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STOCHASTIC COMPARISONS OF COHERENT SYSTEMS UNDER DIFFERENT RANDOM ENVIRONMENTS

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Abstract

For many practical situations in reliability engineering, components in the system are usually dependent since they generally work in a collaborative environment. In this paper we build sufficient conditions for comparing two coherent systems under different random environments in the sense of the usual stochastic, hazard rate, reversed hazard rate, and likelihood ratio orders. Applications and numerical examples are provided to illustrate all the theoretical results established here.

Keywords: Coherent system; random environment; stochastic order; distortion function; dependent component

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1. Introduction

A system is said to be coherent if it has no irrelevant component and the structure function of the system is nondecreasing in each component meaning that an improvement of a component cannot lead to a deterioration in the performance of the system. In the context of reliability theory, it is commonly assumed that the components' lifetimes are independent and identically distributed (i.i.d). However, the components are usually dependent in many practical situations since they generally work in a collaborative environment. In this viewpoint, we think of the components of a coherent system as dependent through an environmental random variable. For more information on system reliability and replacement policies subjected to random environments, we refer the interested reader to [16], [24], and [33]–[35].

Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be the lifetime vector of the components in a coherent system. Denote the joint distribution function of $\boldsymbol{X}$ by

$$F(x_1, \ldots, x_n) := \mathbb{P}(X_1 \leq x_1, \ldots, X_n \leq x_n).$$

Let Θ be an environmental random variable with distribution function $W(\cdot)$ and density function $w(\cdot)$ with support in $\chi \subseteq \mathbb{R}_+ := [0, +\infty)$. Furthermore, let $\boldsymbol{X}(\theta) := (X_1(\theta), \ldots, X_n(\theta))$ denote the lifetime vector of the components conditioned on an environment profile $\Theta = \theta$, where $X_i(\theta)$ has marginal distribution function $F_i(\cdot \mid \theta)$ for $i = 1, \ldots, n$. Then the joint

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reliability function of $\boldsymbol{X}(\theta)$ can be represented as

$$\begin{aligned}\bar{F}(x_1, \dots, x_n \mid \theta) &= \mathbb{P}(X_1(\theta) > x_1, \dots, X_n(\theta) > x_n) \\ &= \widehat{K}(\bar{F}_1(x_1 \mid \theta), \dots, \bar{F}_n(x_n \mid \theta)), \end{aligned} \tag{1.1}$$

where $\widehat{K}(\cdot)$, the survival copula, is a multivariate survival function supported on $\mathbb{R}_+^n$ and has uniform marginals on [0, 1]. We refer the interested reader to [32] for an elaborate treatment on copula functions. Throughout this paper, we consider $\boldsymbol{X}(\Theta_i) := (X_1(\Theta_i), \dots, X_n(\Theta_i))$ as a random mapping whose distribution function is a mixture of the distribution functions of a random vector $(X_1(\theta_i), \dots, X_n(\theta_i))$ with environmental distribution $W_i(\cdot)$ for $i = 1, 2$.

It is well known that the reliability of a coherent system can be represented as a multivariate dual distortion function of its components' reliability; see [30]. Let $T_i := \tau(\boldsymbol{X}(\Theta_i))$ be the lifetime of a coherent system with joint reliability as in (1.1) under the random environment Θ_i for $i = 1, 2$. If the dual distortion function of this coherent system is denoted by h then the reliability of the coherent system under environment Θ_i can be expressed as

$$\begin{aligned}\bar{F}_{\tau(\boldsymbol{X}(\Theta_i))}(x) &:= \mathbb{P}(\tau(\boldsymbol{X}(\Theta_i)) > x) \\ &= \int_{\chi} h(\bar{F}_1(x \mid \theta), \dots, \bar{F}_n(x \mid \theta)) \, \mathrm{d}W_i(\theta), \qquad i = 1, 2, \end{aligned} \tag{1.2}$$

where $h\colon [0, 1]^n \mapsto [0, 1]$ depends on the structure function of the system and the survival copula $\widehat{K}$. Moreover, h is an increasing continuous function in $[0, 1]^n$ such that $h(0, \dots, 0) = 0$ and $h(1, \dots, 1) = 1$, and is usually called the dual distortion or domination function; see [12] and [28]. Moreover, the distribution function of $\tau(\boldsymbol{X}(\Theta_i))$ is given by

$$F_{\tau(\boldsymbol{X}(\Theta_i))}(x) := \mathbb{P}(\tau(\boldsymbol{X}(\Theta_i)) \le x) = \int_{\chi} \tilde{h}(F_1(x \mid \theta), \dots, F_n(x \mid \theta)) \, \mathrm{d}W_i(\theta), \qquad i = 1, 2,$$

where $\tilde{h}(u_1, \dots, u_n) = 1 - h(1 - u_1, \dots, 1 - u_n)$. For the $\bar{F}(x \mid \theta) = \bar{F}_1(x \mid \theta) = \cdots = \bar{F}_n(x \mid \theta)$ case, (1.2) reduces to

$$\bar{F}_{\tau(\boldsymbol{X}(\Theta_i))}(x) = \int_{\chi} h(\bar{F}(x \mid \theta)) \, \mathrm{d}W_i(\theta) \quad \text{for } i = 1, 2.$$

In what follows, we call the function h in (1.2) the distortion function for convenience. Since the lifetime of a coherent system can be obtained from its minimal path sets, it can be seen that the distortion function h depends on such minimal path sets and the survival copula function $\widehat{K}(\cdot)$ of the components' joint lifetimes. For instance, the lifetime of a series system can be written as $T = \min\{X_1(\theta), \dots, X_n(\theta)\}$ and the minimal path set for such system is $\mathcal{P} = \{1, 2, \dots, n\}$. Therefore, by using a copula representation, the reliability function of such a system with dependent components can be written as

$$\bar{F}_T(x) = h(\bar{F}(x \mid \theta)) = \widehat{K}(\bar{F}(x \mid \theta), \dots, \bar{F}(x \mid \theta)).$$

It is worth mentioning that if the component lifetimes are independent then the survival copula function is of the form of $\widehat{K}(u_1, \dots, u_n) = \prod_{i=1}^n u_i$. In this case, the reliability function in (1.1) includes many important multivariate mixture models studied in the literature. For example,

- multivariate mixture scale model:

$$\widehat{K}(\bar{F}_1(x \mid \theta), \dots, \bar{F}_n(x \mid \theta)) = \prod_{i=1}^n \bar{F}_i\left(\frac{x}{\theta}\right);$$

- multivariate mixture frailty model:

$$\widehat{K}(\bar{F}_1(x \mid \theta), \ldots, \bar{F}_n(x \mid \theta)) = \prod_{i=1}^{n} \bar{F}_i^{\theta}(x);$$

- multivariate general mixture model:

$$\widehat{K}(\bar{F}_1(x \mid \theta), \ldots, \bar{F}_n(x \mid \theta)) = \prod_{i=1}^{n} \bar{F}_i(x \mid \theta).$$

The above three models are special cases of (1.1) and, thus, (1.2) reduces to the lifetime of a series system with components working under the random environment Θ_i.

These three mixture models have received considerable attention with regard to studying the effects of various random environments and/or different baseline random variables with respect to univariate and multivariate stochastic orders; see, for example, [1]–[3], [6], [10], [15], [17], [19], [21], and [22].

The last few decades have seen significant developments on stochastic comparisons of coherent systems. For instance, Kochar *et al.* [18] showed that the lifetimes of two coherent systems with i.i.d. components are stochastically ordered when their signatures are stochastically ordered. Navarro and Rubio [26] compared two coherent systems in terms of stochastic precedence order. With the help of the survival signature and distortion function, Samaniego and Navarro [36] compared coherent systems comprising independent heterogeneous components. Navarro *et al.* [30] obtained some order preservation properties for general coherent systems through the definition of generalized distortion distributions. For further discussions on this topic, we refer the interested readers to [20], [27], and [28] and the references therein.

However, to the best of the authors' knowledge, there is no result in the literature on comparisons of coherent systems under random environments. For this reason, in this paper, we study the effects of random environments on the lifetimes of coherent systems by considering model (1.2). Sufficient conditions are established for comparing $\tau(\boldsymbol{X}(\Theta_1))$ and $\tau(\boldsymbol{X}(\Theta_2))$ according to the usual stochastic, hazard rate, reversed hazard rate, and likelihood ratio orders.

The rest of this paper is organized as follows. In Section 2 we present some pertinent definitions and notions used in the rest of the paper. In Section 3, stochastic comparisons are conducted on two coherent systems under different random environments in the sense of various stochastic orders. In Section 4 we conclude the paper with applications for scale and frailty models.

2. Preliminaries

Throughout the paper, *increasing* means *nondecreasing* and *decreasing* means *nonincreasing*. All involved integrals and expectations are assumed to exist whenever they appear. For a multivariate function $h(\boldsymbol{u})$, the partial derivative of $h(\boldsymbol{u})$ with respective its ith coordinate is denoted by $\partial h(\boldsymbol{u})/\partial u_i$ for $i = 1, \ldots, n$.

Definition 2.1. Let X and Y be two nonnegative random variables with density functions f and g, distribution functions F and G, survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, hazard rate functions $h_X = f/\bar{F}$ and $h_Y = g/\bar{G}$, and reversed hazard rate functions $r_X = f/F$ and $r_Y = g/G$, respectively. Then X is said to be smaller than Y in

- likelihood ratio order (denoted by $X \leq_{\mathrm{lr}} Y$) if $g(x)/f(x)$ is increasing in $x \in \mathbb{R}_+$;
- hazard rate order (denoted by $X \leq_{\mathrm{hr}} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in $x \in \mathbb{R}_+$ or, equivalently, $h_X(x) \geq h_Y(x)$ for all $x \in \mathbb{R}_+$;

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- reversed hazard rate order (denoted by $X \le_{\rm rh} Y$) if $G(x)/F(x)$ is increasing in $x \in \mathbb{R}_+$ or, equivalently, $r_X(x) \le r_Y(x)$ for all $x \in \mathbb{R}_+$;
- usual stochastic order (denoted by $X \le_{\rm st} Y$) if $\bar{F}(x) \le \bar{G}(x)$ for all $x \in \mathbb{R}_+$ or, equivalently, $\mathbb{E}[\phi(X)](\le [\ge])\mathbb{E}[\phi(Y)]$ for any increasing (decreasing) function $\phi\colon \mathbb{R} \to \mathbb{R}$.

It is well known that

$$X \le_{\rm lr} Y \quad \Longrightarrow \quad X \le_{\rm hr[rh]} Y \quad \Longrightarrow \quad X \le_{\rm st} Y,$$

but neither reversed hazard rate order nor hazard rate order implies the other. For comprehensive discussions on stochastic orders, we refer the reader to [23] and [37].

Next we present several definitions of ageing properties that are useful in establishing characterization properties of components or systems in reliability theory.

Definition 2.2. A random variable X is said to have

- increasing (decreasing) failure rate (IFR (DFR)) if $\bar{F}(x+t)/\bar{F}(x)$ is decreasing (increasing) in $x \in \mathbb{R}_+$ for each $t \in \mathbb{R}_+$ or, equivalently, $h_X(x)$ is increasing (decreasing) in $x \in \mathbb{R}_+$;
- increasing (decreasing) reversed failure rate (IRFR (DRFR)) if $F(x+t)/F(x)$ is increasing (decreasing) in $x \in \mathbb{R}_+$ for each $t \in \mathbb{R}_+$ or, equivalently, $r_X(x)$ is increasing (decreasing) in $x \in \mathbb{R}_+$;
- increasing (decreasing) likelihood ratio (ILR (DLR)) if $f(x+t)/f(x)$ is decreasing (increasing) in $x \in \mathbb{R}_+$ for each $t \in \mathbb{R}_+$ or, equivalently, f is log-concave (log-convex).

According to [4, Theorem 5.2], it is known that a mixture of DFR distributions is also DFR. The following implications are well known:

$$\text{DLR} \quad \Longrightarrow \quad \text{DFR}, \qquad \text{ILR} \quad \Longrightarrow \quad \text{IFR}.$$

For more discussions on ageing concepts, we refer the reader to [4], [8], [9], and [25].

In the following definition, we introduce the notion of totally positive of order 2 (TP_2) and the reverse regular of order 2 (RR_2).

Definition 2.3. A function $\psi(x,\theta)$ is said to be TP_2 (RR_2) in (x,θ) if $\psi(x,\theta) \ge 0$ and

$$\psi(x_1,\theta_1)\psi(x_2,\theta_2)(\ge [\le])\psi(x_1,\theta_2)\psi(x_2,\theta_1)$$

whenever $x_2 \ge x_1$ and $\theta_2 \ge \theta_1$.

We point out that the TP_2 (RR_2) property of $\psi(x,\theta)$ is equivalent to $\psi(x,\theta_2)/\psi(x,\theta_1)$ is increasing (decreasing) in x whenever $\theta_1 \le \theta_2$. The following lemma is useful for proving our main results in the next section.

Lemma 2.1. (Karlin [13, p. 99].) *Let $\psi(x,\theta)$ be a* TP_2 *(*RR_2*) function in $(\theta,x) \in \mathbb{R} \times \mathbb{R}$, and $w_i(\theta)$ be* TP_2 *in $(i,\theta) \in \{1,2\} \times \mathbb{R}$, where $w_i(\theta)$ is a probability density function in θ for each i. Then the function*

$$\phi_i(x) = \int_{\mathbb{R}} \psi(\theta,x) w_i(\theta)\,\mathrm{d}\theta$$

is TP_2 *(*RR_2*) in $(i,x) \in \{1,2\} \times \mathbb{R}$.*

We refer the interested reader to [13] and [14] for more details on various dependence notions and their properties.

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3. Main results

In this section we obtain various ordering results for the lifetimes of coherent systems under different random environments. Hereafter, it is assumed that the survival functions of the lifetimes of coherent systems with components having lifetimes $\boldsymbol{X}$ under random environments Θ_i are in the form of (1.2), where h is the distortion function associated with the system's structure and joint lifetimes among components.

First, we establish some sufficient conditions for the usual stochastic order in order to compare two coherent systems having different environments.

Theorem 3.1. *Suppose that*

(i) $X_i(\theta_1) \le_{\text{st}} X_i(\theta_2)$ *for* $\theta_1 \le \theta_2$, $i = 1, \dots, n$*;*

(ii) $\Theta_1 \le_{\text{st}} \Theta_2$.

Then we have $\tau(\boldsymbol{X}(\Theta_1)) \le_{\text{st}} \tau(\boldsymbol{X}(\Theta_2))$.

Proof. As a distortion function, $h(u_1, \dots, u_n)$ is clearly increasing in u_i for $i = 1, \dots, n$. From the assumption that $X_i(\theta_1) \le_{\text{st}} X_i(\theta_2)$, we know that $\bar{F}_i(x \mid \theta)$ is increasing in θ for $i = 1, \dots, n$. Then we obtain

$$h(\bar{F}_i(x \mid \theta_1), \dots, \bar{F}_i(x \mid \theta_1)) \le h(\bar{F}_i(x \mid \theta_2), \dots, \bar{F}_i(x \mid \theta_2)),$$

which implies that $\tau(\boldsymbol{X}(\theta_1)) \le_{\text{st}} \tau(\boldsymbol{X}(\theta_2))$. The desired result then follows by applying [37, Theorem 1.A.6]. □

Define a class of distortion functions as

$$\mathcal{D} = \{\Phi \colon \alpha_i^{\Phi}(u_1, \dots, u_n) \text{ is decreasing in } \boldsymbol{u} \in (0,1)^n,\ i = 1, \dots, n\},$$

where

$$\alpha_i^{\Phi}(u_1, \dots, u_n) = \frac{u_i}{\Phi(u_1, \dots, u_n)} \frac{\partial}{\partial u_i} \Phi(u_1, \dots, u_n)$$

for any differentiable function $\Phi \colon [0,1]^n \mapsto [0,1]$. In the next result we use the hazard rate order to compare the lifetimes of coherent systems under different random environments.

Theorem 3.2. *Suppose that*

(i) $h \in \mathcal{D}$*;*

(ii) $X_i(\theta_1) \le_{\text{hr}} X_i(\theta_2)$ *for* $\theta_1 \le \theta_2$, $i = 1, \dots, n$*;*

(iii) $\Theta_1 \le_{\text{hr}} \Theta_2$.

Then we have $\tau(\boldsymbol{X}(\Theta_1)) \le_{\text{hr}} \tau(\boldsymbol{X}(\Theta_2))$.

Proof. Assume that $\theta_1 \le \theta_2$. The survival function of $\tau(\boldsymbol{X}(\theta_i))$ is given by

$$\bar{F}_{\tau(\boldsymbol{X}(\theta_i))}(x) = h(\bar{F}_1(x_1 \mid \theta_i), \dots, \bar{F}_n(x_1 \mid \theta_i)).$$

Let $u_i = \bar{F}_i(x \mid \theta_1)$, $u_i^* = \bar{F}_i(x \mid \theta_2)$, $\boldsymbol{u} = (u_1, \dots, u_n)$, and $\boldsymbol{u}^* = (u_1^*, \dots, u_n^*)$. Since $X_i(\theta_1) \le_{\text{hr}} X_i(\theta_2)$, it follows that $u_i \le u_i^*$ for all $i = 1, \dots, n$. The assumption that $h \in \mathcal{D}$ is

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equivalent to that of $\alpha_i^h(\boldsymbol{u})$ is decreasing in $\boldsymbol{u} \in [0, 1]^n$. Thus, applying the result of Navarro *et al.* [30, Proposition 2.3(ii)], it follows that

$$\tau(\boldsymbol{X}(\theta_1)) \leq_{\rm hr} \tau(\boldsymbol{X}(\theta_2)). \tag{3.1}$$

Then, from (3.1) and the assumption that $\Theta_1 \leq_{\rm hr} \Theta_2$, and using [37, Theorem 1.B.14], it follows that $\tau(\boldsymbol{X}(\Theta_1)) \leq_{\rm hr} \tau(\boldsymbol{X}(\Theta_2))$, yielding the desired result. □

We point out that the assumption of Theorem 3.2(i) has also been used by Navarro *et al.* [29, Proposition 2.5] to present sufficient conditions for the preservation of IFR and DFR reliability classes under the formation of general coherent systems. In the following example we provide an illustration for the assumption of Theorem 3.2(i).

Example 3.1. The function $\widehat{K}(u_1, \dots, u_n)$ that corresponds to the well-known Gumbel–Barnett copula has the following analytic expression:

$$\widehat{K}(u_1, \dots, u_n) = u_1 \cdots u_n \mathrm{e}^{-\alpha \log u_1 \cdots \log u_n}, \qquad \alpha > 0,\ 0 < u_i < 1.$$

Let $\tau(\boldsymbol{X})$ be the lifetime of a series system with two heterogeneous dependent components having lifetimes X_1 and X_2. Then, the distortion function of $\tau(\boldsymbol{X})$ is given by

$$h(\boldsymbol{u}) = \widehat{K}(u_1, u_2) = u_1 u_2 \mathrm{e}^{-\alpha \log u_1 \log u_2}, \qquad 0 < u_i < 1,\ i = 1, 2.$$

It is clear that $h(\boldsymbol{u})$ is exchangeable. Furthermore, it can be shown that, for all $\alpha > 0$, the function

$$\frac{u_1}{h(\boldsymbol{u})} \frac{\partial h(\boldsymbol{u})}{\partial u_1} = 1 - \alpha \log u_2$$

is clearly decreasing in u_1 and u_2.

For coherent systems with homogeneous components, the assumption of Theorem 3.2(i) is equivalent to the function $uh'(u)/h(u)$ being decreasing in $u \in (0, 1)$. Thus, we obtain the following corollary.

Corollary 3.1. *Suppose that* $X \stackrel{\rm st}{=} X_i$ *for* $i = 1, \dots, n$, *where* '$\stackrel{\rm st}{=}$' *denotes equal in the usual stochastic order, and*

(i) $uh'(u)/h(u)$ *is decreasing in* $u \in (0, 1)$;

(ii) $X(\theta_1) \leq_{\rm hr} X(\theta_2)$ *for* $\theta_1 \leq \theta_2$;

(iii) $\Theta_1 \leq_{\rm hr} \Theta_2$.

Then we have $\tau(\boldsymbol{X}(\Theta_1)) \leq_{\rm hr} \tau(\boldsymbol{X}(\Theta_2))$.

Define

$$\mathcal{D}^* = \{\Phi \colon \beta_i^{\Phi}(u_1, \dots, u_n) \text{ is increasing in } \boldsymbol{u} \in (0, 1)^n,\ i = 1, \dots, n\},$$

where

$$\beta_i^{\Phi}(u_1, \dots, u_n) = \frac{(1 - u_i)}{1 - \Phi(u_1, \dots, u_n)} \frac{\partial}{\partial u_i} \Phi(u_1, \dots, u_n).$$

We use the following lemma to connect the classes $\mathcal{D}$ and $\mathcal{D}^*$.

Lemma 3.1. *We say that* $\tilde{h} \in \mathcal{D}$ *is equivalent to* $h \in \mathcal{D}^*$.

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Proof. Note that $\tilde{h} \in \mathcal{D}$ is equivalent to

$$\alpha_i^{\tilde{h}}(u_1, \dots, u_n) = \frac{u_i}{\tilde{h}(u_1, \dots, u_n)} \frac{\partial}{\partial u_i} \tilde{h}(u_1, \dots, u_n)$$

being decreasing in $\boldsymbol{u} \in (0, 1)^n$. We can then write

$$\begin{aligned}
\alpha_i^{\tilde{h}}(u_1, \dots, u_n) &= \frac{u_i}{\tilde{h}(u_1, \dots, u_n)} \frac{\partial}{\partial u_i} \tilde{h}(u_1, \dots, u_n) \\
&= \frac{u_i}{1 - h(1-u_1, \dots, 1-u_n)} \frac{\partial}{\partial u_i} [1 - h(1-u_1, \dots, 1-u_n)] \\
&= -\frac{u_i}{1 - h(1-u_1, \dots, 1-u_n)} \frac{\partial}{\partial u_i} h(1-u_1, \dots, 1-u_n) \\
&= -\frac{1-v_i}{1 - h(v_1, \dots, v_n)} \frac{\partial}{\partial (1-v_i)} h(v_1, \dots, v_n) \\
&= \frac{1-v_i}{1 - h(v_1, \dots, v_n)} \frac{\partial}{\partial v_i} h(v_1, \dots, v_n) \\
&= \beta_i^h(v_1, \dots, v_n),
\end{aligned} \tag{3.2}$$

where $v_i = 1 - u_i$, $i = 1, \dots, n$. Then, from (3.2), it follows that $\alpha_i^{\tilde{h}}(u_1, \dots, u_n)$ is decreasing in $\boldsymbol{u} \in (0, 1)^n$, which is equivalent to $\beta_i^h(v_1, \dots, v_n)$ being increasing in $\boldsymbol{v} \in (0, 1)^n$. Thus, we have $h \in \mathcal{D}^*$. □

Next we compare the reversed hazard rate functions of two coherent systems having different random environments, which describes the translation of the reversed hazard rate order between the random environments into stochastic comparisons on $\tau(\boldsymbol{X}(\Theta_1))$ and $\tau(\boldsymbol{X}(\Theta_2))$.

Theorem 3.3. *Suppose that*

(i) $h \in \mathcal{D}^*$;

(ii) $X_i(\theta_1) \le_{\text{rh}} X_i(\theta_2)$ *for* $\theta_1 \le \theta_2$, $i = 1, \dots, n$;

(iii) $\Theta_1 \le_{\text{rh}} \Theta_2$.

Then we have $\tau(\boldsymbol{X}(\Theta_1)) \le_{\text{rh}} \tau(\boldsymbol{X}(\Theta_2))$.

Proof. Assume that $\theta_1 \le \theta_2$. The distribution function of $\tau(\boldsymbol{X}(\theta_i))$ is given by

$$F_{\tau(\boldsymbol{X}(\theta_i))}(x) = 1 - h(\bar{F}_1(x_1 \mid \theta_i), \dots, \bar{F}_n(x_1 \mid \theta_i)).$$

Let $u_i = \bar{F}_i(x \mid \theta_1)$, $u_i^* = \bar{F}_i(x \mid \theta_2)$, $\boldsymbol{u} = (u_1, \dots, u_n)$, and $\boldsymbol{u}^* = (u_1^*, \dots, u_n^*)$. From the assumption that $X_i(\theta_1) \le_{\text{rh}} X_i(\theta_2)$, it follows that $u_i \le u_i^*$ for all $i = 1, \dots, n$. The assumption that $h \in \mathcal{D}^*$ is equivalent to $\beta_i^h(\boldsymbol{u})$ being increasing in $\boldsymbol{u} \in (0, 1)^n$ or, equivalently, $\tilde{h} \in \mathcal{D}$ based on Lemma 3.1. Thus, from [30, Proposition 2.3(iii)], it follows that

$$\tau(\boldsymbol{X}(\theta_1)) \le_{\text{rh}} \tau(\boldsymbol{X}(\theta_2)). \tag{3.3}$$

By (3.3), the assumption that $\Theta_1 \le_{\text{hr}} \Theta_2$ and [37, Theorem 1.B.52], it follows that $\tau(\boldsymbol{X}(\Theta_1)) \le_{\text{rh}} \tau(\boldsymbol{X}(\Theta_2))$, as required. □

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It is important to note that the assumption of Theorem 3.3(i) has been exploited by Navarro *et al.* [29, Proposition 2.6] to study preservation of the DRFR class for coherent systems consisting of homogeneous dependent components.

Note that Theorem 3.3(i) is equivalent to the function $(1-u)h'(u)/(1-h(u))$ increasing in $u \in (0,1)$ when the components are homogeneous. Therefore, the following result can be obtained for two coherent systems with different environments compared in the sense of the reversed hazard rate order.

Corollary 3.2. *Suppose that* $X \stackrel{\text{st}}{=} X_i$ *for* $i = 1, \dots, n$, *and*

(i) $(1-u)h'(u)/(1-h(u))$ *is increasing in* $u \in (0,1)$*;*

(ii) $X(\theta_1) \le_{\text{rh}} X(\theta_2)$ *for* $\theta_1 \le \theta_2$*;*

(iii) $\Theta_1 \le_{\text{rh}} \Theta_2$.

Then we have $\tau(\boldsymbol{X}(\Theta_1)) \le_{\text{rh}} \tau(\boldsymbol{X}(\Theta_2))$.

Let $\tau_{k\,|\,n}$ be the lifetime of a k-out-of-n system comprising homogeneous dependent components, and let $\bar{F}_{k\,|\,n}$ be the survival function of $\tau_{k\,|\,n}$. Then we have

$$\begin{aligned}\bar{F}_{k\,|\,n}(t) &= \sum_{j=0}^{n-k}(-1)^{n-k-j}\binom{n}{j}\binom{n-j-1}{k-1}\bar{F}_{1:\,n-j}(t)\\ &= \sum_{j=0}^{n-k}(-1)^{n-k-j}\binom{n}{j}\binom{n-j-1}{k-1}\widehat{K}(\underbrace{1,\dots,1}_{j},\underbrace{\bar{F}(t),\dots,\bar{F}(t)}_{n-j}),\end{aligned} \tag{3.4}$$

where $\widehat{K}$ is an exchangeable survival copula. By setting $u := \bar{F}(t)$, the distortion function of $\tau_{k\,|\,n}$ (denoted by $h_{k\,|\,n}$) can be obtained from (3.4) as

$$h_{k\,|\,n}(u) = \sum_{j=0}^{n-k}(-1)^{n-k-j}\binom{n}{j}\binom{n-j-1}{k-1}\widehat{K}(\underbrace{1,\dots,1}_{j},\underbrace{u,\dots,u}_{n-j}). \tag{3.5}$$

In the following example we illustrate Corollary 3.2(i).

Example 3.2. The function $\widehat{K}(u_1, \dots, u_n)$ corresponding to the Farlie–Gumbel–Morgenstern (FGM) copula is expressed as

$$\widehat{K}(u_1,\dots,u_n) = \prod_{i=1}^{n} u_i + \alpha\prod_{i=1}^{n} u_i(1-u_i),$$

where $\alpha \in [-1,1]$. Consider a 2-out-of-3 system with three homogeneous dependent components. Based on (3.5), the distortion function of this system is given by

$$h_{2\,|\,3}(u) = 3u^2 - 2u^3(1+\alpha(1-u)^3).$$

Zhang *et al.* [39, Example 3.7] showed that, for all $\alpha \in [-1,1]$, $(1-u)h'_{2\,|\,3}(u)/(1-h_{2\,|\,3}(u))$ is an increasing function in $u \in (0,1)$ no matter whether these three components are positively upper orthant dependent for $\alpha \in (0,1]$ (see [31]) or negatively upper orthant dependent for $\alpha \in [-1,0)$.

Now we establish some sufficient conditions for comparing the lifetimes of two coherent systems with homogeneous components in the sense of the likelihood ratio order.

Theorem 3.4. *Suppose that there exists some point $v \in (0, 1)$ such that*

(i) $uh''(u)/h'(u)$ *is decreasing and nonnegative for all* $u \in (0, v]$*, and*

(ii) $(1-u)h''(u)/h'(u)$ *is decreasing and nonpositive for all* $u \in (v, 1)$.

Assume further that

(iii) $X(\theta_1) \le_{\mathrm{lr}} X(\theta_2)$ *for* $\theta_1 \le \theta_2$*;*

(iv) $\Theta_1 \le_{\mathrm{lr}} \Theta_2$.

Then we have $\tau(\boldsymbol{X}(\Theta_1)) \le_{\mathrm{lr}} \tau(\boldsymbol{X}(\Theta_2))$.

Proof. The density function of $\tau(\boldsymbol{X}(\Theta_i))$ is given by

$$f_{\tau(\boldsymbol{X}(\Theta_i))}(x) = \int_{\chi} f(x \mid \theta) h'(\bar{F}(x \mid \theta)) w_i(\theta)\, \mathrm{d}\theta, \qquad i = 1, 2.$$

To prove the desired result, it suffices to show that $f_{\tau(\boldsymbol{X}(\Theta_2))}(x)/f_{\tau(\boldsymbol{X}(\Theta_1))}(x)$ is increasing in $x \in \mathbb{R}_+$ or, equivalently, $f_{\tau(\boldsymbol{X}(\Theta_i))}(x)$ is TP_2 in $(i, x) \in \{1, 2\} \times \mathbb{R}_+$.

From the assumption that $\Theta_1 \le_{\mathrm{lr}} \Theta_2$, it follows that $w_i(\theta)$ is TP_2 in $(i, \theta) \in \{1, 2\} \times \mathbb{R}_+$. Let

$$\psi(x, \theta) = f(x \mid \theta) h'(\bar{F}(x \mid \theta)).$$

Applying Lemma 2.1, it is enough to show that $\psi(x, \theta)$ is a TP_2 function in $(x, \theta) \in \mathbb{R}_+ \times \mathbb{R}_+$, and this amounts to showing that

$$\frac{\psi(x, \theta_2)}{\psi(x, \theta_1)} = \frac{f(x \mid \theta_2)}{f(x \mid \theta_1)} \frac{h'(\bar{F}(x \mid \theta_2))}{h'(\bar{F}(x \mid \theta_1))} \quad \text{is increasing in } x \in \mathbb{R}_+.$$

On the one hand, from assumption (iii), it follows that $f(x \mid \theta)$ is TP_2 in (x, θ). Thus, we need only prove that

$$\Psi(x) := \frac{h'(\bar{F}(x \mid \theta_2))}{h'(\bar{F}(x \mid \theta_1))} \quad \text{is increasing in } x \in \mathbb{R}_+.$$

Let $x_v(\theta)$ be the solution of $v = \bar{F}(x \mid \theta)$ for some specified $\Theta = \theta$.

We consider two cases.

Case 1: $x \ge x_v(\theta)$. In this case, we have $\bar{F}(x \mid \theta) \in (0, v)$. Let $u_1 = \bar{F}(x \mid \theta_1)$ and $u_2 = \bar{F}(x \mid \theta_2)$, $h_X(x \mid \theta)$ be the hazard rate function of $X(\theta)$, and $\tilde{r}_X(x \mid \theta)$ be the reversed hazard rate function of $X(\theta)$. Then, assumption (iii) implies that $u_1 \le u_2$. In light of assumption (i), by taking the derivative of $\Psi(x)$ with respect to x, we obtain

$$\begin{aligned}
\Psi'(x) &\overset{\mathrm{sgn}}{=} \frac{f(x \mid \theta_1) h''(\bar{F}(x \mid \theta_1))}{h'(\bar{F}(x \mid \theta_1))} - \frac{f(x \mid \theta_2) h''(\bar{F}(x \mid \theta_2))}{h'(\bar{F}(x \mid \theta_2))} \\
&= h_X(x \mid \theta_1) \frac{\bar{F}(x \mid \theta_1) h''(\bar{F}(x \mid \theta_1))}{h'(\bar{F}(x \mid \theta_1))} - h_X(x \mid \theta_2) \frac{\bar{F}(x \mid \theta_2) h''(\bar{F}(x \mid \theta_2))}{h'(\bar{F}(x \mid \theta_2))} \\
&\ge h_X(x \mid \theta_2) \left[\frac{u_1 h''(u_1)}{h'(u_1)} - \frac{u_2 h''(u_2)}{h'(u_2)} \right] \\
&\ge 0,
\end{aligned}$$

where '$\overset{\mathrm{sgn}}{=}$' denotes equality of sign, which implies the desired result.

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Case 2: $0 < x < x_v$. Note that $\bar{F}(x \mid \theta) \in (v, 1)$. By using assumption (ii), it is easy to see that

$$\begin{aligned}\Psi'(x) &\overset{\text{sgn}}{=} \frac{f(x \mid \theta_1)h''(\bar{F}(x \mid \theta_1))}{h'(\bar{F}(x \mid \theta_1))} - \frac{f(x \mid \theta_2)h''(\bar{F}(x \mid \theta_2))}{h'(\bar{F}(x \mid \theta_2))} \\ &= \tilde{r}_X(x \mid \theta_1)\frac{(1-\bar{F}(x \mid \theta_1))h''(\bar{F}(x \mid \theta_1))}{h'(\bar{F}(x \mid \theta_1))} - \tilde{r}_X(x \mid \theta_2)\frac{(1-\bar{F}(x \mid \theta_2))h''(\bar{F}(x \mid \theta_2))}{h'(\bar{F}(x \mid \theta_2))} \\ &\geq \tilde{r}_X(x \mid \theta_2)\left[\frac{(1-u_1)h''(u_1)}{h'(u_1)} - \frac{(1-u_2)h''(u_2)}{h'(u_2)}\right] \\ &\geq 0.\end{aligned}$$

Thus, the desired result follows. □

Franco *et al.* [11, Corollary 2] showed that conditions (i) and (ii) of Theorem 3.4 always hold for the case of a k-out-of-n system with i.i.d. components.

Next we present a numerical example to illustrate conditions (i) and (ii) of Theorem 3.4 for a k-out-of-n system with homogeneous and dependent components.

Example 3.3. Let $\tau(\boldsymbol{X}) = \min\{\max\{X_1, X_2\}, \max\{X_1, X_3\}, \max\{X_2, X_3, X_4\}\}$ be the lifetime of a coherent system with five dependent and homogeneous components whose lifetimes have an FGM survival copula. Thus, the distortion function of $\tau(\boldsymbol{X})$ is given by

$$h(u) = 4u^2 - 4u^3 + u^4 + \alpha u^4(1-u)^4, \qquad \alpha \in [-1, 1].$$

We calculate that

$$\begin{aligned}\Delta_1(u) &:= \frac{uh''(u)}{h'(u)} \\ &= \frac{u(8 - 24u + 12u^2 + 12\alpha(1-u)^4u^2 - 32\alpha(1-u)^3u^3 + 12\alpha(1-u)^2u^4)}{8u - 12u^2 + 4u^3 + 4\alpha(1-u)^4u^3 - 4\alpha(1-u)^3u^4}\end{aligned}$$

and

$$\begin{aligned}\Delta_2(u) &:= \frac{(1-u)h''(u)}{h'(u)} \\ &= \frac{(1-u)(8 - 24u + 12u^2 + 12\alpha(1-u)^4u^2 - 32\alpha(1-u)^3u^3 + 12\alpha(1-u)^2u^4)}{8u - 12u^2 + 4u^3 + 4\alpha(1-u)^4u^3 - 4\alpha(1-u)^3u^4}.\end{aligned}$$

With $\alpha = 0.2$, in Figure 1 we present plots of the functions $\Delta_1(u)$ and $\Delta_2(u)$ for $u \in (0, 0.421)$ and $u \in (0.421, 1)$, respectively. We see that $\Delta_1(u)$ is decreasing and nonnegative on $u \in (0, 0.421)$, while $\Delta_2(u)$ is decreasing and nonpositive on $u \in (0.421, 1)$, which agrees with conditions (i) and (ii) of Theorem 3.4.

In the next example we illustrate the result of Theorem 3.4.

Example 3.4. Suppose that Θ_1 and Θ_2 have density functions

$$f_{\Theta_1}(\theta) = \left(\frac{1}{\sqrt{\theta}} + 1\right)\mathrm{e}^{-2\sqrt{\theta}-\theta} \quad \text{and} \quad f_{\Theta_2}(\theta) = \theta\mathrm{e}^{-\theta}, \qquad \theta \in \mathbb{R}_+.$$

It is easy to see that $\Theta_1 \leq_{\text{lr}} \Theta_2$. Under the setup of Example 3.3, let $\tau(\boldsymbol{X}(\theta))$ be the lifetime of the coherent system with dependent component lifetimes having exponential lifetimes with common hazard rate $1/\theta$. In Figure 2 we present the plots of the ratio of the density functions of $\tau(\boldsymbol{X}(\Theta_2))$ and $\tau(\boldsymbol{X}(\Theta_1))$, from which we see that $\tau(\boldsymbol{X}(\Theta_1)) \leq_{\text{lr}} \tau(\boldsymbol{X}(\Theta_2))$.

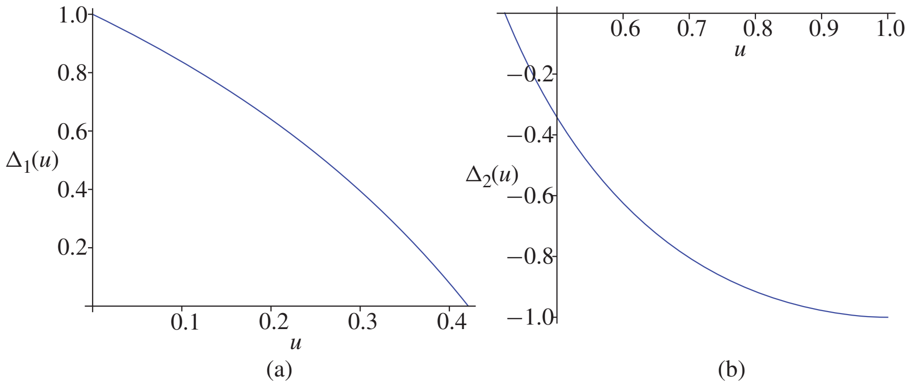

FIGURE 1: Plots of (a) $\Delta_1(u)$ and (b) $\Delta_2(u)$ for $u \in (0, 1)$.

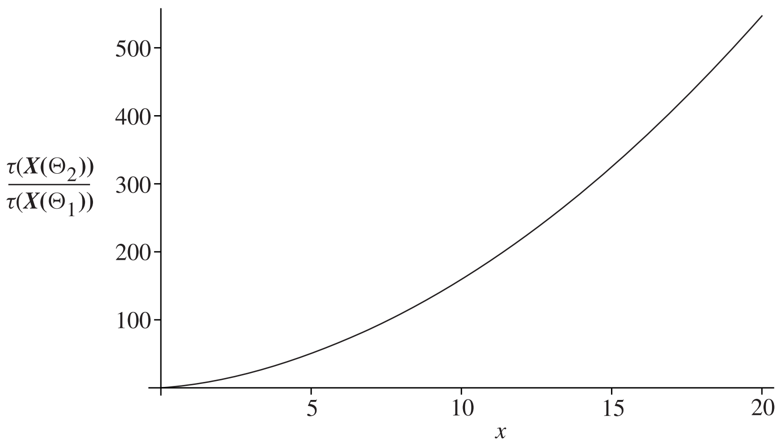

FIGURE 2: Plot of the ratio of the density functions of $\tau(\boldsymbol{X}(\Theta_2))$ and $\tau(\boldsymbol{X}(\Theta_1))$.

4. Concluding remarks

In this paper we have established some sufficient conditions for comparing two coherent systems with different random environments by means of the usual stochastic, hazard rate, reversed hazard rate, and likelihood ratio orders. We provided two important statistical models in order to illustrate the results of Theorems 3.2–3.4.

Let $\boldsymbol{X}(\theta) = (\theta X_1, \dots, \theta X_n)$ be the lifetimes of a set of heterogeneous dependent random variables given $\Theta = \theta$. Then we have $\tau(\boldsymbol{X}(\Theta_i)) = \tau(\Theta_i \boldsymbol{X})$, $i = 1, 2$. The survival function of the lifetime of the coherent system is given by

$$\bar{F}_{\tau(\Theta_i \boldsymbol{X})}(x) = \int_{\chi} h\left(\bar{F}_1\left(\frac{x}{\theta}\right), \dots, \bar{F}_n\left(\frac{x}{\theta}\right)\right) \mathrm{d}W_i(\theta), \qquad i = 1, 2.$$

Then the following result can be derived from the main results established in the last section.

Proposition 4.1. (i) Under the assumptions of (i) and (iii) of Theorem 3.2 (Theorem 3.3), if X_i is DFR (DRFR) for $i = 1, \dots, n$, we have $\tau(\Theta_1 \boldsymbol{X}) \le_{\text{hr[rh]}} \tau(\Theta_2 \boldsymbol{X})$.

(ii) Under the assumptions of (i), (ii), and (iv) of Theorem 3.4, if X_i is ILR then we have $\tau(\Theta_1 \boldsymbol{X}) \leq_{\rm lr} \tau(\Theta_2 \boldsymbol{X})$.

For given $\Theta = \theta$, we assume that $\bar{F}_i(x \mid \theta) = \bar{F}_i^{1/\theta}(x)$, $i = 1, 2, \dots, n$. Then the survival function of the lifetime of the coherent system is given by

$$\bar{F}_{\tau(\boldsymbol{X}(\Theta_i))}(x) = \int_{\chi} h(\bar{F}_1^{1/\theta}(x), \dots, \bar{F}_n^{1/\theta}(x))\, \mathrm{d}W_i(\theta), \qquad i = 1, 2.$$

For further discussions on frailty models, we refer the reader to [22] and [38]. In the case of the frailty model, we clearly have $X_i(\theta_1) \leq_{\rm lr} X_i(\theta_2)$ for $i = 1, 2, \dots, n$. So the following result can be obtained from Section 3.

Proposition 4.2. (i) Under the assumptions of (i) and (ii) of Theorem 3.2 (Theorem 3.3), it follows that $\tau(\boldsymbol{X}(\Theta_1)) \leq_{\rm hr[rh]} \tau(\boldsymbol{X}(\Theta_2))$.

(ii) Under the assumptions of (i), (ii), and (iv) of Theorem 3.4, we have $\tau(\boldsymbol{X}(\Theta_1)) \leq_{\rm lr} \tau(\boldsymbol{X}(\Theta_2))$.

Boland and Samaniego [7] defined a mixed coherent system as a mixture of systems with different structures but the same components. It should be noted that (1.2) can be regarded as the survival function of another type of mixed coherent system if the distribution of the random environment is treated as the distribution of the weights of some coherent subsystems having a common distortion function but different components. Therefore, the results established in Theorems 3.1–3.4 can be utilized in order to compare new kinds of mixed systems having different weights.

Further work is needed to obtain sufficient conditions on the distortion function in order to extend Theorem 3.4 to the case of coherent systems having heterogeneous components under different random environments. Motivated by Belzunce *et al.* [5], in which the authors studied the preservation of orderings between the components under the formation of coherent systems with different structures, it will be of natural interest to establish sufficient conditions for comparing two coherent systems with different structures under different environments.

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