J. Inst. Math. Jussieu (2016) **15**(2), 407–443
doi:10.1017/S1474748014000334

DERIVED ALGEBRAIC COBORDISM

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(Received 14 January 2014; revised 29 September 2014; accepted 29 September 2014; first published online 30 October 2014)

Abstract We construct a cohomology theory using quasi-smooth derived schemes as generators and an analog of the bordism relation using derived fiber products as relations. This theory has pull-backs along all morphisms between smooth schemes independent of any characteristic assumptions. We prove that, in characteristic zero, the resulting theory agrees with algebraic cobordism as defined by Levine and Morel. We thus obtain a new set of generators and relations for algebraic cobordism.

Keywords: algebraic cobordism; virtual fundamental classes; quasi-smooth schemes

2010 *Mathematics subject classification*: Primary 14F43
Secondary 14C40; 55N22

1. Introduction

In his treatise on the universality of the formal group law of complex oriented cobordism [13], Quillen introduced a geometric set of generators and relations for complex oriented cobordism. He viewed complex cobordism as the universal contravariant functor on smooth manifolds endowed with Gysin homomorphisms for proper oriented maps. The distinctive feature of complex cobordism is that it comes equipped with a first Chern class operator for complex line bundles, and this Chern class operator satisfies the universal formal group law. Quillen's construction of this theory is surprisingly simple. In his geometric description, cobordism classes in the cobordism group $U^q(X)$ can be represented by proper complex oriented maps $Z \to X$, and two such maps give the same cobordism class if they arise as fibers of a proper complex oriented map $W \to X \times \mathbb{R}$. The construction of pull-backs in this geometric description for all maps between smooth manifolds relies on Thom's transversality theorem. Since any two maps between manifolds can be moved by a homotopy until they are transversal, the pull-back of any generator $Z \to X$ will again be a proper complex oriented map between smooth manifolds.

A corresponding algebraic theory was introduced by Levine and Morel [6]. Compared to complex cobordism, it has a distinctly different flavor in two ways. First of all, the formal group law controlling the first Chern class operator does not appear in a natural way. In the algebraic setting, the formal group law corresponds to a rule on how to take a divisor consisting of several components apart. Surprisingly enough, as shown by Levine and Pandharipande in [7], imposing a rule on the simplest possible configuration of divisors is already sufficient to obtain a formal group law for the first Chern class operator. Alternatively, the formal group law can be imposed formally, as done in [6]. On this issue, derived algebraic geometry does not offer any insight. The second difference to the topological theory arises in the construction of pull-backs along all morphisms between smooth schemes. Since algebraic geometry is by nature much more rigid, two morphisms between smooth schemes can in general not be arranged to be transversal. Taking smooth schemes as representatives of algebraic cobordism classes, it therefore is difficult to obtain pull-backs in algebraic cobordism. In fact, this is by far the most difficult part in the construction of [6], and it is only carried out in characteristic zero. After these difficulties are out of the way, the resulting theory has a close resemblance to Quillen's theory of complex oriented cobordism. For instance, algebraic cobordism satisfies a similar universal property, and the cobordism ring of a point is isomorphic to the Lazard ring.

A natural idea how to obtain a cohomology theory having pull-backs along all morphisms between smooth schemes is to enlarge the class of schemes allowed as generators of the theory. Staying in the realm of algebraic geometry, it is unclear though which class of schemes to pick. The least complicated schemes after smooth schemes are local complete intersections, and these are also not stable under pull-backs between morphisms of smooth schemes. Here, derived algebraic geometry offers a natural solution. The class of quasi-smooth derived schemes is a mild generalization of smooth schemes and local complete intersections. It satisfies excellent stability properties under all derived pull-backs of morphisms between smooth schemes, so that any cohomology theory defined with quasi-smooth derived schemes as generators will automatically have pull-backs along such morphisms. The class of quasi-smooth schemes offers a natural algebraic–geometric analog of moving by a homotopy to transversal intersection; see, for example, the work of Ciocan-Fontanine and Kapranov in [2]. Quasi-smooth schemes are also not too far a generalization from schemes. For instance, the simplicial commutative rings giving the local theory of quasi-smooth schemes only have finitely many homotopy groups. Furthermore, locally every quasi-smooth derived scheme can be written as the derived fiber product of underived schemes.

The aim of this paper is to study the theory obtained by using an algebraic version of Quillen's construction with quasi-smooth derived schemes replacing smooth manifolds as representatives of bordism classes, and by replacing deformation to transversal intersection by derived fiber products in the definition of the bordism relation. The resulting theory is called *derived algebraic cobordism*. By construction, the resulting cohomology theory will have pull-backs along all morphisms between smooth manifolds, independent of any characteristic assumption on the base field. In particular, we obtain a first Chern class operator for line bundles. As in the algebraic theory of Levine and Morel, it is too much to expect that this operator will satisfy a formal group law. After imposing this formal group law, we obtain a theory that resembles the theory of Levine and Morel. In characteristic zero, we prove that there is a natural comparison map between algebraic bordism and derived algebraic bordism, and this map is in fact an isomorphism. We thus obtain a new set of generators and relations for algebraic cobordism.

The motivation came from finding an analog of Spivak's [15] and Joyce's [5] results in derived differential geometry and d-manifolds, respectively. Using generators and relations as prescribed above, Joyce and Spivak define derived cobordism for any smooth manifold, and both are able to prove that derived cobordism agrees with classical cobordism. Our result is slightly stronger in that it shows that, for any quasi-projective derived scheme (including both singular and derived), the aforementioned isomorphism exists.

This result also provides a conceptual explanation why the virtual fundamental classes of Behrend and Fantechi [1] and Li and Tian [8] exist. It implies that every quasi-smooth derived scheme of virtual dimension d is in fact bordant to a smooth scheme of dimension d.

The method of proof employed here is remarkably similar to that of the proofs of Joyce and Spivak. For those acquainted with the proof, we have the following rough analogies.

Differential geometric	Algebro geometric
Manifold	Smooth scheme
Derived manifold	Quasi-smooth derived scheme
Tubular neighborhood	Deformation to the normal cone
Thom's transversality	Levine's moving lemma

More precisely, following ideas from [11], we introduce orientations for quasi-smooth morphisms in algebraic bordism. In the case of a quasi-smooth derived scheme mapping to a point, this gives exactly the virtual fundamental class. Using the tools summarized in the above table, we then prove a Grothendieck–Riemann–Roch result stating that these orientations are compatible with the orientations defined for derived algebraic bordism (Corollary 5.11). As a consequence, we obtain that the natural transformation

$$\vartheta_{\Omega} : d\Omega_* \longrightarrow \Omega_*$$

obtained from the universal property of $d\Omega_*$ is an isomorphism.

An immediate consequence of this Grothendieck–Riemann–Roch result is that derived bordism and classical bordism coincide as homology theories on the category of derived quasi-projective schemes.

Theorem 5.12. *For all $X \in \mathbf{dQPr}_k$ the morphism*

$$\vartheta_{d\Omega} : \Omega_*(X) \longrightarrow d\Omega_*(X)$$

is an isomorphism.

To emphasize the analogy between derived fiber products and deformation to transversal intersection, we prove the following formula for the intersection product in derived algebraic cobordism.

Theorem 6.1. *Let $X \in \mathbf{Sm}_k$. Then the intersection product is represented by the homotopy fiber product:*

$$[Y] \cdot [Z] = [Y \times^h_X Z] \in d\Omega^*(X).$$

We briefly review the argument and the contents of the paper.

In § 2, we introduce the abstract notion of an oriented Borel–Moore functor of geometric type with quasi-smooth pull-backs. We then define derived algebraic bordism in § 3, and show that it is the universal oriented Borel–Moore functor of geometric type with quasi-smooth pull-backs. In § 4, we study some properties of the underived bordism Ω_*. We first extend Ω_* to a homology theory on derived quasi-projective schemes and verify that is the universal oriented Borel–Moore homology theory of geometric type. This yields a natural transformation $\vartheta_{d\Omega} : \Omega_* \to d\Omega_*$. The remainder of the section is devoted to constructing pull-backs along quasi-smooth morphisms in Ω_*. Once this is done, we obtain, using the universal property of $d\Omega_*$, a natural transformation $\vartheta_{\Omega_*} : d\Omega_* \to \Omega_*$ in the opposite direction. Finally, in § 5, we show that these natural transformations are inverse to each other using a Grothendieck–Riemann–Roch type theorem.

Notation. We will work throughout over a base field k. Starting from § 4, this field will be of characteristic zero.

For any scheme or derived scheme X we will denote by $\mathrm{QCoh}(X)$ the ∞-category of quasi-coherent sheaves on X introduced by Lurie in [10]. Roughly, objects of $\mathrm{QCoh}(X)$ correspond to possibly unbounded complexes of quasi-coherent sheaves on X.

We have adopted using $\mathbb{L}$ for the Lazard ring and L_X for the cotangent complex of a scheme X.

In the text, various categories of schemes and derived schemes are encountered. We have used bold font with hopefully self-explanatory names for these categories. For instance, $\mathbf{dQPr}_k$ denotes the category of derived quasi-projective schemes over k, $\mathbf{Sm}_k$ denotes the category of smooth quasi-projective schemes over k, and so forth.

Given a scheme X and the data of an effective Cartier divisor on X, D, we do not distinguish between the Cartier divisor and the natural subscheme it generates. We let $\mathcal{O}_X(D)$ be the associated line bundle on X, with the natural section implicit. We let $|D|$ denote the support of the Cartier divisor. This is the reduced subscheme of D.

Similar to Fulton's convention [4, Convention 1.4], if $i : Y \hookrightarrow X$ is a closed embedding and $\alpha \in d\Omega_*(Y)$, when no confusion can arise we write $\alpha \in d\Omega_*(X)$ rather than $i_*\alpha$.

Throughout this text, $f : X \to Y$ is transverse to $g : Z \to Y$ if they are Tor-independent and $X \times_Y Z \to Z$ is smooth. Here, Tor-independence means that $\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Z) = 0$ for all $i > 0$.

2. Oriented Borel–Moore functors on derived schemes

In this section, we introduce oriented Borel–Moore functors on quasi-projective derived schemes which have quasi-smooth pull-backs. We believe this is the right setting for studying virtual fundamental classes, since for any quasi-projective quasi-smooth derived scheme X we can then define the virtual fundamental class via $\pi_X^*[1]$, where $\pi_X : X \to \mathrm{pt}$ is the structure morphism.

Following Levine and Morel [6], we first introduce oriented Borel–Moore functors with product. In these theories, one has no control over the behavior of the first Chern class operator. We later pass to oriented Borel–Moore functors of geometric type. There, the first Chern class satisfies a formal group law.

2.1. Oriented Borel–Moore functors with product

The definition of an oriented Borel–Moore functor with product on derived schemes is an immediate generalization of the original definition of Levine and Morel in [6] for schemes. Let $\mathbf{dQPr}_k$ denote the category of quasi-projective derived schemes over k. Let $\mathbf{dQPr}'_k$ denote the subcategory of $\mathbf{dQPr}_k$ with proper morphisms.

Definition 2.1. An oriented Borel–Moore functor with product on $\mathbf{dQPr}_k$ is given by the following.

(D1) An additive functor $A_* : \mathbf{dQPr}'_k \to \mathbf{Ab}_*$.

(D2) For each smooth morphism $f : X \to Y$ in $\mathbf{dQPr}_k$ of pure relative dimension d, a homomorphism of graded abelian groups

$$f^* : A_*(Y) \longrightarrow A_{*+d}(X).$$

(D3) For each line bundle L on X, a homomorphism of graded abelian groups

$$c_1(L) : A_*(X) \longrightarrow A_{*-1}(X).$$

(D4) For each pair (X, Y) in $\mathbf{dQPr}_k$, a bilinear graded pairing

$$\times : A_*(X) \times A_*(Y) \longrightarrow A(X \times Y)$$

which is commutative, associative, and admits a distinguished element $1 \in A_0(\mathrm{pt})$ as a unit.

These data are required to satisfy the following axioms.

(A1) Let $f : X \to Y$ and $g : Y \to Z$ be smooth morphisms in $\mathbf{dQPr}_k$ of pure relative dimension. Then

$$(g \circ f)^* = f^* \circ g^*.$$

Moreover, $\mathrm{id}_X^* = \mathrm{id}_{A_*(X)}$.

(A2) Let $f : X \to Z$ and $g : Y \to Z$ be morphisms in $\mathbf{dQPr}_k$, where f is proper and g is smooth of pure relative dimension. Let

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

be the resulting Cartesian square. Then

$$g^* f_* = f'_* g'^*.$$

(A3) Let $f : X \to Y$ be proper, and let $L \to Y$ be a line bundle. Then

$$f_* \circ c_1(f^* L) = c_1(L) \circ f_*.$$

https://doi.org/10.1017/S1474748014000334 Published online by Cambridge University Press

(A4) Let $f : X \to Y$ be smooth of pure relative dimension, and let $L \to Y$ be a line bundle. Then

$$c_1(f^*L) \circ f^* = f^* \circ c_1(L).$$

(A5) For all line bundles L and M on $X \in \mathbf{dQPr}_k$, we have

$$c_1(L) \circ c_1(M) = c_1(M) \circ c_1(L).$$

If L and M are isomorphic as line bundles on X, then $c_1(L) = c_1(M)$ holds.

(A6) For proper morphisms f and g, we have

$$\times \circ (f_* \times g_*) = (f \times g)_* \circ \times.$$

(A7) For smooth morphisms f and g, we have

$$\times \circ (f^* \times g^*) = (f \times g)^* \circ \times.$$

(A8) For $X, Y \in \mathbf{dQPr}_k$ and $L \to X$ a line bundle, we have

$$(c_1(L)(\alpha)) \times \beta = c_1(p_1^*(L))(\alpha \times \beta).$$

Remark 2.2. It might seem surprising that in (A2) the square is only required to be Cartesian and not homotopy Cartesian. But in this case the Cartesian product is a homotopy fiber product since g is assumed to be smooth and thus flat.

The following lemma follows immediately from the definitions.

Lemma 2.3. *An oriented Borel–Moore functor with product on* $\mathbf{dQPr}_k$ *restricted to* $\mathbf{QPr}_k$ *is a Borel–Moore functor with product on* $\mathbf{QPr}_k$.

For a general oriented Borel–Moore functor, one has no control over the behavior of the first Chern class operator. The next better behaved type of homology theory introduced by Morel and Levine is an oriented Borel–Moore functor of geometric type. For such a theory, the first Chern class operator is controlled by a formal group law. Since the conditions only involve smooth schemes and a smooth derived scheme is always classical, the definition carries over immediately to derived schemes. We briefly recall the notion for the reader's convenience.

Recall that an *oriented Borel–Moore* R_**-functor with product* is an oriented Borel–Moore functor A_* equipped with a graded ring homomorphism $R_* \to A_*(k)$. Here, R_* is graded unital commutative ring.

Definition 2.4. Let $\mathbb{L}_*$ be the Lazard ring, graded such that the universal formal group law has total degree -1. An *oriented Borel–Moore functor of geometric type* is an oriented Borel–Moore $\mathbb{L}_*$-functor A_* on $\mathbf{dQPr}_k$ satisfying the following additional axioms.

(Dim) For a smooth scheme X and line bundles $L_1, \ldots, L_r$ on X with $r > \dim(X)$, we have

$$c_1(L_1) \circ \cdots \circ c_1(L_r)(1_X) = 0 \in A_*(X).$$

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(Sect) For a smooth scheme X and a section $s : X \to L$ of a line bundle L on X transverse to the zero-section, we have

$$c_1(L)(1_X) = i_*(1_Z),$$

where Z is the zero-set of s and $i : Z \to X$ is the inclusion.

(FGL) for a smooth scheme X and line bundles L, M on X, we have

$$F_A(c_1(L), c_1(M))(1_X) = c_1(L \otimes M)(1_X) \in A_*(X),$$

where F_A is the image of the universal formal group law on $\mathbb{L}$ under the homomorphism given by the $\mathbb{L}_*$-structure.

Remark 2.5. In [6], Borel–Moore functors satisfying more axioms than those of a functor of geometric type are considered. One of the axioms for a weak homology theory is a weak localization axiom closely related to axiom (Sect) of a functor of geometric type.

(Loc) Let L be a line bundle on a scheme X admitting a section $s : X \to L$ which is transverse to zero. Let $i : Z \to X$ be the inclusion of the zero-set of s. Then the image of $c_1(L) : A_*(X) \to A_{*-1}(X)$ is contained in the image of $i_* : A_{*-1}(Z) \to A_{*-1}(X)$.

Again, the following lemma is immediate from the definitions.

Lemma 2.6. *An oriented Borel–Moore functor of geometric type on* $\mathbf{dQPr}_k$ *restricted to* $\mathbf{QPr}_k$ *is a Borel–Moore functor of geometric type on* $\mathbf{QPr}_k$*.*

If, in addition to axiom (Loc) of Remark 2.5, one requires the projective bundle theorem, and an extended homotopy relation, then one obtains the definition of a *weak homology theory*. From there, one can further ask for pull-back along locally complete intersection morphisms and certain cellular decomposition formulas to obtain a well-behaved functor called a *Borel–Moore homology theory*.

2.2. Borel–Moore functors with quasi-smooth pull-backs

To introduce virtual fundamental classes in an oriented Borel–Moore functor it is necessary to define orientations, i.e. pull-backs, for quasi-smooth morphisms. We first recall some basic notions on quasi-smooth morphisms.

Definition 2.7. A morphism $f : X \to Y$ of derived schemes is *quasi-smooth* if f is locally of finite presentation and the relative cotangent complex $L_{X/Y}$ is of Tor-amplitude $\leqslant 1$.

Example 2.8. Let $f : X \to Y$ be a local complete intersection morphism of schemes. Then f is quasi-smooth.

Remark 2.9. Note that being locally of finite presentation as a morphism of derived schemes is in general a stronger condition than requiring the underlying morphism of schemes to be locally of finite presentation in the usual sense. For instance, being locally of finite presentation as a morphism of derived schemes implies that the relative cotangent complex is perfect. This is only true for local complete intersection morphisms of classical schemes.

For any point $p : \operatorname{Spec} A \to X$ the above condition implies that $p^* L_{X/Y}$ is locally isomorphic to a two-term complex of vector bundles. This leads to the definition of virtual dimension.

Definition 2.10. Let $f : X \to Y$ be a quasi-smooth morphism, and let $p : \operatorname{Spec} k \to X$ be a k-point of X. Then the *virtual dimension of f at p* is defined as

$$\operatorname{virdim}(f, p) = H_0(p^* L_{X/Y}) - H_1(p^* L_{X/Y}).$$

Remark 2.11. The virtual dimension of f at $p \in X$ is a locally constant function.

For later uses we will introduce quasi-smooth morphisms that admit a factorization into a quasi-smooth embedding followed by a smooth morphism. In the case of local complete intersection morphisms of schemes this is often built into the definition. As a rule of thumb, in English texts a local complete intersection has a global factorization by definition (see, e.g., [4, 6]), whereas in French texts this is an additional property (see, e.g., [3]).

Definition 2.12. We say that a morphism $f : X \to Y$ of derived schemes is *smoothable* if it factors as closed embedding followed by smooth morphism. Such a factorization is called a *smoothing*.

Remark 2.13. Let $f : X \to Y$ be a quasi-smooth morphism, and let $X \overset{i}{\hookrightarrow} M \xrightarrow{p} Y$ be a smoothing. It then follows that $i : X \hookrightarrow M$ is a quasi-smooth embedding.

Remark 2.14. Locally a smoothing exists for any quasi-smooth morphism.

Example 2.15. Any morphism $f : X \to Y$ with $X, Y \in \mathbf{dQPr}_k$ admits a smoothing.

Once we have pull-backs along quasi-smooth morphism, we will, in particular, have fundamental classes for local complete intersections. We need a normalization property for these fundamental classes to ensure compatibility with fundamental classes for local complete intersections in other homology theories. This normalization is automatically satisfied for Borel–Moore homology theories.

To state the desired normalization property, we first borrow notation from [6, § 3.1]. Given a formal group law $F(u, v) \in R[[u, v]]$ for some commutative ring R, there exists the *difference group law*

$$F^-(u, v) \in R[[u, v]].$$

If we let $\chi(u)$ denote the unique inverse power series satisfying $F(u, \chi(u)) = 0$, this difference group law is defined by the equation

$$F(u, v) = F^-(u, \chi(v)).$$

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Often the suggestive notation $+_F$ is used in place of the formal group law and $-_F$ for the difference. Given an integer n, denote

$$[n]_F \cdot u := \begin{cases} u +_F \ldots +_F u & n \geqslant 0, \\ u -_F \ldots -_F u & n < 0, \end{cases}$$

where the operation is performed $|n|$ times, and given integers $n_1, \ldots, n_m$,

$$F^{n_1,\ldots,n_m}(u_1, \ldots, u_m) := [n_1]_F \cdot u_1 +_F [n_2]_F \cdot u_2 \cdots +_F [n_m]_F \cdot u_m.$$

One can guess many identities among the formal power series with this notation. For instance,

$$(u +_F v) - (0 +_F v) = u$$

translates to the formal power series identity

$$F^-(F(u, v), F(0, v)) = u$$

used in Lemma 3.17.

Let E be a strict normal crossing divisor on a smooth scheme X with support $|E|$. Following [6], if A_* is any Borel–Moore functor with first Chern classes obeying a formal group law and having proper push-forwards (this included Borel–Moore functors of geometric type), there exists a class $[E \to |E|] \in A_*(|E|)$ defined as follows. Writing $E = \sum_{j=1}^m n_j E_j$ with each E_j integral, for any index $J = (j_1, \ldots, j_m)$ with $\|J\| \leqslant 1$,[1] let $E^J := \bigcap_{i, j_i=1} E_i$ be the Jth face, and let $i^J : E^J \to |E|$ be the natural inclusion. If $L_i = \mathcal{O}_X(E_i)$ and $L_i^J = (i^J)^* L_i$, define

$$[E \to |E|] = \sum_{J, \|J\| \leqslant 1} i_*^J \left([F_J^{n_1,\ldots,n_m}(L_1^J, \ldots, L_m^J)] \right). \tag{1}$$

With the notation now set, we define the central notion of this section.

Definition 2.16. An *oriented Borel–Moore functor with quasi-smooth pull-backs* (also referred to as 'with quasi-smooth orientations', or 'with quasi-smooth Gysin-maps') on $\mathbf{dQPr}_k$ consists of an oriented Borel–Moore functor A_* of geometric type equipped with the following.

(B1) For each equi-dimensional quasi-smooth morphism $f : Y \to X$ of relative virtual dimension d, a homomorphism of graded abelian groups

$$f^* : A_*(X) \longrightarrow A_{*+d}(Y).$$

These pull-backs should satisfy the following axiom.

(QS1) Let $s : L \to X$ be a section of a line bundle. Then $c_1(L) = s^* s_*$.

(QS2) Let $f : X \to Y$ and $g : Y \to Z$ be quasi-smooth morphisms of pure relative virtual dimension. Then

$$(g \circ f)^* = f^* g^*.$$

[1] Here, $\|J\| := \mathrm{Sup}_i(j_i)$.

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(QS3) Let $f : X \to Z$ and $g : Y \to Z$ be morphisms giving the homotopy Cartesian square

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

with f proper and g quasi-smooth and virtually equi-dimensional. Then

$$g^* f_* = f'_* g'^*.$$

(QS4) For quasi-smooth morphisms f and g, we have

$$\times \circ (f^* \times g^*) = (f \times g)^* \circ \times.$$

(QS5) For any strict normal crossing divisor E in a smooth scheme X, we have

$$\pi_E^*([1]) = (\zeta_E)_*([E \to |E|]).$$

Here, $\pi_E : E \to \mathrm{pt}$ is the structure morphism of E, which is a local complete intersection morphism, and $\zeta_E : |E| \to E$ is the natural embedding.

Remark 2.17. Axiom (QS5) is likely equivalent to the extended homotopy property. In particular, by [6, Chapter 7], it is satisfied by all Borel–Moore homology functors. As mentioned previously, it is directly related to the universal morphism from regular bordism commuting with locally complete intersection pull-back (a priori, it only commutes with smooth pull-back). Since we want our derived bordism group to extend bordism (it will in fact be an isomorphism), it is a requirement. This hypothesis will likely be relegated irrelevant with more investigation in properties of the 'naive' derived bordism groups.

The pull-back f^* is called an orientation of the quasi-smooth morphism. In any oriented Borel–Moore functor with quasi-smooth pull-backs or orientations we can define virtual fundamental classes.

Definition 2.18. Let X be a quasi-projective quasi-smooth derived scheme, and let A_* be an oriented Borel–Moore functor with quasi-smooth pull-backs. Then the *virtual fundamental class* is defined as

$$\pi_X^*([1]) \in A_*(X).$$

Here, $\pi_X : X \to \mathrm{pt}$ is the structure morphism, and π_X^* is quasi-smooth pull-back.

The following definition summarizes all desirable properties a homology theory with quasi-smooth pull-backs should have.

Definition 2.19. An *oriented Borel–Moore homology theory with quasi-smooth pull-backs* on $\mathbf{dQPr}_k$ is given by an additive functor $A_* : \mathbf{dQPr}'_k \to \mathbf{Ab}_*$ equipped with quasi-smooth pull-backs and an external product, such that the following hold.

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(BM1) Axioms (QS2), (QS3), and (QS4) hold.

(BM2) The projective bundle theorem of [6, Definition 5.1.3] holds.

(BM3) The extended homotopy relation of [6, Definition 5.1.3] holds.

(BM4) The cellular decomposition relation of [6, Definition 5.1.3] holds.

3. Derived algebraic bordism

In this section, we define derived algebraic bordism by generators and relations. Derived algebraic bordism will turn out to be the universal oriented Borel–Moore functor of geometric type with orientations for quasi-smooth morphisms. Since a quasi-smooth morphism of schemes is a local complete intersection morphism, derived algebraic bordism has pull-backs for local complete intersection morphisms. Besides all the axioms necessary for an oriented Borel–Moore functor of geometric type, derived algebraic bordism additionally satisfies axiom (Loc) of Remark 2.5, although we will not use this.

We begin with the generators of our theory.

Definition 3.1. Let X be a quasi-projective derived scheme over k. Denote by $\mathcal{M}_n(X)^+$ the free abelian group generated by proper morphisms

$$f : Y \longrightarrow X,$$

where $Y \in \mathbf{QSm}_k$ is irreducible and of virtual dimension n. We will refer to elements of $\mathcal{M}_*(X)$ as *derived bordism cycles.*

We next introduce relations among the generators.

Definition 3.2. Let $X \in \mathbf{dQPr}_k$, and denote by $p : X \times \mathbb{P}^1 \to \mathbb{P}^1$ the projection onto the second factor. Let $Y \in \mathbf{dQPr}_k$ be quasi-smooth of pure virtual dimension, and let

$$g : Y \to X \times \mathbb{P}^1$$

be a proper morphism. We can then form the homotopy Cartesian square

$$\begin{array}{ccccc} Y_0 & \longrightarrow & Y & \longleftarrow & Y_\infty \\ \downarrow & & \downarrow{\scriptstyle g} & & \downarrow \\ X & \longrightarrow & X \times \mathbb{P}^1 & \longleftarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} & & \downarrow \\ 0 & \longrightarrow & \mathbb{P}^1 & \longleftarrow & \infty. \end{array}$$

The associated *homotopy fiber relation* then is

$$[Y_0 \to X] - [Y_\infty \to X].$$

Let $\mathcal{R}_*(X) \subset \mathcal{M}_*(X)^+$ be the subgroup generated by all homotopy fiber relations.

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Remark 3.3. It follows from basic properties of homotopy fiber products that Y_0 and Y_∞ are quasi-smooth derived schemes without any assumptions on 0 and ∞ being regular values of $p \circ g$.

We can now define a naive version of derived algebraic bordism.

Definition 3.4. Let $X \in \mathbf{dQPr}_k$. Then *naive derived algebraic bordism* is defined by

$$d\Omega_*^{\text{naive}}(X) = \mathcal{M}_*(X)^+ / \langle \mathcal{R}_*(X) \rangle.$$

Remark 3.5. Let $[Y \to X]$ and $[Y' \to X]$ be generators of $d\Omega_*^{\text{naive}}(X)$, where Y is weakly equivalent to Y'. Using the homotopy fiber relation, it follows that $[Y \to X] = [Y' \to X]$ in $d\Omega_*^{\text{naive}}(X)$.

The functor $d\Omega_*^{\text{naive}}$ already has some of the requisite structure of an oriented Borel–Moore functor. In particular, we can immediately define push-forward along proper maps and pull-back along quasi-smooth morphisms (which includes smooth morphisms). Let $g : X \to X'$ be a proper map in $\mathbf{dQPr}_k$. The map

$$g_* : \mathcal{M}_*(X)^+ \longrightarrow \mathcal{M}_*(X')^+$$

is given by

$$g_*([f : Y \to X]) = [g \circ f : Y \to X'].$$

It is immediate that this descends to a functorial push-forward

$$g_* : d\Omega_*^{\text{naive}}(X) \to d\Omega_*^{\text{naive}}(X').$$

Definition 3.6. Let $g : X \to X'$ be a quasi-smooth morphism between quasi-projective derived schemes. The *quasi-smooth pull-back*

$$g^* : \mathcal{M}_*(X')^+ \longrightarrow \mathcal{M}_*(X)^+$$

is given by

$$g^*([f : Y \to X']) = [Y \times_{X'}^h X \to X].$$

Again, it is immediate that this descends to a functorial pull-back

$$g^* : d\Omega_*^{\text{naive}}(X') \longrightarrow d\Omega_*^{\text{naive}}(X).$$

Remark 3.7. When g is smooth and thus flat, the usual fiber product is a homotopy fiber product, and we arrive at

$$g^*([f : Y \to X']) = [Y \times_{X'} X \to X].$$

Since we have defined pull-backs for quasi-smooth morphisms, we automatically have first Chern class operators.

Definition 3.8. Let $X \in \mathbf{dQPr}_k$, let L be a line bundle on X, and let $s_0 : X \to L$ be the zero-section. We can then define the first Chern class operator via

$$c_1(L)([Y \to X]) = s_0^*(s_0)_*([Y \to X]).$$

More generally, for any vector bundle $E \to X$ of rank r, we define the *Euler class* or *top Chern class* as

$$c_r(L)([Y \to X]) = s_0^*(s_0)_*([Y \to X]).$$

Lemma 3.9. *Given a line bundle L on $X \in \mathbf{QPr}_k$ and any $s : X \to L$, define $c_1^s(L) : d\Omega_*(X) \to d\Omega_*(X)$ as $s_0^* s_*([Y \xrightarrow{f} X])$. Then $c_1^s(L) = c_1^{s_0}(L) = c_1(L)$.*

Proof. Let us first assume that L has two non-zero-sections, s and s'. Define $Z \to X \times \mathbb{P}^1$ to be the derived scheme

$$\begin{CD} Z @>>> Y \times \mathbb{P}^1 \\ @V{\tilde{f}}VV @VV{\tilde{s}\circ(f\times \mathrm{id})}V \\ X \times \mathbb{P}^1 @>{s_0}>> L \boxtimes \mathcal{O}_{\mathbb{P}^1}(1), \end{CD}$$

where $\tilde{s} = sx_0 + s'x_\infty$, and where x_0, x_∞ are the Cartier divisors corresponding to 0 and ∞ (they are both sections of $\mathcal{O}_{\mathbb{P}^1}(1)$). It is clear that Z is quasi-smooth and that $\tilde{f}|_0 = c_1^s(L) \cap [Y \xrightarrow{f} X]$ and $\tilde{f}|_\infty = c_1^{s'}(L) \cap [Y \xrightarrow{f} X]$, thus proving the claim.

For the case that $s = s_0$, note that, if $p : L \to X$ is the natural projection, then $p^*L \to L$ has two natural sections: the canonical section $\tilde{s}$ and p^*s'. These sections are both non-zero-sections of the same line bundle, and thus, from the above, we have the following diagram for any $[X \xrightarrow{f} Y]$:

$$\begin{CD} Y'' @>>> Y' @>>> Y \times_X L \times \mathbb{P}^1 \\ @VVV @VVV @VVV \\ Z' @>>> Z @>>> L \times \mathbb{P}^1 \\ @VVV @VVV @VVV \\ X \times \mathbb{P}^1 @>>> L \times \mathbb{P}^1 @>>> L \boxtimes \mathcal{O}_{\mathbb{P}^1}. \end{CD}$$

The bottom morphisms are quasi-smooth, and thus the quasi-smoothness of Y implies that Y'' is quasi-smooth. We then have

$$[Y''|_0 \to X] = c_1^{s_0}(L) \cap [Y \to X]$$

and

$$[Y''|_\infty \to X] = c_1^{s'}(L) \cap [Y \to X],$$

and thus the desired bordism. □

Remark 3.10. The same proof works also works for any vector bundle of rank $e \geqslant 1$.

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To promote $d\Omega_*^{\text{naive}}$ to an oriented Borel–Moore functor with product, we define the external product by

$$\begin{aligned} \times : \mathcal{M}_*(X) \times \mathcal{M}_*(X') &\longrightarrow \mathcal{M}(X \times X') \\ [Y \to X] \times [Y' \to X'] &\longmapsto [Y \times_k Y' \to X \times_k X']. \end{aligned}$$

Clearly this descends to $d\Omega_*^{\text{naive}}$.

Proposition 3.11. *$d\Omega_*^{\text{naive}}$ is an oriented Borel–Moore functor with product.*

Proof. We have defined the projective push-forward, smooth pull-back, first Chern classes, and the external product. The proof then is a long but simple check of the axioms using properties of homotopy Cartesian squares. For instance, axiom (A2) follows from the fact that, given a diagram

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ W & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z, \end{array}$$

where the two inner squares are homotopy Cartesian, then the outer is also homotopy Cartesian. □

With these definitions, we can prove that $d\Omega_*^{\text{naive}}$ partially satisfies the properties of an oriented Borel–Moore functor of geometric type.

Proposition 3.12. *$d\Omega_*^{\text{naive}}$ satisfies axiom (Sect).*

Proof. Let X be a smooth scheme, let L be a line bundle on X, and let $s : X \to L$ be a section which is transverse to the zero-section. Since the first Chern class operator is independent of the choice of section, we can take $c_1(L)(1_X) = 0^* s_*(1_X)$, where $0 : X \to L$ is the zero-section. Since the section is assumed to be transverse to zero, the first Chern class operator is then given by the Cartesian square

$$\begin{array}{ccc} Z & \longrightarrow & X \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ Z & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow[0]{} & L. \end{array}$$

Here, Z is the zero-set of s. We now have

$$\begin{aligned} c_1(L)(1_X) = 0^* s_*(1_X) &= 0^*([X \xrightarrow{\mathrm{id}} X \xrightarrow{s} L]) \\ &= [Z \xrightarrow{\mathrm{id}} Z \xrightarrow{i} X] = i_*([Z \xrightarrow{\mathrm{id}} Z]). \end{aligned}$$ □

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Remark 3.13. In fact, replacing $[X \to X]$ with $[Y \to X]$ in the proof above shows that $d\Omega_*^{\text{naive}}$ satisfies axiom (Loc) of Remark 2.5.

Remark 3.14. Using the homotopy zero-set, Proposition 3.12 is even true without the transverse to zero assumption.

This is as far as one can get by using the homotopy fiber relation. It seems impossible to show further properties of the first Chern class operator of Definition 3.8, for example, that it satisfies a formal group law and axiom (Dim). To go any further, one must artificially impose a formal group law. This requires that we first know that c_1 acts nilpotently.

In the following propositions, following the arguments of [7], we will show that it is enough to show nilpotence of c_1 for globally generated bundles. Once this is shown, it then is legal to impose the formal group law for globally generated bundles. We are then left to prove that axioms (FGL) and (Dim) hold for all line bundles.

Proposition 3.15. *Let X be a smooth scheme, and let $L_1, \dots, L_r$ be globally generated bundles with* $\dim(X) < r$. *Then*

$$\prod_{i=1}^{r} c_1(L_i) = 0.$$

Proof. By Proposition 3.12, $c_1(L_1) \circ c_1(L_2) \circ \cdots \circ c_1(L_r)(1_X)$ set-theoretically can be arranged to be the empty set. Any morphism factoring through the empty set is zero. □

We now impose the formal group law on these globally generated line bundles. Recall that $\mathbb{L}_*$ denotes the Lazard ring, graded such that the universal formal group law has total degree -1. Define a new oriented Borel–Moore functor by

$$X \longmapsto \mathbb{L}_* \otimes_{\mathbb{Z}} d\Omega_*^{\text{naive}}(X).$$

For any smooth scheme X, set

$$\mathcal{R}_*^{\text{FGL}}(X) \subset \mathbb{L}_* \otimes_{\mathbb{Z}} d\Omega_*^{\text{naive}}(X)$$

to be the subset of elements of the form

$$F_{\mathbb{L}}(c_1(L), c_1(M))(1_X) - c_1(L \otimes M)(1_X)$$

for globally generated line bundles L, M on a smooth scheme X. Here, $F_{\mathbb{L}}$ is the universal formal group law.

Recall from [6, § 2.1.5] that, given an oriented Borel–Moore functor with product A_* and a subset of homogeneous elements $\mathcal{R}_*(X) \subset A_*(X)$ compatible with the external product, it is possible to define the quotient Borel–Moore homology functor $A_*/\mathcal{R}_*$.

Definition 3.16. Define *derived algebraic pre-bordism* as

$$d\Omega_*^{\text{pre}} = d\Omega_*^{\text{naive}} / \langle \mathcal{R}_*^{\text{FGL}} \rangle.$$

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Using the notation for the formal group law set in the previous section, we now have to show that (Dim) and (FGL) hold for all line bundles. The proofs proceed exactly as in [7].

Lemma 3.17. *Let $X \in \mathbf{Sm}_k$, let L be a line bundle on X, and let M be a globally generated line bundle such that $L \otimes M$ is globally generated. Then*

$$c_1(L) = F_{\mathbb{L}}^{-}(c_1(L \otimes M), c_1(M)).$$

Proof. The proof follows from the power series identity

$$F^{-}(F(u, v), F(0, v)) = F^{-}(u, 0) = u. \qquad \square$$

Proposition 3.18. *$d\Omega_*^{\mathrm{pre}}$ satisfies axiom (Dim).*

Proof. Let X be a quasi-projective scheme, and let $L_1, \ldots, L_r$ be a sequence of line bundles with $r > \dim(X)$. By the previous lemma, we have

$$c_1(L_i) = F_{\mathbb{L}}^{-}(c_1(L_i \otimes M), c_1(M)),$$

where M is globally generated and chosen such that $L_i \otimes M$ is globally generated. The proof then is the same as [7, Lemma 9.3]. $\square$

Proposition 3.19. *$d\Omega_*^{\mathrm{pre}}$ satisfies axiom (FGL).*

Proof. Let X be a smooth scheme, and let L, M be line bundles on X. Choose globally generated bundles N_1, N_2 such that $L \otimes N_1$ and $L \otimes N_2$ are globally generated. The proof then follows from the formal power series identity

$$F^{-}\left(F(u_1, v_1), F(u_2, v_2)\right) = F\left(F^{-}(u_1, u_2), F^{-}(v_1, v_2)\right)$$

combined with Lemma 3.17 and the formal group law for globally generated bundles. $\square$

Lastly, we ensure that strict normal crossing divisors in a smooth scheme have the correct fundamental class. For any smooth scheme X and any strict normal crossing divisor E in X, set

$$\mathcal{R}_*^{\mathrm{SNC}}(E) \subset d\Omega_*^{\mathrm{pre}}(E)$$

to be the subset of elements of the form

$$\pi_E^*([1]) - (\zeta_E)_*[E \to |E|].$$

Here, $\pi_E : E \to \mathrm{pt}$ is the structure morphism of the support of E, $\zeta_E : |E| \to E$ is the natural closed embedding, and $[E \to |E|]$ is the class defined in (1).

Definition 3.20. Define *derived algebraic bordism* as

$$d\Omega_* = d\Omega_*^{\mathrm{pre}} / \langle \mathcal{R}_*^{\mathrm{SNC}} \rangle.$$

Remark 3.21. Instead of taking the axiomatic approach to obtaining a formal group law, it should be possible to impose some form of the double point relations of [7] to obtain the formal group law and the correct fundamental class for strict normal crossing divisors in one step.

We are now ready to prove the universality of derived algebraic bordism.

Theorem 3.22. *$d\Omega_*$ is the universal oriented Borel–Moore functor with quasi-smooth pull-backs of geometric type.*

Proof. By Propositions 3.11, 3.12, 3.18, and 3.19, $d\Omega_*$ is an oriented Borel–Moore functor of geometric type and has orientations for quasi-smooth morphisms. The normalization property (QS5) is clear by construction. We are left to show universality. The proof of universality is the same as that for [13, Proposition 1.10]. Let A_* be another oriented Borel–Moore functor of geometric type with quasi-smooth pull-backs. Then we can define a natural transformation

$$\vartheta_A : d\Omega_* \longrightarrow A_*$$

by

$$\vartheta_A([Y \xrightarrow{f} X]) = f_* \pi_Y^*[1].$$

Here, as before, $\pi_Y : Y \to \mathrm{pt}$ is the structure morphism of Y. The proof that ϑ is compatible with the structures of an oriented Borel–Moore functor of geometric type is straightforward. □

Remark 3.23. Further examples of oriented Borel–Moore functors with quasi-smooth pull-backs are discussed in [9]. There, using the quasi-smooth pull-backs defined by [11], Chow homology is extended to derived schemes. This theory has the additive formal group law. As another example, G-theory with quasi-smooth pull-backs is introduced. This carries the multiplicative formal group law.

4. Quasi-smooth pull-backs in algebraic bordism

We now restrict consideration to the case that k is of characteristic zero. We begin by extending the classical underived bordism Ω_* to a functor on the category of derived quasi-projective schemes $\mathbf{dQPr}_k$ with proper morphisms using the same generators and relations as for classical schemes. By abuse of notation, we will still denote this extended functor by Ω_*. Given a derived quasi-projective scheme X, we then have the closed embedding of the underlying scheme $\iota_X : t_0(X) \hookrightarrow X$. In particular, this is a proper map, and there exists a push-forward on associated bordism groups.

Lemma 4.1. *Let $X \in \mathbf{dQPr}_k$. Then the inclusion of the classical part $\iota_X : t_0(X) \hookrightarrow X$ induces an isomorphism*

$$\iota_{X*} : \Omega_*(t_0(X)) \longrightarrow \Omega_*(X).$$

Proof. This is clear, since the generators of $\Omega_*(X)$ are given by morphisms $f : Y \to X$ with Y smooth, and every such map factors through the truncation $t_0(X)$. □

Using the above lemma, it is immediate to equip the extended functor Ω_* with pull-backs along smooth morphisms of derived quasi-projective schemes. Since the truncation of a smooth morphism $f : X \to Y$ is again smooth, we can define f^* by the composition

$$\Omega_*(Y) \cong \Omega_*(t_0(Y)) \overset{t_0(f)^*}{\to} \Omega_*(t_0(X)) \cong \Omega_*(X).$$

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Along the same lines, we can define a first Chern class operator for a line bundle L on a derived scheme X for the extended functor Ω_* by the composition

$$\Omega_*(X) \cong \Omega_*(t_0(X)) \overset{c_1(\iota_X^* L)}{\to} \Omega_{*-1}(t_0(X)) \cong \Omega_{*-1}(X).$$

Almost by definition, this makes the extended functor Ω_* an oriented Borel–Moore functor of geometric type. The same proof as in the discrete case shows that it is in fact the universal such functor.

Corollary 4.2. *There is a canonical classifying morphism*

$$\vartheta_{d\Omega} : \Omega_* \longrightarrow d\Omega_*.$$

Proof. Since $d\Omega_*$ is an oriented Borel–Moore functor of geometric type on $\mathbf{dQPr}_k$, and since by the preceding discussion Ω_* is the universal such Borel–Moore functor of geometric type, we obtain a natural transformation

$$\vartheta_{d\Omega} : \Omega_* \longrightarrow d\Omega_*$$

given by

$$\vartheta_{d\Omega}([Y \xrightarrow{f} X]) = f_* \pi_Y^*[1].$$

Here, $\pi_Y : Y \to \mathrm{pt}$ is the structure morphism of Y, and the pull-back and push-forward morphisms on the right-hand side are in $d\Omega_*$. □

The remainder of this section is devoted to constructing a classifying map in the opposite direction. To construct this morphism, we want to use the universal property of $d\Omega_*$. Since $d\Omega_*$ is universal with respect to quasi-smooth pull-backs, we first have to construct quasi-smooth pull-backs in Ω_*. This will be done in this section.

More generally, it is possible to show this for any Borel–Moore homology theory that has intersections with pseudo-divisors defined (in such a way as to commute with smooth pull-back and proper push-forward) and that satisfies a homotopy invariance property.

In order to construct these quasi-smooth pull-backs, we first review some background material on quasi-smooth embeddings.

4.1. Background on quasi-smooth embeddings

Let $f : X \hookrightarrow Y$ be a quasi-smooth closed embedding. Using the truncation functor, we have a commutative diagram in $\mathbf{dSch}_k$:

$$\begin{array}{ccc} X & \overset{f}{\hookrightarrow} & Y \\ {\scriptstyle \iota_X}\uparrow & & \uparrow{\scriptstyle \iota_Y} \\ t_0(X) & \underset{t_0(f)}{\hookrightarrow} & t_0(Y). \end{array}$$

We have the following basic result.

Lemma 4.3. *Let $f : X \hookrightarrow Y$ be a quasi-smooth embedding of virtual codimension d. Then $\iota_X^* L_{X/Y}[-1]$ is a locally free sheaf of rank d on $t_0(X)$.*

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Proof. Since f is quasi-smooth, $L_{X/Y}$ is perfect of Tor-amplitude $\leqslant 1$. Furthermore, since f is an embedding, $L_{X/Y}$ is in $\mathrm{QCoh}(X)_{\geqslant 1}$ (homological grading). It thus follows that $\iota_X^* L_{X/Y}[-1]$ is a locally free sheaf. The claim on the rank is clear from the definition of virtual codimension. □

Definition 4.4. Let $f : X \hookrightarrow Y$ be a quasi-smooth embedding. Define the *virtual normal bundle* $N_X Y \to t_0(X)$ to be the geometric vector bundle corresponding to $\iota_X^* L_{X/Y}[-1]^\vee$.

We now compare the virtual normal bundle $N_X Y$ to the normal cone $C_X Y$ of the underlying embedding $t_0(f) : t_0(X) \hookrightarrow t_0(Y)$. Recall that the normal cone is the scheme over $t_0(X)$ given by $\mathrm{Spec}_{\mathcal{O}_{t_0(X)}}\left(\bigoplus_{n\geqslant 0} I^n/I^{n+1}\right)$, where I is the ideal sheaf of $t_0(X)$ in $t_0(Y)$.

Using the functoriality properties of the cotangent complex, we have a morphism

$$\iota_X^* L_{X/Y} \longrightarrow L_{t_0(X)/t_0(Y)}.$$

Let I be the ideal sheaf of $t_0(X)$ in $t_0(Y)$. It is a classical fact that $\pi_1(L_{t_0(X)/t_0(Y)}) = I/I^2$. We thus obtain a morphism

$$i : C_X Y \to N_X Y.$$

We have the following result, which will be the basis for the construction of quasi-smooth pull-backs in Ω_*.

Lemma 4.5. *The morphism*

$$i : C_X Y \longrightarrow N_X Y$$

is a closed embedding.

Proof. The claim is local and will follow once it is shown that $j_X^* L_{X/Y} \longrightarrow I/I^2$ is surjective. Let $f : A \to B$ be a 0-connective morphism of simplicial commutative k-algebras. We then have the commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ \pi_0 A & \longrightarrow & \pi_0 B. \end{array}$$

This gives the following diagram of cofiber sequences:

$$\begin{array}{ccccc} L_A \otimes_A \pi_0 B & \longrightarrow & L_B \otimes_B \pi_0 B & \longrightarrow & L_{B/A} \otimes_B \pi_0 B \\ \downarrow & & \downarrow & & \downarrow \\ L_{\pi_0 A} \otimes_{\pi_0 A} \pi_0 B & \longrightarrow & L_{\pi_0 B} & \longrightarrow & L_{\pi_0 B/\pi_0 A} \\ \downarrow & & \downarrow & & \downarrow \\ L_{\pi_0 A/A} \otimes_{\pi_0 A} \pi_0 B & \longrightarrow & L_{\pi_0(B)/B} & \longrightarrow & M. \end{array}$$

We have to show that $M \in \mathrm{QCoh}(\pi_0 B)_{\geqslant 2}$. Now recall that, since $A \to \pi_0 A$ is 1-connective, $L_{\pi_0 A/A}$ is in $\mathrm{QCoh}(\pi_0 A)_{\geqslant 2}$. Likewise, $L_{\pi_0 B/B}$ is 2-connective. Since M is the cofiber of 2-connective objects, the claim follows. □

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4.2. Deformation to the normal cone

We now introduce deformation to the normal cone for a quasi-smooth embedding $X \hookrightarrow Y$.

We begin by reviewing the Rees construction for filtered simplicial modules over a field k. Given a simplicial k-module M with filtration $\{M_i\}_{i\in\mathbb{Z}}$ such that $M_i \subset M_{i+1}$ and $M = \cup M_i$, we can form a simplicial graded $k[t]$-module $\bigoplus M_i t^i$. We call this association the Rees construction.

We now review some facts from commutative algebra of simplicial rings [12]. Let $f : R \to S$ be a morphism of simplicial commutative k-algebras which is levelwise surjective. We equip the categories of simplicial commutative k-algebras and simplicial k-modules with their standard model structures. We then obtain a fiber sequence

$$I \to R \to S$$

of simplicial k-modules, where I is a simplicial ideal in R. Now factor this morphism in the model category of simplicial commutative k-algebras as $R \to P \to S$, where $R \to P$ is a cofibration and $P \to S$ is a weak equivalence. Setting $Q = P \otimes_R S$, we obtain a multiplication map $m : Q \to S$. This gives a fiber sequence

$$J \to Q \to S$$

in the category of simplicial k-modules, where J is a simplicial ideal in Q. The cotangent complex $L_{S/R}$ can then be identified with J/J^2 in the homotopy category of simplicial S-modules.

By identifying S with $S \otimes_R R$, it follows that the multiplication $m : Q \to S$ has a section $s = \mathrm{id} \otimes f$ in the homotopy category of simplicial Q-modules. In particular, we obtain an identification $J \simeq I \otimes_R^{\mathbb{L}} S[1]$ as Q-modules. By adjunction, we have an identification as S-modules $J \otimes_Q^{\mathbb{L}} S \simeq I \otimes_R^{\mathbb{L}} S[1]$. In particular, we have $J/J^2 = I/I^2[1]$. Combining with the formula for the cotangent complex in the paragraph above, we obtain the identification $L_{S/R} = I/I^2[1]$.

Now assume that A is a smooth discrete simplicial k-algebra and that $f : A \to B$ is quasi-smooth morphism which induces a surjective morphism $\pi_0 A \to \pi_0 B$. By first factoring $A \to B$ in the model category of simplicial k-algebras as $A \to B' \to B$, where $A \to B'$ is a cofibration and $B' \to B$ is a weak equivalence, and then factoring $A \to B'$ as $A \to A' \to B'$, where $A \to A'$ is a cofibration and a weak equivalence and $A' \to B'$ is a fibration, we can assume that $f' : A' \to B'$ is levelwise surjective, and that A' and B' are both levelwise smooth k-algebras. In particular, in the fiber sequence

$$I \to A' \xrightarrow{f'} B',$$

the simplicial ideal I is levelwise a regular ideal. Furthermore, since $A' \to B'$ is quasi-smooth, using the identification $L_{B'/A'}[-1] \simeq I/I^2$ obtained above it follows that I/I^2 is a projective A'-module.

We now filter A' by powers of the simplicial ideal I to obtain a filtered simplicial k-algebra (A', F). Applying the Rees construction gives a simplicial graded $k[t]$-algebra $\bigoplus_{n\in\mathbb{Z}} F^i t^i$. The fiber over zero can be identified with the associated graded simplicial algebra $\bigoplus_{n\geqslant 0} I^n/I^{n+1}$. Since the simplicial ideal I is levelwise regular and I/I^2 is

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projective, we obtain an identification $\mathrm{Sym}^*_B(I/I^2) \simeq I^n/I^{n+1}$. As a consequence, we can identify the fiber over zero of the Rees-algebra $\bigoplus_{n\in\mathbb{Z}} F^i t^i$ with $\mathrm{Sym}^*_B(L_{B/A}[-1])$.

The previous discussion immediately generalizes to schemes, giving us a deformation to the normal cone space $M^\circ_X Y$. By the fiberwise criterion for flatness, this space is flat over $Y \times \mathbb{A}^1$. The fiber over zero of the truncation $t_0(M^\circ_X Y)$ is a Cartier divisor on $t_0(M^\circ_X Y)$. When $X \hookrightarrow Y$ is a quasi-smooth embedding, the fiber over zero is given by the vector bundle corresponding to the locally free sheaf $j^*_X L_{X/Y}[-1]$, and thus is the virtual normal bundle N of Definition 4.4.

4.3. Pull-back along quasi-smooth embeddings

We will define the pull-back along a quasi-smooth embedding $f : X \hookrightarrow Y$ using the deformation to the normal cone space introduced in § 4.2. We will abbreviate the space $t_0(M^\circ_X Y)$ to M°. Recall that this scheme is flat over $Y \times \mathbb{A}^1$, with the virtual normal bundle N of Definition 4.4 as fiber over zero.

Applying the localization sequence for algebraic bordism in characteristic zero, we obtain an exact sequence

$$\Omega_*(N) \xrightarrow{i_*} \Omega_*(M^\circ) \xrightarrow{j^*} \Omega_*(Y \times (\mathbb{A}^1 \setminus 0)) \longrightarrow 0.$$

Since $i^* i_* = 0$, we thus obtain a morphism

$$s_{X/Y} : \Omega_*(Y \times (\mathbb{A}^1 \setminus 0)) \longrightarrow \Omega_{*-1}(N).$$

Finally, we can define the specialization morphism as the composition

$$\sigma_{X/Y} : \Omega_*(Y) \xrightarrow{\mathrm{pr}^*} \Omega_{*+1}(Y \times \mathbb{A}^1) \xrightarrow{s_{X/Y}} \Omega_*(N).$$

In complete analogy to the case of pull-backs along regular embeddings, we can then define pull-backs along quasi-smooth morphisms.

For the following definition, recall that, since algebraic cobordism satisfies the extended homotopy property, for any scheme X over k and any rank n vector bundle $p : E \to X$, the morphism $p^* : \Omega_*(X) \to \Omega_{*+n}(E)$ is an isomorphism. This allows us to form the inverse $(p^*)^{-1}$. In particular, we can apply this to the virtual normal bundle. Note that the same reasoning does not apply if the normal cone fails to be a vector bundle. This is precisely what makes it impossible to define pull-backs along embeddings that are not local complete intersections or more generally quasi-smooth.

Definition 4.6. Let $f : X \hookrightarrow Y$ be a quasi-smooth embedding of virtual codimension d with $Y \in \mathbf{QPr}_k$, and let $p : N \to t_0(X)$ be the projection of the virtual normal bundle. We define quasi-smooth pull-back as the composition

$$f^* : \Omega_*(Y) \xrightarrow{\sigma_{X/Y}} \Omega_*(N) \xrightarrow{(p^*)^{-1}} \Omega_{*-d}(t_0(X)) \xrightarrow{(\iota_X)_*} \Omega_{*-d}(X).$$

Remark 4.7. Unraveling the definition of $\sigma_{X/Y}(u)$ for a class $u \in \Omega_*(Y)$, we see that we first have to pull back u to $Y \times (\mathbb{A}^1 \setminus 0)$, giving us a class $\mathrm{pr}^* u \in \Omega_{*+1}(\mathbb{A}^1 \setminus 0)$, then choose

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a preimage $\tilde{u}$ of $\mathrm{pr}^* u$ in $\Omega_{*+1}(M^\circ)$, and finally intersect with the Cartier divisor N to obtain $i_N^* \tilde{u} \in \Omega_*(N)$.

Thus once we have chosen a lifting $\tilde{u} \in \Omega_{*+1}(M^\circ)$, quasi-smooth pull-back is given by $(\iota_X)_* \circ (p^*)^{-1} \circ i_N^*$. This relates the general quasi-smooth pull-back to intersecting with an effective Cartier divisor and intersecting with a zero-section.

We now relate this to an alternative definition, which is closer to the definition of virtual pull-backs of [11]. Denote by $M^\circ_{t_0(X)} Y$ the deformation to the normal cone space of the inclusion $t_0(X) \hookrightarrow Y$. The fiber over 0 of $M^\circ_{t_0(X)} Y$ is $C_{t_0(X)} Y$, the normal cone of $t_0(X)$ in Y. This is an effective Cartier divisor on $M^\circ_{t_0(X)} Y$. Again using that $i^* i_* = 0$ for this divisor and the exact localization sequence for algebraic bordism, there is a specialization morphism

$$\sigma_{t_0(X)/Y} : \Omega_*(Y) \longrightarrow \Omega_{*+1}(Y \times (\mathbb{A}^1 \setminus 0)) \longrightarrow \Omega_*(C_{t_0(X)} Y).$$

By Lemma 4.5, we have a closed immersion $j : C_{t_0(X)} Y \hookrightarrow N_X Y$ of the normal cone of $t_0(X)$ in Y into the virtual normal bundle.

Lemma 4.8. *Let $f : X \to Y$ be a quasi-smooth embedding. Denote by $j : C_{t_0(X)} Y \to N_X Y$ the inclusion of Lemma 4.5. Then*

$$\sigma_{X/Y} = j_* \circ \sigma_{t_0(X)/Y}.$$

Proof. By [6, Lemma 6.2.1], intersection with a Cartier divisor commutes with proper push-forward. □

Corollary 4.9. *Let $f : X \hookrightarrow Y$ be a quasi-smooth embedding. Then*

$$f^*(u) = \iota_{X*} \circ (p^*)^{-1} \circ j_* \circ \sigma_{X/Y}(u)$$

for all $u \in d\Omega_*(Y)$.

Remark 4.10. By Corollary 4.9, the pull-back for quasi-smooth embeddings differs from the pull-back for regular embeddings only by the inclusion of the normal cone into the virtual normal bundle.

The formula for quasi-smooth pull-backs given in Corollary 4.9 is the most convenient form to prove the expected properties of the orientations defined. The remainder of this section closely follows the construction of pull-backs for local complete intersections of Verdier [3]. We first recall some basic properties of the specialization homomorphism.

Lemma 4.11. *Let*

$$\begin{array}{ccc} X' & \overset{i'}{\hookrightarrow} & Y' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X & \overset{i}{\hookrightarrow} & Y \end{array}$$

be a Cartesian diagram in $\mathbf{QPr}_k$ *with i a closed embedding. Let $g : C_{X'} Y' \to C_X Y$ be the induced morphism of normal cones.*

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(a) *Assume that f in the above diagram is proper. Then*

$$\begin{array}{ccc} \Omega_*(Y') & \xrightarrow{\sigma_{X'/Y'}} & \Omega_*(C_{X'}Y') \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ \Omega_*(Y) & \xrightarrow[\sigma_{X/Y}]{} & \Omega_*(C_X Y) \end{array}$$

commutes.

(b) *Assume that f in the above diagram is smooth of pure relative dimension d. Then*

$$\begin{array}{ccc} \Omega_*(Y) & \xrightarrow{\sigma_{X/Y}} & \Omega_*(C_X Y) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ \Omega_{*+d}(Y') & \xrightarrow[\sigma_{X'/Y'}]{} & \Omega_{*+d}(C_{X'}Y') \end{array}$$

commutes.

Proof. By functoriality of the deformation to the normal cone space, we have a morphism $h : M^\circ_{X'}Y' \to M^\circ_X Y$ which induces $g : C_{X'}Y' \to C_X Y$ on the exceptional divisors.

Assume that f is proper. Intersection with a pseudo-divisor commutes with proper push-forward [6, Lemma 6.2.1], so the diagram

$$\begin{array}{ccc} \Omega_*(Y' \times (\mathbb{A}^1 \setminus 0)) & \xrightarrow{\sigma^*_{X'/Y'}} & \Omega_*(C_{X'}Y') \\ \downarrow & & \downarrow{\scriptstyle g_*} \\ \Omega_*(Y \times (\mathbb{A}^1 \setminus 0)) & \xrightarrow[\sigma^*_{X/Y}]{} & \Omega_*(C_X Y) \end{array}$$

commutes, and the claim follows.

For the case that f is smooth, one can explicitly compute that $M^\circ_{X'}Y' = Y' \times \mathbb{A}^1 \times_{Y \times \mathbb{A}^1} M^\circ_X Y$ and that h is the natural projection. In particular, by base change, the morphism g is smooth. The proof now proceeds as above, since intersection with a pseudo-divisor commutes with smooth pull-back [6, Lemma 6.2.1]. □

Remark 4.12. Lemma 4.11 also admits an indirect proof. Assume that there exists some Cartesian diagram such that the diagrams in (a) and (b) do not commute. Then one has an immediate contradiction to the functoriality properties of the refined pull-backs of Ω_* stated in [6, Proposition 6.6.3].

Lemma 4.13. *Let*

$$\begin{array}{ccc} X' & \overset{i'}{\hookrightarrow} & Y' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X & \underset{i}{\hookrightarrow} & Y \end{array}$$

be a homotopy Cartesian diagram in $\mathbf{dQPr}_k$ *with i a quasi-smooth closed embedding.*

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(a) *Assume that in the above diagram f is proper. Then*

$$i^* f_* = f'_* i'^*.$$

(b) *Assume that in the above diagram f is smooth. Then*

$$i'^* f^* = f'^* i^*.$$

Proof. Since the truncation functor t_0 takes homotopy Cartesian diagrams to Cartesian diagrams, we can apply Lemma 4.11 to the diagram

$$\begin{array}{ccc} t_0(X') & \xhookrightarrow{t_0(i')} & t_0(Y') \\ {\scriptstyle t_0(f')}\downarrow & & \downarrow{\scriptstyle t_0(f)} \\ t_0(X) & \xhookrightarrow{t_0(i)} & t_0(Y). \end{array}$$

Using basic functoriality properties of the cotangent complex, we obtain a further commutative diagram:

$$\begin{array}{ccc} C_{X'}Y' & \longrightarrow & N_{X'}Y' \\ {\scriptstyle g}\downarrow & & \downarrow \\ C_X Y & \longrightarrow & N_X Y. \end{array}$$

Here, g is the morphism of normal cones induced by the base change used in Lemma 4.11.

Now assume that f is proper. Combining Lemma 4.11 with the above diagram, it follows that

$$\begin{array}{ccccccc} \Omega_*(Y') & \xrightarrow{\sigma_{X'/Y'}} & \Omega_*(C_{X'/Y'}) & \longrightarrow & \Omega_*(N_{X'}Y') & \longrightarrow & \Omega_*(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle g_*} & & \downarrow & & \downarrow{\scriptstyle f'_*} \\ \Omega_*(Y) & \xrightarrow[\sigma_{X/Y}]{} & \Omega_*(C_{X/Y}) & \longrightarrow & \Omega_*(N_X Y) & \longrightarrow & \Omega_*(X) \end{array}$$

commutes. Since the composition of the horizontal morphisms is respectively i^* and i'^*, the claim follows. The case of f smooth is analogous. □

Our next task is to verify functoriality of the orientation for quasi-smooth closed embeddings. Apart from the fact that we have to keep track of the embeddings of the normal cones in the virtual normal bundles, the proof proceeds in exact analogy to the case of regular embeddings of discrete schemes.

Proposition 4.14. *Let $X \overset{i}{\hookrightarrow} Y \overset{j}{\hookrightarrow} Z$ be quasi-smooth embeddings. Then*

$$(j \circ i)^* = i^* j^*.$$

Proof. Let us first assume that $t_0(j) : t_0(Y) \hookrightarrow t_0(Z)$ is given by the embedding

$$t_0(Y) \hookrightarrow C_Y Z \hookrightarrow N_Y Z$$

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of Y in the virtual normal bundle. Let $p : N_Y Z \to t_0(Y)$ be the projection. We have an isomorphism [4, Proof of Theorem 6.5]

$$C_X(N_Y Z) \simeq C_X Y \times_{t_0(X)} (N_Y Z \times_{t_0(Y)} t_0(X))$$

inducing a morphism

$$q' : C_X(N_Y Z) \longrightarrow C_X Y.$$

Additionally assume that the virtual normal bundle $N_X Z$ splits as $N_Y Z \oplus N_X Y$, and let $q : N_X Y \oplus N_Y Z \to N_X Y$ be the projection, giving us a commutative diagram

$$\begin{array}{ccc} C_X(N_Y Z) & \longrightarrow & N_X Y \oplus N_Y Z \\ \downarrow{\scriptstyle q'} & & {\scriptstyle q}\downarrow \\ C_X Y & \longrightarrow & N_X Y. \end{array}$$

Here, the lower horizontal morphism is the canonical inclusion of the normal cone in the virtual normal bundle. In this situation, we have an isomorphism of the open deformation to the normal cone spaces $M^{\circ}_{t_0(X)/N_Y Z} \cong M^{\circ}_{t_0(X)/t_0(Y)} \times_{t_0(Y)} N_Y Z$ compatible with the projections to $\mathbb{A}^1$ [3, Corollaire 2.18]. We thus obtain a commutative diagram:

$$\begin{array}{ccccc} \Omega_*(Y) & \xrightarrow{\sigma_{X/Y}} & \Omega_*(C_X Y) & \longrightarrow & \Omega_*(N_X Y) \\ {\scriptstyle p^*}\downarrow & & {\scriptstyle q'^*}\downarrow & & {\scriptstyle q^*}\downarrow \\ \Omega_{*+d}(Z) & \xrightarrow[\sigma_{X/Z}]{} & \Omega_{*+d}(C_X Z) & \longrightarrow & \Omega_{*+d}(N_X Y \oplus N_Y Z). \end{array}$$

To prove our claim, since $j^* = (p^*)^{-1}$, it suffices to prove that

$$(j \circ i)^* p^* = i^*.$$

Let $\pi : N_X Y \to X$ and $\rho : N_X Y \oplus N_Y Z \to X$ be the structure morphisms. Since $q^* \circ \pi^* = \rho^*$ is an isomorphism, it suffices to prove that

$$(\pi \circ q)^* \circ (j \circ i)^* \circ p^* = q^* \circ \pi^* \circ i^*.$$

But this is immediate from the commutativity of the above diagram. The reduction of the general case to the special case treated here is a standard application of the double deformation space

$$M^{\circ}_{X \times \mathbb{A}^1 / M^{\circ}_{Y/Z}} \longrightarrow \mathbb{A}^1 \times \mathbb{A}^1$$

obtained by applying the deformation to the normal cone construction to the embedding $X \times \mathbb{A}^1 \hookrightarrow M^{\circ}_{Y/Z}$, as in [11, Theorem 4.8], [6, Theorem 6.6.5], or [3, Theoreme 4.4]. □

4.4. Pull-backs for quasi-smooth morphisms

After having defined orientations for quasi-smooth closed embeddings, we now move to general quasi-smooth morphisms. Since we are only working with quasi-projective derived schemes, a smoothing is always available.

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Lemma 4.15. *Let $f : X \to Y$ be a quasi-smooth morphism in $\mathbf{dQPr}_k$. Assume that $f = p \circ i = p' \circ i'$ are two smoothings of f. Then*

$$i^* p^* = i'^* p'^*.$$

Proof. Let $X \overset{i}{\hookrightarrow} P \overset{p}{\to} Y$ and $X \overset{i'}{\hookrightarrow} P' \overset{p'}{\to} Y$ be the two smoothings. Using the diagonal embedding $X \to P \times_Y P'$ and the Cartesian diagram

$$\begin{CD} P' @>>> P \times_Y P' \\ @VVV @VVV \\ X @>>> P, \end{CD}$$

one quickly reduces to the case where $f : X \hookrightarrow Y$ is a quasi-smooth embedding, and one has a smoothing

$$\begin{array}{ccc} & & P \\ & \overset{i}{\nearrow} & \downarrow{\scriptstyle p} \\ X & \underset{f}{\hookrightarrow} & Y. \end{array}$$

We then have the Cartesian and homotopy Cartesian diagram

$$\begin{CD} P' @>{f'}>> P \\ @A{s}AA @VV{p}V \\ X @>>{f}> Y \end{CD} \qquad (p' : P' \to X)$$

with $f' \circ s = i$. Since $p' \circ s = \mathrm{id}$ and p' is smooth, we obtain that L_s is of Tor-dimension $\leqslant 1$. The morphism s is an embedding, since f' and $i = f' \circ s$ are embeddings (in particular, f' is separated), and thus s is a quasi-smooth embedding. By Proposition 4.14, we have

$$i^* p^* = s^* f'^* p^*.$$

Using Lemma 4.13(a), the right-hand side is equal to $s^* p'^* f^*$. Since $s^* p'^* = \mathrm{id}$, the claim follows. □

This allows us to define orientations for quasi-smooth morphisms in Ω_*.

Definition 4.16. Let $f : X \to Y$ be a quasi-smooth morphism in $\mathbf{dQPr}_k$, and let $X \overset{i}{\hookrightarrow} P \overset{p}{\to} Y$ be a smoothing of f with p of relative dimension n and i of virtual codimension m. Let $d = n - m$. We then define *quasi-smooth pull-back* as the composition

$$f^* : \Omega_*(Y) \xrightarrow{p^*} \Omega_{*+n}(P) \xrightarrow{i^*} \Omega_{*+d}(X).$$

Here, i^* is the quasi-smooth pull-back for quasi-smooth embeddings of Definition 4.6.

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Theorem 4.17. *Let k be a field of characteristic zero. Then Ω_* is an oriented Borel–Moore functor on $\mathbf{dQPr}_k$ of geometric type with quasi-smooth pull-backs.*

Proof. It remains to verify the axioms of Definition 2.16. To prove functoriality and thus (QS2), choose smoothings as in [6, Remark 5.1.2] such that we obtain a ladder of smoothings

$$\begin{array}{ccccc} X & \longrightarrow & P_1 & \longrightarrow & P \\ & \searrow & \downarrow & & \downarrow \\ & & Y & \longrightarrow & P_2 \\ & & & \searrow & \downarrow \\ & & & & Z \end{array}$$

with the square Cartesian and homotopy Cartesian. To prove axiom (QS3), factor f as $p \circ i$. The statement then immediately follows from the individual statements for p and i. For p, the statement is clear, since p is smooth; for i, this is Lemma 4.13(b). The proof of (QS4) is straightforward from the definitions. Since axiom (QS5) only applies to smooth and thus underived schemes, we can deduce it from the corresponding statement in [6, Proposition 7.2.2]. □

Since $d\Omega_*$ is the universal Borel–Moore functor with orientations for quasi-smooth morphisms, we obtain the following corollaries.

Corollary 4.18. *Let k be a field of characteristic zero. We then obtain a classifying morphism*

$$\vartheta_\Omega : d\Omega_* \to \Omega_*.$$

Proof. This immediately follows from the universal property of $d\Omega_*$ as the universal Borel-Moore homology theory on $\mathbf{dQPr}_k$ with quasi-smooth pull-backs. Since Ω_* is also such a theory, the existence of the classifying morphism ϑ_Ω follows. □

Corollary 4.19. *For all $X \in \mathbf{dQPr}_k$, the natural transformation*

$$\vartheta_{d\Omega} : \Omega_*(X) \longrightarrow d\Omega_*(X)$$

is injective.

Proof. We want to show that ϑ_Ω is a left inverse. To prevent confusion, in the following we will decorate pull-backs and push-forwards with the homology theory they are taken in. Let $f : Y \to X$ be a bordism cycle. Since ϑ_Ω commutes with proper push-forwards and smooth pull-backs, we have

$$\begin{aligned} \vartheta_\Omega \circ \vartheta_{d\Omega}([Y \to X]) &= \vartheta_\Omega(f_*^{d\Omega} \pi_{Y,d\Omega}^*[1]) \\ &= f_*^{\Omega} \pi_{Y,\Omega}^* \vartheta_\Omega[1] \\ &= [Y \to X]. \end{aligned}$$

□

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5. Spivak's theorem

We now have two Borel–Moore functors of geometric type on $\mathbf{dQPr}_k$ at our disposal: derived algebraic bordism $d\Omega_*$ and algebraic bordism Ω_*. Both are equipped with orientations for quasi-smooth morphisms. We have also constructed natural transformations $\vartheta_{d\Omega} : \Omega_* \to d\Omega_*$ and $\vartheta_\Omega : d\Omega_* \to \Omega_*$ between these theories. Here, $\vartheta_{d\Omega}$ is induced by the universal property of Ω_* as the universal Borel-Moore homology theory of geometric type on $\mathbf{dQPr}_k$, and ϑ_Ω is induced by the universal property of $d\Omega$ as the universal Borel-Moore homology theory on $\mathbf{dQPr}_k$ with quasi-smooth pull-backs.

The goal of this section is to compare how the orientations for quasi-smooth morphisms of these two theories interact. In the following, we will prove a Grothendieck–Riemann–Roch-type result stating that $\vartheta_{d\Omega}$ in fact commutes with these orientations. As a direct consequence of this result, we obtain an algebraic version of Spivak's theorem for derived cobordism in the differentiable setting.

The strategy for proving our Grothendieck–Riemann–Roch-type result is the same as in the classical case treated in [3] or [4]. We first check the compatibility of the orientations in specific settings, and later reduce the general case to the known setting.

A word on notation is in order. In the following, we will often encounter formulas involving pull-backs and push-forwards in different homology theories. Except in special cases, we have chosen not to decorate these operations with the homology theories they are taken in. We hope that this does not lead to confusion.

5.1. Preliminaries on $d\Omega_*$

For certain types of bordism classes, we want to reduce pull-back along a quasi-smooth embedding in $d\Omega_*$ to intersecting with a effective Cartier divisor. The key ingredient in this step is again deformation to the normal cone.

Recall from Corollary 4.19 that $\vartheta_{d\Omega}$ is injective. We let $d\Omega_*^{\mathrm{cl}}$ denote the image of $\vartheta_{d\Omega}$. For any $X \in \mathbf{dQPr}_k$, classes in $d\Omega_*^{\mathrm{cl}}$ are thus given by proper morphisms $[Y \to X]$, where Y is assumed to be smooth.

Lemma 5.1. *Let $f : X \hookrightarrow Y$ be a quasi-smooth embedding, and let $g : Y' \to Y$ be a morphism of derived schemes such that g factors over the normal bundle $N_X Y$. Then we have an equivalence of derived schemes*

$$Y' \times^h_Y X \simeq Y' \times^h_{N_X Y} X.$$

Proof. This follows immediately from the criterion that a morphism of derived schemes is an equivalence if it induces an isomorphism on the truncation t_0 and an equivalence of the cotangent complexes. □

Proposition 5.2. *Let $f : X \to Y$ be a quasi-smooth embedding, and let $u \in d\Omega_*^{\mathrm{cl}}(Y)$. Then there exists a class $u' \in d\Omega_*^{\mathrm{cl}}(N)$ such that*

$$f^*(u) = s^*(u').$$

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Proof. We let M be the truncation $t_0(M_X Y)$ of the deformation to the normal cone space of X in Y, and let $\mathrm{pr} : Y \times \mathbb{A}^1 \to Y$ be the projection on the first factor. We then obtain a class $\mathrm{pr}^* u \in d\Omega_{*+1}(Y \times \mathbb{A}^1)$. Identify $Y \times \mathbb{A}^1$ with an open subset of M. Since $\mathrm{pr}^* u$ is in $d\Omega^{\mathrm{cl}}_{*+1}(Y \times \mathbb{A}^1)$, we can extend it to a class $\tilde{u}$ in $d\Omega^{\mathrm{cl}}_{*+1}(M)$. Since the fiber over zero of M is the virtual normal bundle $N_X Y$, the class $\tilde{u}$ provides a bordism between $u \in d\Omega^{\mathrm{cl}}_*(X)$ and a class $u' \in d\Omega_*(N)$. The remaining claim follows from Lemma 5.1. □

Remark 5.3. Once we haven chosen a lifting $\tilde{u}$, we obtain the following explicit formula for quasi-smooth pull-back:

$$f^*(u) = s^* \circ i^*(\tilde{u}).$$

This reduces all questions about quasi-smooth pull-backs to intersection with a Cartier divisor followed by pull-back along a zero-section.

We need to compare our Euler classes with more traditionally defined Chern classes in cases of overlap. The injectivity of $\vartheta_{d\Omega}$ allows us to define Chern class operators on $d\Omega^{\mathrm{cl}}_*$ as follows. With $X \in \mathbf{QPr}_k$, let $E \to X$ be a vector bundle of rank $e+1$ with associated projective bundle $q : \mathbb{P}(E) \to X$. We denote the universal line bundle as $\mathcal{O}(1)$, and let ξ be its first Chern class operator on $d\Omega_*(\mathbb{P}(E))$. Composing with pull-back along q, we have operators

$$\phi_j := \xi^j \circ q^* : d\Omega_*(X) \longrightarrow d\Omega_{*+e-j}(X).$$

Let

$$\Phi : \bigoplus_{j=0}^{n} d\Omega_{*-e+j}(X) \longrightarrow d\Omega_*(\mathbb{P}(E))$$

be the sum over the operators ϕ_j.

Lemma 5.4. *The morphism*

$$\Phi : \bigoplus_{j=0}^{n} d\Omega^{\mathrm{cl}}_{*-e+j}(X) \longrightarrow d\Omega^{\mathrm{cl}}_*(\mathbb{P}(E))$$

is an isomorphism.

Proof. Since $d\Omega^{\mathrm{cl}}_*$ is precisely the image of $\vartheta_{d\Omega}$, the proof follows from the fact that $\vartheta_{d\Omega}$ commutes with smooth pull-back and first Chern class operators, and that the projective bundle theorem holds in Ω_*. □

The following corollary reduces many questions about vector bundles to sums of line bundles via the splitting principle.

Corollary 5.5. *The morphism*

$$q^* : d\Omega^{\mathrm{cl}}_*(X) \longrightarrow d\Omega^{\mathrm{cl}}_{*+e}(\mathbb{P}(E))$$

is injective.

Using the method of [6, § 4.1.7], as an immediate consequence of the projective bundle theorem we obtain the existence of uniquely defined operators

$$\tilde{c}_i(E) : d\Omega_*^{\mathrm{cl}}(X) \longrightarrow d\Omega_{*-i}^{\mathrm{cl}}(X)$$

satisfying the relations

$$\sum_{i=0}^{n} (-1)^i c_1(\mathcal{O}(1))^{n-i} \circ q^* \circ \tilde{c}_i(E) = 0.$$

Note that, by construction, $c_1 = \tilde{c}_1$. These Chern classes satisfy all the expected properties, e.g., the Whitney product formula. In particular, if $E = \bigoplus_{i=0}^{e} L_i$ is a direct sum of line bundles, then the jth Chern class of E is the jth symmetric polynomial in the $c_1(L_i)$.

A further immediate consequence of Lemma 5.4 is an explicit formula for the pull-back along a zero-section of a vector bundle.

Corollary 5.6. *Let $X \in \mathbf{QPr}_k$, let $p : E \to X$ be a rank r vector bundle, and let $q : \mathbb{P}(E \oplus 1) \to X$ be the corresponding projective bundle with universal quotient bundle ξ. Let $i : E \hookrightarrow \mathbb{P}(E \oplus 1)$ denote the canonical open embedding. Then, for any $u \in d\Omega_*^{\mathrm{cl}}(\mathbb{P}(E \oplus 1))$, we have*

$$(p^*)^{-1}(i^* u) = q_*(c_r(\xi) \cap u).$$

We show compatibility with the definition of the top Chern class given in Definition 3.8.

Lemma 5.7. *Let $X \in \mathbf{QPr}_k$, and let $E \to X$ be a vector bundle of rank e. Then*

$$c_e(E) \cap u = \tilde{c}_e(E) \cap u$$

for all $u \in d\Omega_*^{\mathrm{cl}}(X)$.

Proof. Let $q : \mathbb{P}(E) \to X$ be the projection. By the injectivity of q^*, we can apply the splitting principle and assume that E is a direct sum of line bundles $\bigoplus_{i=1}^{e} L_i$. Then $\tilde{c}_e(E) = c_1(L_1) \circ \cdots \circ c_1(L_e)$ by the Whitney product formula.

We prove the claim by induction on rank. Let $u = [Y \to X]$. For $e = 1$, the claim holds by construction of the Chern classes. Unraveling the definitions, the left-hand side is given by the homotopy fiber product

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \Big\downarrow & & \Big\downarrow \\ & & X \\ & & \Big\downarrow{\scriptstyle s_0} \\ X & \xrightarrow[s_0]{} & L_1 \oplus \cdots \oplus L_e. \end{array} \tag{2}$$

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By induction, the right-hand side is given by the homotopy fiber product

$$
\begin{array}{ccccc}
Y'' & \longrightarrow & Y' & \longrightarrow & Y \\
\downarrow & & \downarrow & & \downarrow \\
\downarrow & & \downarrow & & X \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle s_0} \\
\downarrow & & X & \xrightarrow{s_0} & L_1 \oplus \cdots L_{e-1} \\
\downarrow & & \downarrow{\scriptstyle s_0} & & \\
X & \xrightarrow[s_0]{} & L_e. & &
\end{array}
$$

Using the universal property of the homotopy fiber product (2), we obtain a morphism $Y'' \to Y'$. Since we have always used the zero-sections, this is an isomorphism on the underlying classical schemes. By a straightforward calculation, it also induces an equivalence of the cotangent complexes. It thus is an equivalence. □

5.2. A Grothendieck–Riemann–Roch result

We next show that $\vartheta_{d\Omega}$ commutes with Chern class operators and intersection with effective Cartier divisors.

Lemma 5.8. *Let $X \in \mathbf{QPr}_k$ and let $E \to X$ be a vector bundle. Then the following diagram commutes:*

$$
\begin{array}{ccc}
\Omega_*(X) & \xrightarrow{\vartheta_{d\Omega}} & d\Omega_*(X) \\
{\scriptstyle c_p(E)\bigcap\_}\downarrow & & \downarrow{\scriptstyle c_p(E)\bigcap\_} \\
\Omega_{*-r}(X) & \xrightarrow{\vartheta_{d\Omega}} & d\Omega_{*-r}(X).
\end{array}
$$

Proof. Let $q : \mathbb{P}(E) \to X$ be the projection. Since q^* is injective both for Ω_* and $d\Omega_*^{\mathrm{cl}}$, we can apply the splitting principle and assume that E is a direct sum of line bundles L_i. Then, in both Ω_* and $d\Omega_*$, the operation $c_p(E)$ is the pth symmetric polynomial in the $c_1(L_i)$. Since $\vartheta_{d\Omega}$ commutes with first Chern classes, the claim follows. □

For the proof of the next lemma, we have to recall one of the key technical results in [6]. Levine and Morel prove that, for any finite-type scheme X over k and any pseudo-divisor D on X, there exists an isomorphism between $\Omega_*(X)$ and a group $\Omega_*(X)_D$, whose generators are classes $[Y \xrightarrow{f} X]$ such that either $f(Y)$ is contained in D or f^*D is a strict normal crossing divisor. We call this result *Levine's moving lemma* [6, Theorem 6.4.12].

Lemma 5.9. *Let $i_D : D \hookrightarrow X$ be an effective Cartier divisor on a scheme X. Then the following diagram commutes:*

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$$\begin{array}{ccc} \Omega_*(X) & \xrightarrow{\vartheta_{d\Omega}} & d\Omega_*(X) \\ {\scriptstyle i_D^*}\downarrow & & \downarrow{\scriptstyle i_D^*} \\ \Omega_{*-1}(D) & \xrightarrow{\vartheta_{d\Omega}} & d\Omega_{*-1}(D). \end{array} \tag{3}$$

Proof. By Levine's moving lemma, it suffices to treat the cases of bordism cycles $f : Y \to X$ such that f factors through D or the fiber product of f and i_D is a strict normal crossing divisor of Y.

We first assume that f factors through D. Let $f^D : Y \to D$ be the induced morphism, i.e., $f = i_D \circ f^D$. By definition,

$$\vartheta_{d\Omega} \circ i_D^* = \vartheta_{d\Omega}(f_*^D(c_1(f^*\mathcal{O}_X(D)) \cap 1_Y)).$$

On the other hand, by Lemma 5.1,

$$i_D^* \circ \vartheta_{d\Omega} = c_1(i_D^*\mathcal{O}_X(D)) \cap [Y \xrightarrow{f^D} D].$$

The claim follows by applying the projection formula (A3) of the axioms of Borel–Moore functor and observing that $\vartheta_{d\Omega}$ commutes with proper push-forward and first Chern classes.

Let us now assume that $D' := D \times_X Y$ is a strict normal crossing divisor of Y. This implies that we have a Tor-independent[2] diagram:

$$\begin{array}{ccc} D' & \xrightarrow{i_{D'}} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ D & \xrightarrow[i_D]{} & X. \end{array}$$

In this case,

$$\vartheta_{d\Omega} \circ i_D^* = \vartheta_{d\Omega}(f'_*(\zeta_{D*}([D' \to |D'|]))).$$

Recall that the class $[D' \to |D'|]$ is not given by a morphism, but instead by equation (1). Since the above diagram is Tor-independent, it is a homotopy fiber diagram, and thus

$$\begin{aligned} i_D^* \circ \vartheta_{d\Omega}([Y \to X]) &= i_D^* f_*(1_Y) \\ &= f'_* i_{D'}^* 1_Y \\ &= f'_* 1_{D'}. \end{aligned}$$

The claim then follows by applying relation $\mathcal{R}_*^{\mathrm{SNC}}$ and observing that $\vartheta_{d\Omega}$ commutes with proper push-forwards and first Chern classes, and is compatible with the formal group law. □

[2]Recall that this means that $\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_D) = 0$ for $i > 0$.

Theorem 5.10. *Let $f : X \hookrightarrow Y$ be a quasi-smooth embedding of virtual codimension d in* $\mathbf{dQPr}_k$ *with Y a scheme. Then the following diagram commutes:*

$$\begin{array}{ccc} \Omega_*(Y) & \xrightarrow{\vartheta_{d\Omega}} & d\Omega_*(Y) \\ {\scriptstyle f^*_{\Omega}}\downarrow & & \downarrow{\scriptstyle f^*_{d\Omega}} \\ \Omega_{*-d}(X) & \xrightarrow{\vartheta_{d\Omega}} & d\Omega_{*-d}(X). \end{array}$$

Proof. Let M be the truncation of the deformation to the normal cone space $M^\circ_X Y$. By Remark 4.7 and Remark 5.3, after we have chosen a lifting $\tilde{u} \in \Omega_{*+1}(M)$ of $\mathrm{pr}^* u \in \Omega_{*+1}(Y \times \mathbb{A}^1)$, pull-back along f is given by $s^* \circ i^*$ both in Ω_* and in $d\Omega_*$. Here, $i : N \to M$ is the inclusion of the virtual normal bundle N as Cartier divisor on M, and $s : t_0(X) \to N$ is the zero-section. Note that by the localization theorem for Ω_* a suitable lifting $\tilde{u}$ always exists.

The case of intersection with a Cartier divisor was treated in Lemma 5.9. It remains to prove that $\vartheta_{d\Omega}$ commutes with intersection with the zero-section. So let $X \in \mathbf{QPr}_k$, let $p : E \to X$ be a vector bundle, let $q : \mathbb{P}(E \oplus 1)$ be the associated projective bundle with open embedding $i : E \to \mathbb{P}(E \oplus 1)$ and universal quotient bundle ξ, and let $u \in \Omega_*(E)$. By the localization theorem for Ω_*, there exists a class $\tilde{u} \in \Omega_*(\mathbb{P}(E \oplus 1))$ such that $i^*(\tilde{u}) = u$. Since the projective bundle theorem holds for Ω_*, going around the left-hand side of our diagram is given by

$$\vartheta_{d\Omega} \circ s^*(u) = \vartheta_{d\Omega} \circ q_*(c_d(\xi) \cap \tilde{u}).$$

On the other hand, since $\vartheta_{d\Omega}(u)$ by definition lies in $d\Omega_*^{\mathrm{cl}}$, we can apply Corollary 5.6 to compute going around the right-hand side:

$$s^* \circ \vartheta_{d\Omega}(u) = q_*(c_d(\xi) \cap \vartheta_{d\Omega}(\tilde{u})).$$

Since $\vartheta_{d\Omega}$ commutes with Chern classes by Lemma 5.8 and with proper push-forward, the claim follows. □

We can now use the previous theorem to obtain a Grothendieck–Riemann–Roch-type result for the natural transformation $\vartheta_{d\Omega}$.

Corollary 5.11 (Grothendieck–Riemann–Roch for $\vartheta_{d\Omega}$). *Let $f : X \to Y$ be a quasi-smooth morphism of relative virtual dimension d in* $\mathbf{dQPr}_k$. *Then the following diagram commutes:*

$$\begin{array}{ccc} \Omega_*(Y) & \xrightarrow{\vartheta_{d\Omega}} & d\Omega_*(Y) \\ {\scriptstyle f^*_{\Omega}}\downarrow & & \downarrow{\scriptstyle f^*_{d\Omega}} \\ \Omega_{*+d}(X) & \xrightarrow{\vartheta_{d\Omega}} & d\Omega_{*+d}(X). \end{array}$$

Proof. Let $u = [V \xrightarrow{g} Y] \in \Omega_*(Y)$ be a bordism cycle. By definition, $u = g_* 1_V$, where $g : V \to Y$ is proper and V is smooth. By factoring f as a quasi-smooth embedding and

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a smooth morphism, we can assume that f is an embedding (the case of f smooth is clear). We can now form the homotopy fiber product

$$\begin{array}{ccc} W & \xrightarrow{f'} & V \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

Note that $f' : W \to V$ is a quasi-smooth embedding with V smooth. Thus Theorem 5.10 is applicable to f'. By axiom (QS3), $f^*u = g_* f^* 1_V = g'_* f'^* 1_V$. Applying $\vartheta_{d\Omega}$ to this equation

$$\begin{aligned} \vartheta_{d\Omega}(f^*u) &= \vartheta_{d\Omega}(g'_* f'^* 1_V) \\ &= g'_* \vartheta_{d\Omega}(f'^* 1_V) \\ &= g'_* f'^* \vartheta_{d\Omega}(1_V) \\ &= f^* g_* \vartheta_{d\Omega}(1_V) \\ &= f^* \vartheta_{d\Omega}(g_* 1_V) \\ &= f^* \vartheta_{d\Omega}(u) \end{aligned}$$

the result immediately follows. □

We are now able to deduce an algebraic version of Spivak's theorem from the Grothendieck–Riemann–Roch theorem for $\vartheta_{d\Omega}$.

Theorem 5.12 (Algebraic Spivak theorem). *For all $X \in \mathbf{dQPr}_k$, the morphism*

$$\vartheta_{d\Omega} : \Omega_*(X) \longrightarrow d\Omega_*(X)$$

is an isomorphism.

Proof. By Lemma 4.19, $\vartheta_{d\Omega}$ is a right inverse to ϑ_{Ω}. We have to show that it is also a left inverse. Let $f : Y \to X$ be a derived bordism cycle. By Corollary 5.11, $\vartheta_{d\Omega}$ commutes with pull-backs along quasi-smooth morphisms. We then have

$$\begin{aligned} \vartheta_{d\Omega} \circ \vartheta_{\Omega}([Y \to X]) &= \vartheta_{d\Omega}(f_*^{\Omega} \pi_{Y,\Omega}^*[1]) \\ &= f_*^{d\Omega} \vartheta_{d\Omega}(\pi_{Y,\Omega}^*[1]) \\ &= f_*^{d\Omega} \pi_{Y,d\Omega}^* \vartheta_{d\Omega}([1]) \\ &= [Y \to X]. \end{aligned}$$

□

Since there exist non-trivial derived schemes of negative virtual dimension, we by no means know a priori that $d\Omega_n(X) = 0$ for $n < 0$. This is immediately implied by the previous theorem.

Corollary 5.13. *Let $n < 0$. For all $X \in \mathbf{dQPr}_k$, we have*

$$d\Omega_n(X) = 0.$$

Theorem 5.12 provides a geometric explanation why virtual fundamental classes exist for quasi-smooth derived schemes. It implies that every quasi-smooth projective derived scheme bords to a smooth projective scheme. We make this precise in the following corollary. We write $[Y]$ for the class $[Y \to \mathrm{pt}]$ in $d\Omega_*(k)$.

Corollary 5.14. *For every projective quasi-smooth derived scheme X of virtual dimension n there exists a smooth projective scheme Y of dimension n such that*

$$[Y] = [X] \in d\Omega_n(k).$$

Proof. By Theorem 5.12, the morphism $\vartheta_{d\Omega} : \Omega_n(k) \to d\Omega_n(k)$ is an isomorphism. Since the image of $\vartheta_{d\Omega}$ consists of smooth projective schemes, the claim follows. □

A further consequence of the comparison theorem is that the projective bundle theorem, the extended homotopy relation, and the cellular decomposition relation hold for $d\Omega_*$.

Corollary 5.15. *$d\Omega_*$ is an oriented Borel-Moore homology theory with quasi-smooth pull-backs as in Definition 2.19.*

Remark 5.16. In differential geometry, the corresponding result of Theorem 5.12 admits a direct proof [5, 15]. There every derived manifold in either the sense of Spivak or Joyce admits a global presentation as zero-set of a section of a vector bundle. Such a derived manifold can be made bordant to a manifold by arranging the section to be transverse.

6. Derived Algebraic Cobordism

In this final section, we want to study the cohomology theory $d\Omega^*$ on smooth projective schemes $\mathbf{Sm}_k$ associated to the Borel–Moore homology theory $d\Omega_*$.

For $X \in \mathbf{Sm}_k$ of pure dimension d, we set $d\Omega^n(X) := d\Omega_{d-n}(X)$. The cohomology theory $d\Omega^*$ is then equipped with an external product. Since for a smooth scheme X the diagonal embedding $\delta : X \to X \times X$ is a local complete intersection morphism, we can define an intersection product via

$$d\Omega^*(X) \otimes d\Omega^*(X) \longrightarrow d\Omega^*(X \times X) \xrightarrow{\delta^*} d\Omega^*(X).$$

Here, the first arrow is the external product. Given two cobordism classes $[Y \to X]$ and $[Z \to X]$, we will denote the intersection product by $[Y] \cdot [Z]$. The intersection product turns $d\Omega^*(X)$ into a commutative graded ring with unit $[X \xrightarrow{\mathrm{id}} X]$. Unraveling the definitions, the intersection product is represented by the homotopy fiber product

$$\begin{array}{ccc} (Y \times Z) \times^h_{X \times X} X & \longrightarrow & Y \times Z \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times X. \end{array} \tag{4}$$

For the proof of next theorem, recall that, for two subsets U and V of a set X, and denoting by Δ the diagonal of X in $X \times X$, we have the set-theoretic identity $(U \times V) \cap \Delta = U \cap V$. In homological algebra, given a k-algebra A and two A-modules M and N,

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this translates to the identity $(M \otimes_k N) \otimes_{A \otimes A} A \simeq M \otimes_A N$. We refer to this as *reduction to the diagonal*. Using this, we can then prove the following result on the intersection product.

Theorem 6.1. *Let $X \in \mathbf{Sm}_k$. Then the intersection product is represented by the homotopy fiber product:*

$$[Y] \cdot [Z] = [Y \times_X^h Z] \in d\Omega^*(X).$$

Proof. Since the base k is a field, the proof follows from reduction to the diagonal, as in Serre [14, V.B.1]. □

In particular, since the natural transformation ϑ_Ω commutes with the intersection product, we obtain the formula

$$[Y] \cdot [Z] = \vartheta_\Omega([Y \times_X^h Z])$$

in algebraic cobordism $\Omega^*(X)$ for algebraic cobordism classes $[Y \to X]$ and $[Z \to X]$.

Acknowledgements. The authors would like to thank Barbara Fantechi for explaining the virtual pull-backs of [11] and Gabriele Vezzosi for the suggestion of imposing the formal group law instead of trying to prove it exists, which the authors tried unsuccessfully for quite some time. We would also like to thank David Ben-Zvi for helpful comments on various drafts of this work, and Bertrand Toën for some help with deformation to the normal cone for derived schemes.

Finally, we would like to thank the Mathematical Institute of the University of Bonn for its hospitality, and the SFB/TR 45 'Periods, Moduli Spaces and Arithmetic of Algebraic Varieties' of the DFG (German Research Foundation) for financial support.

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