

OPTIMAL POLICY IN RATIONAL EXPECTATIONS MODELS: NEW SOLUTION ALGORITHMS

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This paper develops methods to solve for optimal discretionary policies and optimal commitment policies in rational expectations models. These algorithms, which allow the optimization constraints to be conveniently expressed in second-order structural form, are more general than existing methods and are simple to apply. We use several New Keynesian business cycle models to illustrate their application. Simulations show that the procedures developed in this paper can quickly solve small-scale models and that they can be usefully and effectively applied to medium- and large-scale models.

Keywords: Optimal Policy, Commitment, Discretion

1. INTRODUCTION

Central banks and other policy makers are commonly modeled as agents whose objective is to minimize some loss function subject to constraints that contain forward-looking rational expectations [see Woodford (2002), Clarida, Galí, and Gertler (1999), or the Taylor (1999) conference volume, for example]. But of course Kydland and Prescott (1977) showed that in the absence of some commitment mechanism optimal policies were time-inconsistent. The modern policy literature addresses this time-consistency problem either by simply assuming that the policymaker can commit to not reoptimize or by explicitly modeling the strategic interactions that occur between the various economic agents in the model.¹ When the former approach is taken the policy maker commits to undertake a single optimization and implements the chosen policy in all subsequent periods. When the latter approach is pursued the technique is to solve for a subgame-perfect equilibrium. Although other time-consistent equilibria could be studied,² it is most common to solve for Markov-perfect Stackelberg-Nash equilibria in which the policy maker is the Stackelberg leader and private-sector agents and future policy makers are Stackelberg followers.

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Several methods for solving for optimal commitment policies and for Markov-perfect Stackelberg-Nash equilibria (optimal discretionary policies in what follows) have been developed. For commitment, available algorithms include those developed in Currie and Levine (1985, 1993) and Söderlind (1999), which are based on Hamiltonian and Lagrangian methods, respectively, and Backus and Driffill (1986), which transforms the system to a dynamic programming problem to obtain the commitment policy. For discretion, two popular algorithms are those developed by Oudiz and Sachs (1985) and Backus and Driffill (1986). Both Oudiz and Sachs (1985) and Backus and Driffill (1986) solve for optimal discretionary rules using dynamic programming, but Oudiz and Sachs (1985) solve for the stationary feedback rule by taking the limit as $t \rightarrow -\infty$ of a finite horizon problem, whereas Backus and Driffill (1986) solve the asymptotic problem directly.³ By construction optimal discretionary policies are time-consistent.

A feature common to all of these solution methods is that they require the optimization constraints to be written in state-space form. An attractive feature of the state-space form is that it provides a compact and efficient representation of an economic model in a form that contains only first-order dynamics. However, reliance on the state-space form also has disadvantages. In particular, use of the state-space representation forces the distinction between predetermined and nonpredetermined variables and often requires considerable effort to manipulate the model in the required form. Both of these requirements, but especially the latter, mean that these solution methods come with “overhead” costs that can be considerable.

In this paper, we develop algorithms to solve for optimal commitment policies and optimal discretionary policies that differ from existing algorithms in that they allow the constraints to be written in structural form rather than in state-space form. As a broadbrush characterization, for small-scale models that have only a few nonpredetermined variables, it is generally not too difficult to express the model in state-space form, in which case solving the model using a state-space solution method may be convenient. However, even for models that are only modestly complex it is often much easier to write the model in structural form, and the algorithms presented here have been developed with these models in mind.⁴ Although of interest in their own right, the algorithms developed also here have several other useful attributes. First, they do not require the predetermined variables to be separated from the nonpredetermined variables; second, they can be easily applied to models whose optimization constraints contain the expectation of next period’s policy instrument(s); and, third, they supply the Euler equation for the optimal discretionary policy, which makes them particularly convenient when a “targeting rule” rather than an “instrument rule” (see Svensson, 2003) is sought.

Although the relative strength of these algorithms is that they can be easily applied to models that are difficult to manipulate into a state-space form, to illustrate their use we apply them to several, reasonably simple, New Keynesian models and compare their properties to existing solution methods. For this comparison, we take

the models developed by Galí and Monacelli (2005) and Erceg, Henderson, and Levin (2000) and solve for optimal commitment policies and optimal discretionary policies for a range of policy regimes. Although the algorithms developed in this paper exploit less model structure than the state-space methods, they are still reasonably efficient. To demonstrate that they can usefully solve larger models, we also apply them to the model developed by Fuhrer and Moore (1995).⁵

The remainder of this paper is structured as follows. The following section introduces the components of the policy maker's optimization problem, describing the objective function and the equations that constrain the optimization process. Section 3 shows how to solve for optimal commitment rules. Section 4 turns to discretion. The discretionary problem is formulated in terms of an unconstrained optimization, which contrasts with the more usual dynamic programming or Lagrangian-based approaches. In Section 5, we employ two New Keynesian models to calculate the computation times of a range of solution methods. Section 5 also discusses situations for which convergence problems can be encountered when solving for optimal policies. Section 6 concludes while appendices contain technical details.

2. THE OPTIMIZATION PROBLEM

Let y_t be an $(n \times 1)$ vector of endogenous variables and x_t be a $(p \times 1)$ vector of policy instruments. For convenience, the variables in y_t and x_t represent deviations from nonstochastic steady-state values. The policy maker sets x_t to minimize the loss function⁶

$$\text{Loss}(0, \infty) = E_0 \sum_{t=0}^{\infty} \beta^t [y_t' W y_t + x_t' Q x_t], \quad (1)$$

where $\beta \in (0, 1)$ is the discount factor and W ($n \times n$) and Q ($p \times p$) are symmetric, positive semidefinite, matrices containing policy preferences, or policy tastes. Here E_0 represents the mathematical expectations operator conditional on period 0 information. Loss functions like equation (1) are widely employed in the literature because, with linear constraints, they lead to linear decision rules. Furthermore, as is now well documented, this loss function can represent a second-order approximation to a representative agent's utility function [see Diaz-Gimenez (1999), for example] and in certain situations it collapses to an expression containing just output and inflation (Woodford, 2002).

The policy maker minimizes (1) subject to the following system of dynamic constraints⁷

$$A_0 y_t = A_1 y_{t-1} + A_2 E_t y_{t+1} + A_3 x_t + A_4 E_t x_{t+1} + A_5 v_t, \quad (2)$$

where $v_t \sim iid [0, \Omega]$ is an $(s \times 1, s \leq n)$ vector of innovations and the matrices $A_0, A_1, A_2, A_3, A_4,$ and A_5 contain the model's structural parameters, which are conformable with $y_t, x_t,$ and $v_t,$ as necessary. The dating convention in equation (2)

is such that every element in y_{t-1} has a known value as of the beginning of period t . A_0 is assumed to be nonsingular.

It is not necessary to explicitly include the matrix A_5 in equation (2) because it is always possible to include v_t within y_t . However, its presence allows us to easily accommodate models where the innovations are scaled by structural parameters without having to expand the state vector or redefine the innovations. An important feature of equation (2) is that it contains not just the contemporaneous policy instruments but also the expectation of next period’s instrument vector, which is useful for models that contain an interest rate term structure, for example.⁸

3. COMMITMENT

Under commitment the policymaker optimizes once and never reoptimizes. The assumption is that either the policy maker can adhere to its chosen policy or that the discount factor is sufficiently large to allow the chosen policy to be supported as a reputation equilibrium [see Currie and Levine (1993, Chapter 5)]. To solve for optimal commitment policies, we observe that from the definition of rational expectations we can write

$$y_{t+1} = E_t y_{t+1} + \eta_{t+1} \tag{3}$$

$$x_{t+1} = E_t x_{t+1} + \mu_{t+1}, \tag{4}$$

where the expectation errors, η_{t+1} and μ_{t+1} , are martingale difference sequences. Substituting equations (3) and (4) into equation (2) gives

$$A_0 y_t = A_1 y_{t-1} + A_2 y_{t+1} + A_3 x_t + A_4 x_{t+1} + \rho_t \tag{5}$$

$$\rho_t = A_5 v_t - A_2 \eta_{t+1} - A_4 \mu_{t+1}. \tag{6}$$

To solve the optimization problem from the standpoint of period t_0 , we form the Lagrangian

$$L = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{(t-t_0)} [y_t' W y_t + x_t' Q x_t + 2\lambda_t' \times (A_0 y_t - A_1 y_{t-1} - A_2 y_{t+1} - A_3 x_t - A_4 x_{t+1} - \rho_t)], \tag{7}$$

and take derivatives with respect to x_t , y_t , and λ_t , giving⁹

$$\frac{\partial L}{\partial x_t} = Q x_t - A_3' \lambda_t - \beta^{-1} A_4' \lambda_{t-1} = 0, \quad t > t_0 \tag{8}$$

$$\frac{\partial L}{\partial y_t} = W y_t + A_0' \lambda_t - \beta^{-1} A_2' \lambda_{t-1} - \beta A_1' E_t \lambda_{t+1} = 0, \quad t > t_0 \tag{9}$$

$$\frac{\partial L}{\partial \lambda_t} = A_0 y_t - A_1 y_{t-1} - A_2 E_t y_{t+1} - A_3 x_t - A_4 E_t x_{t+1} - A_5 v_t = 0, \quad t \geq t_0 \tag{10}$$

$$\frac{\partial L}{\partial \mathbf{x}_t} = \mathbf{Q}\mathbf{x}_t - \mathbf{A}'_3\boldsymbol{\lambda}_t = \mathbf{0}, \quad t = t_0 \tag{11}$$

$$\frac{\partial L}{\partial \mathbf{y}_t} = \mathbf{W}\mathbf{y}_t + \mathbf{A}'_0\boldsymbol{\lambda}_t - \beta\mathbf{A}'_1E_t\boldsymbol{\lambda}_{t+1} = \mathbf{0}, \quad t = t_0, \tag{12}$$

where $\mathbf{y}_{t_0-1} = \bar{\mathbf{y}}$. The fact that the policy maker behaves differently in the first period than it does in subsequent periods is reflected in the fact that equations (11) and (12) apply in the initial period ($t = t_0$), whereas equations (8) and (9) apply in all subsequent periods. However, the recursive nature of the system can be restored by employing equations (8)–(10), for all $t \geq t_0$, but with the initial conditions $\mathbf{y}_{t_0-1} = \bar{\mathbf{y}}$ and $\boldsymbol{\lambda}_{t_0-1} = \mathbf{0}$.¹⁰ The Lagrange multipliers equal zero at the start of the initial period because, as implied by equations (11) and (12), the policy maker exploits agent’s expectations in the initial period, while promising never to do so in the future (Kydland and Prescott, 1980). It is worth noting that it is the very fact that the policy maker behaves differently in the initial period that provides the starting values for the Lagrange multipliers that are needed for the first-order conditions to have a unique, stable, rational expectations equilibrium. The presence of the expected future instrument vector in the optimization constraints presents no special difficulties because the policy maker optimizes only once and the dynamics arising from this term can easily be accounted for during that optimization.

Given $\mathbf{y}_{t_0-1} = \bar{\mathbf{y}}$ and $\boldsymbol{\lambda}_{t_0-1} = \mathbf{0}$, equations (8)–(10), can be solved in a number of ways. One possibility is to write them in second-order form

$$\begin{bmatrix} \mathbf{0} & \mathbf{A}_0 & -\mathbf{A}_3 \\ \mathbf{A}'_0 & \mathbf{W} & \mathbf{0} \\ -\mathbf{A}'_3 & \mathbf{0} & \mathbf{Q} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_t \\ \mathbf{y}_t \\ \mathbf{x}_t \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{A}_1 & \mathbf{0} \\ \beta^{-1}\mathbf{A}'_2 & \mathbf{0} & \mathbf{0} \\ \beta^{-1}\mathbf{A}'_4 & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_{t-1} \\ \mathbf{y}_{t-1} \\ \mathbf{x}_{t-1} \end{bmatrix} \tag{13}$$

$$+ \begin{bmatrix} \mathbf{0} & \mathbf{A}_2 & \mathbf{A}_4 \\ \beta\mathbf{A}'_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} E_t \begin{bmatrix} \boldsymbol{\lambda}_{t+1} \\ \mathbf{y}_{t+1} \\ \mathbf{x}_{t+1} \end{bmatrix} + \begin{bmatrix} \mathbf{A}_5 \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} [\mathbf{v}_t], \tag{14}$$

and to apply an undetermined-coefficient technique, such as these developed by Binder and Pesaran (1995), McCallum (1999), or Uhlig (1999). An alternative approach is to write equations (8)–(10) in first-order form

$$\mathbf{M}\boldsymbol{\Gamma}_t = \mathbf{N}E_t\boldsymbol{\Gamma}_{t+1} + \boldsymbol{\Upsilon}_t, \tag{15}$$

where

$$\mathbf{M} = \begin{bmatrix} \mathbf{0} & -\mathbf{A}_1 & \mathbf{0} & \mathbf{A}_0 & -\mathbf{A}_3 \\ -\beta^{-1}\mathbf{A}'_2 & \mathbf{0} & \mathbf{A}'_0 & \mathbf{W} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ -\beta^{-1}\mathbf{A}'_4 & \mathbf{0} & -\mathbf{A}'_3 & \mathbf{0} & \mathbf{Q} \end{bmatrix}, \quad \boldsymbol{\Gamma}_t = \begin{bmatrix} \boldsymbol{\lambda}_{t-1} \\ \mathbf{y}_{t-1} \\ \boldsymbol{\lambda}_t \\ \mathbf{y}_t \\ \mathbf{x}_t \end{bmatrix},$$

$$N = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & A_2 & A_4 \\ \mathbf{0} & \mathbf{0} & \beta A'_1 & \mathbf{0} & \mathbf{0} \\ I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \Upsilon_t = \begin{bmatrix} A_5 \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} [v_t],$$

and to solve the system using an eigenvalue method, such as Anderson and Moore (1985), Klein (2000), King and Watson (2002), Christiano (2002), or Sims (2002). Notice that the vectors in equations (15) have been allocated such that the predetermined variables, λ_{t-1} and y_{t-1} , enter at the top of the system, the “stable” block, whereas the jump variables, λ_t , y_t , and x_t , enter at the bottom, the “unstable” block. The order of the variables within these vectors is irrelevant because by construction every element in λ_{t-1} and y_{t-1} has a known value at the beginning of period t . Although the order of the variables within the system is an issue for many solution methods, here one simply orders vectors, which makes these Euler equations particularly easy to solve.

Both undetermined-coefficient methods and eigenvalue methods return solutions in the form

$$\begin{bmatrix} \lambda_t \\ y_t \\ x_t \end{bmatrix} = \begin{bmatrix} \theta_{11} & \theta_{12} & \mathbf{0} \\ \theta_{21} & \theta_{22} & \mathbf{0} \\ \varphi_1 & \varphi_2 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \lambda_{t-1} \\ y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \theta_{13} \\ \theta_{23} \\ \varphi_3 \end{bmatrix} [v_t]. \tag{16}$$

The optimal commitment rule can be seen in equation (16) to depend on the vector of lagged Lagrange multipliers. These Lagrange multipliers enter the rule to ensure that today’s policy validates how private sector expectations were formed in the past.

Remark 1: The Euler equations for optimal commitment policy, equations (8)–(10), do not depend on the covariance matrix of the shocks, Ω . Consequently, optimal commitment policies are certainty equivalent.

Remark 2: Let $z_t \equiv [\lambda'_t \ y'_t \ x'_t]'$,

$$\widehat{K} \equiv \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & W & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & Q \end{bmatrix},$$

and, in obvious notation, equation (16) be written as $z_t = H z_{t-1} + G v_t$. Then, the loss function can be written as $\text{Loss}(t, \infty) = [z'_{t-1} H' \widehat{P} H z_{t-1} + v'_t G' \widehat{P} G v_t + \frac{\beta}{1-\beta} \text{tr}(G' \widehat{P} G \Omega)]$, where $\widehat{P} \equiv \widehat{K} + \beta H' \widehat{P} H$ (see Appendix A.2).

Remark 3: A consequence of Remark 2 is that $\lim_{\beta \uparrow 1} (1 - \beta) \text{Loss}(t, \infty) = \text{tr}(\mathbf{K}\Phi)$ where Φ is the unconditional variance-covariance matrix of $[\mathbf{y}'_t \mathbf{x}'_t]'$ and

$$\mathbf{K} \equiv \begin{bmatrix} \mathbf{W} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} \end{bmatrix}$$

(see Appendix A.3).

4. DISCRETION

In this section, we consider discretion and develop a numerical procedure that solves for Markov-perfect Stackelberg-Nash equilibria in which the policy maker optimizing today is the Stackelberg leader and private sector agents and future policymakers are Stackelberg followers. Broadly, the innovations in this section are that we allow expected future instruments to enter the optimization constraints and that we base the algorithm on a framework in which the optimization constraints are written in structural form, not in state-space form. The outcome is a solution procedure that is easy to apply and that eliminates the need to represent the optimization constraints in state-space form. The solution method also returns the Euler equations associated with the optimal discretionary policy, not just the decisions rules, which makes it particularly well suited for studying “targeting rules” (Svensson, 2003).

Because the system’s state variables are \mathbf{y}_{t-1} and \mathbf{v}_t , in the equilibrium that we seek the endogenous variables and the policy instruments will all be functions of these variables. Assume, then, that a stationary solution to the policy maker’s optimization problem exists and is given by

$$\mathbf{y}_t = \mathbf{H}_1\mathbf{y}_{t-1} + \mathbf{H}_2\mathbf{v}_t \tag{17}$$

$$\mathbf{x}_t = \mathbf{F}_1\mathbf{y}_{t-1} + \mathbf{F}_2\mathbf{v}_t. \tag{18}$$

It is the time-invariant matrices \mathbf{H}_1 , \mathbf{H}_2 , \mathbf{F}_1 , and \mathbf{F}_2 that we seek.

Substituting equations (17) and (18) into equation (2) gives

$$\mathbf{D}\mathbf{y}_t = \mathbf{A}_1\mathbf{y}_{t-1} + \mathbf{A}_3\mathbf{x}_t + \mathbf{A}_5\mathbf{v}_t \tag{19}$$

where

$$\mathbf{D} \equiv \mathbf{A}_0 - \mathbf{A}_2\mathbf{H}_1 - \mathbf{A}_4\mathbf{F}_1. \tag{20}$$

The matrix \mathbf{D} embeds how future policy makers respond to movements in \mathbf{y}_t and the policy maker setting \mathbf{x}_t today allows for \mathbf{D} , rather than just \mathbf{A}_0 , when calculating the impact of its policy decisions. Thus, future policy makers are followers with respect to current policy makers.

Using equations (17) and (18), we can express the loss function in terms of H_1 , H_2 , F_1 , F_2 , y_t and the instrument vector, x_t as (see Appendix B.1)

$$\text{Loss}(t, \infty) = y_t' P y_t + x_t' Q x_t + \frac{\beta}{1 - \beta} \text{tr}[(F_2' Q F_2 + H_2' P H_2) \Omega], \tag{21}$$

where

$$P \equiv W + \beta F_1' Q F_1 + \beta H_1' P H_1. \tag{22}$$

Although a Lagrangian could be used, it is just as simple to substitute equation (19) into equation (21) and transform what was a constrained optimization problem into an unconstrained optimization problem. Following this substitution we have¹¹

$$\begin{aligned} \text{Loss}(t, \infty) &= (A_1 y_{t-1} + A_3 x_t + A_5 v_t)' D^{-1} P D^{-1} (A_1 y_{t-1} + A_3 x_t + A_5 v_t) \\ &+ x_t' Q x_t + \frac{\beta}{1 - \beta} \text{tr}[(F_2' Q F_2 + H_2' P H_2) \Omega]. \end{aligned} \tag{23}$$

Differentiating equation (23) with respect to x_t and setting the resulting derivative equal to zero gives

$$\frac{\partial \text{Loss}(t, \infty)}{\partial x_t} = A_3' D^{-1} P D^{-1} (A_1 y_{t-1} + A_5 v_t) + (Q + A_3' D^{-1} P D^{-1} A_3) x_t = 0. \tag{24}$$

Equation (24) can alternatively be expressed in terms of endogenous variables as

$$Q x_t + A_3' D^{-1} P y_t = 0, \tag{25}$$

which is useful when a “targeting rule” rather than an “instrument rule” (Svensson, 2003) is of interest. Solving equation (24) for x_t produces

$$\begin{aligned} x_t &= - (Q + A_3' D^{-1} P D^{-1} A_3)^{-1} A_3' D^{-1} P D^{-1} (A_1 y_{t-1} + A_5 v_t) \\ &\equiv F_1 y_{t-1} + F_2 v_t, \end{aligned} \tag{26}$$

which when inserted back into equation (19) leads to

$$\begin{aligned} y_t &= D^{-1} (A_1 + A_3 F_1) y_{t-1} + D^{-1} (A_5 + A_3 F_2) v_t \\ &\equiv H_1 y_{t-1} + H_2 v_t. \end{aligned} \tag{27}$$

Of course, both P and D are implicit functions of H_1 , H_2 , F_1 , and F_2 . Thus, equations (26) and (27) must be solved for a fixed point to obtain the desired solution. The numerical procedure is as follows

- Step 1. Initialize H_1 , H_2 , F_1 , and F_2 .
- Step 2. Solve for D according to equation (20) and for P from equation (22), iterating “backward through time” using a method such as the doubling algorithm.¹²

Step 3. Update $H_1, H_2, F_1,$ and F_2 according to

$$F_1 = -(Q + A_3' O^{-1} P D^{-1} A_3)^{-1} A_3' D^{-1} P D^{-1} A_1, \tag{28}$$

$$F_2 = -(Q + A_3' D'^{-1} P D^{-1} A_3)^{-1} A_3' D'^{-1} P D^{-1} A_5, \tag{29}$$

$$H_1 = D^{-1}(A_1 + A_3 F_1), \tag{30}$$

$$H_2 = D^{-1}(A_5 + A_3 F_2). \tag{31}$$

Step 4. Iterate over Steps 2 and 3 until convergence.

As should be clear from these four steps, the procedure is easy to implement, requiring no more than standard matrix operations. It is worth noting that the procedure does not require the predetermined variables to be separated from the non-predetermined variables, thereby avoiding the matrix partitioning required by state-space methods.

Remark 4: The feedback parameter matrices, F_1 and F_2 , are independent of Ω and thus optimal discretionary policies are certainty equivalent. Moreover, P and D , and hence $H_1, H_2, F_1,$ and F_2 , are independent of the economy's initial state.

Remark 5: Let $z_t \equiv [y_t' \ x_t']'$ and

$$K \equiv \begin{bmatrix} W & \mathbf{0} \\ \mathbf{0} & Q \end{bmatrix}.$$

For given $F_1, F_2,$ and H_2 matrices, and an H_1 matrix whose spectral radius is less than one, the transition equation for z_t is $z_t = H z_{t-1} + G v_t$, where H and G are formed from $F_1, F_2, H_2,$ and H_1 in a straightforward way. Then equation (1) can be written as $Loss(t, \infty) = z_{t-1}' H_1' \tilde{P} H_1 z_{t-1} + v_t' G' \tilde{P} G v_t + \frac{\beta}{1-\beta} \text{tr}(G' \tilde{P} G \Omega)$, where $\tilde{P} \equiv K + \beta H' \tilde{P} H$ (see Appendix B.2).

Remark 6: As a consequence of Remark 5, $\lim_{\beta \uparrow 1} (1-\beta) Loss(t, \infty) = \text{tr}(K \Phi)$, where Φ is the unconditional covariance matrix of z_t (see Appendix B.3).

5. COMPARING SOLUTION METHODS¹³

The previous two sections showed how to solve for optimal commitment policies and optimal discretionary policies when the optimization problem is formulated with the constraints in structural form. This contrasts with the approach taken in Oudiz and Sachs (1985), Currie and Levine (1985, 1993), Backus and Driffill (1986), and Söderlind (1999), which is to minimize

$$Loss(t, \infty) = E_t \sum_{j=0}^{\infty} \beta^j [p'_{t+j} S p_{t+j} + 2p'_{t+j} U x_{t+j} + x'_{t+j} R x_{t+j}], \tag{32}$$

subject to

$$\begin{bmatrix} \mathbf{p}_{1t+1} \\ E_t \mathbf{p}_{2t+1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{p}_{1t} \\ \mathbf{p}_{2t} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} [x_t] + \begin{bmatrix} s_{t+1} \\ \mathbf{0} \end{bmatrix}, \quad (33)$$

where $\mathbf{p}_t \equiv [\mathbf{p}'_{1t} \mathbf{p}'_{2t}]'$ combines the predetermined variables, \mathbf{p}_{1t} , and the nonpredetermined variables, \mathbf{p}_{2t} . The key difference between this formulation and that employed in this paper lies largely in how the constraints are expressed, structural form, equation (2), or state-space form, equation (33).

In this section we apply the algorithms developed in Sections 3 and 4 to two optimization-based New Keynesian models and compare their computation times to state-space solution methods. Because optimal commitment policies and optimal discretionary policies are often studied in the monetary policy literature, the models that we analyze are drawn from that literature. We study the small open-economy model developed by Galí and Monacelli (2005) and closed-economy model developed by Erceg, Henderson, and Levin (2000) (EHL), both are representative of the models used to examine monetary policy issues. For discretion, we compare the algorithm developed in Section 4 to those developed in Oudiz and Sachs (1985) and Backus and Driffill (1986). For commitment, we compare the algorithm presented in Section 3 to those developed by Backus and Driffill (1986) and Söderlind (1999). Subsequently, we apply our procedures to the larger Fuhrer and Moore (1995) model and discuss situations in which convergence problems can sometimes be encountered when solving for optimal policies.

Solving for optimal commitment policies using the approach of Section 3 requires solving a rational expectations (RE) model and there are numerous techniques available to perform this task. We take the opportunity to compare two RE solution methods, the “brute force” undetermined-coefficient method presented in Binder and Pesaran (1995) and the QZ-decomposition method presented in Klein (2000). For the latter, we used both the *real* QZ decomposition and the *complex* QZ decomposition¹⁴; however, only the computation times for the *real* QZ decomposition are reported since they were uniformly faster than those for the *complex* QZ decomposition.¹⁵

5.1. A Small-Scale Open-Economy Model

In the open-economy model developed by Galí and Monacelli (2005), households consume an aggregate of domestically produced and imported goods. The law-of-one-price applies to imported goods. Domestic firms are monopolistically competitive and subject to Calvo-style price rigidities (Calvo, 1983). Financial assets are internationally tradable and nominal uncovered interest parity holds. The law-of-one-price together with the definition for consumer price inflation implies that the real exchange rate and the terms-of-trade are perfectly correlated, whereas nominal uncovered interest parity coupled with the law-of-one-price implies that real uncovered interest parity holds.

For our purposes, the relevant model equations are¹⁶

$$\begin{aligned}
 y_t &= E_t y_{t+1} - \frac{\omega_\alpha}{\sigma} (i_t - E_t \pi_{t+1}) + g_t \\
 \pi_t &= \beta E_t \pi_{t+1} + \kappa_\alpha y_t + u_t \\
 \pi_t^c &= \pi_t + \frac{\alpha}{1 - \alpha} (q_t - q_{t-1}) \\
 q_t &= E_t q_{t+1} - (1 - \alpha) (i_t - E_t \pi_{t+1} - i_t^* + E_t \pi_{t+1}^*) + (1 - \alpha) \epsilon_t,
 \end{aligned}$$

where y_t denotes the output gap, π_t denotes domestic goods' inflation, π_t^c denotes consumer price inflation rate, q_t denotes the real exchange rate (an increase in q_t represents a real depreciation), i_t and i_t^* denote domestic and foreign nominal interest rates, respectively, π_t^* denotes foreign inflation, and g_t , u_t , and ϵ_t are white noise shocks.¹⁷

The central bank sets the nominal interest rate, i_t , to minimize the loss function

$$\text{Loss}(t, \infty) = E_t \sum_{j=0}^{\infty} \beta^j [(\pi_{t+j}^c)^2 + \lambda y_{t+j}^2 + \nu (i_{t+j} - i_{t+j-1})^2], \quad (34)$$

where the policy preference parameters, λ and ν , are restricted to be nonnegative. In the experiments that follow, we consider a range of policy preference parameters, allowing λ to take on the values 0, 1, and 3 and ν to take on the values 0, $\frac{1}{2}$, and 1.

5.1.1. Solution times. We first assume that policy is set with discretion. Table 1 reports the time it takes to solve the Galí-Monacelli model using the algorithm in Section 4 and the algorithms in Oudiz and Sachs (1985), Backus and Driffill (1986), and Söderlind (1999).¹⁸ The procedure presented in Söderlind (1999) is

TABLE 1. Solution times under discretion (hundreds of seconds)

Regime	Section 4	Oudiz-Sachs	Backus-Driffill/ Söderlind
$(\lambda, \nu) = (0, 0)$	0.09	0.16	0.16
$(\lambda, \nu) = (0, \frac{1}{2})$	0.17	0.16	0.17
$(\lambda, \nu) = (0, 1)$	0.19	0.17	0.18
$(\lambda, \nu) = (1, 0)$	0.15	0.11	0.11
$(\lambda, \nu) = (1, \frac{1}{2})$	0.14	0.12	0.12
$(\lambda, \nu) = (1, 1)$	0.16	0.12	0.12
$(\lambda, \nu) = (3, 0)$	0.08	0.08	0.08
$(\lambda, \nu) = (3, \frac{1}{2})$	0.12	0.09	0.10
$(\lambda, \nu) = (3, 1)$	0.12	0.10	0.10

TABLE 2. Solution times under commitment (hundreds of seconds)

Regime	Section 3		Backus-Driffill	Söderlind
	Binder-Pesaran	Real-QZ	Dyn. Prog.	Real-QZ
$(\lambda, \nu) = (0, 0)$	—	—	—	—
$(\lambda, \nu) = (0, \frac{1}{2})$	0.10	0.13	0.07	0.09
$(\lambda, \nu) = (0, 1)$	0.12	0.12	0.09	0.09
$(\lambda, \nu) = (1, 0)$	0.15	0.11	—	0.05
$(\lambda, \nu) = (1, \frac{1}{2})$	0.21	0.12	0.14	0.08
$(\lambda, \nu) = (1, 1)$	0.19	0.12	0.14	0.09
$(\lambda, \nu) = (3, 0)$	0.27	0.11	—	0.05
$(\lambda, \nu) = (3, \frac{1}{2})$	0.29	0.12	0.21	0.08
$(\lambda, \nu) = (3, 1)$	0.30	0.12	0.21	0.08

the same as Backus and Driffill (1986) and for this reason both algorithms are attributed the same computation time.

The first point to note is that all of the algorithms are able to quickly obtain the optimal discretionary rule for each parameter combination considered. At the same time, the results indicate that the Oudiz-Sachs algorithm is the fastest for this model, but only marginally; the difference between the fastest algorithm (Oudiz-Sachs) and the slowest algorithm (Section 4) is just 0.00012 second on average. It takes slightly longer to solve the model when the constraints are in structural form because the structural form does not exploit as much of the model's structure as the state-space form does. No convergence problems were encountered with any of the algorithms.

Turning to the commitment solution methods, Table 2 shows that for the algorithms that use recursive methods—Binder-Pesaran and Backus-Driffill—the solution time is increasing in λ and ν . The algorithms that obtain the commitment policy using the *real* QZ decomposition are both quicker in general than the recursive methods and have solution times that are largely invariant to the loss function's parameterization. As with discretion, use of the structural form weighs on the computation time slightly. Table 2 also shows that for this model the Backus-Driffill algorithm cannot obtain the solution when $\nu = 0$, regardless of the value for λ .¹⁹ In fact, none of the algorithms are able to obtain a solution when $\lambda = \nu = 0$. For the Binder-Pesaran method the matrix recursion used to solve the RE model does not converge, although the iterations do converge when ν is small but nonzero. The procedures that rely on eigenvalue methods determine the correct number of unstable eigenvalues, but a singularity prevents the jump variables from being related to the predetermined variables in a way that eliminates the unstable dynamics.²⁰ The fact that the problem occurs regardless of whether the model is in structural or state-space form shows that it is not idiosyncratic to one particular formulation of the optimization problem.

5.2. A Small-Scale Closed-Economy Model

The second model that we examine is that developed by Erceg, Henderson, and Levin (2000) (EHL). EHL assume that both prices and wages are subject to Calvo-contracts, but the model is otherwise a standard New Keynesian business cycle model with monopolistically competitive firms. When log-linearized about the model’s Pareto optimal equilibrium, the relevant equations are:²¹

$$\begin{aligned}
 g_t &= E_t g_{t+1} - \frac{1}{\sigma l_c} (i_t - E_t \pi_{t+1} - r_t^*) \\
 \pi_t &= \beta E_t \pi_{t+1} + \kappa_\rho (\bar{w}_t - mpl_t) \\
 w_t &= \beta E_t w_{t+1} + \kappa_\rho (mrs_t - \bar{w}_t) \\
 mpl_t &= \bar{w}_t^* - \lambda_{mpl} g_t \\
 mrs_t &= \bar{w}_t^* + \lambda_{mrs} g_t \\
 \bar{w}_t &= \bar{w}_{t-1} + w_t - \pi_t \\
 r_t^* &= \sigma l_c (E_t y_{t+1}^* - y_t^*) + \sigma l_q (E_t q_{t+1} - q_t) \\
 \bar{w}_t^* &= w_x^* x_t + w_q^* q_t + w_z^* z_t \\
 y_t^* &= y_x^* x_t + y_q^* q_t + y_z^* z_t \\
 x_t &= \rho x_{t-1} + \varepsilon_{\pi t},
 \end{aligned}$$

where g_t denotes the output gap, i_t denotes the short-term nominal interest rate, π_t denotes price inflation, \bar{w}_t denotes the real wage, mpl_t denotes the marginal product of labor, w_t denotes wage inflation, mrs_t denotes the marginal rate of substitution between consumption and leisure, and r_t^* , \bar{w}_t^* , and y_t^* denote the Pareto optimal real interest rate, real wage, and level of output, respectively. Finally, x_t , z_t , and q_t , denote a technology shock, a leisure preference shock, and a consumption preference shock.²²

For the purposes of this section, the central bank’s loss function is assumed to be

$$\text{Loss}(t, \infty) = E_t \sum_{j=0}^{\infty} \beta^j [\pi_{t+j}^2 + \lambda g_{t+j}^2 + v(i_{t+j} - i_{t+j-1})^2]$$

with the nominal interest rate, i_t , serving as the central bank’s policy instrument.

5.2.1. *Solution times.* Table 3 reports the time taken to solve the model under discretion for different parameterizations of the policy objective function.²³

As with the Galí-Monacelli model, none of the algorithms had any difficulty solving the model for any of the parameterizations considered. Aside from two notable outliers, the Oudiz-Sachs algorithm is again the most efficient while the algorithm developed in Section 4 is typically the slowest, taking on average

TABLE 3. Solution times under discretion (hundreds of seconds)

Regime	Section 4	Oudiz-Sachs	Backus-Driffill/ Söderlind
$(\lambda, \nu) = (0, 0)$	0.38	0.51	0.51
$(\lambda, \nu) = (0, \frac{1}{2})$	1.67	4.02	2.07
$(\lambda, \nu) = (0, 1)$	1.83	4.03	2.14
$(\lambda, \nu) = (1, 0)$	4.54	2.40	2.52
$(\lambda, \nu) = (1, \frac{1}{2})$	4.92	2.34	2.48
$(\lambda, \nu) = (1, 1)$	5.03	2.33	2.49
$(\lambda, \nu) = (3, 0)$	4.49	2.51	2.63
$(\lambda, \nu) = (3, \frac{1}{2})$	4.99	2.48	2.60
$(\lambda, \nu) = (3, 1)$	5.16	2.48	2.62

1.5 hundreds of a second longer the obtain the solution. Although all of the algorithms are notably slower when solving the EHL model than when solving the Galí-Monacelli model, the relative inefficiency of the structural form method is more apparent here because the EHL model contains a larger number of non-predetermined variables, whose lagged values are all treated as candidate state variables by the algorithm.

Table 4 reports computation times for the EHL model when solving for optimal commitment policies; several interesting results emerge. First, the computation time is longer when a recursive method is used, such as the Backus-Driffill algorithm or Binder and Pesaran’s RE solution method. Second, optimal commitment policies can generally be obtained more quickly than optimal discretionary policies, even if a recursive method is used to solve for the optimal commitment policy. Third, as earlier, the Backus-Driffill algorithm is unable to solve for the optimal commitment policy when $\nu = 0$, but now none of the other algorithms have problems.

TABLE 4. Solution times under commitment (hundreds of seconds)

Regime	Section 3		Backus-Driffill Dyn. Prog.	Söderlind Real-QZ
	Binder-Pesaran	Real-QZ		
$(\lambda, \nu) = (0, 0)$	2.81	0.26	—	0.10
$(\lambda, \nu) = (0, \frac{1}{2})$	0.91	0.32	1.18	0.16
$(\lambda, \nu) = (0, 1)$	1.07	0.35	1.18	0.15
$(\lambda, \nu) = (1, 0)$	2.06	0.27	—	0.06
$(\lambda, \nu) = (1, \frac{1}{2})$	1.82	0.29	1.23	0.12
$(\lambda, \nu) = (1, 1)$	1.77	0.30	2.08	0.13
$(\lambda, \nu) = (3, 0)$	2.93	0.25	—	0.07
$(\lambda, \nu) = (3, \frac{1}{2})$	3.22	0.30	1.48	0.12
$(\lambda, \nu) = (3, 1)$	3.26	0.31	1.51	0.12

5.3. A Larger Model

The final model that we consider is the well-known model developed by Fuhrer and Moore (1995). The particular specification that we analyze comes from Fuhrer (1997, Table IV). The model equations are given by

$$\begin{aligned}
 y_t &= 1.45y_{t-1} - 0.47y_{t-2} - 0.34\rho_{t-1} + \epsilon_t \\
 \rho_t &= \frac{40}{41}E_t\rho_{t+1} + \frac{1}{41}(i_t - E_t\pi_{t+1}) \\
 p_t &= 0.42w_t + 0.31w_{t-1} + 0.19w_{t-2} + 0.08w_{t-3} \\
 \pi_t &= 4(p_t - p_{t-1}) \\
 v_t &= 0.42\bar{w}_t + 0.31\bar{w}_{t-1} + 0.19\bar{w}_{t-2} + 0.08\bar{w}_{t-3} \\
 \bar{w}_t &= 0.42v_t + 0.31E_tv_{t+1} + 0.19E_tv_{t+2} + 0.08E_tv_{t+3} \\
 &\quad + 0.002(0.42y_t + 0.31E_ty_{t+1} + 0.19E_ty_{t+2} + 0.08E_ty_{t+3}) + \epsilon_t,
 \end{aligned}$$

where y_t represents the output gap, p_t represent the price level, w_t represents the nominal wage, π_t represents inflation, \bar{w}_t represents the real wage, ρ_t represents the real yield on a 10-year bond, and i_t , the short-term nominal interest rate, serves as the central bank's policy instrument.

Unlike the optimization-based models analyzed earlier, which each contained only two endogenous state variables, the Fuhrer-Moore model contains seven endogenous state variables, and it also contains a relatively large number of non-predetermined variables. Despite its larger size, the Fuhrer-Moore model can easily be written in second-order structural form and solved using the algorithms developed in Sections 3 and 4. As earlier, the central bank's loss function takes the form

$$\text{Loss}(t, \infty) = E_t \sum_{j=0}^{\infty} \beta^j [\pi_{t+j}^2 + \lambda y_{t+j}^2 + \nu (i_{t+j} - i_{t+j-1})^2].$$

Table 5 reports the time taken (in hundreds of seconds) to solve the Fuhrer-Moore model using the algorithms presented in Sections 3 and 4 for different values of λ and ν . The discount factor, β , is set to 0.99.²⁴

Table 5 underscores the point that it is much more efficient to solve for optimal commitment policies the *real* QZ decomposition than it is to use the Binder-Pesaran solution method, which employs recursive iterations, and that optimal commitment policies can be obtained more quickly than optimal discretionary policies. Nevertheless, averaging across the parameterizations for which the solution could be obtained, it takes less than one-tenth of a second to solve for the optimal discretionary policy. Perhaps the most notable feature of Table 5, however, is that when $\lambda = \nu = 0$ the optimal commitment policy is obtained when using the *real* QZ decomposition but not (to sufficient accuracy) when

TABLE 5. Solution times (hundreds of seconds)

Regime	Discretion	Commitment	
		Binder-Pesaran	Real-QZ
$(\lambda, \nu) = (0, 0)$	—	—	1.40
$(\lambda, \nu) = (0, \frac{1}{2})$	4.36	8.37	1.63
$(\lambda, \nu) = (0, 1)$	5.17	9.20	1.63
$(\lambda, \nu) = (1, 0)$	11.70	8.94	1.47
$(\lambda, \nu) = (1, \frac{1}{2})$	5.35	6.17	1.64
$(\lambda, \nu) = (1, 1)$	5.17	5.57	1.65
$(\lambda, \nu) = (3, 0)$	16.24	7.26	1.48
$(\lambda, \nu) = (3, \frac{1}{2})$	9.26	7.53	1.66
$(\lambda, \nu) = (3, 1)$	8.73	7.21	1.72

using Binder-Pesaran. Moreover, for this parameterization of the loss function, the optimal discretionary policy could not be obtained.

5.4. Convergence Issues

For the three models analyzed here, one or more of the solution algorithms could not obtain the optimal (commitment/discretionary) policy for some parameterizations of the loss function. Looking closely at the offending parameterizations, it is notable that they are all parameterizations for which no penalty is placed on the policy instruments, that is cases where the central bank is not concerned about interest rate smoothing/stabilization. Similar instances are documented in Svensson (2000) and Dennis and Söderström (Forthcoming), and, together, these occurrence highlight the fact that the properties of the solution algorithms, particularly those for obtaining discretionary policies, have not been explored. In fact, when discussing their algorithm for obtaining optimal discretionary policies, Oudiz and Sachs (1985, p. 311) comment “We do not know of any general result concerning the convergence of this process. However in our empirical applications we have not run into major problems.” Similarly, Söderlind (1999, p. 819), notes that “The general properties of this algorithm are unknown. Practical experience suggests that it is often harder to find the discretionary equilibrium than the commitment equilibrium. It is unclear if this is due to the algorithm.”

The approach taken in this paper provides insights as to why, and under what circumstances, convergence problems may be encountered when solving for optimal discretionary policies. For this purpose, the key equations in the algorithm are

$$P = W + \beta F_1' Q F_1 + \beta H_1' P H_1 \tag{35}$$

$$D = A_0 - A_2 H_1 - A_4 F_1 \tag{36}$$

$$H_1 = D^{-1} (I - A_3 (Q + A_3' D^{-1} P D^{-1} A_3)^{-1} A_3' D'^{-1} P D^{-1}) A_1 \tag{37}$$

where equations (35) and (36) are taken directly from Sections 4 and (37) is obtained by substituting equation (28) into equation (30). By inspection, difficulties obtaining the optimal discretionary policy can arise when:

1. The discount factor is close to one and the H_1 matrix associated with the optimal discretionary policy has a spectral radius close to one. In this situation, it takes a long time for recursive methods to obtain the fixed point P . If this problem occurs, it may be useful to solve for P using a nonrecursive method, such as the Hessenberg-Schur algorithm described in Anderson et al. (1996).
2. The iterations produce a D matrix that is singular. Because A_2 and A_4 are typically sparse and A_0 has full rank, a singularity in D is unlikely to occur.
3. The iterations produce a $(Q + A_3 D^{-1} P D^{-1} A_3)$ matrix that is singular. This will occur if both Q and $P D^{-1} A_3$ equal null matrices, that is, if the loss function does not penalize movements in the policy instrument(s) and the target variables are contemporaneously unaffected by movements in the policy instruments. When both $P D^{-1} A_3$ and Q equal null matrices the optimal discretionary policy is not uniquely determined by setting the derivative of the loss function with respect to the policy instrument(s) to zero, see equation (24).

Among these three situations, the first will likely increase the computation time, the second is very unlikely to occur, and the third accounts for the problems encountered above, in Svensson (2000), and in Dennis and Söderström (Forthcoming), which all involve a loss function that does not penalize the policy instrument. In fact, the assumption that Q is positive definite—rather than just positive semi-definite—which would preclude this problem, is standard in linear-quadratic control theory (Rustem and Zarrop, 1979; Anderson et al. 1996, p. 175). We did not restrict Q to be positive definite in our analysis because policies where the central bank does not smooth or stabilize interest rates are of broad interest and can be obtained for many models.

6. CONCLUSION

This paper has presented new algorithms to solve for optimal commitment rules and optimal discretionary rules in rational expectation models. These algorithms differ from those developed by Oudiz and Sachs (1985), Backus and Driffill (1986), and Söderlind (1999) in that they do not require the optimization constraints to be written in state-space form. This paper has also extended the class of models that can be analyzed, allowing expectations of future policy instruments to enter the optimization constraints, which can be useful when solving models that contain an interest rate term structure, and showed how to derive the Euler equation for the optimal discretionary policy, which is needed to study targeting rules.

After setting up the optimization problem in Section 2, the approaches taken to solve for optimal commitment policies and optimal discretionary policies were described in Sections 3 and 4, respectively. Section 5 took two optimization-based New Keynesian models and used a range of solution algorithms to solve them for optimal commitment policies and optimal discretionary policies for a variety of

policy objective functions. For these models it was feasible to analytically derive state-space representations. Although the relative advantage of the solution methods developed in this paper is that they save the user from having to manipulate a model into state-space form, computational experiments on these small-scale models showed that the state-space solution methods are generally more efficient, precisely because they exploit more of the model's structure. Nevertheless, the methods developed here can be employed to solve larger models and for small models they are efficient enough to be used in estimation exercises, which may involve a model being solved many times within a hill-climbing routine [see Dennis (2004b) for an application]. Section 5 also solved the Fuhrer-Moore model, a model that can be much more quickly expressed in structural form than in a state-space form.

The paper also has discussed some of the computational problems that can be encountered when solving for optimal policies and showed why, in practice, placing a small penalty on movements in the policy instrument(s), invariably overcomes these problems. These experiments also showed that it is generally more efficient to solve for optimal commitment policies by solving the underlying rational expectations model using the *real* QZ-decomposition rather than either the *complex* QZ-decomposition or the Binder-Pesaran algorithm.

NOTES

1. An alternative approach is to assume that the policy maker sets policy according to a "timeless perspective" (Woodford, 1999). According to the timeless perspective the policymaker optimizing today behaves as it would have chosen to if it had optimized at a time far in the past. Useful discussions and analyses of timelessly optimal policies can be found in Giannoni and Woodford (2002) and Jensen and McCallum (2002).

2. See Holly and Hughes-Hallett (1989), Amman and Kendrick (1999), Chen and Zadorozney (2002), and Blake (2004), for other approaches.

3. Söderlind (1999) provides a popular implementation of the Backus and Driffill (1986) algorithm. Krusell, Quadrini, and Rios-Rull (1997) provide an alternative, guess-and-verify, approach to solving for time-consistent policies.

4. Dennis (2004a) shows how to solve for optimal simple rules when the optimization constraints are written in structural form.

5. Dennis and Söderström (Forthcoming) apply the methods developed in this paper to the macro-policy model developed by Orphanides and Wieland (1998), which is somewhat more complicated than the Fuhrer and Moore (1995) model.

6. The omission of terms interacting y_t and x_t in equation (1) is without loss of generality. When the constraints are in structural form it is always possible to include x_t within y_t with W containing the relevant penalty terms.

7. Binder and Pesaran (1995) show how a wide class of models, which can contain quite general lead-lag structures and expectations formed with varying information sets, can be written in terms of equation (2).

8. Svensson (2000) also solves a model that contains the expected future value of the policy instrument. However, Svensson (2000) considers discretion but not commitment.

9. Because W and Q are symmetric positive semidefinite and the policy constraints are convex the Lagrangian is also convex and the first-order conditions are sufficient for locating the minimum.

10. More precisely, the constraints on λ_{t-1} are $[A_2' A_4'] \lambda_{t-1} = \mathbf{0}$. Because $\dim[\ker([A_2' A_4'])]$ is not necessarily zero, the restriction in the text, $\lambda_{t-1} = \mathbf{0}$, is sufficient but stronger than necessary.

11. This transformation requires that D have full rank, which is invariably satisfied because A_0 has full rank.

12. A description of the doubling algorithm along with other methods for solving matrix Sylvester equations can be found in Anderson, Hansen, McGrattan, and Sargent (1996).

13. All simulations were conducted in Gauss 6.0.17 using a Pentium IV 2.6Ghz processor with 1.5GB RAM.

14. Routines to perform the *real* QZ decomposition and the *complex* QZ decomposition in Gauss are available from Paul Söderlind's Web site. These excellent routines were employed in this study.

15. Söderlind's (1999) method of solving for commitment equilibria also utilizes the *real* QZ decomposition.

16. The structural form and the state-space form used for the Galí-Monacelli model are contained in a technical appendix that is available on request.

17. We follow Galí and Monacelli (2005) and set $\eta = 1$, $\alpha = 0.4$, $\beta = 0.99$, $\theta = 0.75$, $\sigma = 1$, $\varphi = 3$, $\omega_\alpha = 1 + \alpha(2 - \alpha)(\sigma\eta - 1)$, and $\kappa_\alpha = (1 - \theta)(1 - \beta\theta)/\theta(\varphi + \frac{\sigma}{\omega_\alpha})$. We condition on the foreign variables and normalize their values to zero.

18. The model was solved 100,000 times using each algorithm from which average solution times were computed. The convergence criteria were set so that the algorithms all returned solutions that were identical to 13 significant figures.

19. The source of this difficulty is a singularity in the transformation the algorithm uses to rotate the dynamic programming solution into state/co-state space.

20. This instability differs from the standard form of instability encountered in rational expectations models, which is that the system contains more unstable eigenvalues than there are nonpredetermined variables.

21. The structural form and the state-space form used for the EHL model are contained in a technical appendix that is available on request.

22. We parameterize the model according to Erceg, Henerson, and Levin (2000), setting $\sigma = \chi = 1.5$, $\alpha = 0.3$, $\theta_w = \theta_p = \frac{1}{3}$, $\zeta_w = \zeta_p = 0.75$, $\rho = 0.95$, $\beta = 0.99$, $\bar{C} = 3.163$, $\bar{Q} = 0.3163$, $\bar{N} = 0.27$, $\bar{Z} = 0.03$, $l_c = \bar{C}/(\bar{C} - \bar{Q})$, $l_q = \bar{Q}/(\bar{C} - \bar{Q})$, $l_n = \bar{N}/(1 - \bar{N} - \bar{Z})$, $l_z = \bar{Z}/(1 - \bar{N} - \bar{Z})$, $\Lambda = \alpha + \chi l_n + (1 - \alpha)\sigma l_c$, $\lambda_{mpl} = \alpha/(1 - \alpha)$, $\lambda_{mrs} = \sigma l_c + \chi l_n/(1 - \alpha)$, $\kappa_p = (1 - \zeta_p)(1 - \beta\zeta_p)/\zeta_p$, $\kappa_w = (1 - \zeta_w)(1 - \beta\zeta_w)/[(1 + \chi l_n(\frac{1 + \theta_w}{\theta_w}))\zeta_w]$, $w_x^* = (\chi l_n + \alpha l_c)/\Lambda$, $w_q^* = -\alpha\sigma l_q/\Lambda$, $w_z^* = \alpha\chi l_z/\Lambda$, $y_x^* = (1 + \chi l_n)/\Lambda$, $y_q^* = (1 - \alpha)\sigma l_q/\Lambda$, $y_z^* = -(1 - \alpha)\chi l_z/\Lambda$.

23. The model was solved 100,000 times using each algorithm from which average solution times were computed. The convergence criteria were set so that the algorithms all returned solutions that were identical to 12 significant figures.

24. The model was solved 10,000 times using each algorithm from which average solution times were computed.

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APPENDIX A

A.1. MATHEMATICAL PRELIMINARY

Assume that $0 < \beta < 1$ and that the spectral radius of the matrix θ is less than one, then the infinite series $S = \sum_{j=0}^{\infty} \beta^j \theta^j W \theta^j$ is convergent. Consequently, $\beta \theta' S \theta = \sum_{j=1}^{\infty} \beta^j \theta^j W \theta^j = S - W$, and S can be found as the fixed point of $S = W + \beta \theta' S \theta$.

A.2. ESTABLISHING REMARK 2

Let

$$z_t \equiv \begin{bmatrix} \lambda_t \\ y_t \\ x_t \end{bmatrix}$$

and

$$\widehat{K} \equiv \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & W & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & Q \end{bmatrix}.$$

From equation (27) the recursive equilibrium law of motion for z_t can be expressed as $z_t = H z_{t-1} + G v_t$. Given this law of motion, we have

$$\begin{aligned} \text{Loss}(t, \infty) &= E_t \sum_{j=0}^{\infty} \beta^j (z'_{t+j} K z_{t+j}) \\ &= \left[z'_t \left(\sum_{j=0}^{\infty} \beta^j H^j K H^j \right) z_t + \frac{\beta}{1-\beta} \left(\sum_{j=0}^{\infty} \beta^j \text{tr}(G' H^j K H^j G \Omega) \right) \right], \end{aligned}$$

which using the result regarding convergent infinite series in Appendix A.1 can be simplified to

$$\text{Loss}(t, \infty) = \left[z_t' \widehat{P} z_t + \frac{\beta}{1 - \beta} \text{tr}(G' \widehat{P} G \Omega) \right],$$

where $\widehat{P} \equiv \widehat{K} + \beta H' \widehat{P} H$. Finally, again using the law of motion for z_t we obtain

$$\text{Loss}(t, \infty) = \left[z_{t-1}' H' \widehat{P} H z_{t-1} + v_t' G' \widehat{P} G v_t + \frac{\beta}{1 - \beta} \text{tr}(G' \widehat{P} G \Omega) \right],$$

as required.

A.3. ESTABLISHING REMARK 3

From A.2 the loss function under commitment can be written as

$$\text{Loss}(t, \infty) = \left[z_t' \widehat{P} z_t + \frac{\beta}{1 - \beta} \text{tr}(G' \widehat{P} G \Omega) \right], \tag{A.1}$$

where $\widehat{P} \equiv \widehat{K} + \beta H' \widehat{P} H$, and the economy's evolution is given by

$$z_t = H z_{t-1} + G v_t. \tag{A.2}$$

From Appendix A.2, the unconditional variance-covariance matrix for z_t is the fixed point of

$$\Sigma = H \Sigma H' + G \Omega G', \tag{A.3}$$

from which the unconditional variance-covariance matrix for $[y_t' x_t']'$, Φ , can be obtained. Multiplying (A.1) through by $(1 - \beta)$ gives

$$\begin{aligned} (1 - \beta) \text{Loss}(t, \infty) &= (1 - \beta) z_t' \widehat{P} z_t + \beta \text{tr}(G' \widehat{P} G \Omega) \\ &= (1 - \beta) z_t' \widehat{P} z_t + \beta \text{tr}(\widehat{P} G \Omega G'). \end{aligned}$$

Now employing (A.3) we have

$$\begin{aligned} (1 - \beta) \text{Loss}(t, \infty) &= (1 - \beta) z_t' \widehat{P} z_t + \beta \text{tr}[\widehat{P}(\Sigma - H \Sigma H')] \\ &= (1 - \beta) z_t' \widehat{P} z_t + \beta \text{tr}(\widehat{P} \Sigma) - \beta \text{tr}(H' \widehat{P} H \Sigma) \\ &= (1 - \beta) z_t' \widehat{P} z_t + \beta \text{tr}(\widehat{P} \Sigma) - \text{tr}[(\widehat{P} - \widehat{K}) \Sigma] \\ &= (1 - \beta) z_t' \widehat{P} z_t - (1 - \beta) \text{tr}(\widehat{P} \Sigma) + \text{tr}(\widehat{K} \Sigma). \end{aligned}$$

Provided the spectral radius of H is less than one, \widehat{P} will remain bounded even in the limit as $\beta \uparrow 1$. Therefore, $\lim_{\beta \uparrow 1} \text{Loss}(t, \infty) = \text{tr}(\widehat{K} \Sigma)$. However, defining

$$K \equiv \begin{bmatrix} W & 0 \\ 0 & Q \end{bmatrix}$$

so that

$$\widehat{K} \equiv \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix}$$

only the elements in Σ associated with y_t and x_t (i.e., Φ) are relevant. Consequently, we have $\lim_{\beta \uparrow 1} \text{Loss}(t, \infty) = \text{tr}(\mathbf{K}\Phi)$, as required.

APPENDIX B

B.1. DERIVING EQUATION (21)

We show that

$$\begin{aligned} \text{Loss}(t, \infty) &= E_t \sum_{j=0}^{\infty} \beta^j (y'_{t+j} \mathbf{W}y_{t+j} + x'_{t+j} \mathbf{Q}x_{t+j}) \\ &= y'_t \mathbf{P}y_t + x'_t \mathbf{Q}x_t + \frac{\beta}{1-\beta} \text{tr}[(\mathbf{F}'_2 \mathbf{Q} \mathbf{F}_2 + \mathbf{H}'_2 \mathbf{P} \mathbf{H}_2) \Omega], \end{aligned}$$

where $\mathbf{P} \equiv \mathbf{S} + \beta \mathbf{R} = \mathbf{W} + \beta \mathbf{F}'_1 \mathbf{Q} \mathbf{F}_1 + \beta \mathbf{H}'_1 \mathbf{P} \mathbf{H}_1$.

To establish this equality we exploit the properties of convergent geometric series, and the requirement that, in equilibrium, $y_{t+j} = \mathbf{H}_1 y_{t+j-1} + \mathbf{H}_2 v_{t+j}$, and $x_{t+j} = \mathbf{F}_1 y_{t+j-1} + \mathbf{F}_2 v_{t+j}$, $\forall j > 0$. First write

$$\begin{aligned} E_t \sum_{j=0}^{\infty} \beta^j (y'_{t+j} \mathbf{W}y_{t+j} + x'_{t+j} \mathbf{Q}x_{t+j}) \\ = x'_t \mathbf{Q}x_t + E_t \sum_{j=0}^{\infty} \beta^j (y'_{t+j} \mathbf{W}y_{t+j}) + E_t \sum_{j=1}^{\infty} \beta^j (x'_{t+j} \mathbf{Q}x_{t+j}). \end{aligned} \tag{B.1}$$

The first term on the RHS of (B.1) is in the appropriate form. We will treat the second and third terms on the RHS separately, beginning with the second term. We can write

$$E_t \sum_{j=0}^{\infty} \beta^j (y'_{t+j} \mathbf{W}y_{t+j}) = y'_t \left(\sum_{j=0}^{\infty} \beta^j \mathbf{H}'_1{}^j \mathbf{W} \mathbf{H}_1^j \right) y_t + \beta \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \beta^{(l+j)} \text{tr}(\mathbf{H}'_2 \mathbf{H}_1{}^l \mathbf{W} \mathbf{H}_1^j \mathbf{H}_2 \Omega).$$

Assuming that the spectral radius of \mathbf{H}_1 is less than one, the result in Appendix A.1 leads to

$$E_t \sum_{j=0}^{\infty} \beta^j (y'_{t+j} \mathbf{W}y_{t+j}) = y'_t \mathbf{S}y_t + \frac{\beta}{1-\beta} \text{tr}(\mathbf{H}'_2 \mathbf{S} \mathbf{H}_2 \Omega), \tag{B.2}$$

where $\mathbf{S} \equiv \mathbf{W} + \beta \mathbf{H}'_1 \mathbf{S} \mathbf{H}_1$.

Turning to the third term on the RHS of (B.1)

$$E_t \sum_{j=1}^{\infty} \beta^j (x'_{t+j} \mathbf{Q}x_{t+j}) = \beta y'_t \left(\sum_{j=0}^{\infty} \beta^j \mathbf{H}'_1{}^j \mathbf{F}'_1 \mathbf{Q} \mathbf{F}_1 \mathbf{H}_1^j \right) y_t + \beta \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \beta^{(l+j)} \text{tr}(\mathbf{H}'_2 \mathbf{F}'_1{}^l \mathbf{Q} \mathbf{F}_1 \mathbf{H}_2^j \Omega).$$

Again, provided the spectral radius of H_1 is less than one, the result in Appendix A.1 gives $R \equiv F_1' Q F_1 + \beta H_1' R H_1$. Therefore,

$$E_t \sum_{j=1}^{\infty} \beta^j (x'_{t+j} Q x_{t+j}) = \beta y_t' R y_t + \frac{\beta}{1-\beta} \text{tr}(F_2' Q F_2 \Omega) + \frac{\beta^2}{1-\beta} \text{tr}(H_2' R H_2 \Omega). \tag{B.3}$$

Substituting (B.2) and (B.3) back into (B.1) gives

$$\text{Loss}(t, \infty) = y_t'(S + \beta R)y_t + x_t' Q x_t + \frac{\beta}{1-\beta} \text{tr}(F_2' Q F_2 \Omega) + \frac{\beta}{1-\beta} \text{tr}[H_2'(S + \beta R)H_2 \Omega].$$

Finally, let $P \equiv S + \beta R = W + \beta F_1' Q F_1 + \beta H_1' P H_1$ giving

$$\text{Loss}(t, \infty) = y_t' P y_t + x_t' Q x_t + \frac{\beta}{1-\beta} \text{tr}(F_2' Q F_2 \Omega) + \frac{\beta}{1-\beta} \text{tr}(H_2' P H_2 \Omega),$$

or

$$\text{Loss}(t, \infty) = y_t' P y_t + x_t' Q x_t + \frac{\beta}{1-\beta} \text{tr}[(F_2' Q F_2 + H_2' P H_2) \Omega],$$

as required.

B.2. ESTABLISHING REMARK 5

Let $z_t \equiv [y_t' \ x_t']'$, then the policy objective function, (B.1), can be written as

$$\text{Loss}(t, \infty) = E_t \sum_{j=0}^{\infty} \beta^j z_{t+j}' K z_{t+j}, \tag{B.4}$$

where

$$K \equiv \begin{bmatrix} W & 0 \\ 0 & Q \end{bmatrix},$$

and

$$z_{t+j} = \begin{bmatrix} H_1 & 0 \\ F_1 & 0 \end{bmatrix} z_{t+j-1} + \begin{bmatrix} H_2 \\ F_2 \end{bmatrix} v_{t+j} \equiv H z_{t+j-1} + G v_{t+j}, \quad \forall j > 0. \tag{B.5}$$

Employing (B.5) in (B.4) gives

$$\begin{aligned} \text{Loss}(t, \infty) &= E_t \sum_{j=0}^{\infty} \beta^j z_{t+j}' K z_{t+j} \\ &= z_t' \left(\sum_{j=0}^{\infty} \beta^j H^j K H^j \right) z_t + \frac{\beta}{1-\beta} \sum_{j=0}^{\infty} \beta^j \text{tr}(G' H^j K H^j G \Omega). \end{aligned}$$

Now, exploiting the properties of convergent geometric series from Appendix A.1, we have

$$\text{Loss}(t, \infty) = z_t' \tilde{P} z_t + \frac{\beta}{1-\beta} \text{tr}(G' \tilde{P} G \Omega),$$

or

$$\text{Loss}(t, \infty) = z'_{t-1} \mathbf{H}' \tilde{\mathbf{P}} \mathbf{H} z_{t-1} + v'_t \mathbf{G}' \tilde{\mathbf{P}} \mathbf{G} v_t + \frac{\beta}{1 - \beta} \text{tr}(\mathbf{G}' \tilde{\mathbf{P}} \mathbf{G} \Omega),$$

where $\tilde{\mathbf{P}} \equiv \mathbf{K} + \beta \mathbf{H}' \tilde{\mathbf{P}} \mathbf{H}$, as necessary.

B.3. ESTABLISHING REMARK 6

In this appendix we establish that $\lim_{\beta \uparrow 1} (1 - \beta) \text{Loss}(t, \infty) = \text{tr}(\mathbf{K} \Phi)$. Recall that the policy objective function can be written as

$$\text{Loss}(t, \infty) = y'_t \mathbf{P} y_t + x'_t \mathbf{Q} x_t + \frac{\beta}{1 - \beta} \text{tr}[(\mathbf{F}'_2 \mathbf{Q} \mathbf{F}_2 + \mathbf{H}'_2 \mathbf{P} \mathbf{H}_2) \Omega],$$

where $\mathbf{P} \equiv \mathbf{W} + \beta \mathbf{F}'_1 \mathbf{Q} \mathbf{F}_1 + \beta \mathbf{H}'_1 \mathbf{P} \mathbf{H}_1$. The components of the unconditional variance-covariance matrices for y_t and x_t are given by

$$\Phi_y = \mathbf{H}_1 \Phi_y \mathbf{H}'_1 + \mathbf{H}_2 \Omega \mathbf{H}'_2 \tag{B.6}$$

$$\Phi_x = \mathbf{F}_1 \Phi_y \mathbf{F}'_1 + \mathbf{F}_2 \Omega \mathbf{F}'_2 \tag{B.7}$$

$$\Phi_{yx} = \mathbf{H}_1 \Phi_y \mathbf{F}'_1 + \mathbf{H}_2 \Omega \mathbf{F}'_2$$

$$\Phi_{xy} = \mathbf{F}_1 \Phi_y \mathbf{H}'_1 + \mathbf{F}_2 \Omega \mathbf{H}'_2.$$

Scaling the policy objective function by $(1 - \beta)$ gives

$$(1 - \beta) \text{Loss}(t, \infty) = (1 - \beta)(y'_t \mathbf{P} y_t + x'_t \mathbf{Q} x_t) + \beta \text{tr}[(\mathbf{F}'_2 \mathbf{Q} \mathbf{F}_2 + \mathbf{H}'_2 \mathbf{P} \mathbf{H}_2) \Omega].$$

Exploiting the properties of the trace operator results in

$$(1 - \beta) \text{Loss}(t, \infty) = (1 - \beta)(y'_t \mathbf{P} y_t + x'_t \mathbf{Q} x_t) + \beta \text{tr}(\mathbf{Q} \mathbf{F}'_2 \Omega \mathbf{F}_2 + \mathbf{P} \mathbf{H}_2 \Omega \mathbf{H}'_2).$$

Employing (B.6) and (B.7) gives

$$\begin{aligned} (1 - \beta) \text{Loss}(t, \infty) &= (1 - \beta)(y'_t \mathbf{P} y_t + x'_t \mathbf{Q} x_t) \\ &+ \beta \text{tr}[\mathbf{Q}(\Phi_x - \mathbf{F}_1 \Phi_y \mathbf{F}'_1) + \mathbf{P}(\Phi_y - \mathbf{H}_1 \Phi_y \mathbf{H}'_1)] \\ &= (1 - \beta)(y'_t \mathbf{P} y_t + x'_t \mathbf{Q} x_t) + \beta \text{tr}[\mathbf{Q} \Phi_x + \mathbf{W} \Phi_y - (1 - \beta)(\mathbf{F}'_1 \mathbf{Q} \mathbf{F}_1 + \mathbf{H}'_1 \mathbf{P} \mathbf{H}_1)]. \end{aligned}$$

Because the spectral radius of \mathbf{H}_1 is less than one, \mathbf{P} remains bounded even as $\beta \uparrow 1$. Thus, $\lim_{\beta \uparrow 1} (1 - \beta)(y'_t \mathbf{P} y_t + x'_t \mathbf{Q} x_t) = \mathbf{0}$ and $\lim_{\beta \uparrow 1} (1 - \beta)(\mathbf{F}'_1 \mathbf{Q} \mathbf{F}_1 + \mathbf{H}'_1 \mathbf{P} \mathbf{H}_1) = \mathbf{0}$, which gives us the result that $\lim_{\beta \uparrow 1} (1 - \beta) \text{Loss}(t, \infty) = \text{tr}(\mathbf{Q} \Phi_x + \mathbf{W} \Phi_y) = \text{tr}(\mathbf{K} \Phi)$, where

$$\Phi \equiv \begin{bmatrix} \Phi_y & \Phi_{yx} \\ \Phi_{xy} & \Phi_x \end{bmatrix}.$$