that component k is not in a failed state at time t if and only if $A_t(k) < \widehat{L}(k)$ where $\widehat{L}(k)$ is the intrinsic lifetime of component k. These also imply that component k fails at time

$$L(k) = \inf\{t \in \mathbb{R}_+; A_t(k) > \hat{L}(k)\}$$

when its intrinsic age exceeds its intrinsic lifetime. Furthermore, since the intrinsic lifetimes $\{\widehat{L}(k)\}$ are independent and identically distributed random variables that have the exponential distribution with rate 1, we can write

$$\begin{aligned} P\{L(k) > t | A_0(k) = a, X_0 = i\} &= P\left\{\widehat{L}(k) > A_t(k) | A_0(k) = a, X_0 = i\right\}, \\ &= E\left[e^{-(A_t(k) - a)} | A_0(k) = a, X_0 = i\right]. \end{aligned} \tag{5}$$

Let $B = \{0, 1\}$ be the binary set, $h(i, a, t)$ denote the vector with elements $h_k(i, a(k), t)$ and

$$I_{\{\text{condition}\}} = \begin{cases} 1 & \text{if condition holds,} \\ 0 & \text{otherwise} \end{cases}$$

be the indicator function for any condition, for example, $a < b$, $x \in A$, etc. We let ψ_i be the structure function of the system defined on $\mathcal{F}$ during phase i such that

$$\psi_i(a) = \begin{cases} 1 & \text{if the system is in working condition at intrinsic age } a, \\ 0 & \text{otherwise} \end{cases}$$

for all $a \in \mathcal{F}$. It can be determined using the reliability structure of the whole system. For instance, if the system is a series one with m components during phase i, then

$$\psi_i(a) = \prod_{k=1}^{m} I_{\{a(k)<+\infty\}},$$

for all $a \in \mathcal{F}$, where $a(k)$ is the intrinsic age of component k. Similarly, if the system is a parallel one during phase i, then

$$\psi_i(a) = 1 - \prod_{k=1}^{m} \left(1 - I_{\{a(k)<+\infty\}}\right)$$

for all $a \in \mathcal{F}$. More generally, if we have a coherent structure with some structure function ϕ_i defined on B^m during phase i, then it suffices to take

$$\psi_i(a) = \phi_i \left(I_{\{a(1)<+\infty\}}, I_{\{a(2)<+\infty\}}, \ldots, I_{\{a(m)<+\infty\}}\right).$$

In this study, we assume that $\psi_i(a) = \psi_i(a_1, a_2, \ldots, a_m)$ is nonincreasing in a_k for every k. Note that if the system structure is coherent in all phases, this condition is satisfied trivially.

We assume that $P\{T_1 \leq t, X_1 = j | X_0 = i, A_0 = a\} = Q(i, j, t)$ for all $a \in F$, so that the age of the system does not affect the mission that will be performed. Moreover, we now let

$$\overline{p}^k_{ia(k)}(s, db(k)) = P\{A_s(k) \in db | Y = i, A_0 = a\} \tag{6}$$

denote the probability that the intrinsic age of component k will be in $db(k)$ at time s given that the initial age of the system is a and the mission consists of phase i only. It now follows from the construction of the intrinsic aging process that

$$\overline{p}^k_{ia(k)}(s, db(k)) = \begin{cases} e^{-(b(k)-a(k))} & \text{if } a(k) < +\infty, b(k) = h_k\left(i, a(k), s\right) < +\infty, \\ 1 - e^{-(h_k(i,a(k),s)-a(k))} & \text{if } a(k) < +\infty, b(k) = +\infty, \\ 1 & \text{if } a(k) = +\infty, b(k) = +\infty, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

We suppose that the condition $\{Y = i\}$ in Eq. (6) can be extended further so that $\overline{p}^k_{ia(k)}(t, db(k)) = P\{A_t(k) \in db | Y, A_0 = a\}$ for any realization of the mission process $Y = \{Y_s; s \geq 0\}$ so long as $Y_s = i$ for all $0 \leq s \leq t$. If the system is at age a initially, the probability that the age of the system will be in db after s units of time during phase i is

$$\widetilde{p}\left(i, a, s, db\right) = P\left\{A_t \in db | Y = i, A_0 = a\right\} = \prod_{k=1}^{m} \overline{p}^k_{ia(k)}(s, db(k)). \tag{8}$$

Note that Eq. (8) follows from our assumption that the aging of the components are independent during any given phase. As long as the phase of the mission is fixed, aging of component k occurs deterministically according to $h_k\left(i, a\left(k\right), t\right)$ for all $k \in S$ and any component fails as soon as the age exceeds the exponential threshold.

In the foregoing text, unless otherwise specified, any vector c is a column vector. If a and b are vectors with the same size, we will use $a \geq (\leq) \, b$, $a \neq b$, and $a \lneqq b$ when $a\left(k\right) \geq (\leq) \, b\left(k\right)$ for every k, $a(k) \neq b(k)$ for at least one k, and $a \leq b$ with $a \neq b$ respectively. For any vectors $x, y \in \mathcal{F}$ with $x = (x(1), \ldots, x(m))$ and $y = (y(1), \ldots, y(m))$, the arithmetic operations xy, $x+y$, $x-y$, and x/y define the vectors whose ith entries are given by $x(i)\,y(i)$, $x(i)+y(i)$, $x(i)-y(i)$, and $x(i)/y(i)$. We let $\mathbf{0}$ and $\mathbf{1}$ denote column vectors with all entries being equal to 0 and 1, respectively. We also assume that $r_k(i,a)$ is increasing in a and it is strictly positive. Finally, all costs are discounted at some rate $\alpha > 0$. For a technical reason which will be clear shortly, we further assume that $K = \sup_{i\in E} E\left[e^{-\alpha T_1} | X_0 = i\right] < 1$.

3. OPTIMAL REPLACEMENT PROBLEM

In this section, we will analyze a quite complex maintenance problem for a phased mission system with some structure function ψ_i during phase i. We assume that the system is observed only at the beginning of each phase. After an observation, a decision is made for each component to replace or not to replace it by considering the intrinsic age vector of the system. Then, the system starts to perform the next phase. We also assume that the duration of the replacement activity is negligible (or included in the phase durations).

We let B^m be the set of all replacement policies so that for any $r \in B^m$, if $r(k) = 1(0)$, component k will (will not) be replaced. If the next phase is i and the intrinsic age vector of the system is a, then the cost of applying the replacement policy r is $c_m(i,a;r)$. The cost of performing phase i with an initial intrinsic age a is $c(i,a)$, which is increasing in a; and if the system fails during phase i, the failure cost f_i is incurred. We assume that $\sup_{i\in E, a\in\mathcal{F}} c(i,a) = C < +\infty$ and $\sup_{i\in E} f_i = f < +\infty$.

Assumption 1: *The maintenance cost function $c_m : E \times \mathcal{F} \times B^m \to \mathbb{R}_+$ satisfies*

(i) $c_m(i,a;\mathbf{0}) = 0$;
(ii) $r,s \in B^m$ *with* $r \geq s$ *implies* $c_m(i,a;r) \geq c_m(i,a;s)$;
(iii) $r,s \in B^m$ *with* $rs = \mathbf{0}$ *implies that* $c_m(i,a;r+s) \leq c_m(i,a;r) + c_m(i,a;s)$;
(iv) $c_m(i,a;r)$ *is independent of* a_k *if* $r_k = 0$ *for all* k;
(v) $\sup_{i\in E, a\in\mathcal{F}} c_m(i,a;\mathbf{1}) = C_m < +\infty$;
(vi) $c_m(i,a;r)$ *is increasing in* a_k *for all* k.

The conditions imposed on c_m by Assumption 1 are quite important and reasonable. Conditions (i) and (ii) simply state that no cost is incurred if there is no replacement and the replacement cost increases as more components are replaced. By condition (iii), if we consider two replacement policies which do not replace the same components, the cost of applying both policies at the same time is less than the sum of the individual costs. This is very reasonable if there is a fixed cost associated with each replacement activity. Condition (iv) asserts that the cost of a replacement policy is not affected by the age of a component that is not replaced. The cost of replacing older components is higher by condition (vi). This is also reasonable since the salvage value of older components is lower.

Our purpose is to find a replacement policy which minimizes the expected total discounted cost. Let $v(i,a)$ denote the minimum expected total discounted cost if the initial phase is i, and the device is at age a. Then, v satisfies the dynamic programming equation (DPE)

$$v(i,a) = \min_{r\in B^m} \left\{c_m(i,a;r) + c(i,a(1-r)) + \Gamma v(i,a(1-r))\right\} \quad \textbf{(9)}$$

where the operator $\Gamma : \mathfrak{B} \to \mathfrak{B}$ is defined by

$$\Gamma g\left(i,a\right) = \sum_{j \in E} \int_0^{+\infty} Q\left(i,j,ds\right) e^{-\alpha s} \left\{ \int_{\mathcal{F}} \widetilde{p}\left(i,a,s,db\right) \left[g\left(j,b\right) + \left(1-\psi_i\left(b\right)\right) f_i\right] \right\} \quad \textbf{(10)}$$

for any function g in the set $\mathfrak{B}$ of all bounded nonnegative real-valued functions defined on $E \times \mathcal{F}$. Note that we implicitly assume in Eq. (10) that if the system fails during a phase, the failure cost is incurred at the end of the phase.

We will first show that the DPE Eq. (9) has a unique solution, which is increasing in the initial age of the system. We will next give several results explaining the structure of the optimal replacement policy. The next theorem shows the existence of the optimal replacement policy utilizing Banach's contraction mapping theorem.

Theorem 2: *There is a unique function v^* in $\mathfrak{B}$ which satisfies the DPE given in Eq. (9).*

We next show that v^* is an increasing function of the age of the system for a fixed phase. The following lemma is very important in proving this. Define

$$B_k = \{c \in \mathcal{F} : c\left(k\right) < +\infty\}, \quad \textbf{(11)}$$

so that its complement is $\overline{B}_k = \{c \in \mathcal{F} : c\left(k\right) = +\infty\}$.

Lemma 3: *If $g\left(i,a\right)$ is increasing in a for every $i \in E$, then*

$$\int_{\overline{B}_k} \prod_{j \neq k} \overline{p}^j_{ia(j)}(s,dc(j)) f\left(i,j,c\right) \geq \int_{\substack{B_k; \\ c(k)=h_k(i,a(k),s)}} \prod_{j \neq k} \overline{p}^j_{ia(j)}(s,dc(j)) f\left(i,j,c\right)$$

for all $i \in E, k \in S, a \in \mathcal{F}$, and $s \in \mathbb{R}_+$ where

$$f\left(i,j,c\right) = g\left(j,c\right) + \left(1-\psi_i\left(c\right)\right) f_i.$$

Now, the following main result can be proved by using the previous technical lemma.

Theorem 4: *Let v^* be the optimal solution in Theorem 2, then*

(i) $0 \leq v^* \leq (C_m + C + Kf)/(1-K)$;
(ii) *$v^*\left(i,a\right)$ is increasing in a.*

We introduce some new notation for simplicity. If r^* is the optimal policy, we let

$$C\left(i,a\right) = \{k; r^*_k\left(i,a\right) = 1\}$$
$$R\left(i,A\right) = \{a; C\left(i,a\right) = A\}$$

for every $i \in E, a \in \mathcal{F}$, and $A \subset S$. Here, $C\left(i,a\right)$ denotes the set of components which are optimally replaced if the age of the system is a during phase i, and $R\left(i,A\right)$ denotes the set of ages at which the optimal decision is to replace the components in A during phase i.

The following theorem simply states that the optimal decision is do-nothing for any state reached just after a maintenance. In other words, it is always preferred to replace a set of components at the same time, rather than replacing two disjoint subsets of the same components successively. This is an intuitive consequence of (iii) in Assumption 1 which is quite reasonable if there is a fixed cost involved in the repair procedure.

THEOREM 5: *There is an optimal replacement policy satisfying DPE given in Eq. (9) such that*

(i). $r_k^*(i,a) = 0$ *if* $a_k = 0$,
(ii). $a(1 - r^*(i,a)) \in R(i, \varnothing)$.

PROOF: The proof of the first statement trivially follows from Eq. (9) since $c_m(i,a;r)$ is increasing in r and kth entry of $a(1-r)$ is 0 independent of r. To prove the second statement, first note that $r_k^*(i,a) = 1$ implies that

$$r_k^*(i, a(1 - r^*(i,a))) = 0$$

for any $k \in C(i,a)$ by the first statement. By taking the contrapositive, $r_k^*(i,a(1 - r^*(i,a))) = 1$ implies that $r_k^*(i,a) = 0$. Therefore,

$$r^*(i,a)\, r^*(i, a(1 - r^*(i,a))) = \mathbf{0}$$

and

$$c_m(i, a; r^*(i, a(1 - r^*(i,a)))) = c_m(i, a(1 - r^*(i,a)); r^*(i, a(1 - r^*(i,a))))$$

by Assumption 1. By defining

$$\widehat{r}(i,a) = r^*(i,a) + r^*(i, a(1 - r^*(i,a)))$$

we have

$$\begin{aligned}
v^*(i,a) &\le c_m(i,a;\widehat{r}(i,a)) + c(i, a(1-\widehat{r}(i,a))) + \Gamma v^*(i, a(1-\widehat{r}(i,a))) \\
&\le c_m(i,a;r^*(i,a)) + c(i, a(1-\widehat{r}(i,a))) + \Gamma v^*(i, a(1-\widehat{r}(i,a))) \\
&\quad + c_m(i, a(1-r^*(i,a)); r^*(i, a(1-r^*(i,a)))) \\
&= c_m(i,a;r^*(i,a)) + c_m(i, a(1-r^*(i,a)); r^*(i, a(1-r^*(i,a)))) \\
&\quad + c(i, a(1-r^*(i,a))(1 - r^*(i, a(1-r^*(i,a))))) \\
&\quad + \Gamma v^*(i, a(1-r^*(i,a))(1 - r^*(i, a(1-r^*(i,a))))) \\
&= c_m(i,a;r^*(i,a)) + v^*(i, a(1-r^*(i,a))) \\
&\le c_m(i,a;r^*(i,a)) + c(i, a(1-r^*(i,a))) + \Gamma v^*(i, a(1-r^*(i,a))) \\
&= v^*(i,a).
\end{aligned}$$

In this chain of implications, the first inequality directly follows from Eq. (9) since $\widehat{r} \in B^m$. The second inequality follows from (iii) in Assumption 1. The first equality follows from the fact that

$$\begin{aligned}
1 - \widehat{r}(i,a) &= 1 - r^*(i,a) - r^*(i, a(1-r^*(i,a))) + r^*(i,a)\, r^*(i, a(1-r^*(i,a))) \\
&= 1 - r^*(i,a) - (1 - r^*(i,a))\, r^*(i, a(1-r^*(i,a))) \\
&= (1 - r^*(i,a))(1 - r^*(i, a(1-r^*(i,a))))
\end{aligned}$$

since

$$r^*(i,a)\, r^*(i, a(1-r^*(i,a))) = \mathbf{0}.$$

The second equality follows from the fact that

$$\begin{aligned} v^*\left(i, a\left(1-r^*\left(i, a\right)\right)\right) = {} & c_m\left(i, a\left(1-r^*\left(i, a\right)\right); r^*\left(i, a\left(1-r^*\left(i, a\right)\right)\right)\right) \\ & + c\left(i, a\left(1-r^*\left(i, a\right)\right)\left(1-r^*\left(i, a\left(1-r^*\left(i, a\right)\right)\right)\right)\right) \\ & + \Gamma v^*\left(i, a\left(1-r^*\left(i, a\right)\right)\left(1-r^*\left(i, a\left(1-r^*\left(i, a\right)\right)\right)\right)\right) \end{aligned}$$

and the third inequality follows from the fact that $c_m\left(i, a, x; \mathbf{0}\right) = 0$ and

$$v^*\left(i, a\left(1-r^*\left(i, a\right)\right)\right) \leq c\left(i, a\left(1-r^*\left(i, a\right)\right)\right) + \Gamma v^*\left(i, a\left(1-r^*\left(i, a\right)\right)\right). \qquad \textbf{(12)}$$

Finally, the last equality is trivial since $r^*\left(i, a\right)$ minimizes the right-hand side of Eq. (9). Therefore, all of these inequalities must be equalities which means that

$$c_m\left(i, a; \widehat{r}\left(i, a\right)\right) = c_m\left(i, a; r^*\left(i, a\right)\right) + c_m\left(i, a\left(1-r^*\left(i, a\right)\right); r^*\left(i, a\left(1-r^*\left(i, a\right)\right)\right)\right)$$

and Eq. (12) is also an equality. But this implies that we can define r^* at $a\left(1-r^*\left(i, a\right)\right)$ such that $r^*\left(i, a\left(1-r^*\left(i, a\right)\right)\right) = \mathbf{0}$ and $\widehat{r}\left(i, a\right) = r^*\left(i, a\right)$ with $a\left(1-r^*\left(i, a\right)\right) \in R\left(i, \varnothing\right)$. ■

The following two results express the monotonic structure of the optimal replacement policy. If the optimal decision for a component is replacement at a decision epoch where the component age is a, then the same decision is still optimal when the age of the component is larger than a in the same phase.

THEOREM 6: *Suppose that $c_m\left(i, a; r\right)$ is independent of a. Then, there is an optimal replacement policy satisfying DPE given in Eq. (9) such that*

(i) *If $b_k \geq a_k$ for $k \in A \subset C\left(i, a\right)$ and $b_k = a_k$ for $k \notin A$, then $r^*\left(i, b\right) = r^*(i, a)$;*
(ii) *If $b_k < a_k$ for $k \in A \subset S \backslash C(i, a)$ and $b_k = a_k$ for $k \notin A$, then there exists $k \in A$ such that $r_k^*(i, b) = 0$.*

PROOF: Using Theorem 4, we have $v^*\left(i, b\right) \geq v^*\left(i, a\right)$ and, hence,

$$\begin{aligned} & c_m\left(i, b; r^*\left(i, b\right)\right) + c\left(i, b\left(1-r^*\left(i, b\right)\right)\right) + \Gamma v^*\left(i, b\left(1-r^*\left(i, b\right)\right)\right) \\ & \quad \geq c_m\left(i, a; r^*\left(i, a\right)\right) + c\left(i, a\left(1-r^*\left(i, a\right)\right)\right) + \Gamma v^*\left(i, a\left(1-r^*\left(i, a\right)\right)\right) \\ & = c_m\left(i, b; r^*\left(i, a\right)\right) + c\left(i, b\left(1-r^*\left(i, a\right)\right)\right) + \Gamma v^*\left(i, b\left(1-r^*\left(i, a\right)\right)\right). \end{aligned}$$

The last equality follows from the main assumption. This result implies that at age b, if we apply the optimal policy at age a, we have the same optimal cost. Therefore, in the optimal policy at age b, we can apply the optimal replacement policy at age a and this proves (i). To prove (ii) by contradiction, suppose that $r_k^*\left(i, b\right) = 1$ for every $k \in A$. Then, $a\left(k\right) \geq b\left(k\right)$ for $k \in A \subset C\left(i, b\right)$ and $a\left(k\right) = b\left(k\right)$ for $k \notin A$. Applying Theorem 6 (i), we have $r^*\left(i, a\right) = r^*\left(i, b\right)$. This implies that $r_k^*\left(i, a\right) = 1$ for every $k \in A \subset S \backslash C\left(i, a\right)$. Clearly, this is a contradiction. ■

An immediate corollary of this theorem is the following.

COROLLARY 7: *Let r^* be the optimal policy of Theorem 6. Then,*

(i) *$r_k^*\left(i, a\right)$ is increasing in a_k for all $k \in S$;*
(ii) *If $a \in R\left(i, S\right)$, then $b \in R\left(i, S\right)$ for all $b \geq a$;*

(iii) *If* $b(k) < a(k)$ *and* $b(j) = a(j)$ *for every* $j \neq k$ *with* $r_k^*(i,a) = 0$, *then* $r_k^*(i,b) = 0$;

(iv) *If* $a \in R(i, \varnothing)$, $b(k) < a(k)$ *and* $b(j) = a(j)$ *for every* $j \neq k$, *then* $r_k^*(i,b) = 0$.

PROOF: To prove (i), it suffices to show that if $r_k^*(i,a) = 1$ for some $a \in \mathcal{F}$, then $r_k^*(i,b) = 1$ for $b \in \mathcal{F}$ with $b(k) \geq a(k)$ and $b(j) = a(j)$ for every $j \neq k$. This follows from Theorem 6 trivially. To prove the second statement suppose that $a \in R(i,S)$ and choose $b \in \mathcal{F}$ such that $b \geq a$. Then, since $r_k^*(i,a) = 1$ for every $k \in S$, $r^*(i,b) = r^*(i,a)$ using Theorem 6. This trivially implies that $b \in R(i,S)$. The proofs of (iii) and (iv) follow trivially from (ii) in Theorem 6 by taking $A = \{k\}$. ■

Even though the structure of the optimal replacement policy of the whole system is too complex to be characterized using some simple rules, the optimal replacement policy of a component in the system is monotone. This also implies that the optimal policy for a component in the system adopts a control-limit rule, where the threshold value is a function of the phase of the mission and the ages of the other components in the system. In other words, there exists a threshold value above which the optimal decision for a component in the system is "replace", and below which to do nothing is optimal if the phase of the mission and the ages of the other components are fixed.

4. OPTIMAL REPAIR PROBLEM

In the previous section, there are only two decision alternatives at each decision epoch: to replace a component by a brand new one or to let it operate during the next phase. In many applications, however, it is also possible to repair a component so that its age is decreased to a lower level by some technical maintenance operations or by simply replacing the old component by one that is younger, if not brand new. We will use the settings and probabilities constructed in the previous section once more.

The decision maker observes the system at the beginning of each phase and makes a repair decision. If the next phase is i and the intrinsic age of the system is a at the end of a phase, then the decision maker chooses an action $y(i,a)$ from the set $\{b \in \mathcal{F}; b \leq a\}$. The cost of repairing the system from age a to b during phase i is $C_i(a;b)$ where $b \leq a$.

ASSUMPTION 8: *The repair cost function* $C_i : \left\{(a,b) \in \mathcal{F}^2; b \leq a\right\} \to \mathbb{R}_+$ *satisfies*

(i) $C_i(a;b)$ *is increasing in* a_k *and decreasing in* b_k *for every* $k \in S$;

(ii) $C_i(a;a) = 0$;

(iii) $\sup_{i \in E, a \in \mathcal{F}} C_i(a;\mathbf{0}) = C_r < +\infty$.

Condition (i) simply states that the cost of repair increases as the amount of improvement increases. Condition (ii) asserts that if the system does not experience any maintenance, then no cost will be incurred. It is clear that these are very reasonable assumptions.

We also suppose that if there are more than one optimal repair action, then the alternative with lower final age will be chosen. Our purpose is to find a repair policy which minimizes the expected total discounted cost. Let $v(i,a)$ denote the minimum expected total discounted cost if the initial phase is i, and the device is at age a. Then, v satisfies the DPE

$$v\,(i,a) = \inf_{b;b\leq a} \{C_i\,(a;b) + c\,(i,b) + \Gamma v\,(i,b)\}\,, \tag{13}$$

where the operator $\Gamma : \mathfrak{B} \to \mathfrak{B}$ is defined by

$$\Gamma g\,(i,a) = \sum_{j\in E} \int_0^{+\infty} Q\,(i,j,ds)\, e^{-\alpha s} \left\{ \int_{\mathcal{F}} \widetilde{p}\,(i,a,s,db)\,[g\,(j,b) + (1-\psi_i\,(b))\,f_i] \right\} \tag{14}$$

for any function g in $\mathfrak{B}$.

The following result shows that an optimal solution for the DPE Eq. (13) exists.

THEOREM 9: *There is a unique function v^* in $\mathfrak{B}$ which satisfies the DPE given in Eq. (13).*

The next result shows that the DPE Eq. (13) corresponding to the optimal repair problem has a bounded solution which is increasing in the ages of the components. The only restriction for being increasing is that the repair cost function must be non-decreasing in the ages of the unmaintained components. This is very reasonable since the repair cost function is generally independent of the ages of the unmaintained components in real-life applications, and the condition of the result is satisfied in this case.

THEOREM 10: *Let v^* be the optimal value function of Theorem 9. Then,*

(i) $0 \leq v^* \leq (C_r + C + Kf)/(1-K)$;

(ii) *If $C_i\,(a;b) \geq C_i\,(\overline{a};\overline{b})$ whenever $b_k = a_k \geq \overline{a}_k = \overline{b}_k$ for some k and $\overline{a}_j = a_j$, $\overline{b}_j = b_j$ for every $j \neq k$, then $v^*\,(i,a)$ is increasing in a_k for every k.*

We will let $y^*\,(i,a)$ denote the optimal decision at state (i,a) which provides the minimum to the right-hand side of Eq. (13). The structure of the repair cost function C is too general to obtain useful characterizations of the optimal policy. We will therefore impose additional restrictions on C which lead to some simplifications.

The first case that we consider assumes that for the same amount of improvement, the cost of two successive repairs is larger than the cost of a direct repair. Under this cost structure, it is always preferred to repair a system at some age a directly to a lower age b, rather than repairing it first to an intermediate age c $(b < c < a)$ and then to age b. If there is a fixed cost associated with each repair action, then this assumption is quite reasonable. A quite intuitive and expected consequence of this assumption is the following theorem.

THEOREM 11: *If $C_i\,(a;b) \leq C_i\,(a;c) + C_i\,(c;b)$ for all $a \geq c \geq b \geq \mathbf{0}$ and $i \in E$, then there is an optimal policy such that $y^*\,(i, y^*\,(i,a)) = y^*\,(i,a)$.*

PROOF: Choose arbitrary $i \in E$, and $a \in \mathcal{F}$. Suppose that $y^*\,(i,a) = \alpha$, and choose some $b \leq \alpha$. Using the main hypothesis, we have

$$C_i\,(a;b) \leq C_i\,(a;\alpha) + C_i\,(\alpha;b)$$

and

$$C_i\,(a;b) - C_i\,(a;\alpha) \leq C_i(\alpha;b). \tag{15}$$

Since $y^*\,(i,a) = \alpha$,

$$C_i\,(a;\alpha) + c\,(i,\alpha) + \Gamma v^*\,(i,\alpha) \leq C_i\,(a;b) + c\,(i,b) + \Gamma v^*\,(i,b)$$

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and

$$c\,(i,\alpha)+\Gamma v^{*}\,(i,\alpha)\leq C_{i}\,(a;b)-C_{i}\,(a;\alpha)+c\,(i,b)+\Gamma v^{*}(i,b).$$

This implies that

$$c\,(i,\alpha)+\Gamma v^{*}\,(i,\alpha)\leq C_{i}\,(\alpha;b)+c\,(i,b)+\Gamma v^{*}(i,b) \tag{16}$$

using Eq. (15). Since b is an arbitrary value satisfying $b\leq\alpha$, using Eq. (16), we have

$$c\,(i,\alpha)+\Gamma v^{*}\,(i,\alpha)\leq \inf_{b;b\leq\alpha}\{C_{i}\,(\alpha;b)+c\,(i,b)+\Gamma v^{*}\,(i,b)\}.$$

Since $C_i\,(\alpha;\alpha)=0$, we can conclude that we can choose $y^{*}\,(i,\alpha)=\alpha$ in the optimal repair policy and, hence, $y^{*}\,(i,y^{*}\,(i,a))=y^{*}\,(i,a)$. ■

The next result states that if there is no fixed cost associated with a repair action, and if the optimal decision at age $a+u$ is to repair the system to age a, then the optimal decision for the all intermediate ages, from a to $a+u$, is to repair the system to the same age. This clearly implies that the optimal repair policy is monotone, as is proved in the subsequent corollary.

THEOREM 12: *Choose some $a\in\mathcal{F}$ and $i\in E$. Suppose that $C_i\,(b;d)=C_i\,(b;c)+C_i\,(c;d)$ for all $a+u\geq b\geq c\geq d\geq \mathbf{0}$ for some $u\geq\mathbf{0}$. Then, $y^{*}\,(i,a+u)\leq a$ implies that there is an optimal policy such that $y^{*}\,(i,a+z)=y^{*}\,(i,a)$ for all $0\leq z\leq u$.*

PROOF: Choose arbitrary $z\neq\mathbf{0}$. By the main hypothesis, it is clear that there is a repair decision at $(i,a+u)$ and, hence,

$$\begin{aligned}v^{*}\,(i,a+u)&=\inf_{b;b\leq a+u}\{C_{i}\,(a+u;b)+c\,(i,b)+\Gamma v^{*}\,(i,b)\}\\&=\inf_{b;b\leq a+z}\{C_{i}\,(a+u;b)+c\,(i,b)+\Gamma v^{*}\,(i,b)\}\\&=\inf_{b;b\leq a}\{C_{i}\,(a+u;b)+c\,(i,b)+\Gamma v^{*}\,(i,b)\}\,.\end{aligned}$$

This implies that

$$\begin{aligned}\inf_{b;b\leq a+z}\{C_{i}\,(a+u;b)+c\,(i,b)+\Gamma v^{*}\,(i,b)\}&=C_{i}\,(a+u;a+z)\\&\quad+\inf_{b;b\leq a+z}\{C_{i}\,(a+z;b)+c\,(i,b)+\Gamma v^{*}\,(i,b)\}\\&=C_{i}\,(a+u;a+z)\\&\quad+\inf_{b;b\leq a}\{C_{i}\,(a+z;b)+c\,(i,b)+\Gamma v^{*}\,(i,b)\}\end{aligned}$$

and

$$v^{*}\,(i,a+z)=\inf_{b;b\leq a+z}\{C_{i}\,(a+z;b)+c\,(i,b)+\Gamma v^{*}\,(i,b)\} \tag{17}$$

$$=\inf_{b;b\leq a}\{C_{i}\,(a+z;b)+c\,(i,b)+\Gamma v^{*}\,(i,b)\}. \tag{18}$$

Therefore, there is an optimal policy with $y^*(i, a+z) \leq a$. Now, choose $b \leq a$. Then,

$$\begin{aligned}
&C_i(a+z; y^*(i,a)) + c(i, y^*(i,a)) + \Gamma v^*(i, y^*(i,a)) \\
&= C_i(a+z; a) + C_i(a; y^*(i,a)) + c(i, y^*(i,a)) + \Gamma v^*(i, y^*(i,a)) \\
&\leq C_i(a+z; a) + C_i(a; b) + c(i,b) + \Gamma v^*(i,b) \\
&= C_i(a+z; b) + c(i,b) + \Gamma v^*(i,b).
\end{aligned}$$

This and Eq. (17) imply that

$$\begin{aligned}
&C_i(a+z; y^*(i,a)) + c(i, y^*(i,a)) + \Gamma v^*(i, y^*(i,a)) \\
&\quad \leq \inf_{b; b \leq a} \{C_i(a+z; b) + c(i,b) + \Gamma v^*(i,b)\} \\
&\quad = \inf_{b; b \leq a+z} \{C_i(a+z; b) + c(i,b) + \Gamma v^*(i,b)\}
\end{aligned}$$

and there is an optimal policy such that $y^*(i, a+z) = y^*(i,a)$. ■

An immediate corollary of the theorem is the following, stating that if there is no fixed cost of a repair action, then the optimal repair policy is monotone.

Corollary 13: *Suppose that $C_i(a;b) = C_i(a;c) + C_i(c;b)$ for every $a \geq c \geq b \geq \mathbf{0}$ and $i \in E$. Then, there is an optimal policy such that $y_k^*(i,a)$ is increasing in a_k.*

Proof: Suppose that $y^*(i,a) = c$ and $y^*(i, a+u) = b$ where $u_k > 0$ and $u_j = 0$ for every $j \neq k$. If $b_k \geq a_k$, then $b_k \geq c_k$ since $c_k \leq a_k$ trivially. Now, suppose that $b_k < a_k$. Then, we have $b \leq a$ and $y^*(i,a) = y^*(i, a+u)$ using Theorem 12 and this completes the proof. ■

The following result gives a sufficient condition for the optimal repair policy to be a replacement policy. The sufficient condition is that the total repair and state occupancy cost is increasing in the final age of the system occurring as a result of a repair activity. This condition is only possible when the marginal increase in the state occupancy cost $c(i,b)$ is larger than the marginal decrease in the repair cost $C_i(a,b)$ as b increases for all $a \in \mathcal{F}$. If this condition holds, then the optimal repair policy has the structure that the optimal decision is either "do nothing" or system replacement. This result is very intuitive since repairing the device to a smaller age is always cheaper under this cost structure.

Proposition 14: *Suppose that*

$$C_i(a;b) + c(i,b) \geq C_i(a;c) + c(i,c) \tag{19}$$

whenever $b \geq c$ and $b \neq a$. Then, $v^*(i,a)$ is increasing in a and there is an optimal policy such that $y^*(i,a) \in \{\mathbf{0}, a\}$. If Eq. (19) also holds for $b = a$, then there is an optimal policy such that $y^*(i,a) = \mathbf{0}$.

Proof: Following the same steps as in the proof of Theorem 4, it can be shown that $\Gamma g(i,a)$ is increasing in a. Choose $a, c \in \mathcal{F}$ such that $c(k) > a(k)$ and $c(j) = a(j)$ for every $j \neq k$. We need to show that $\Upsilon g(i,c) \geq \Upsilon g(i,a)$, where

$$\Upsilon g(i,a) = \inf_{b; b \leq a} \{C_i(a;b) + c(i,b) + \Gamma g(i,b)\} \tag{20}$$

for all $i \in E, a \in \mathcal{F}$.

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Choose $b \leq a$. Then, trivially $b \leq c$ and, hence,

$$C_i(c;b) + c(i,b) + \Gamma g(i,b) \geq C_i(a;b) + c(i,b) + \Gamma g(i,b) \geq \Upsilon g(i,a)$$

since $C_i(a;b)$ is increasing in a. Now, choose $b \leq c$ with $c(k) > b(k) > a(k)$. Define $\bar{b}$ such that $\bar{b}(k) = a(k)$ and $\bar{b}(j) = b(j)$ for every $j \neq k$. Then, $\bar{b} \leq b$ and $\bar{b} \leq a$. This implies that

$$\begin{aligned} C_i(c;b) + c(i,b) + \Gamma g(i,b) &\geq C_i\left(c;\bar{b}\right) + c\left(i,\bar{b}\right) + \Gamma g\left(i,\bar{b}\right) \\ &\geq C_i\left(a;\bar{b}\right) + c\left(i,\bar{b}\right) + \Gamma g\left(i,\bar{b}\right) \\ &\geq \Upsilon g(i,a). \end{aligned}$$

If $b = c$,

$$\begin{aligned} C_i(c;b) + c(i,b) + \Gamma g(i,b) &= c(i,c) + \Gamma g(i,c) \\ &\geq c(i,a) + \Gamma g(i,a) \\ &\geq \Upsilon g(i,a). \end{aligned}$$

Then,

$$\begin{aligned} \Upsilon g(i,c) = \min\Bigg\{ &\inf_{b;b\leq a}\{C_i(c;b) + c(i,b) + \Gamma g(i,b)\}, \\ &\inf_{b;b\leq c, b(k)>a(k)}\{C_i(c;b) + c(i,b) + \Gamma g(i,b)\}\Bigg\} \\ &\geq \min\{\Upsilon g(i,a), \Upsilon g(i,a)\} = \Upsilon g(i,a). \end{aligned}$$

Using the main hypothesis,

$$C_i(a;\mathbf{0}) + c(i,\mathbf{0}) + \Gamma v(i,\mathbf{0}) \leq C_i(a;b) + c(i,b) + \Gamma v(i,b)$$

for every $b \lneqq a$. Therefore,

$$\begin{aligned} v^*(i,a) &= \inf_{b;b\leq a}\{C_i(a;b) + c(i,b) + \Gamma v^*(i,b)\} \\ &= \min\left\{\inf_{b\lneqq a}\{C_i(a;b) + c(i,b) + \Gamma v^*(i,b)\}, c(i,a) + \Gamma v^*(i,a)\right\} \\ &= \min\{C_i(a;\mathbf{0}) + c(i,\mathbf{0}) + \Gamma v^*(i,\mathbf{0}), c(i,a) + \Gamma v^*(i,a)\}. \end{aligned}$$

This trivially implies that there is an optimal policy such that $y^*(i,a) \in \{\mathbf{0}, a\}$. Suppose that Eq. (19) also holds for $b = a$. It is sufficient to show that $C_i(a;\mathbf{0}) + c(i,\mathbf{0}) + \Gamma v^*(i,\mathbf{0}) \leq c(i,a) + \Gamma v^*(i,a)$. This follows from $\Gamma v^*(i,a) \geq \Gamma v^*(i,\mathbf{0})$ and

$$c(i,a) = C_i(a;a) + c(i,a) \geq C_i(a;\mathbf{0}) + c(i,\mathbf{0}).$$

■

An interesting special case is when the repair action corresponds to selling the old device at hand and replacing it with a younger one purchased from the market. Let $c_i(a)$ and $s_i(a)$ be the purchase cost and salvage value, respectively, of a device with intrinsic age a. Then, $C_i(a;b) = c_i(b) - s_i(a)$ whenever $b \leq a$ with $a \neq b$ and, as usual, $C_i(a;a) = 0$. Thereafter,

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this cost model will be called sell–purchase cost model. We assume that c_i and s_i are both decreasing in a_k for every k with $c_i \geq s_i$. It is easy to show that the hypothesis of Theorem 11 is satisfied under this cost structure and hence $y^*(i, y^*(i,a)) = y^*(i,a)$ for every a.

Under the sell–purchase cost model, Eq. (13) simplifies to

$$v(i,a) = \min\left\{c(i,a) + \Gamma v(i,a) + s_i(a), \inf_{b \lneqq a}\{c_i(b) + c(i,b) + \Gamma v(i,b)\}\right\} - s_i(a) \quad \textbf{(21)}$$

for all $i \in E$ and $a \in \mathcal{F}$.

The following two results characterizes the structure of the optimal repair policy under the sell–purchase cost model.

Theorem 15: *Let $y^*(i,a)$ be the optimal repair policy of DPE given in Eq. (21). Then, if $y^*(i,a) \neq a$, $y^*(i,a+u) \lneqq a$ for some $u \geq \mathbf{0}$ and $y^*(i,a+z) \neq a+z$ for some $\mathbf{0} \leq z \leq u$, then there is an optimal policy such that $y^*(i,a+z) = y^*(i,a)$.*

Proof: Since $y^*(i,a) \neq a$,

$$v^*(i,a) = \inf_{b \lneqq a}\{c_i(b) + \Gamma v^*(i,b)\} - s_i(a).$$

If $u = \mathbf{0}$, then there is nothing to prove. Suppose that $u \neq \mathbf{0}$. By the main hypothesis, it is clear that there is a repair decision at $(i, a+u)$ and, hence,

$$\begin{aligned} v^*(i,a+u) &= \inf_{b \lneqq a+u}\{c_i(b) + c(i,b) + \Gamma v^*(i,b)\} - s_i(a+u) \\ &= \inf_{b \lneqq a}\{c_i(b) + c(i,b) + \Gamma v^*(i,b)\} - s_i(a+u) \end{aligned} \quad \textbf{(22)}$$

where the last equality follows from $y^*(i,a+u) \lneqq a$. Now, choose arbitrary $z \neq \mathbf{0}$ such that $y^*(i,a+z) \neq a+z$. Then, we have

$$v^*(i,a+z) = \inf_{b \lneqq a+z}\{c_i(b) + c(i,b) + \Gamma v^*(i,b)\} - s_i(a+z)$$

and

$$\{b; b \lneqq a\} \subset \{b; b \lneqq a+z\} \subset \{b; b \lneqq a+u\}.$$

This implies that

$$\begin{aligned} \inf_{b \lneqq a}\{c_i(b) + c(i,b) + \Gamma v^*(i,b)\} &\geq \inf_{b \lneqq a+z}\{c_i(b) + c(i,b) + \Gamma v^*(i,b)\} \\ &\geq \inf_{b \lneqq a+u}\{c_i(b) + c(i,b) + \Gamma v^*(i,b)\} \end{aligned}$$

and using Eq. (22),

$$\inf_{b \lneqq a+z}\{c_i(b) + c(i,b) + \Gamma v^*(i,b)\} = \inf_{b \lneqq a}\{c_i(b) + c(i,b) + \Gamma v^*(i,b)\}$$

which implies that we can choose $y^*(i,a+z) = y^*(i,a)$. ■

An immediate corollary of the theorem is the following, stating that the optimal repair policy is nondecreasing over the set of ages for which the optimal decision is not do nothing.

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This also implies that when the optimal decision changes from do nothing to repair, the optimal repair policy might be decreasing, that is, it is possible that $y^*(i, a^+) < y^*(i, a) = a$.

COROLLARY 16: *Let $y^*(i, a)$ be the optimal repair policy of DPE given in Eq. (21). Then, there is an optimal policy such that $y_k^*(i, \overline{a}) \geq y_k^*(i, a)$ provided that $\overline{a}_k > a_k$ for some k, $\overline{a}_j = a_j$ for every $j \neq k$ and $y^*(i, a) \neq a$.*

PROOF: Note that if $y_k^*(i, \overline{a}) = \overline{a}$, then there is nothing to prove. Choose a, $\overline{a}$ and k such that $\overline{a}_k > a_k, \overline{a}_j = a_j$ for every $j \neq k$, $y^*(i, a) = c \neq a$, and $y^*(i, \overline{a}) = b \neq \overline{a}$. If $b_k \geq a_k$, then $b_k \geq c_k$ since $c_k \leq a_k$. Now, assume that $b_k < a_k$. This implies that $b \lneqq a$. If $u = \overline{a} - a$, $y^*(i, a) \neq a$, $y^*(i, a+u) \lneqq a$ and $y^*(i, a+u) \neq a+u$ by Theorem 15. This implies that $y^*(i, a+u) = y^*(i, a)$ and, hence, $y^*(i, \overline{a}) = y^*(i, a)$ which completes the proof. ■

In some cases, the purchase cost and the salvage value of a system may be equal. Then, $C_i(a; b) = c_i(b) - c_i(a)$ and Eq. (13) simplifies to

$$v(i, a) = \inf_{b; b \leq a} \{c_i(b) + c(i, b) + \Gamma v(i, b)\} - c_i(a). \tag{23}$$

THEOREM 17: *Let $y^*(i, a)$ be the optimal repair policy of DPE given in Eq. (23). Then, there is an optimal policy such that*

(i). *If $y^*(i, a+u) \leq a$, for some $u \geq \mathbf{0}$, then $y^*(i, a+z) = y^*(i, a)$ for all $\mathbf{0} \leq z \leq u$,*

(ii). *$y_k^*(i, a)$ is increasing in a_k.*

PROOF: Choose arbitrary $a \geq c \geq b \geq \mathbf{0}$. Then,

$$\begin{aligned} C_i(a; c) + C_i(c; b) &= c_i(c) - c_i(a) + c_i(b) - c_i(c) \\ &= c_i(b) - c_i(a) = C_i(a; b). \end{aligned}$$

Then, the results trivially follow from Theorem 12 and Corollary 13. ■

The previous theorem characterizes the structure of the optimal repair policy when the purchase cost and the salvage value of a system are equal. The optimal repair policy is increasing under such a cost structure. Moreover, if $y^*(i, b) = a < b$, then the optimal decision is to repair the system to age a for all intermediate ages between a and b. This can be trivially proved by letting $u = b - a$ in Theorem 17, and applying Theorem 11.

In addition, if there is no salvage value, that is, $s_i = 0$, the DPE given in Eq. (13) can be rewritten as

$$v(i, a) = \min\left\{c(i, a) + \Gamma v(i, a), \inf_{b \lneqq a} \{c_i(b) + c(i, b) + \Gamma v(i, b)\}\right\}. \tag{24}$$

THEOREM 18: *Let $y^*(i, a)$ be the optimal repair policy of DPE given in Eq. (24). Then, there is an optimal policy such that*

(i). *If $y^*(i, a) \neq a$, $y^*(i, a+u) = b$ with $b \lneqq a$ for some $u \geq \mathbf{0}$ and $y^*(i, a+z) \neq a+z$ for some $\mathbf{0} \leq z \leq u$, then $y^*(i, a+z) = y^*(i, a)$,*

(ii). *$y_k^*(i, \overline{a}) \geq y_k^*(i, a)$ provided that $\overline{a}_k > a_k$ for some k, $\overline{a}_j = a_j$ for every $j \neq k$ and $y^*(i, a) \neq a$.*

PROOF: The results follow trivially from Theorem 15 and Corollary 16 since $c_i \geq s_i = 0$. ■

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APPENDIX A

A.1 Proof of Theorem 2

For any $g \in \mathfrak{B}$, we define the operator $\Upsilon : \mathfrak{B} \to \mathfrak{B}$ so that

$$\Upsilon g(i,a) = \min_{r \in B^m} \{c_m(i,a;r) + c(i,a(1-r)) + \Gamma g(i,a(1-r))\} \quad \textbf{(A.1)}$$

for all $i \in E, a \in \mathcal{F}$.

We will use Banach's contraction mapping theorem. Choose two functions $f, g \in \mathfrak{B}$ and suppose $\|\cdot\|$ is the usual supremum norm on $\mathfrak{B}$ such that $\|g\| = \sup_{i \in E, a \in \mathcal{F}} |g(i,a)|$. Note that

$$\begin{aligned}\Upsilon g(i,a) - \Upsilon f(i,a) &= \min_{r \in B^m} \{c_m(i,a;r) + c(i,a(1-r)) + \Gamma g(i,a(1-r))\} \\ &\quad - \min_{r \in B^m} \{c_m(i,a;r) + c(i,a(1-r)) + \Gamma f(i,a(1-r))\}. \end{aligned} \quad \textbf{(A.2)}$$

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Let $\overline{r}$ be the replacement policy which minimizes the second term on the right-hand side of Eq. (A.2). Then,

$$\begin{aligned}
\Upsilon g\left(i,a\right)-\Upsilon f\left(i,a\right) &= \min_{r\in B^m}\left\{c_m\left(i,a;r\right)+c\left(i,a\left(1-r\right)\right)+\Gamma g\left(i,a\left(1-r\right)\right)\right\}\\
&\quad -c_m\left(i,a;\overline{r}\right)-c\left(i,a\left(1-\overline{r}\right)\right)-\Gamma f\left(i,a\left(1-\overline{r}\right)\right)\\
&\le c_m\left(i,a;\overline{r}\right)+c\left(i,a\left(1-\overline{r}\right)\right)+\Gamma g\left(i,a\left(1-\overline{r}\right)\right)\\
&\quad -c_m\left(i,a;\overline{r}\right)-c\left(i,a\left(1-\overline{r}\right)\right)-\Gamma f\left(i,a\left(1-\overline{r}\right)\right)\\
&=\Gamma g\left(i,a\left(1-\overline{r}\right)\right)-\Gamma f\left(i,a\left(1-\overline{r}\right)\right)\\
&=\sum_{j\in E}\int_0^{+\infty}Q\left(i,j,ds\right)e^{-\alpha s}\int_{\mathcal{F}}\widetilde{p}\left(i,a\left(1-\overline{r}\right),s,db\right)\left[g\left(j,b\right)-f\left(j,b\right)\right]\\
&\le\sum_{j\in E}\int_0^{+\infty}Q\left(i,j,ds\right)e^{-\alpha s}\int_{\mathcal{F}}\widetilde{p}\left(i,a\left(1-\overline{r}\right),s,db\right)\left\|g-f\right\|\\
&\le K\left\|g-f\right\|.
\end{aligned}$$

Similarly, it can be shown that $\Upsilon f\left(i,a\right)-\Upsilon g\left(i,a\right)\le K\left\|g-f\right\|$ for any $i\in E$ and $a\in\mathcal{F}$. Thus, we have $\left\|\Upsilon g-\Upsilon f\right\|\le K\left\|g-f\right\|$. Since $K<1$, Υ is a contraction mapping on $\mathfrak{B}$ and it has a unique fixed point $v^*=\Upsilon v^*$ which is the unique solution of DPE given in Eq. (9). ∎

A.2 Proof of Lemma 3

Choose arbitrary $c\in B_k$ such that $c\left(k\right)=h_k\left(i,a(k),s\right)$. Then, there exists $c^*\in\overline{B}_k$ such that $c^*\left(k\right)=+\infty$ and $c^*\left(j\right)=c\left(j\right)$ for every $j\ne k$. If $\psi_i\left(c\right)=0$, then $\psi_i\left(c^*\right)=0$ since ψ_i is non-increasing in c_k. Then, $f\left(i,j,c\right)=f_i+g\left(j,c\right)$, $f\left(i,j,c^*\right)=f_i+g\left(j,c^*\right)$ and, hence, $f\left(i,j,c^*\right)\ge f\left(i,j,c\right)$. Now, suppose that $\psi_i\left(c\right)=1$. Then,

$$f\left(i,j,c\right)=g\left(j,c\right)\le g\left(j,c^*\right)+\left(1-\psi_i\left(c^*\right)\right)f_i=f\left(i,j,c^*\right).$$

Thus, for every $c\in B_k$ with $c\left(k\right)=h_k\left(i,a(k),s\right)$, we can find $c^*\in\overline{B}_k$ such that $f\left(i,j,c^*\right)\ge f\left(i,j,c\right)$ and $c^*\left(j\right)=c\left(j\right)$ for every $j\ne k$. This completes the proof. ∎

A.3 Proof of Theorem 4

It suffices to show that Υg is increasing in a and $0\le\Upsilon g\le(C_m+C+Kf)/(1-K)$ if $0\le g\le(C_m+C+Kf)/(1-K)$ and g is increasing in a. It is clear that $0\le\Gamma g\le K(C_m+C+f)/(1-K)$. Then, using Eq. (A.1) we have $0\le\Upsilon g\le(C_m+C+Kf)/(1-K)$. It is clear that

$$\begin{aligned}
\frac{dh_k\left(i,a_k,s\right)}{da_k}&=\frac{dH_k\left(i,H_k^{-1}\left(i,a_k\right)+s\right)}{da_k}=\frac{dH_k\left(i,t\right)}{dt}\Big|_{t=H_k^{-1}\left(i,a_k\right)+s}\frac{d\left(H_k^{-1}\left(i,a_k\right)+s\right)}{da_k}\\
&=r_k\left(i,H_k\left(i,H_k^{-1}\left(i,a_k\right)+s\right)\right)\frac{dH_k^{-1}\left(i,a_k\right)}{da_k}\\
&=r_k\left(i,H_k\left(i,H_k^{-1}\left(i,a_k\right)+s\right)\right)\frac{1}{\frac{dH_k\left(i,t\right)}{dt}\Big|_{t=H_k^{-1}\left(i,a_k\right)}}\\
&=\frac{r_k\left(i,H_k\left(i,H_k^{-1}\left(i,a_k\right)+s\right)\right)}{r_k\left(i,a_k\right)}\ge 1,
\end{aligned}$$

if component k and the system are in working condition since r_k is always positive and increasing. Therefore, $h(i,a,s)$ and $h(i,a,s)-a$ are increasing in a. Choose $a,b \in \mathcal{F}$ such that $a(k)<b(k)$ and $b(j)=a(j)$ for every $j \neq k$ for some k. We need to show that $\Upsilon g(i,b) \geq \Upsilon g(i,a)$. Define

$$\widetilde{p}^k(i,a,s,dc)=\prod_{j\neq k}\overline{p}^j_{ia(j)}(s,dc(j)).$$

Then, since $c_m(i,a;r)$ and $c(i,a(1-r))$ are increasing in a for every r, it is sufficient to show that

$$\int_{\mathcal{F}}\widetilde{p}(i,b,s,dc)\,f(i,j,c)\geq\int_{\mathcal{F}}\widetilde{p}(i,a,s,dc)\,f(i,j,c)$$

for a given s where $f(i,j,c)=g(j,c)+(1-\psi_i(c))\,f_i$. Let

$$q=\int_{\mathcal{F}}(\widetilde{p}(i,b,s,dc)-\widetilde{p}(i,a,s,dc))\,f(i,j,c)\,.$$

Then, we need to show that $q\geq 0$. Suppose that $b(k)<+\infty$. Then,

$$\begin{aligned}
q&=\int_{B_k}(\widetilde{p}(i,b,s,dc)-\widetilde{p}(i,a,s,dc))\,f(i,j,c)\\
&\quad+\int_{\overline{B}_k}(\widetilde{p}(i,b,s,dc)-\widetilde{p}(i,a,s,dc))\,f(i,j,c)\\
&=\int_{\substack{B_k;\\ c(k)=h_k(i,b(k),s)}}\widetilde{p}^k(i,a,s,dc)\,e^{-(h_k(i,b(k),s)-b(k))}f(i,j,c)\\
&\quad-\int_{\substack{B_k;\\ c(k)=h_k(i,a(k),s)}}\widetilde{p}^k(i,a,s,dc)\,e^{-(h_k(i,a(k),s)-a(k))}f(i,j,c)\\
&\quad\quad+\int_{\overline{B}_k}\widetilde{p}^k(i,a,s,dc)\left(e^{-(h_k(i,a(k),s)-a(k))}-e^{-(h_k(i,b(k),s)-b(k))}\right)f(i,j,c)\\
&\geq\int_{\substack{B_k;\\ c(k)=h_k(i,a(k),s)}}\widetilde{p}^k(i,a,s,dc)\left(e^{-(h_k(i,b(k),s)-b(k))}-e^{-(h_k(i,a(k),s)-a(k))}\right)f(i,j,c)\\
&\quad+\int_{\overline{B}_k}\widetilde{p}^k(i,a,s,dc)\left(e^{-(h_k(i,a(k),s)-a(k))}-e^{-(h_k(i,b(k),s)-b(k))}\right)f(i,j,c)\\
&=\left(e^{-(h_k(i,b(k),s)-b(k))}-e^{-(h_k(i,a(k),s)-a(k))}\right)\\
&\quad\times\left[\int_{\substack{B_k;\\ c(k)=h_k(i,a(k),s)}}\widetilde{p}^k(i,a,s,dc)\,f(i,j,c)-\int_{\overline{B}_k}\widetilde{p}^k(i,a,s,dc)\,f(i,j,c)\right]\\
&\geq 0,
\end{aligned}$$

where the last inequality follows from Lemma 3.

Now, suppose that $b(k) = +\infty$. Then,

$$
\begin{aligned}
q &= \int_{\overline{B}_k} \left(\widetilde{p}(i,b,s,dc) - \widetilde{p}(i,a,s,dc)\right) f(i,j,c) - \int_{B_k} \widetilde{p}(i,a,s,dc)\, f(i,j,c) \\
&= \int_{\overline{B}_k} \widetilde{p}^k(i,a,s,dc) \left(e^{-(h_k(i,a(k),s)-a(k))}\right) f(i,j,c) \\
&\quad - \int_{\substack{B_k;\\ c(k)=h_k(i,a(k),s)}} \widetilde{p}^k(i,a,s,dc)\, e^{-(h_k(i,a(k),s)-a(k))} f(i,j,c) \\
&= e^{-(h_k(i,a(k),s)-a(k))} \\
&\quad \times \left[\int_{\overline{B}_k} \widetilde{p}^k(i,a,s,dc)\, f(i,j,c) - \int_{\substack{B_k;\\ c(k)=h_k(i,a(k),s)}} \widetilde{p}^k(i,a,s,dc)\, f(i,j,c)\right] \\
&\geq 0,
\end{aligned}
$$

where the last inequality follows from Lemma 3. ■

A.4 Proof of Theorem 9

The same proof as for Theorem 2 applies by defining the operator $\Upsilon : \mathfrak{B} \to \mathfrak{B}$ for any $g \in \mathfrak{B}$ as

$$\Upsilon g(i,a) = \inf_{b;b\leq a} \{C_i(a;b) + c(i,b) + \Gamma g(i,b)\} \tag{A.3}$$

for all $i \in E, a \in \mathcal{F}$. ■

A.5 Proof of Theorem 10

To simplify the notation, we let $(a_k, b) = c$ where $c(k) = a(k)$ and $c(j) = b(j)$ for every $j \neq k$.

We need to show that Υg is increasing in a and $0 \leq \Upsilon g \leq (C_r + C + Kf)/(1-K)$ if $0 \leq g \leq (C_r + C + Kf)/(1-K)$ and g is increasing in a. Following the same steps as in the proof of Theorem 4, it can be shown that $0 \leq v^* \leq (C_r + C + Kf)/(1-K)$ and $\Gamma g(i,a)$ is increasing in a. Now choose $a, c \in \mathcal{F}$ such that $c(k) > a(k)$ and $c(j) = a(j)$ for every $j \neq k$. We need to show that $\Upsilon g(i,c) \geq \Upsilon g(i,a)$. Choose $b \leq a$. Then, trivially $b \leq c$ and, hence,

$$C_i(c;b) + c(i,b) + \Gamma g(i,b) \geq C_i(a;b) + c(i,b) + \Gamma g(i,b) \geq \Upsilon g(i,a)$$

since $C_i(a;b)$ is increasing in a. Now, choose $b \leq c$ with $b(k) > a(k)$. Then,

$$
\begin{aligned}
C_i(c;b) + c(i,b) + \Gamma g(i,b) &\geq C_i((b_k,c);b) + c(i,b) + \Gamma g(i,b) \\
&\geq C_i((a_k,(b_k,c));(a_k,b)) + c(i,(a_k,b)) + \Gamma g(i,(a_k,b)) \\
&= C_i(a;(a_k,b)) + c(i,(a_k,b)) + \Gamma g(i,(a_k,b)) \\
&\geq \Upsilon g(i,a),
\end{aligned}
$$

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where the first inequality follows from the fact that $c \geq (b_k, c)$ and the second inequality follows from the main hypothesis and $b \geq (a_k, b)$. Then,

$$\begin{aligned}\Upsilon g\left(i,c\right) = \min\Bigg\{ & \inf_{b;b\leq a}\left\{C_i\left(c;b\right)+c\left(i,b\right)+\Gamma g\left(i,b\right)\right\}, \\ & \inf_{b;b\leq c, b(k)>a(k)}\left\{C_i\left(c;b\right)+c\left(i,b\right)+\Gamma g\left(i,b\right)\right\}\Bigg\} \\ \geq & \min\left\{\Upsilon g(i,a), \Upsilon g\left(i,a\right)\right\} = \Upsilon g\left(i,a\right).\end{aligned}$$

■

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