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AN INVARIANCE PRINCIPLE AND A LARGE DEVIATION PRINCIPLE FOR THE BIASED RANDOM WALK ON $\mathbb{Z}^d$

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Abstract

We establish an invariance principle and a large deviation principle for a biased random walk RW_λ with $\lambda \in [0, 1)$ on $\mathbb{Z}^d$. The scaling limit in the invariance principle is not a d-dimensional Brownian motion. For the large deviation principle, its rate function is different from that of a drifted random walk, as may be expected, though the reflected biased random walk evolves like the drifted random walk in the interior of the first quadrant and almost surely visits coordinate planes finitely many times.

Keywords: Biased random walk; invariance principle; large deviation principle

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Secondary 60F05; 60F10

1. Introduction

The biased random walk RW_λ with parameter $\lambda \in [0, \infty)$ was introduced to design a Monte Carlo algorithm for the self-avoiding walk by Berretti and Sokal [5]. The idea was refined and developed in [12, 20, 22]. In the 1990s, Lyons [13, 14, 15] and Lyons *et al.* [16, 17] made fundamental advances in the study of biased random walks on Cayley graphs and trees. In particular, they proved remarkable results on transience/recurrence phase transition for RW_λ when λ changes. The critical value λ_c is equal to the exponential growth rate of a Cayley graph or the branching number of a tree. The phase transition and speed for RW_λ were also obtained on Galton–Watson trees in [16]. In recent years, problems of RW_λ on Galton–Watson trees continue to attract much interest, where some variants of the biased random walk are possibly modified (see, for example, [1, 2, 3, 4, 10]). For the invariance principle of RW_λ, Bowditch [6] proved a quenched invariance principle for RW_λ on a supercritical Galton–Watson tree where the corresponding scaling limit is a one-dimensional Brownian motion. On a typical Cayley graph, Euclidean lattice $\mathbb{Z}^d$, the speed and spectral radius for RW_λ were studied in [21]. To understand the limit behavior of RW_λ on $\mathbb{Z}^d$ is one of the motivations of this paper.

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We first introduce the notion of a biased random walk. Let $\mathbf{0}=(0, \ldots, 0) \in \mathbb{Z}^d$ and $|x| = \sum_{i=1}^d |x_i|$ denote the graph distance between $x=(x_1, \ldots, x_d) \in \mathbb{Z}^d$ and $\mathbf{0}$. Write $\mathbb{N}$ for the set of natural numbers, and let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. For any $n \in \mathbb{Z}_+$, define

$$B(n) = \{x \in \mathbb{Z}^d : |x| \le n\}, \qquad \partial B(n) = \{x \in \mathbb{Z}^d : |x| = n\}.$$

Let $\lambda \in [0, \infty)$. If an edge $e=\{x, y\}$ is at distance n from $\mathbf{0}$, i.e. $\min(|x|, |y|) = n$, its conductance is defined as $c(e) = c(x, y) = \lambda^{-n}$. Denote by RW_λ the random walk associated with such conductances, and call it the biased random walk with parameter λ. In other words, RW_λ is the Markov chain on $\mathbb{Z}^d$ with the following transition probabilities: for $v, u \in \mathbb{Z}^d$ with $|v-u|=1$,

$$p(v, u) = \frac{c(v, u)}{\sum_{w:|w-v|=1} c(v, w)} = \begin{cases} \frac{1}{d_v} & \text{if } v=\mathbf{0}, \\ \frac{\lambda}{d_v+(\lambda-1)d_v^-} & \text{if } u \in \partial B(|v|-1) \text{ and } v \ne \mathbf{0}, \\ \frac{1}{d_v+(\lambda-1)d_v^-} & \text{otherwise.} \end{cases} \tag{1}$$

Here, $d_v = 2d$ is the degree of the vertex v, and d_v^- (resp. d_v^+) is the number of edges connecting v to $\partial B(|v|-1)$ (resp. $\partial B(|v|+1)$). When $\lambda = 1$, RW_λ is the usual simple random walk (SRW).

Let $(X_n)_{n=0}^\infty = (X_n^1, \ldots, X_n^d)_{n=0}^\infty$ be RW_λ on $\mathbb{Z}^d$. It is known that $(X_n)_{n=0}^\infty$ is transient for $0 \le \lambda < 1$ and positive recurrent for $\lambda > 1$ (see [15] and [18, Theorem 3.10]). In the latter case, the related central limit theorem and invariance principle can be derived from [11] and [19] straightforwardly. In this paper we will focus on the case $0 \le \lambda < 1$.

Note that we have, for $v=(v_1, \ldots, v_d) \in \mathbb{Z}^d$,

$$d_v^+ = d + \kappa(v), \qquad d_v^- = d - \kappa(v),$$

where $\kappa(v)$ is the number of $1 \le i \le d$ such that $v_i = 0$. It is easy to see from (1) that $(X_n)_{n=0}^\infty$ is closely related to a drifted random walk on $\mathbb{Z}^d$. More precisely, before leaving the first open orthant, $(X_n)_{n=0}^\infty$ has the same transition probabilities as those of the drifted random walk on $\mathbb{Z}^d$ with step distribution ν given by

$$\nu(e_1) = \cdots = \nu(e_d) = \frac{1}{d(1+\lambda)} \quad \text{and} \quad \nu(-e_1) = \cdots = \nu(-e_d) = \frac{\lambda}{d(1+\lambda)}, \tag{2}$$

where $\{e_1, \ldots, e_d\}$ is the standard basis in $\mathbb{Z}^d$. See Figure 1 for an illustration of transition probabilities for biased and drifted random walks on $\mathbb{Z}^2$.

Using this connection to the drifted random walk, it is proved in [21] that, for $\lambda \in (0, 1)$, $(X_n)_{n=0}^\infty$ almost surely (a.s.) visits the union of axial hyperplanes

$$\mathcal{X} = \{x = (x_1, \ldots, x_d) \in \mathbb{Z}^d : x_i = 0 \text{ for some } 1 \le i \le d\} \tag{3}$$

only finitely many times, and the following strong law of large numbers holds:

$$\lim_{n\to\infty} \frac{1}{n}(|X_n^1|, \ldots, |X_n^d|) \to \frac{1-\lambda}{1+\lambda}\left(\frac{1}{d}, \ldots, \frac{1}{d}\right) \quad \text{a.s.} \tag{4}$$

In this paper we prove the invariance principle (IP) and central limit theorem (CLT) for the reflected RW_λ $\{(|X_n^1|, \ldots, |X_n^d|)\}_{n=0}^\infty$ in Section 2, and the large deviation principle (LDP) for the scaled reflected RW_λ $\{\frac{1}{n}(|X_n^1|, \ldots, |X_n^d|)\}_{n=0}^\infty$ in Section 3. The aforementioned connection

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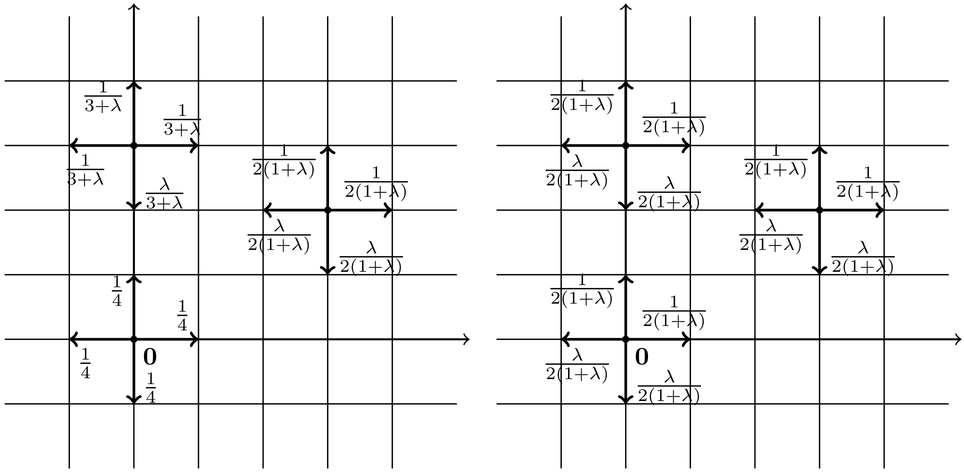

FIGURE 1: Illustration of transition probabilities for biased and drifted random walks at **0**, on axis, and in the first open quadrant. Left: biased random walk. Right: drifted random walk.

between biased and drifted random walks will also play an important role in the proofs of our main results. We will see in the proof that the scaling limit in the IP of the reflected RW_λ is the same as that of the drifted random walk, and hence is not a d-dimensional Brownian motion. However, their rate functions in the LDP are quite different from each other. Here, we can only calculate the rate function Λ^* when $d \in \{1, 2\}$ and $\lambda \in (0, 1)$ as well as $d \geq 2$ and $\lambda = 0$. To obtain an explicit rate function when $d \geq 3$ and $\lambda \in (0, 1)$ remains an open problem.

2. CLT and IP for reflected RW_λ with $\lambda \in [0, 1)$

In this section we use the martingale CLT for $\mathbb{R}^d$-valued martingales [9, Chapter 7, Theorem 1.4] and the martingale characterization of Markov chains to prove the IP for the reflected RW_λ $\{(|X_n^1|, |X_n^2|, \ldots, |X_n^d|)\}$ (Theorem 2.1). The CLT for the reflected RW_λ is a consequence of the corresponding IP.

To describe our main result we need to introduce some notation. For any nonnegative definite symmetric $d \times d$ matrix A, let $\mathcal{N}(\mathbf{0}, A)$ be the normal distribution with mean $\mathbf{0}$ and covariance matrix A. Define the following positive definite symmetric $d \times d$ matrix $\Sigma = (\Sigma_{ij})_{1 \leq i,j \leq d}$:

$$\Sigma_{ii} = \frac{1}{d} - \frac{(1-\lambda)^2}{d^2(1+\lambda)^2}, \qquad \Sigma_{ij} = -\frac{(1-\lambda)^2}{d^2(1+\lambda)^2}, \qquad 1 \leq i \neq j \leq d.$$

For a random sequence $(Y_n)_{n=0}^{\infty}$ and a random variable X in the Skorokhod space $D([0, \infty), \mathbb{R}^d)$, $Y_n \to X$ means that as $n \to \infty$, Y_n converges in distribution to X in $D([0, \infty), \mathbb{R}^d)$. For any $a \in \mathbb{R}$, let $\lfloor a \rfloor$ be the integer part of a. Put

$$v = \left(\frac{1-\lambda}{d(1+\lambda)}, \ldots, \frac{1-\lambda}{d(1+\lambda)} \right) \in \mathbb{R}^d.$$

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Theorem 2.1. (IP and CLT.) *Let $0 \le \lambda < 1$, and $(X_m)_{m=0}^{\infty}$ be* RW$_\lambda$ *on $\mathbb{Z}^d$ starting at $x \in \mathbb{Z}^d$. Then, on $D([0, \infty), \mathbb{R}^d)$,*

$$\left(\frac{(|X^1_{\lfloor nt \rfloor}|, \ldots, |X^d_{\lfloor nt \rfloor}|) - nvt}{\sqrt{n}}\right)_{t\ge 0} \tag{5}$$

converges in distribution to $\frac{1}{\sqrt{d}}[I - \frac{1-\rho_\lambda}{d}E]W_t$ as $n \to \infty$, where $(W_t)_{t\ge 0}$ is the d-dimensional Brownian motion starting at **0**, *I is the identity matrix, and E denotes the $d \times d$ matrix whose entries are all equal to* 1. *Here, $\rho_\lambda = 2\sqrt{\lambda}/(1+\lambda)$ is the spectral radius of $(X_n)_{n\ge 1}$ (see* [21, *Theorem* 1.1]).

In particular, we have

$$\frac{(|X^1_n|, \ldots, |X^d_n|) - nv}{\sqrt{n}} \to \mathcal{N}(\mathbf{0}, \Sigma)$$

in distribution as $n \to \infty$.

Remark 2.1.
(i) Note that $\Sigma = 0$ when $d = 1$ and $\lambda = 0$. Hence, for $d = 1$ and $\lambda = 0$, both

$$\frac{|X_n| - nv}{\sqrt{n}} = \frac{|X_0| + n - nv}{\sqrt{n}} \to \mathcal{N}(\mathbf{0}, \Sigma)$$

and

$$\left(\frac{|X_{\lfloor nt \rfloor}| - nvt}{\sqrt{n}}\right)_{t\ge 0}$$

converging in distribution to $Y = (Y_t)_{t\ge 0}$ are not interesting.

From Theorem 2.1, the following holds: For RW$_\lambda$ $(X_m)_{m=0}^{\infty}$ with $\lambda < 1$ on $\mathbb{Z}^d$ which starts at any fixed vertex, as $n \to \infty$,

$$\left(\frac{|X_{\lfloor nt \rfloor}| - n\frac{1-\lambda}{1+\lambda}t}{\sqrt{n}}\right)_{t\ge 0}$$

converges in distribution to $(\rho_\lambda B_t)_{t\ge 0}$ on $D([0, \infty), \mathbb{R})$ and $(B_t)_{t\ge 0}$ is the 1-dimensional Brownian motion starting at 0.

(ii) The limit process is not a d-dimensional Brownian motion since the cross-variation process of $M^{n,i}$ and $M^{n,j}$ (as below) doesn't converge to **0** as $n \to \infty$, $1 \le i \ne j \le d$. Note that by the martingale CLT, the drifted random walk defined in (2) has the same limit process in IP, which is not a d-dimensional Brownian motion.

Proof of Theorem 2.1. We only prove IP for

$$\left(\frac{(|X^1_{\lfloor nt \rfloor}|, \ldots, |X^d_{\lfloor nt \rfloor}|) - nvt}{\sqrt{n}}\right)_{t\ge 0}.$$

Let $\mathcal{F}_k = \sigma(X_0, X_1, \ldots, X_k)$, $k \in \mathbb{Z}_+$. For $1 \le i \le d$, define functions $f_i\colon \mathbb{Z}^d \to \mathbb{R}$ by

$$f_i(x) = \begin{cases} \frac{2}{d+\kappa(x)+\lambda(d-\kappa(x))} & \text{if } x_i = 0, \\ \frac{1-\lambda}{d+\kappa(x)+\lambda(d-\kappa(x))} & \text{if } x_i \ne 0. \end{cases}$$

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Note that, given $\mathcal{F}_{k-1}$, the conditional distribution of X_k is $p(X_{k-1}, \cdot)$, where $p(v, u)$ is the transition probability of $(X_n)_{n=0}^{\infty}$ defined in (1). By a direct calculation, we have, for $k \in \mathbb{N}$,

$$f_i(X_{k-1}) = \mathrm{E}(|X_k^i| - |X_{k-1}^i| \mid \mathcal{F}_{k-1}).$$

Therefore, $\{|X_k^i| - |X_{k-1}^i| - f_i(X_{k-1})\}_{k\geq 1}$ is an $\mathcal{F}_k$-adapted martingale-difference sequence, and so is

$$\left\{\xi_{n,k}^i := \frac{1}{\sqrt{n}}(|X_k^i| - |X_{k-1}^i| - f_i(X_{k-1}))\right\}_{k\geq 1} \tag{6}$$

for any $n \in \mathbb{N}$. Define $M^n = (M_t^n)_{t\geq 0}$ and $A_n = (A_n(t))_{t\geq 0}$ as follows:

$$M_t^n = (M_t^{n,1}, \ldots, M_t^{n,d}) = \sum_{k=1}^{\lfloor nt\rfloor} \xi_{n,k},$$

$$A_n(t) = (A_n^{i,j}(t))_{1\leq i,j\leq d} = \sum_{k=1}^{\lfloor nt\rfloor} (\xi_{n,k}^i \xi_{n,k}^j)_{1\leq i,j\leq d}, \qquad t \geq 0. \tag{7}$$

Then each $(M_t^{n,i})_{t\geq 0}$ is an $\mathcal{F}_{\lfloor nt\rfloor}$-martingale with $M_0^{n,i} = 0$, and, further, each M^n is an $\mathbb{R}^d$-valued $\mathcal{F}_{\lfloor nt\rfloor}$-martingale with $M_0 = \mathbf{0}$.

Note that, for any $t > 0$,

$$|M_t^n - M_{t-}^n| \leq |\xi_{n,\lfloor nt\rfloor}| = \sum_{i=1}^{d} |\xi_{n,\lfloor nt\rfloor}^i| \leq \frac{2d}{\sqrt{n}},$$

and for any $T \in (0, \infty)$,

$$\lim_{n\to\infty} \mathrm{E}\Big[\sup_{t\leq T} |M_t^n - M_{t-}^n|\Big] \leq \lim_{n\to\infty} \frac{2d}{\sqrt{n}} = 0. \tag{8}$$

By [9, Chapter 7, Theorem 1.4], to show the convergence of (5) it suffices to prove that

$$A_n^{i,j}(t) = [M_\cdot^{n,i}, M_\cdot^{n,j}]_t \tag{9}$$

and

$$\lim_{n\to\infty} A_n(t) = t\Sigma, \tag{10}$$

where $[M_\cdot^{n,i}, M_\cdot^{n,j}]$ is the cross-variation process of $M_\cdot^{n,i}$ and $M_\cdot^{n,j}$.

We first prove (9). Fix any $1 \leq i, j \leq d$ and $0 \leq s < t < \infty$. By the martingale property,

$$\mathrm{E}[M_s^{n,j}(M_t^{n,i} - M_s^{n,i}) \mid \mathcal{F}_{\lfloor ns\rfloor}] = M_s^{n,j}\mathrm{E}[M_t^{n,i} - M_s^{n,i} \mid \mathcal{F}_{\lfloor ns\rfloor}] = 0,$$

and similarly $\mathrm{E}[M_s^{n,i}(M_t^{n,j} - M_s^{n,j}) \mid \mathcal{F}_{\lfloor ns\rfloor}] = 0$. Additionally,

$$\begin{aligned}
&\mathrm{E}[(M_t^{n,i} - M_s^{n,i})(M_t^{n,j} - M_s^{n,j}) \mid \mathcal{F}_{\lfloor ns\rfloor}] \\
&= \mathrm{E}\left[\sum_{k=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \xi_{n,k}^i \xi_{n,k}^j \,\middle|\, \mathcal{F}_{\lfloor ns\rfloor}\right] + \mathrm{E}\left[\sum_{\lfloor ns\rfloor+1\leq r<k\leq\lfloor nt\rfloor} \xi_{n,k}^i \xi_{n,r}^j \,\middle|\, \mathcal{F}_{\lfloor ns\rfloor}\right] \\
&\quad + \mathrm{E}\left[\sum_{\lfloor ns\rfloor+1\leq r<k\leq\lfloor nt\rfloor} \xi_{n,k}^j \xi_{n,r}^i \,\middle|\, \mathcal{F}_{\lfloor ns\rfloor}\right]
\end{aligned}$$

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$$
\begin{aligned}
&= \mathrm{E}\Bigg[\sum_{k=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \xi^i_{n,k}\xi^j_{n,k} \;\Bigg|\; \mathcal{F}_{\lfloor ns\rfloor}\Bigg] + \sum_{\lfloor ns\rfloor+1\le r<k\le\lfloor nt\rfloor} \mathrm{E}[\mathrm{E}[\xi^i_{n,k}\xi^j_{n,r} \mid \mathcal{F}_r] \mid \mathcal{F}_{\lfloor ns\rfloor}] \\
&\quad + \sum_{\lfloor ns\rfloor+1\le r<k\le\lfloor nt\rfloor} \mathrm{E}[\mathrm{E}[\xi^j_{n,k}\xi^i_{n,r} \mid \mathcal{F}_r] \mid \mathcal{F}_{\lfloor ns\rfloor}] \\
&= \mathrm{E}\Bigg[\sum_{k=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \xi^i_{n,k}\xi^j_{n,k} \;\Bigg|\; \mathcal{F}_{\lfloor ns\rfloor}\Bigg] \\
&= \mathrm{E}[A^{i,j}_n(t) - A^{i,j}_n(s) \mid \mathcal{F}_{\lfloor ns\rfloor}].
\end{aligned}
$$

Since

$$
\begin{aligned}
&M^{n,i}_t M^{n,j}_t - A^{i,j}_n(t) \\
&\qquad = M^{n,i}_s M^{n,j}_s - A^{i,j}_n(s) + M^{n,i}_s(M^{n,j}_t - M^{n,j}_s) + M^{n,j}_s(M^{n,i}_t - M^{n,i}_s) \\
&\qquad\qquad + (M^{n,i}_t - M^{n,i}_s)(M^{n,j}_t - M^{n,j}_s) - (A^{i,j}_n(t) - A^{i,j}_n(s)),
\end{aligned}
$$

we see that

$$
\mathrm{E}[M^{n,i}_t M^{n,j}_t - A^{i,j}_n(t) \mid \mathcal{F}_{\lfloor ns\rfloor}] = M^{n,i}_s M^{n,j}_s - A^{i,j}_n(s),
$$

which implies (9).

It remains to prove (10). Recall from the introduction that $(X_n)_{n=0}^{\infty}$ visits the union of axial hyperplanes $\mathcal{X}$ defined by (3) finitely many times. Thus, for large enough k, X_{k-1} belongs to one of the 2^d open orthants, and hence, for any $1 \le i \ne j \le d$, $f_i(X_{k-1}) = f_j(X_{k-1}) = \frac{1-\lambda}{d(1+\lambda)}$. As a consequence, we have

$$
\frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor}(|X^j_k| - |X^j_{k-1}|) = \frac{1}{n}(|X^j_{\lfloor nt\rfloor}| - |X^j_0|) \to \frac{(1-\lambda)t}{d(1+\lambda)},
$$

which also holds with j replaced by i. Recall the definition of $\xi^i_{n,k}$ from (6). We have that

$$
\begin{aligned}
\sum_{k=1}^{\lfloor nt\rfloor} \xi^i_{n,k}\xi^j_{n,k} &= \sum_{k=1}^{\lfloor nt\rfloor} \frac{(|X^i_k| - |X^i_{k-1}| - f_i(X_{k-1}))}{\sqrt{n}} \frac{(|X^j_k| - |X^j_{k-1}| - f_j(X_{k-1}))}{\sqrt{n}} \\
&= \frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor}\{(|X^i_k| - |X^i_{k-1}|)(|X^j_k| - |X^j_{k-1}|) - (|X^i_k| - |X^i_{k-1}|) f_j(X_{k-1}) \\
&\qquad - (|X^j_k| - |X^j_{k-1}|) f_i(X_{k-1}) + f_i(X_{k-1}) f_j(X_{k-1})\} \\
&= \frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor}\{-(|X^i_k| - |X^i_{k-1}|) f_j(X_{k-1}) - (|X^j_k| - |X^j_{k-1}|) f_i(X_{k-1}) \\
&\qquad + f_i(X_{k-1}) f_j(X_{k-1})\} \\
&\to -\frac{(1-\lambda)^2}{d^2(1+\lambda)^2} t.
\end{aligned} \tag{11}
$$

In the second equality, we used the fact that X^i and X^j with $i \ne j$ cannot jump together to ensure that $(|X^i_k| - |X^i_{k-1}|)(|X^j_k| - |X^j_{k-1}|) = 0$.

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On the other hand, for any fixed $1 \le i \le d$, construct the following martingale-difference sequence $(\zeta^i_{n,k})_{k\ge 1}$:

$$\zeta^i_{n,k} = (\sqrt{n}\xi^i_{n,k})^2 - \mathrm{E}[(\sqrt{n}\xi^i_{n,k})^2 \mid \mathcal{F}_{k-1}], \qquad k \in \mathbb{N}.$$

Then, for any $1 \le k < \ell < \infty$, $\mathrm{E}[\zeta^i_{n,k}] = \mathrm{E}[\zeta^i_{n,\ell}] = 0$ and

$$\mathrm{E}[\zeta^i_{n,k}\zeta^i_{n,\ell}] = \mathrm{E}[\zeta^i_{n,k}\mathrm{E}[\zeta^i_{n,\ell} \mid \mathcal{F}_{\ell-1}]] = 0,$$

which implies that $(\zeta^i_{n,k})_k$ is a sequence of uncorrelated random variables. Notice that, for any $k \in \mathbb{N}$, $|\zeta^i_{n,k}| \le 4$ and hence $\mathrm{Var}(\zeta^i_{n,k}) \le 16$. By the strong law of large numbers for uncorrelated random variables ([18, Theorem 13.1]), we have that almost surely, as $n \to \infty$,

$$\frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor} \zeta^i_{n,k} = \sum_{k=1}^{\lfloor nt\rfloor}\{(\xi^i_{n,k})^2 - \mathrm{E}[(\xi^i_{n,k})^2 \mid \mathcal{F}_{k-1}]\} \to 0. \tag{12}$$

Due to (4) and (6), almost surely, as $n \to \infty$,

$$\begin{aligned}
\sum_{k=1}^{\lfloor nt\rfloor} \mathrm{E}[(\xi^i_{n,k})^2 \mid \mathcal{F}_{k-1}] &= \sum_{k=1}^{\lfloor nt\rfloor} \mathrm{E}\bigg[\bigg(\frac{|X^i_k| - |X^i_{k-1}| - f_i(X_{k-1})}{\sqrt{n}}\bigg)^2 \,\bigg|\, \mathcal{F}_{k-1}\bigg] \\
&= \frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor}\{\mathrm{E}[(|X^i_k| - |X^i_{k-1}|)^2 \mid \mathcal{F}_{k-1}] \\
&\qquad - 2\mathrm{E}[(|X^i_k| - |X^i_{k-1}|)f_i(X^i_{k-1}) \mid \mathcal{F}_{k-1}] \\
&\qquad + \mathrm{E}[(f_i(X^i_{k-1}))^2 \mid \mathcal{F}_{k-1}]\} \\
&= \frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor}\{\mathrm{E}[(|X^i_k| - |X^i_{k-1}|)^2 \mid \mathcal{F}_{k-1}] - (f_i(X^i_{k-1}))^2\} \\
&\to t\frac{1}{d} - t\frac{(1-\lambda)^2}{d^2(1+\lambda)^2}.
\end{aligned}$$

Together with (12), we have

$$\sum_{k=1}^{\lfloor nt\rfloor} (\xi^i_{n,k})^2 \to t\frac{1}{d} - t\frac{(1-\lambda)^2}{d^2(1+\lambda)^2}. \tag{13}$$

By the definition of A_n in (7), (11), and (13), we complete the proof of (10).

In view of (8), (9), and (10), we have from [9, Chapter 7, Theorem 1.4] that, on $D([0, \infty), \mathbb{R}^d)$, as $n \to \infty$, M^n converges in distribution to an $\mathbb{R}^d$-valued process $Y : Y_t = \frac{1}{\sqrt{d}}[I - \frac{1-\rho_\lambda}{d}E]W_t$ with independent Gaussian increments such that $Y_0 = \mathbf{0}$ and $Y_{t+s} - Y_s$ has the law $\mathcal{N}(\mathbf{0}, t\Sigma)$ for any $0 \le s, t < \infty$. Then it is easy to verify that on $D([0, \infty), \mathbb{R}^d)$,

$$\bigg(\frac{(|X^1_{\lfloor nt\rfloor}|, \ldots, |X^d_{\lfloor nt\rfloor}|) - nvt}{\sqrt{n}}\bigg)_{t\ge 0}$$

converges in distribution to Y, which completes the proof. □

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3. LDP for scaled reflected RW$_\lambda$ with $\lambda \in [0, 1)$

In this section we prove the LDP for the sequence $\{\frac{1}{n}(|X_n^1|, \dots, |X_n^d|)\}_{n=0}^{\infty}$. Before stating our main result in this section, we recall the definition of LDP from [7, Chapter 1].

For $n \geq 1$, let W_n be a random variable in $\mathbb{R}^d$, with distribution μ_n. We say that the sequence $\{W_n\}_{n=0}^{\infty}$ satisfies the LDP with a good rate function I if the following inequalities hold:

(i) For any closed set $F \subseteq \mathbb{R}^d$,

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mu_n(F) \leq -\inf_{x\in F} I(x).$$

(ii) For any open set $G \subseteq \mathbb{R}^d$,

$$\liminf_{n\to\infty} \frac{1}{n} \ln \mu_n(G) \geq -\inf_{x\in G} I(x).$$

Here, a function $I : \mathbb{R}^d \to [0, \infty]$ is said to be a good rate function if it is lower semicontinuous and if all the level sets $\{x \in \mathbb{R}^d : I(x) \leq \alpha\}$ for $\alpha \in [0, \infty)$ are compact.

Recall from [21] that $\rho_\lambda = 2\sqrt{\lambda}/(1+\lambda)$ is the spectral radius of $(X_n)_{n=0}^{\infty}$. Let $s_0 := s_0(\lambda) = \frac{1}{2}\ln\lambda$. For $s = (s_1, \dots, s_d)$ and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, define $(s, x) = \sum_{i=1}^d s_i x_i$ and

$$\psi(s) = N(s)\frac{\rho_\lambda}{d} + \frac{1}{d(1+\lambda)} \sum_{i=1}^{d} \{\lambda e^{-s_i} + e^{s_i}\}\, I_{\{s_i \geq s_0\}}, \quad \text{where } N(s) = \sum_{i=1}^{d} I_{\{s_i < s_0\}}. \tag{14}$$

Our main result in this section is as follows.

Theorem 3.1. (LDP.) *Let $\{X_n\}_{n=0}^{\infty}$ be a biased random walk on $\mathbb{Z}^d$ with parameter $\lambda \in [0, 1)$. We assume further that $\lambda > 0$ when $d = 1$. Then $\{\frac{1}{n}(|X_n^1|, \dots, |X_n^d|)\}_{n=0}^{\infty}$ satisfies the LDP with the good rate function given by*

$$\Lambda^*(x) = \sup_{s\in\mathbb{R}^d} \{(s, x) - \ln\psi(s)\}, \qquad x \in \mathbb{R}^d.$$

Our proof of Theorem 3.1 is based on the Gärtner–Ellis theorem (cf. [7, Theorem 2.3.6]), which will be recalled in Subsection 3.1. Then, in Subsection 3.2 we consider the limit for the logarithmic moment generating functions of $\frac{1}{n}(|X_n^1|, \dots, |X_n^d|)$, and we prove Theorem 3.1 in Subsection 3.3.

Remark 3.1.
(i) The effective domain of the rate function Λ^* will be described in Lemmas 3.7 and 3.8. We shall see in (22) that, for any $x = (x_1, \dots, x_d) \in \mathbb{R}_+^d$,

$$\Lambda^*(x) = \frac{1}{2}\sum_{i=1}^{d} x_i \ln\lambda - \ln\rho_\lambda + \overline{\Lambda^*}(x),$$

where

$$\overline{\Lambda^*}(x) = \sup_{y\in\mathbb{R}^d} \ln\left\{\frac{\exp\left[\sum_{i=1}^d y_i x_i\right]}{\frac{1}{2d}\sum_{i=1}^d (e^{-y_i} + e^{y_i})}\right\}, \qquad x \in \mathbb{R}^d,$$

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is the SRW rate function from the Cramér theorem. Therefore, the rate function Λ^* can be calculated explicitly when $d=1$, 2, and $\lambda\in(0,\ 1)$. More precisely, we have for $d=1$ that

$$\Lambda^*(x)=\begin{cases}\frac{x}{2}\ln\lambda-\ln\rho_\lambda+(1+x)\ln\sqrt{1+x}+(1-x)\ln\sqrt{1-x} & x\in[0,\ 1],\\ \infty & \text{otherwise,}\end{cases}$$

and for $d=2$ that

$$\Lambda^*(x)=\begin{cases}\frac{1}{2}(x_1+x_2)\ln\lambda-\ln\rho_\lambda+\overline{\Lambda^*}(x) & (x_1,x_2)\in\mathcal{D}_{\Lambda^*},\\ \infty & \text{otherwise,}\end{cases}$$

where $\mathcal{D}_{\Lambda^*}=\{x=(x_1,\ x_2)\in\mathbb{R}_+^2 : x_1+x_2\le 1\}$ and, for $x\in\mathcal{D}_{\Lambda^*}$,

$$\begin{aligned}\overline{\Lambda^*}(x)=x_1\ln&\left[\frac{2x_1+\sqrt{x_1^2-x_2^2+1}}{\sqrt{(x_1^2-x_2^2)^2+1-2(x_1^2+x_2^2)}}\right]\\ &+x_2\ln\left[\frac{2x_2+\sqrt{x_2^2-x_1^2+1}}{\sqrt{(x_1^2-x_2^2)^2+1-2(x_1^2+x_2^2)}}\right]\\ &-\ln\left[\frac{\sqrt{x_1^2-x_2^2+1}+\sqrt{x_2^2-x_1^2+1}}{\sqrt{(x_1^2-x_2^2)^2+1-2(x_1^2+x_2^2)}}\right]+\ln 2.\end{aligned}$$

When $d\ge 3$ and $\lambda\in(0,\ 1)$ we cannot calculate Λ^* to get an explicit expression; this remains an open problem.

(ii) Assume $\lambda\in[0,\ 1)$ if $d\ge 2$ and $\lambda\in(0,\ 1)$ if $d=1$. The following sample path large deviation principle (Mogulskii type theorem) holds for the reflected RW_λ starting at $\mathbf{0}$. In fact, for any $0\le t\le 1$, let

$$\begin{aligned}Y_n(t)&=\frac{1}{n}(|X^1_{\lfloor nt\rfloor}|,\ \ldots,\ |X^d_{\lfloor nt\rfloor}|),\\ \widetilde{Y}_n(t)&=Y_n(t)+\left(t-\frac{\lfloor nt\rfloor}{n}\right)[(|X^1_{\lfloor nt\rfloor+1}|,\ \ldots,\ |X^d_{\lfloor nt\rfloor+1}|)-(|X^1_{\lfloor nt\rfloor}|,\ \ldots,\ |X^d_{\lfloor nt\rfloor}|)].\end{aligned}$$

Write $L_\infty([0,\ 1])$ for the $\mathbb{R}^d$-valued L_∞ space on interval $[0,\ 1]$, and μ_n (resp. $\widetilde{\mu}_n$) for the law of $Y_n(\cdot)$ (resp. $\widetilde{Y}_n(\cdot)$) in $L_\infty([0,\ 1])$. From [7, Lemma 5.1.4], μ_n and $\tilde{\mu}_n$ are exponentially equivalent. Denote by $\mathcal{AC}'$ the space of nonnegative absolutely continuous $\mathbb{R}^d$-valued functions ϕ on $[0,\ 1]$ such that

$$||\phi||\le 1,\qquad \phi(0)=\mathbf{0},$$

where $||\cdot||$ is the supremum norm. Let

$$C_{\mathbf{0}}([0,\ 1])=\{f:[0,\ 1]\to\mathbb{R}^d\ |\ f\text{ is continuous},\ f(0)=\mathbf{0}\},$$

and

$$\mathcal{K}=\left\{f\in C_{\mathbf{0}}([0,\ 1])\ |\ \sup_{0\le s<t\le 1}\frac{|f(t)-f(s)|}{t-s}\le 1,\ \sup_{0\le t\le 1}|f(t)|\le 1\right\}.$$

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By the Arzelà–Ascoli theorem, $\mathcal{K}$ is compact in $(C_{\mathbf{0}}([0,1]), \|\cdot\|)$. Note that each $\widetilde{\mu}_n$ concentrates on $\mathcal{K}$, which implies exponential tightness in $C_0[0,1]$ equipped with supremum topology. Define $p_j : L^\infty[0,1] \to (\mathbb{R}^d)^{|j|}$ for any $j \in J$, where J denotes all ordered finite sets with each element taking a value in $(0, 1]$, as follows:

$$p_j(f) = (f(t_1), f(t_2), \ldots, f(t_{|j|})),$$

where $j = (t_1, t_2, \ldots, t_{|j|})$ and $0 < t_1 < t_2 < \cdots < t_{|j|} \le 1$. Given that the finite-dimensional LDPs for $\{\mu_n \circ p_j^{-1}\}$ in $(\mathbb{R}^d)^{|j|}$ follow from Theorem 3.1, $\{\mu_n \circ p_j^{-1}\}$ and $\{\tilde{\mu}_n \circ p_j^{-1}\}$ are exponentially equivalent, and the $\{\tilde{\mu}_n\}$ satisfy the LDP in $L_\infty[0,1]$ with pointwise convergence topology by the Dawson–Gärtner theorem [7]. Together with exponential tightness in $C_0[0,1]$, we can lift the LDP of $\{\tilde{\mu}_n\}$ to $L_\infty[0,1]$ with supremum topology with the good rate function

$$I(\phi) = \begin{cases} \int_0^1 \Lambda^*(\dot{\phi}(t))\,\mathrm{d}t & \text{if } \phi \in \mathcal{AC}', \\ \infty & \text{otherwise.} \end{cases}$$

The same holds for $\{\mu_n\}$ due to exponential equivalence. The proof of the above sample path LDP is similar to that of [7, Theorem 5.1.2].

3.1. Gärtner–Ellis theorem

Consider a sequence of probability measures $\{\mu_n\}_{n=0}^\infty$ on $\mathbb{R}^d$. Let $\Lambda_n(s)$ be the logarithmic moment generating function of μ_n, that is,

$$\Lambda_n(s) = \ln \int_{\mathbb{R}^d} \mathrm{e}^{(s,x)} \mu_n(\mathrm{d}x), \qquad s = (s_1, \ldots, s_d) \in \mathbb{R}^d.$$

Assume that, for each $s \in \mathbb{R}^d$, the limit

$$\Lambda(s) = \lim_{n\to\infty} \frac{1}{n} \Lambda_n(ns) \tag{15}$$

exists as a number in $\mathbb{R} \cup \{\infty\}$. Let $\mathcal{D}_\Lambda = \{s \in \mathbb{R}^d : \Lambda(s) < \infty\}$. The Fenchel–Legendre transform Λ^* of Λ is defined by

$$\Lambda^*(x) = \sup_{s\in\mathbb{R}^d} \{(s,x) - \Lambda(s)\}, \qquad x \in \mathbb{R}^d,$$

with domain $\mathcal{D}_{\Lambda^*} = \{x \in \mathbb{R}^d : \Lambda^*(x) < \infty\}$. We say $y \in \mathbb{R}^d$ is an exposed point of Λ^* if, for some $s \in \mathbb{R}^d$ and all $x \ne y$,

$$(y,s) - \Lambda^*(y) > (x,s) - \Lambda^*(x). \tag{16}$$

An s satisfying (16) is called an exposing hyperplane. We recall the Gärtner–Ellis theorem from [7, Theorem 2.3.6].

Theorem 3.2. (Gärtner–Ellis theorem.) *Suppose that the function Λ in (15) is well defined, and that $\mathbf{0} \in \mathcal{D}_\Lambda^o$, the interior of $\mathcal{D}_\Lambda$.*

(a) *For any closed set $F \subseteq \mathbb{R}^d$,*

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mu_n(F) \le -\inf_{x\in F} \Lambda^*(x).$$

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(b) *For any open set $G \subseteq \mathbb{R}^d$,*

$$\liminf_{n\to\infty} \frac{1}{n} \ln \mu_n(G) \geq - \inf_{x\in G\cap\mathbb{F}} \Lambda^*(x),$$

where $\mathbb{F}$ is the set of exposed points of Λ^* whose exposing hyperplane belongs to $\mathcal{D}^o_\Lambda$.

3.2. Logarithmic moment generating function

To apply the Gärtner–Ellis theorem, we need to prove the convergence of the rescaled logarithmic moment generating functions defined as

$$\Lambda_n(s, x) = \ln \mathrm{E}_x \left[\exp\left\{\left(ns, \left(\frac{|X_n^1|}{n}, \ldots, \frac{|X_n^d|}{n}\right)\right)\right\}\right] = \ln \mathrm{E}_x \left[\exp\left\{\sum_{i=1}^d s_i |X_n^i|\right\}\right].$$

Proposition 3.1. *For every $s \in \mathbb{R}^d$ and $x \in \mathbb{Z}^d$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \Lambda_n(s,\ x) = \ln \psi(s), \tag{17}$$

where ψ is defined in (14).

The rest of this subsection is devoted to the proof of Proposition 3.1. Note that, for $s \in \mathbb{R}^d$ with $s_i \geq s_0$ for $1 \leq i \leq d$, $\psi(s)$ coincides with the logarithmic moment generating function of the drifted random walk with step distribution given by (2). The main idea to prove Proposition 3.1 is to compare the transition probabilities of the biased random walk $\{X_n\}_{n=0}^\infty$ with those of the drifted random walk $\{Z_n\}_{n=0}^\infty$; see Lemmas 3.2 and 3.3. As a consequence, $\Lambda_n(s, x)$ is bounded above and below by the logarithmic moment generating function of Z_n for $s \in \mathbb{R}^d$ with $s_i \geq s_0$ for $1 \leq i \leq d$, and the convergence follows. However, for s in other regions we need more precise estimates for the lower bound of $\Lambda_n(s, x)$; see Lemmas 3.4 and 3.6.

We start the proof of Proposition 3.1 with the following lemma, which shows that the limit (17) is independent of the starting point x of the biased random walk.

Lemma 3.1. *For each $s \in \mathbb{R}^d$ and $x \in \mathbb{Z}^d$, there exists a constant $C > 0$ such that*

$$\Lambda_{n-|x|}(s,\ x) - C \leq \Lambda_n(s,\ \mathbf{0}) \leq \Lambda_{n+|x|}(s,\ x) + C, \qquad n > |x|. \tag{18}$$

Proof. By the Markov property, we have

$$\begin{aligned}\mathrm{E}_{\mathbf{0}}\left[\exp\left\{\sum_{i=1}^d s_i|X_n^i|\right\}\right] &\geq \mathrm{E}_0\left[I_{\{X_{|x|}=x\}} \exp\left\{\sum_{i=1}^d s_i|X_n^i|\right\}\right] \\ &\geq \mathrm{P}_0(X_{|x|}=x)\, \mathrm{E}_x\left[\exp\left\{\sum_{i=1}^d s_i|X_{n-|x|}^i|\right\}\right],\end{aligned}$$

and the first inequality in (18) follows. The second inequality is proved similarly. □

Recall the definition of the drifted random walk $\{Z_n = (Z_n^1, \ldots, Z_n^d)\}_{n=0}^\infty$ on $\mathbb{Z}^d$ by (2).

Lemma 3.2. *We have that, for $k \in \mathbb{Z}_+^d$,*

$$\mathrm{P}_{\mathbf{0}}(X_n = k) \leq \mathrm{P}(Z_n = k \mid Z_0 = \mathbf{0}), \qquad n \geq 0.$$

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Proof. For $x, k \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, let $\Gamma_n(x, k)$ be the set of all nearest-neighbor paths in $\mathbb{Z}^d$ from x to k with length n. For a path $\gamma = \gamma_0\gamma_1 \cdots \gamma_n \in \Gamma_n(x, k)$, let

$$p(\gamma) = p(x, \gamma_1)p(\gamma_1, \gamma_2) \cdots p(\gamma_{n-1}, k).$$

Consider the first n steps of RW_λ along the path $\gamma \in \Gamma_n(0, k)$. Note that for each $0 \le i \le n-1$, $p(\gamma_i, \gamma_{i+1})$ is either $\frac{1}{d+m+(d-m)\lambda}$ or $\frac{\lambda}{d+m+(d-m)\lambda}$, with $m = \kappa(\gamma_i)$. The total number of terms of the forms $\frac{1}{d+m+(d-m)\lambda}$ (resp. $\frac{\lambda}{d+m+(d-m)\lambda}$) is exactly $\frac{n+|k|}{2}$ (resp. $\frac{n-|k|}{2}$). Note that $d + m + (d - m)\lambda \ge d(1 + \lambda)$. Therefore, we have, for $\gamma \in \Gamma_n(\mathbf{0}, k)$,

$$p(\gamma) \le \left(\frac{1}{d(1+\lambda)}\right)^{\frac{n+|k|}{2}} \left(\frac{\lambda}{d(1+\lambda)}\right)^{\frac{n-|k|}{2}}.$$

As a consequence,

$$\begin{aligned} \mathrm{P}_{\mathbf{0}}(X_n = k) &= \sum_{\gamma \in \Gamma_n(\mathbf{0}, k)} p(\gamma) \le \sum_{\gamma \in \Gamma_n(\mathbf{0}, k)} \left(\frac{1}{d(1+\lambda)}\right)^{\frac{n+|k|}{2}} \left(\frac{\lambda}{d(1+\lambda)}\right)^{\frac{n-|k|}{2}} \\ &= \mathrm{P}(Z_n = k \mid Z_0 = \mathbf{0}). \end{aligned}$$

□

Lemma 3.3. *For every $k \in \mathbb{Z}_+^d$ and $z \in \mathbb{Z}_+^d \setminus \mathcal{X}$, we have that*

$$\mathrm{P}_z(X_n = k) \ge n^{-d}\mathrm{P}(Z_n = k \mid Z_0 = z).$$

Proof. Recall that $\mathcal{X}$ denotes the union of all axial hyperplanes of $\mathbb{Z}^d$. Define

$$\sigma = \inf\{n : X_n \in \mathcal{X}\}, \qquad \tau = \inf\{n : Z_n \in \mathcal{X}\}.$$

Starting at $z \in \mathbb{Z}_+^d \setminus \mathcal{X}$, the process $(X_n)_{0 \le n \le \sigma}$ has the same distribution as the drifted random walk $(Z_n)_{0 \le n \le \tau}$. Then we have, for $k \in \mathbb{Z}_+^d$,

$$\mathrm{P}_z(X_n = k) \ge \mathrm{P}(Z_n = k,\ n \le \tau \mid Z_0 = z). \tag{19}$$

For $\alpha, \beta \in \mathbb{Z}$ and $n \in \mathbb{N}$, denote by $P_{n,\beta}(\alpha)$ the number of paths $\gamma = \gamma_0\gamma_1 \cdots \gamma_n$ in $\mathbb{Z}$ with $\gamma_0 = \alpha$ and $\gamma_n = \beta$, and by $Q_{n,\beta}(\alpha)$ the number of those paths with the additional property that $\gamma_i \ge \alpha \wedge \beta$ for $0 \le i \le n$ if $\alpha = \beta$ and $\gamma_i > \alpha \wedge \beta$ for $1 \le i \le n-1$ if $\alpha \ne \beta$. Assume $P_{0,\alpha}(\alpha) = Q_{0,\alpha}(\alpha) = 1$ for convenience in this lemma and next lemma. By the Ballot theorem ([8, Theorem 4.3.2]) and the property of the Catalan number, we have, for $n > 0$,

$$Q_{n,\beta}(\alpha) \ge \frac{|\alpha - \beta| \vee 1}{n} P_{n,\beta}(\alpha). \tag{20}$$

For $k = (k_1, \ldots, k_d) \in \mathbb{Z}_+^d$ and $z = (z_1, \ldots, z_d) \in \mathbb{Z}_+^d \setminus \mathcal{X}$, write $a = \frac{n+|k|-|z|}{2}$ and $b = \frac{n-|k|+|z|}{2}$. By (20), we obtain that

$$\begin{aligned} &\mathrm{P}(Z_n = k, n \le \tau \mid Z_0 = z) \\ &\quad \ge \sum_m \binom{n}{m_1, \ldots, m_d} d^{-n} \left(\frac{\lambda}{1+\lambda}\right)^b \left(\frac{1}{1+\lambda}\right)^a \prod_{j=1}^d Q_{m_j, k_j}(z_j) \\ &\quad \ge \sum_m \binom{n}{m_1, \ldots, m_d} d^{-n} \left(\frac{\lambda}{1+\lambda}\right)^b \left(\frac{1}{1+\lambda}\right)^a \prod_{j=1}^d \frac{|k_j - z_j| \vee 1}{m_j} P_{m_j, k_j}(z_j) \end{aligned}$$

$$\geq n^{-d}\sum_{m}\binom{n}{m_1,\,\ldots,\,m_d}d^{-n}\left(\frac{\lambda}{1+\lambda}\right)^b\left(\frac{1}{1+\lambda}\right)^a\prod_{j=1}^{d}P_{m_j,\,k_j}(z_j)$$
$$= n^{-d}\mathrm{P}(Z_n = k \mid Z_0 = z),$$

where the sum is over all tuples $m=(m_1,\,\ldots,\,m_d)\in\mathbb{Z}^d$ such that $|m|=n$ and $m_j\geq|k_j-z_j|$ for $1\leq j\leq d$. In the second inequality we should replace $(|k_j-z_j|\vee 1)/m_j$ by 1 when $m_j=0$. Together with (19), this completes the proof. □

Recall that $s_0=\frac{1}{2}\ln\lambda$. For fixed $s\in\mathbb{R}^d$, let $I_1=\{1\leq i\leq d: s_i<s_0\}$ and $I_2=\{1,\,\ldots,\,d\}\setminus I_1$. Then $N(s)=|I_1|$ from (14). Define $s^{I_1}=(s_i)_{i\in I_1}$; s^{I_2}, $Z_n^{I_1}$, and $Z_n^{I_2}$ are defined similarly.

Lemma 3.4. *Let* $y_0=\mathbf{0}^{I_1}$ *if n is even and* $y_0=(1,\,0,\,\ldots,\,0)\in\mathbb{R}^{N(s)}$ *otherwise. For any* $z\in\mathbb{Z}^d$ *and* $s\in\mathbb{R}^d$*, we have that*

$$\mathrm{E}\left[\exp\left\{\sum_{i=1}^{d}s_iZ_n^i\right\}\prod_{i=1}^{d}I_{\{Z_n^i\geq z_i\}}\mid Z_0=z\right]$$
$$\geq 2^{-|I_2|}(\mathrm{e}^{s_1}\wedge 1)\mathrm{E}\left[\exp\left\{\sum_{i\in I_2}s_iZ_n^i\right\}I_{\{Z_n^{I_1}=z^{I_1}+y_0\}}\mid Z_0=z\right].$$

Proof. Without loss of generality, we assume that $z=\mathbf{0}$. As in the proof of Lemma 3.3, for $\alpha,\,\beta\in\mathbb{Z}$ we denote by $P_{n,\,\beta}(\alpha)$ the number of nearest-neighbor paths in $\mathbb{Z}$ from α to β. Then we have, for $k\in\mathbb{Z}^d$,

$$\mathrm{P}(Z_n=k\mid Z_0=\mathbf{0})$$
$$=\sum_{m}\binom{n}{m_1,\,\ldots,\,m_d}d^{-n}\left(\frac{\lambda}{1+\lambda}\right)^{\sum_{i=1}^{d}\frac{m_i-k_i}{2}}\left(\frac{1}{1+\lambda}\right)^{\sum_{i=1}^{d}\frac{m_i+k_i}{2}}\prod_{j=1}^{d}P_{m_j,\,k_j}(0)$$
$$=\lambda^{\frac{n-\sum_{i=1}^{d}k_i}{2}}\sum_{m}\binom{n}{m_1,\,\ldots,\,m_d}\left(\frac{1}{d(1+\lambda)}\right)^n\prod_{j=1}^{d}P_{m_j,\,k_j}(0), \tag{21}$$

where the sum is over all tuples $m=(m_1,\,\ldots,\,m_d)\in\mathbb{Z}^d$ such that $|m|=n$ and $m_j\geq|k_j|$ for $1\leq j\leq d$. Note that

$$\mathrm{E}\left[\exp\left\{\sum_{i=1}^{d}s_iZ_n^i\right\}I_{\{Z_n\in\mathbb{Z}_+^d\}}\mid Z_0=\mathbf{0}\right]$$
$$\geq(\mathrm{e}^{s_1}\wedge 1)\mathrm{E}\left[\exp\left\{\sum_{i\in I_2}s_iZ_n^i\right\}I_{\{Z_n^{I_1}=y_0,\,Z_n^{I_2}\in\mathbb{Z}_+^{I_2}\}}\mid Z_0=\mathbf{0}\right].$$

For $\varepsilon=(\varepsilon_i)_{i\in I_2}\in\{-1,\;1\}^{I_2}$, let

$$\mathcal{O}_\varepsilon=\{y=(y_i)_{i\in I_2}\in\mathbb{Z}^{I_2}:\varepsilon_iy_i\geq 0,\;\; i\in I_2\}.$$

Note that for every $k\in\mathbb{Z}^d$, $P_{m_j,\,k_j}(0)=P_{m_j,\,\varepsilon_jk_j}(0)$. Therefore, by (21),

$$\mathrm{P}(Z_n=k\mid Z_0=\mathbf{0})=\lambda^{\sum_{i=1}^{d}\frac{1}{2}(\varepsilon_i-1)k_i}\mathrm{P}(Z_n=(\varepsilon_1k_1,\,\ldots,\,\varepsilon_dk_d)\mid Z_0=0).$$

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Applying the fact that $e^{s_i}\lambda^{-1/2} \geq 1$ for $i \in I_2$, we obtain that, for every $\varepsilon \in \{-1,\ 1\}^{I_2}$,

$$\begin{aligned}
&\mathrm{E}\bigg[\exp\bigg\{\sum_{i\in I_2} s_i Z_n^i\bigg\} I_{\{Z_n^{I_1}=y_0,\, Z_n^{I_2}\in\mathbb{Z}_+^{I_2}\}} \mid Z_0=\mathbf{0}\bigg] \\
&\geq \sum_{k\in\mathbb{Z}_+^{I_2}} (e^{s_i}\lambda^{-1/2})^{\sum_{i\in I_2}(1-\varepsilon_i)k_i} \\
&\qquad \times \exp\bigg\{\sum_{i\in I_2} s_i\varepsilon_i k_i\bigg\} \mathrm{P}(Z_n^{I_1}=y_0,\ Z_n^{I_2}=(\varepsilon_i k_i)_{i\in I_2} \mid Z_0=\mathbf{0}) \\
&\geq \mathrm{E}\bigg[\exp\bigg\{\sum_{i\in I_2} s_i Z_n^i\bigg\} I_{\{Z_n^{I_1}=y_0,Z_n^{I_2}\in\mathcal{O}_\varepsilon\}} \mid Z_0=\mathbf{0}\bigg].
\end{aligned}$$

By taking sum over all $\varepsilon \in \{-1,\ 1\}^{I_2}$, we complete the proof of this lemma. □

Recall the definition of ψ from (14).

Lemma 3.5. *For every $s \in \mathbb{R}^d$ and $x \in \mathbb{Z}^d$ we have that*

$$\limsup_{n\to\infty} \frac{1}{n}\Lambda_n(s,\ x) \leq \ln \psi(s).$$

Proof. By Lemma 3.1, we only need to consider the case $x = \mathbf{0}$. Recall that $s_0 := \frac{1}{2}\ln\lambda$. For $s \in \mathbb{R}^d$, define $\tilde{s} := (s_1 \vee s_0, \ldots, s_d \vee s_0)$, which leads to $\psi(s) = \psi(\tilde{s})$. Since $\Lambda_n(s,\ 0)$ is increasing in each coordinate s_i, we have that $\Lambda_n(s,\ 0) \leq \Lambda_n(\tilde{s},\ 0)$. Then, by Lemma 3.2,

$$\begin{aligned}
\mathrm{E}_{\mathbf{0}}\bigg[\exp\bigg\{\sum_{i=1}^d \tilde{s}_i |X_n^i|\bigg\}\bigg] &\leq 2^d \sum_{k\in\mathbb{Z}_+^d} \exp\bigg\{\sum_{i=1}^d \tilde{s}_i k_i\bigg\} \mathrm{P}(Z_n = k \mid Z_0 = \mathbf{0}) \\
&\leq 2^d \mathrm{E}\bigg[\exp\bigg\{\sum_{i=1}^d \tilde{s}_i Z_n^i\bigg\} \mid Z_0=\mathbf{0}\bigg] \\
&= 2^d \prod_{m=1}^n \mathrm{E}\bigg[\exp\bigg\{\sum_{i=1}^d \tilde{s}_i (Z_m^i - Z_{m-1}^i)\bigg\} \mid Z_0=\mathbf{0}\bigg] \\
&= 2^d \bigg(\sum_{i=1}^d \frac{\lambda e^{-\tilde{s}_i} + e^{\tilde{s}_i}}{d(1+\lambda)}\bigg)^n = 2^d(\psi(s))^n,
\end{aligned}$$

where we have used the fact that $\{Z_n - Z_{n-1}\}_{n\in\mathbb{N}}$ are mutually independent and identically distributed. Then the lemma follows immediately. □

Lemma 3.6. *For every $s \in \mathbb{R}^d$ and $x \in \mathbb{Z}^d$, we have that*

$$\liminf_{n\to\infty} \frac{1}{n}\Lambda_n(s,\ x) \geq \ln\psi(s).$$

Proof. It suffices to prove only for n even. We use the same notation as before Lemma 3.4. Fix $x \in \mathbb{Z}_+^d \setminus \mathcal{X}$. By Lemmas 3.3 and 3.4, we have

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$$\begin{aligned}
\mathrm{E}_x\bigg[\exp\bigg\{\sum_{i=1}^d s_i|X_{2n}^i|\bigg\}\bigg] &\geq n^{-d}\sum_{k\in\mathbb{Z}_+^d}\exp\bigg\{\sum_{i=1}^d s_ik_i\bigg\}\mathrm{P}(Z_{2n}=k\mid Z_0=x)\\
&= n^{-d}\mathrm{E}\bigg[\exp\bigg\{\sum_{i=1}^d s_iZ_{2n}^i\bigg\}I_{\{Z_{2n}\in\mathbb{Z}_+^d\}}\mid Z_0=x\bigg]\\
&\geq cn^{-d}\mathrm{E}\bigg[\exp\bigg\{\sum_{i\in I_2} s_iZ_{2n}^i\bigg\}I_{\{Z_{2n}^{I_1}=x^{I_1}\}}\mid Z_0=x\bigg]
\end{aligned}$$

for some constant $c>0$. Let $\Gamma_1(2n)$ be the number of nearest-neighbor paths of length $2n$ in $\mathbb{Z}^{I_1}$ from $\mathbf{0}^{I_1}$ to $\mathbf{0}^{I_1}$. Similarly, for $k\in\mathbb{Z}^{I_2}$, let $\Gamma_2(2n,\ k)$ be the number of paths of length $2n$ in $\mathbb{Z}^{I_2}$ from $\mathbf{0}^{I_2}$ to k. Then we have, for $k\in\mathbb{Z}^{I_2}$,

$$\mathrm{P}(Z_{2n}^{I_1}=x^{I_1},\ Z_{2n}^{I_2}=x^{I_2}+k\mid Z_0=x)=\sum_{m=0}^n\binom{2n}{2m}\frac{\lambda^{n-\sum_{i\in I_2}k_i}}{[d(1+\lambda)]^{2n}}\Gamma_1(2m)\Gamma_2(2n-2m,\ k).$$

Recall that $N(s)=|I_1|$. Let (W_n) be the drifted random walk in $\mathbb{Z}^{I_2}$ starting at $\mathbf{0}$, i.e. the step distribution is defined similarly to (2) with d replaced by $d-N(s)$. Since $\Gamma_1(2m)\geq cm^{-N(s)/2}(2N(s))^{2m}$, we obtain that

$$\begin{aligned}
&\mathrm{P}(Z_{2n}^{I_1}=x^{I_1},\ Z_{2n}^{I_2}=x^{I_2}+k\mid Z_0=x)\\
&\qquad\geq cn^{-N(s)/2}d^{-2n}\sum_{m=0}^n\binom{2n}{2m}(N(s)\rho_\lambda)^{2m}(d-N(s))^{2n-2m}\mathrm{P}(W_{2n-2m}=k).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathrm{E}_x\bigg[\exp\bigg\{\sum_{i=1}^d s_i|X_{2n}^i|\bigg\}\bigg]\\
&\geq cn^{-3d/2}d^{-2n}\sum_{m=0}^n\sum_{k\in\mathbb{Z}^{I_2}}\exp\bigg\{\sum_{i\in I_2}s_i(k_i+x_i)\bigg\}\binom{2n}{2m}(N(s)\rho_\lambda)^{2m}\\
&\qquad\qquad\times(d-N(s))^{2n-2m}\mathrm{P}(W_{2n-2m}=k)\\
&=cn^{-3d/2}d^{-2n}\sum_{m=0}^n\binom{2n}{2m}(N(s)\rho_\lambda)^{2m}(d-N(s))^{2n-2m}\mathrm{E}\bigg[\exp\bigg\{\sum_{i\in I_2}s_iW_{2n-2m}^i\bigg\}\bigg]\\
&=cn^{-3d/2}d^{-2n}\sum_{m=0}^n\binom{2n}{2m}(N(s)\rho_\lambda)^{2m}(d-N(s))^{2n-2m}\bigg(\frac{\sum_{i\in I_2}(\lambda\mathrm{e}^{-s_i}+\mathrm{e}^{s_i})}{(d-N(s))(1+\lambda)}\bigg)^{2n-2m}\\
&\geq\frac{c}{3}n^{-3d/2}(\psi(s))^{2n}.
\end{aligned}$$

The last inequality holds since, for any positive real number a, b,

$$\frac{\sum_{m=0}^n\binom{2n}{2m}a^{2n-2m}b^{2m}}{(a+b)^{2n}}\to\frac{1}{2},\qquad n\to\infty.$$

The proof is complete. □

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Now we are ready to complete the proof of Proposition 3.1.

Proof of Proposition 3.1. The upper bound for the limit in Equation (17) is proved in Lemma 3.5, and the lower bound is proved in Lemma 3.6. □

3.3. Proof of Theorem 3.1

For $s \in \mathbb{R}^d$, define $\Lambda(s) = \ln \psi(s)$ where ψ is given by (14). Let Λ^* be the Fenchel–Legendre transform of Λ. By the Gärtner–Ellis theorem and Proposition 3.1, to complete the proof of Theorem 3.1 it remains to study the effective domain and the set of exposed points of Λ^*.

Lemma 3.7. *For any $d \geq 1$ and $\lambda \in (0, 1)$, the effective domain of $\Lambda^*(\cdot)$ is*

$$\begin{aligned}\mathcal{D}_{\Lambda^*}(\lambda) &:= \{x \in \mathbb{R}^d : \Lambda^*(x) < \infty\}\\ &= \left\{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : \sum_{i=1}^{d} x_i \leq 1,\ 0 \leq x_i \leq 1,\ 1 \leq i \leq d\right\}.\end{aligned}$$

Furthermore, $\Lambda^*(\cdot)$ is strictly convex in $x = (x_1, \ldots, x_d) \in \mathcal{D}_{\Lambda^*}(\lambda)$ with $\sum_{i=1}^d x_i < 1$, and $\Lambda^*(x) = 0$ if and only if $x = \left(\frac{1-\lambda}{d(1+\lambda)}, \ldots, \frac{1-\lambda}{d(1+\lambda)}\right)$.

Proof. Note that $s_0 = \frac{1}{2} \ln \lambda$. First, let

$$\mathcal{D} = \left\{x = (x_1, \ldots, x_d) \in \mathbb{R}^d_+ : \sum_{j=1}^{d} x_j \leq 1\right\}.$$

Then, for any $x \in \mathcal{D}$, we have

$$\begin{aligned}\Lambda^*(x) &= \sup_{s_i \geq s_0, 1 \leq i \leq d} \ln\left\{\frac{1}{2}(1+\lambda)\frac{\exp\{\sum_{i=1}^d s_i x_i\}}{\frac{1}{2d}\sum_{i=1}^d (\lambda e^{-s_i} + e^{s_i})}\right\}\\ &= \sup_{y_i \geq 0, 1 \leq i \leq d} \ln\left\{\frac{1}{2}(1+\lambda)\lambda^{-\frac{1}{2}+\frac{1}{2}\sum_{i=1}^d x_i}\frac{\exp\{\sum_{i=1}^d y_i x_i\}}{\frac{1}{2d}\sum_{i=1}^d (e^{-y_i} + e^{y_i})}\right\}\\ &= \frac{1}{2}\sum_{i=1}^{d} x_i \ln(\lambda) - \ln(\rho_\lambda) + \sup_{y_i \geq 0, 1 \leq i \leq d} \ln\left\{\frac{\exp\{\sum_{i=1}^d y_i x_i\}}{\frac{1}{2d}\sum_{i=1}^d (e^{-y_i} + e^{y_i})}\right\} < \infty. \qquad (22)\end{aligned}$$

Here, $y_i = s_i - s_0$, $1 \leq i \leq d$. The first equality holds since $\sum_{i=1}^d s_i x_i$ is increasing in s and $\psi(s) = \psi(\tilde{s})$ (cf. Lemma 3.5). For $x \in \mathcal{D}^c$, it is easy to verify that $\Lambda^*(x) = \infty$.

By [7, Lemma 2.3.9], Λ^* is a good convex rate function. On

$$\left\{x = (x_1, \ldots, x_d) \in \mathbb{R}^d_+ : \sum_{i=1}^{d} x_i < 1\right\},$$

it is obvious that the Hessian matrix of $\Lambda(s)$ is positive definite, which implies strict concavity of $s \cdot x - \Lambda(s)$; thus the local maximum of $s \cdot x - \Lambda(s)$ exists uniquely and is attained at a finite solution $s = s(x)$, i.e. $\Lambda^*(x) = s(x) \cdot x - \Lambda(s(x))$. Applying the implicit function theorem, we see that Λ^* is strictly convex. By (22), for any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d_+$ with $\sum_{i=1}^d x_i = 1$,

$$\Lambda^*(x) \geq \limsup_{n\to\infty} \Lambda^*\left[x - \frac{1}{n}\left(x - \left(\frac{1-\lambda}{d(1+\lambda)}, \ldots, \frac{1-\lambda}{d(1+\lambda)}\right)\right)\right]. \qquad (23)$$

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By (23), we can verify that $(\frac{1-\lambda}{d(1+\lambda)}, \ldots, \frac{1-\lambda}{d(1+\lambda)})$ is the unique solution of $\Lambda^*(x) = 0$. □

For $\lambda = 0$, we have the following explicit expression.

Lemma 3.8. *For any $d \geq 2$ and $\lambda = 0$,*

$$\mathcal{D}_{\Lambda^*}(0) := \{x \in \mathbb{R}^d : \Lambda^*(x) < \infty\} = \left\{x = (x_1, \ldots, x_d) \in \mathbb{R}_+^d : \sum_{i=1}^d x_i = 1\right\},$$

$$\Lambda^*(x) = \begin{cases} \ln d + \sum_{i=1}^d x_i \ln x_i & x \in \mathcal{D}_{\Lambda^*}(0), \\ +\infty & \text{otherwise}, \end{cases}$$

$$\{x \in \mathbb{R}^d : \Lambda^*(x) = 0\} = \left\{\left(\frac{1}{d}, \ldots, \frac{1}{d}\right)\right\}.$$

Proof. When $\lambda = 0$, we have $\Lambda^*(x) = \sup_{s \in \mathbb{R}^d} \{(s, x) - \ln (\frac{1}{d} \sum_{i=1}^d \mathrm{e}^{s_i})\}$. Consider the supremum over the line $s_1 = s_2 = \cdots = s_d$ in $\mathbb{R}^d$. We have that

$$\Lambda^*(x) \geq \sup_{s_1 \in \mathbb{R}} s_1(x_1 + \cdots + x_d - 1) \geq +\infty,$$

provided $x_1 + \cdots + x_d \neq 1$. Therefore,

$$\mathcal{D}_{\Lambda^*}(0) \subseteq \left\{x = (x_1, \ldots, x_d) \in \mathbb{R}_+^d : \sum_{i=1}^d x_i = 1\right\}.$$

Assume first that $x_i > 0$, $1 \leq i \leq d$. Let $y_i = \mathrm{e}^{s_i}$, $1 \leq i \leq d$; then, by the Jensen inequality,

$$\begin{aligned}(s, x) - \ln\left(\frac{1}{d}\sum_{i=1}^d \mathrm{e}^{s_i}\right) &= \sum_{i=1}^d x_i \ln \frac{y_i}{x_i} - \ln\left(\sum_{i=1}^d y_i\right) + \sum_{i=1}^d x_i \ln x_i + \ln d \\ &\leq \ln\left(\sum_{i=1}^d x_i \frac{y_i}{x_i}\right) - \ln\left(\sum_{i=1}^d y_i\right) + \sum_{i=1}^d x_i \ln x_i + \ln d \\ &= \sum_{i=1}^d x_i \ln x_i + \ln d,\end{aligned}$$

and the inequality in the second line becomes equality only if each $s_i = \ln x_i$.

If there exists some i such that $x_i = 0$, we still have that

$$(s, x) - \ln\left(\frac{1}{d}\sum_{i=1}^d \mathrm{e}^{s_i}\right) \leq \sum_{i=1}^d x_i \ln x_i + \ln d;$$

then $\sup_{s \in \mathbb{R}^d} \{(s, x) - \ln (\frac{1}{d} \sum_{i=1}^d \mathrm{e}^{s_i})\} = \sum_{i=1}^d x_i \ln x_i + \ln d$ by the lower semicontinuity of $\Lambda^*(\cdot)$. Hence the conclusions follow from direct computation. □

Note that $\mathcal{D}_\Lambda^o = \mathbb{R}^d$ by the definition of $\mathcal{D}_\Lambda^o$ in Subsection 3.1. Let $\mathbb{F}$ be the set of exposed points of Λ^*. Then we have the following result.

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Lemma 3.9. *For any open set G of $\mathbb{R}^d$,*

$$\inf_{x\in G\cap\mathbb{F}} \Lambda^*(x) = \inf_{x\in G} \Lambda^*(x).$$

Proof. Assume $\lambda \in (0, 1)$ and $d \geq 1$. By the duality lemma [7, Lemma 4.5.8], we have that

$$\left\{x = (x_1, \ldots, x_d) \in \mathbb{R}_+^d : \sum_{i=1}^{d} x_i < 1\right\} \subseteq \mathbb{F}.$$

By the strict convexity of $\Lambda^*(\cdot)$ and (23), we have that, for any open set G,

$$\inf_{x\in G\cap\mathbb{F}} \Lambda^*(x) = \inf_{x\in G} \Lambda^*(x).$$

Assume $\lambda = 0$ and $d \geq 2$. We have that

$$\mathbb{F} = \left\{x = (x_1, \ldots, x_d) \in \mathbb{R}_+^d : \sum_{i=1}^{d} x_i = 1,\ \text{and } x_i > 0,\ 1 \leq i \leq d\right\}.$$

Then, by Lemma 3.8, for any open set G,

$$\inf_{x\in G\cap\mathbb{F}} \Lambda^*(x) = \inf_{x\in G} \Lambda^*(x).$$

□

Proof of Theorem 3.1. Let μ_n be the law of $\left(\frac{|X_n^1|}{n}, \ldots, \frac{|X_n^d|}{n}\right)$ for any $n \in \mathbb{N}$. From Proposition 3.1 and Lemmas 3.7 and 3.8, the logarithmic moment generating function exists with $\Lambda(s) = \ln \psi(s)$ and

$$\mathbf{0} \in \mathcal{D}_\Lambda^o = \mathbb{R}^d.$$

By (a) and (b) of the Gärtner–Ellis theorem (Theorem 3.2) and Lemma 3.9, we have that, for any closed set $F \subseteq \mathbb{R}^d$,

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mu_n(F) \leq -\inf_{x\in F} \Lambda^*(x),$$

and, for any open set G of $\mathbb{R}^d$,

$$\liminf_{n\to\infty} \frac{1}{n} \ln \mu_n(G) \geq -\inf_{x\in G} \Lambda^*(x).$$

The proof is complete. □

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