Random field representations for stochastic elliptic boundary value problems and statistical inverse problems

A. N O U Y[1] and C. S O I Z E[2]

[1]*Ecole Centrale de Nantes, LUNAM Université, GeM UMR CNRS 6183, 1 rue de la Noe, 44321 Nantes, France*
email: `anthony.nouy@ec-nantes.fr`
[2]*Université Paris-Est, Laboratoire Modélisation et Simulation Multi-Echelle, MSME UMR 8208 CNRS,*
5 bd Descartes, 77454 Marne-la-Valle, France
email: `christian.soize@univ-paris-est.fr`

This paper presents new results allowing an unknown non-Gaussian positive matrix-valued random field to be identified through a stochastic elliptic boundary value problem, solving a statistical inverse problem. A new general class of non-Gaussian positive-definite matrix-valued random fields, adapted to the statistical inverse problems in high stochastic dimension for their experimental identification, is introduced and its properties are analysed. A minimal parameterisation of discretised random fields belonging to this general class is proposed. Using this parameterisation of the general class, a complete identification procedure is proposed. New results of the mathematical and numerical analyses of the parameterised stochastic elliptic boundary value problem are presented. The numerical solution of this parametric stochastic problem provides an explicit approximation of the application that maps the parameterised general class of random fields to the corresponding set of random solutions. This approximation can be used during the identification procedure in order to avoid the solution of multiple forward stochastic problems. Since the proposed general class of random fields possibly contains random fields which are not uniformly bounded, a particular mathematical analysis is developed and dedicated approximation methods are introduced. In order to obtain an algorithm for constructing the approximation of a very high-dimensional map, complexity reduction methods are introduced and are based on the use of sparse or low-rank approximation methods that exploit the tensor structure of the solution which results from the parameterisation of the general class of random fields.

Key words: 60G60,65M32,60H35,62F99,35J70

Notations

A lower-case letter, y, is a real deterministic variable.
A boldface lower-case letter, $\mathbf{y} = (y_1, \ldots, y_N)$ is a real deterministic vector.
An upper-case letter, Y, is a real random variable.
A boldface upper-case letter, $\mathbf{Y} = (Y_1, \ldots, Y_N)$ is a real random vector.
A lower- (or an upper-) case letter between brackets, $[a]$ (or $[A]$), is a real deterministic matrix.

A boldface upper-case letter between brackets, [**A**], is a real random matrix.

E: Mathematical expectation.

$\mathbb{M}_{N,m}(\mathbb{R})$: set of all the $(N \times m)$ real matrices.

$\mathbb{M}_m(\mathbb{R})$: set of all the square $(m \times m)$ real matrices.

$\mathbb{M}_n^S(\mathbb{R})$: set of all the symmetric $(n \times n)$ real matrices.

$\mathbb{M}_n^+(\mathbb{R})$: set of all the symmetric positive-definite $(n \times n)$ real matrices.

$\mathbb{M}_m^{SS}(\mathbb{R})$: set of all the skew-symmetric $(m \times m)$ real matrices.

$\mathbb{M}_n^U(\mathbb{R})$: set of all the upper triangular $(n \times n)$ real matrices with strictly positive diagonal.

$\mathbb{O}(N)$: set of all the orthogonal $(N \times N)$ real matrices.

$\mathbb{V}_m(\mathbb{R}^N)$: compact Stiefel manifold of $(N \times m)$ orthogonal real matrices.

tr: trace of a matrix.

$< \mathbf{y}, \mathbf{z} >_2$: Euclidean inner product of $\mathbf{y}$ with $\mathbf{z}$ in $\mathbb{R}^n$.

$\|\mathbf{y}\|_2$: Euclidean norm of a vector $\mathbf{y}$ in $\mathbb{R}^n$.

$\|A\|_2$: Operator norm subordinated to $\mathbb{R}^n$: $\|A\|_2 = \sup_{\|\mathbf{y}\|_2=1} \|A\mathbf{y}\|_2$.

$\|A\|_F$: Frobenius norm of a matrix such that $\|A\|_F^2 = \text{tr}\{[A]^T [A]\}$.

$[I_{m,n}]$: matrix in $\mathbb{M}_{m,n}(\mathbb{R})$ such that $[I_{m,n}]_{ij} = \delta_{ij}$.

$[I_m]$: identity matrix in $\mathbb{M}_m(\mathbb{R})$.

1 Introduction

The experimental identification of an unknown non-Gaussian positive matrix-valued random field, using partial and limited experimental data for an observation vector related to the random solution of a stochastic elliptic boundary value problem, stays a challenging problem which has not received yet a complete solution, in particular for high stochastic dimension. An example of a challenging application concerns the identification of random fields that model at a mesoscale the elastic properties of complex heterogeneous materials that cannot be described at the level of their microscopic constituents (e.g., biological materials such as the cortical bone), given some indirect observations measured on a collection of material samples (measurements of their elastic response, e.g., through image analysis or other displacement sensors). Even if many results have already been obtained, additional developments concerning the stochastic representations of such random fields and additional mathematical and numerical analyses of the associated stochastic elliptic boundary value problem must yet be produced. This is the objective of the present paper.

Concerning the representation of random fields adapted to the statistical inverse problems for their experimental identification, one interesting type of representation for any non-Gaussian second-order random field is based on the use of the polynomial chaos expansion [8, 69] for which an efficient construction has been proposed in [18, 26, 27], consisting in coupling a Karhunen–Loève expansion (allowing a statistical reduction to be done) with a polynomial chaos expansion of the reduced model. This type of construction has been extended for an arbitrary probability measure [21, 39, 41, 56, 67, 70] and for random coefficients of the expansion [58].

Concerning the identification of random fields by solving stochastic inverse problems, works can be found such as [16, 35, 36, 63] and some methods and formulations have been proposed for the experimental identification of non-Gaussian random fields

[2, 13, 14, 17, 28, 42, 62, 71] in low stochastic dimension. More recently, a more advanced methodology [59, 60] has been proposed for the identification of non-Gaussian positive-definite matrix-valued random fields in high stochastic dimension for the case for which only partial and limited experimental data are available.

Concerning the mathematical analysis of stochastic elliptic boundary value problems and the associated numerical aspects for approximating the random solutions (the forward problem), many works have been devoted to the simple case of uniformly elliptic and uniformly bounded operators [3–5, 15, 24, 40, 43, 50, 68]. For these problems, existence and uniqueness results can be directly obtained using Lax–Milgram theorem and Galerkin approximation methods in classical approximation spaces can be used. More recently, some works have addressed the mathematical and numerical analyses of some classes of non-uniformly elliptic operators [10, 25, 29, 46], including the case of log-normal random fields.

For the numerical solution of stochastic boundary value problems, classical non-adapted choices of stochastic approximation spaces lead to very high-dimensional representations of the random solutions, e.g., when using classical (possibly piecewise) polynomial spaces in the random variables. For addressing this complexity issue, several complexity reduction methods have been recently proposed, such as model reduction methods for stochastic and parametric partial differential equations [7, 48, 49], sparse approximation methods for high-dimensional approximations [6, 11, 47, 53, 65], or low-rank tensor approximation methods that exploit the tensor structure of stochastic approximation spaces and also allow the representation of high-dimensional functions [9, 19, 22, 23, 37, 44, 51].

In this paper, we present new results concerning:

(i) the construction of a general class of non-Gaussian positive-definite matrix-valued random fields which is adapted to the identification in high stochastic dimension, presented in Section 2,

(ii) the parameterisation of the discretised random fields belonging to the general class and an adapted identification strategy for this class, presented in Section 3 and

(iii) the mathematical analysis of the parameterised stochastic elliptic boundary value problems whose random coefficients belong to the parameterised general class of random fields, and the introduction and analysis of dedicated approximation methods, presented in Section 4.

Concerning (i), we introduce a new class of matrix-valued random fields $\{[\mathbf{K}(\mathbf{x})], \mathbf{x} \in D\}$ indexed on a domain D that are expressed as a non-linear function of second-order symmetric matrix-valued random fields $\{[\mathbf{G}(\mathbf{x})], \mathbf{x} \in D\}$. More precisely, we propose and analyse two classes of random fields that use two different non-linear mappings. The first class, which uses a matrix exponential mapping, contains the class of (shifted) log-normal random fields (when $[\mathbf{G}]$ is a Gaussian random field) and can be seen as a generalisation of this classical class. The second class contains (a shifted version of) the class SFE^+ of positive-definite matrix-valued random fields introduced in [57] which arises from a maximum entropy principle given natural constraints on the integrability of random matrices and their inverse.

Concerning (ii), and following [59], we introduce a parameterisation of the second-order random fields [G] using Karhunen–Loeve and Polynomial Chaos expansions, therefore yielding a parameterisation of the class of random fields in terms of the set of coefficients of the Polynomial Chaos expansions of the random variables of the Karhunen–Loeve expansions. Second-order statistical properties of [G] are imposed by enforcing orthogonality constraints on the set of coefficients, therefore yielding a class of random fields parameterised on the compact Stiefel manifold. It finally results in a parameterised class of random fields $\mathscr{K} = \{[\mathbf{K}(\cdot)] = \mathscr{F}(\cdot, \varXi, \mathbf{z}); \mathbf{z} \in \mathbb{R}^\nu\}$, where the parameters $\mathbf{z}$ are associated with a parameterisation of the compact Stiefel manifold, and where $\varXi$ is the Gaussian germ of the Polynomial Chaos expansion. A general procedure is then proposed for the identification of random fields in this new class, this procedure being an adaptation for the present parameterisation of the procedure described in [60]. The parameterisation of the Stiefel manifold, some additional mathematical properties of the resulting parameterised class of random fields and the general identification procedure are introduced in Section 3.

The numerical solution of the parameterised stochastic boundary value problems provides an explicit approximation of the application $u : D \times \mathbb{R}^{N_g} \times \mathbb{R}^\nu \to \mathbb{R}$ that maps the parameterised general class of random fields to the corresponding set of random solutions $\{u(\cdot, \varXi, \mathbf{z}); \mathbf{z} \in \mathbb{R}^\nu\}$. This explicit map can be efficiently used in the identification procedure in order to avoid the solution of multiple stochastic forward problems. Concerning (iii), the general class of random fields possibly contains random fields which are not uniformly bounded, which requires a particular mathematical analysis and the introduction of dedicated approximation methods. In particular, in Section 4, we prove the existence and uniqueness of a weak solution u in the natural function space associated with the energy norm, and we propose suitable constructions of approximation spaces for Galerkin approximations. Also, since the solution of this problem requires the approximation of a very high-dimensional map, complexity reduction methods are required. The solution map has a tensor structure which is inherited from the particular parameterisation of the general class of random fields. This allows the use of complexity reduction methods based on low-rank or sparse tensor approximation methods. The applicability of these reduction techniques in the present context is briefly discussed in Section 4.3. Note that this kind of approach for stochastic inverse problems has been recently analysed in [54] in a particular Bayesian setting with another type of representation of random fields, and with sparse approximation methods for the approximation of the high-dimensional functions.

2 General class of non-Gaussian positive-definite matrix-valued random fields

2.1 Elements for the construction of a general class of random fields

Let $d \geqslant 1$ and $n \geqslant 1$ be two integers. Let D be a bounded open domain of $\mathbb{R}^d$, let ∂D be its boundary and let $\overline{D} = D \cup \partial D$ be its closure.

Let $\mathscr{C}_n^+$ be the set of all the random fields $[\mathbf{K}] = \{[\mathbf{K}(\mathbf{x})], \mathbf{x} \in D\}$, defined on a probability space $(\Theta, \mathscr{T}, \mathscr{P})$, with values in $\mathbb{M}_n^+(\mathbb{R})$. A random field $[\mathbf{K}]$ in $\mathscr{C}_n^+$ corresponds to the coefficients of a stochastic elliptic operator and must be identified from data, by solving a statistical inverse boundary value problem. The objective of this section is to

construct a general representation of [**K**] which allows its identification to be performed using experimental data and solving a statistical inverse problem based on the explicit construction of the solution of the forward (direct) stochastic boundary value problem. For that, we need to introduce some hypotheses for random field [**K**] which requires the introduction of a subset of $\mathscr{C}_n^+$.

In order to normalise random field [**K**], we introduce a function $\mathbf{x} \mapsto [\underline{K}(\mathbf{x})]$ from D into $\mathbb{M}_n^+(\mathbb{R})$ such that, for all $\mathbf{x}$ in D and for all $\mathbf{z}$ in $\mathbb{R}^n$,

$$\underline{k}_0 \|\mathbf{z}\|_2^2 \ \leqslant \ < [\underline{K}(\mathbf{x})]\,\mathbf{z}, \mathbf{z} >_2 \ \leqslant \ n^{-1/2}\widetilde{\underline{k}}_1 \|\mathbf{z}\|_2^2, \tag{2.1}$$

in which $\underline{k}_0$ and $\widetilde{\underline{k}}_1$ are positive real constants, independent of $\mathbf{x}$, such that $0 < \underline{k}_0 < \widetilde{\underline{k}}_1 < +\infty$. The right inequality means that $\|\underline{K}(\mathbf{x})\|_2 \leqslant n^{-1/2}\widetilde{\underline{k}}_1$ and using $\|\underline{K}(\mathbf{x})\|_F \leqslant \sqrt{n}\,\|\underline{K}(\mathbf{x})\|_2$ yields $\|\underline{K}(\mathbf{x})\|_F \leqslant \widetilde{\underline{k}}_1$. Since $0 \leqslant [\underline{K}(\mathbf{x})]_{jj}$ and $[\underline{K}(\mathbf{x})]_{jj}^2 \leqslant \|\underline{K}(\mathbf{x})\|_F^2$, it can be deduced that

$$\mathrm{tr}[\underline{K}(\mathbf{x})] \ \leqslant \ \underline{k}_1, \tag{2.2}$$

in which $\underline{k}_1$ is a positive real constant, independent of $\mathbf{x}$, such that $\underline{k}_1 = n\widetilde{\underline{k}}_1$.

We then introduce the following representation of the lower-bounded random field [**K**] in $\mathscr{C}_n^+$ such that, for all $\mathbf{x}$ in D,

$$[\mathbf{K}(\mathbf{x})] = \frac{1}{1+\varepsilon} [\underline{L}(\mathbf{x})]^T \left\{ \varepsilon [I_n] + [\mathbf{K}_0(\mathbf{x})] \right\} [\underline{L}(\mathbf{x})], \tag{2.3}$$

in which $\varepsilon > 0$ is any fixed positive real number, where $[I_n]$ is the $(n \times n)$ identity matrix and where $[\underline{L}(\mathbf{x})]$ is the upper triangular $(n \times n)$ real matrix such that, for all $\mathbf{x}$ in D, $[\underline{K}(\mathbf{x})] = [\underline{L}(\mathbf{x})]^T [\underline{L}(\mathbf{x})]$ and where $[\mathbf{K}_0] = \{[\mathbf{K}_0(\mathbf{x})], \mathbf{x} \in D\}$ is any random field in $\mathscr{C}_n^+$.

For instance, if function $[\underline{K}]$ is chosen as the mean function of random field [**K**], that is to say, if for all $\mathbf{x}$ in D, $[\underline{K}(\mathbf{x})] = E\{[\mathbf{K}(\mathbf{x})]\}$, then equation (2.3) shows that $E\{[\mathbf{K}_0(\mathbf{x})]\}$ must be equal to $[I_n]$, what shows that random field $[\mathbf{K}_0]$ is normalised.

Lemma 1 *If $[\mathbf{K}_0]$ is any random field in $\mathscr{C}_n^+$, then the random field [**K**], defined by equation (2.3), is such that*

  (i) *for all $\mathbf{x}$ in D,*

$$\|\mathbf{K}(\mathbf{x})\|_F \leqslant \frac{\underline{k}_1}{1+\varepsilon}(\sqrt{n}\,\varepsilon + \|\mathbf{K}_0(\mathbf{x})\|_F) \quad a.s. \tag{2.4}$$

  (ii) *for all $\mathbf{z}$ in $\mathbb{R}^n$ and for all $\mathbf{x}$ in D,*

$$\underline{k}_\varepsilon \|\mathbf{z}\|_2^2 \ \leqslant \ < [\mathbf{K}(\mathbf{x})]\,\mathbf{z}, \mathbf{z} >_2 \quad a.s, \tag{2.5}$$

in which $\underline{k}_\varepsilon = \underline{k}_0\,\varepsilon/(1+\varepsilon)$ is a positive constant independent of $\mathbf{x}$.

  (iii) *for all $\mathbf{x}$ in D,*

$$\|[\mathbf{K}(\mathbf{x})]^{-1}\|_F \leqslant \frac{\sqrt{n}\,(1+\varepsilon)}{\varepsilon}\,\mathrm{tr}[\underline{K}(\mathbf{x})]^{-1} \quad a.s, \tag{2.6}$$

which shows that, for all integer $p \geqslant 1$, $\{[\mathbf{K}(\mathbf{x})]^{-1}, \mathbf{x} \in D\}$ is a p-order random field,

i.e., for all $\mathbf{x}$ *in D,* $E\{\|[\mathbf{K}(\mathbf{x})]^{-1}\|_F^p\} < +\infty$ *and in particular, is a second-order random field.*

Proof (i) Taking the Frobenius norm of the two members of equation (2.3), using $\|\underline{L}(\mathbf{x})\|_F = \|[\underline{L}(\mathbf{x})]^T\|_F = \sqrt{\text{tr}[\underline{K}(\mathbf{x})]}$ and taking into account equation (2.2), we obtain equation (2.4).

(ii) Equation (2.5) can easily be proven using equation (2.3) and the left inequality in equation (2.1).

(iii) We have $\|[\mathbf{K}(\mathbf{x})]^{-1}\|_F \leqslant \|[\underline{L}(\mathbf{x})]^{-1}\|_F \|[\underline{L}(\mathbf{x})]^{-T}\|_F \|[\mathbf{K}_\varepsilon(\mathbf{x})]^{-1}\|_F$ in which $[\mathbf{K}_\varepsilon(\mathbf{x})] = (\varepsilon[I_n] + [\mathbf{K}_0(\mathbf{x})])/(1+\varepsilon)$. Since $[\mathbf{K}_0(\mathbf{x})]$ is positive definite almost surely, for $\mathbf{x}$ fixed in D, we can write $[\mathbf{K}_0(\mathbf{x})] = [\boldsymbol{\Phi}(\mathbf{x})][\Lambda(\mathbf{x})][\boldsymbol{\Phi}(\mathbf{x})]^T$ in which $[\Lambda(\mathbf{x})]$ is the diagonal random matrix of the positive-valued random eigenvalues $\Lambda_1(\mathbf{x}), \ldots, \Lambda_n(\mathbf{x})$ of $[\mathbf{K}_0(\mathbf{x})]$ and $[\boldsymbol{\Phi}(\mathbf{x})]$ is the orthogonal real random matrix made up of the associated random eigenvectors. It can then be deduced that $[\mathbf{K}_\varepsilon(\mathbf{x})]^{-1} = (1+\varepsilon)[\boldsymbol{\Phi}(\mathbf{x})](\varepsilon[I_n]+[\Lambda(\mathbf{x})])^{-1}[\boldsymbol{\Phi}(\mathbf{x})]^T$ and consequently, for $\mathbf{x}$ in D, $\|[\mathbf{K}_\varepsilon(\mathbf{x})]^{-1}\|_F^2 = \text{tr}\{[\mathbf{K}_\varepsilon(\mathbf{x})]^{-2}\} = (1+\varepsilon)^2 \sum_{j=1}^n (\varepsilon + \Lambda_j(\mathbf{x}))^{-2} \leqslant (1+\varepsilon)^2 n/\varepsilon^2$. Since $\|[\underline{L}(\mathbf{x})]^{-1}\|_F \|[\underline{L}(\mathbf{x})]^{-T}\|_F = \text{tr}[\underline{K}(\mathbf{x})]^{-1}$, we deduce equation (2.6). ☐

Lemma 2 *If* $[\mathbf{K}_0]$ *is any random field in* $\mathscr{C}_n^+$, *such that, for all* $\mathbf{x}$ *in D,*

(i) $\|\mathbf{K}_0(\mathbf{x})\|_F \leqslant \beta_0 < \infty$ *almost surely, in which* β_0 *is a positive-valued random variable independent of* $\mathbf{x}$, *then the random field* $[\mathbf{K}]$, *defined by equation (2.3), is such that, for all* $\mathbf{x}$ *in D,*

$$\|\mathbf{K}(\mathbf{x})\|_F \leqslant \beta < +\infty \quad a.s, \tag{2.7}$$

in which β *is the positive-valued random variable independent of* $\mathbf{x}$ *such that* $\beta = \underline{k}_1(\sqrt{n}\,\varepsilon + \beta_0)/(1+\varepsilon)$.

(ii) $E\{\|\mathbf{K}_0(\mathbf{x})\|_F^2\} < +\infty$, *then* $[\mathbf{K}]$ *is a second-order random field,*

$$E\{\|\mathbf{K}(\mathbf{x})\|_F^2\} < +\infty. \tag{2.8}$$

Proof Equations (2.7) and (2.8) are directly deduced from equation (2.4). ☐

Remark 2.1 If β_0 is a second-order random variable, then equation (2.7) implies equation (2.8). However, if equation (2.8) holds for all $\mathbf{x}$ in D, then this equation does not imply the existence of a positive random variable β such that equation (2.7) is verified and *a fortiori*, even if β existed, this random variable would not be, in general, a second-order random variable.

2.2 Representations of random field $[\mathbf{K}_0]$ as transformations of a non-Gaussian second-order symmetric matrix-valued random field $[\mathbf{G}]$

Let $\mathscr{C}_n^{2,S}$ be the set of all the second-order random fields $[\mathbf{G}] = \{[\mathbf{G}(\mathbf{x})], \mathbf{x} \in D\}$, defined on probability space $(\Theta, \mathscr{T}, \mathscr{P})$, with values in $\mathbb{M}_n^S(\mathbb{R})$. Consequently, for all $\mathbf{x}$ in D, we have $E\{\|\mathbf{G}(\mathbf{x})\|_F^2\} < +\infty$.

In this section, we propose two representations of random field $[\mathbf{K}_0]$ belonging to $\mathscr{C}_n^+$, yielding the definition of two different subsets of $\mathscr{C}_n^+$:

- *Exponential-type representation.* The first type representation is written as $[\mathbf{K}_0(\mathbf{x})] = \exp_{\mathbb{M}}([\mathbf{G}(\mathbf{x})])$ with $[\mathbf{G}]$ in $\mathscr{C}_n^{2,S}$ and where $\exp_{\mathbb{M}}$ denotes the exponential of symmetric square real matrices. It should be noted that random field $[\mathbf{G}]$ is not assumed to be Gaussian. If $[\mathbf{G}]$ were a Gaussian random field, then $[\mathbf{K}_0]$ would be a log-normal matrix-valued random field.

- *Square-type representation.* The second type representation is written as $[\mathbf{K}_0(\mathbf{x})] = [\mathbf{L}(\mathbf{x})]^T [\mathbf{L}(\mathbf{x})]$ in which $[\mathbf{L}(\mathbf{x})]$ is an upper triangular $(n \times n)$ real random matrix, for which the diagonal terms are positive-valued random variables, and which is written as $[\mathbf{L}(\mathbf{x})] = [\mathscr{L}([\mathbf{G}(\mathbf{x})])]$ with $[\mathbf{G}]$ in $\mathscr{C}_n^{2,S}$ and where $[G] \mapsto [\mathscr{L}([G])]$ is a well-defined deterministic mapping from $\mathbb{M}_n^S(\mathbb{R})$ into the set of all the upper triangular $(n \times n)$ real deterministic matrices. Again, random field $[\mathbf{G}]$ is not assumed to be Gaussian. If for all $1 \leqslant j \leqslant j' \leqslant n$, $[\mathbf{G}]_{jj'}$ are independent copies of a Gaussian random field and for a particular definition of mapping $[\mathscr{L}([G])]$, then $[\mathbf{K}_0]$ would be the set SFG^+ of the non-Gaussian matrix-valued random field previously introduced in [57].

These two representations are general enough, but the mathematical properties of each one will be slightly different and the computational aspects will be different.

2.2.1 *Exponential-type representation of random field* $[\mathbf{K}_0]$

It should be noted that for all symmetric real matrix $[A]$ in $\mathbb{M}_n^S(\mathbb{R})$, $[B] = \exp_{\mathbb{M}}([A])$ is a well-defined matrix belonging to $\mathbb{M}_n^+(\mathbb{R})$. All matrix $[A]$ in $\mathbb{M}_n^S(\mathbb{R})$ can be written as $[A] = [\Phi] [\mu] [\Phi]^T$ in which $[\mu]$ is the diagonal matrix of the real eigenvalues $\mu_1, \ldots, \mu_n$ of $[A]$ and $[\Phi]$ is the orthogonal real matrix made up of the associated eigenvectors. We then have $[B] = [\Phi] \exp_{\mathbb{M}}([\mu])[\Phi]^T$ in which $\exp_{\mathbb{M}}([\mu])$ is the diagonal matrix in $\mathbb{M}_n^+(\mathbb{R})$ such that $[\exp_{\mathbb{M}}([\mu])]_{jk} = e^{\mu_j} \delta_{jk}$.

For all random field $[\mathbf{G}]$ belonging to $\mathscr{C}_n^{2,S}$, the random field $[\mathbf{K}_0] = \exp_{\mathbb{M}}([\mathbf{G}])$ defined, for all $\mathbf{x}$ in D, by

$$[\mathbf{K}_0(\mathbf{x})] = \exp_{\mathbb{M}}([\mathbf{G}(\mathbf{x})]), \tag{2.9}$$

belongs to $\mathscr{C}_n^+$. If $[\mathbf{K}_0]$ is any random field given in $\mathscr{C}_n^+$, then there exists a unique random field $[\mathbf{G}]$ with values in the $\mathbb{M}_n^S(\mathbb{R})$ such that, for all $\mathbf{x}$ in D,

$$[\mathbf{G}(\mathbf{x})] = \log_{\mathbb{M}}([\mathbf{K}_0(\mathbf{x})]). \tag{2.10}$$

in which $\log_{\mathbb{M}}$ is the reciprocity mapping of $\exp_{\mathbb{M}}$ which is defined on $\mathbb{M}_n^+(\mathbb{R})$ with values in $\mathbb{M}_n^S(\mathbb{R})$, but in general, this random field $[\mathbf{G}]$ is not a second-order random field and therefore, is not in $\mathscr{C}_n^{2,S}$. The following lemma shows that, if a random field $[\mathbf{K}_0]$ in $\mathscr{C}_n^+$ satisfies additional properties, then there exists $[\mathbf{G}]$ in $\mathscr{C}_n^{2,S}$ such that $[\mathbf{K}_0] = \exp_{\mathbb{M}}([\mathbf{G}])$.

Lemma 3 *Let $[\mathbf{K}_0]$ be a random field belonging to $\mathscr{C}_n^+$ such that for all $\mathbf{x} \in D$,*

$$E\{\|\mathbf{K}_0(\mathbf{x})\|_F^2\} < +\infty \quad , \quad E\{\|[\mathbf{K}_0(\mathbf{x})]^{-1}\|_F^2\} < +\infty. \tag{2.11}$$

Then there exists $[\mathbf{G}]$ belonging to $\mathscr{C}_n^{2,S}$ such that $[\mathbf{K}_0] = \exp_{\mathbb{M}}([\mathbf{G}])$.

Proof For **x** fixed in D, we use the spectral representation of $[\mathbf{K}_0(\mathbf{x})]$ introduced in the proof of Lemma 1. The hypotheses introduced in Lemma 3 can be rewritten as $E\{\|\mathbf{K}_0(\mathbf{x})\|_F^2\} = \sum_{j=1}^n E\{\Lambda_j(\mathbf{x})^2\} < +\infty$ and $E\{\|[\mathbf{K}_0(\mathbf{x})]^{-1}\|_F^2\} = \sum_{j=1}^n E\{\Lambda_j(\mathbf{x})^{-2}\} < +\infty$. Similarly, it can easily be proven that $E\{\|\mathbf{G}(\mathbf{x})\|_F^2\} = \sum_{j=1}^n E\{(\log \Lambda_j(\mathbf{x}))^2\}$. Since n is finite, the proof of the lemma will be complete if we prove that, for all j, we have $E\{(\log \Lambda_j(\mathbf{x}))^2\} < +\infty$ knowing that $E\{\Lambda_j(\mathbf{x})^{-2}\} < +\infty$ and $E\{\Lambda_j(\mathbf{x})^2\} < +\infty$. For j and **x** fixed, let $P(d\lambda)$ be the probability distribution of the random variable $\Lambda_j(\mathbf{x})$. We have $E\{(\log \Lambda_j(\mathbf{x}))^2\} = \int_0^1 (\log \lambda)^2 P(d\lambda) + \int_1^{+\infty} (\log \lambda)^2 P(d\lambda)$. For $0 < \lambda < 1$, we have $(\log \lambda)^2 < \lambda^{-2}$, and for $1 < \lambda < +\infty$, we have $(\log \lambda)^2 < \lambda^2$. It can then be deduced that $E\{(\log \Lambda_j(\mathbf{x}))^2\} < \int_0^1 \lambda^{-2} P(d\lambda) + \int_1^{+\infty} \lambda^2 P(d\lambda) < \int_0^{+\infty} \lambda^{-2} P(d\lambda) + \int_0^{+\infty} \lambda^2 P(d\lambda) = E\{\Lambda_j(\mathbf{x})^{-2}\} + E\{\Lambda_j(\mathbf{x})^2\} < +\infty.$ $\square$

Remark 2.2 The converse of Lemma 3 does not hold. If $[\mathbf{G}]$ is any random field in $\mathscr{C}_n^{2,S}$, in general, the random field $[\mathbf{K}_0] = \exp_{\mathbb{M}}([\mathbf{G}])$ is not a second-order random field.

Proposition 1 *Let $[\mathbf{G}]$ be a random field belonging to $\mathscr{C}_n^{2,S}$ and let $[\mathbf{K}_0]$ be the random field belonging to $\mathscr{C}_n^+$ such that, for all **x** in D, $[\mathbf{K}_0(\mathbf{x})] = \exp_{\mathbb{M}}([\mathbf{G}(\mathbf{x})])$. Then the following results hold:*

(i) *For all **x** in D,*

$$\|\mathbf{K}_0(\mathbf{x})\|_F \leqslant \sqrt{n}\, e^{\|\mathbf{G}(\mathbf{x})\|_F} \quad a.s, \tag{2.12}$$

$$E\{\|\mathbf{K}_0(\mathbf{x})\|_F^2\} \leqslant n\, E\{e^{2\|\mathbf{G}(\mathbf{x})\|_F}\}, \tag{2.13}$$

$$E\{\|\log_{\mathbb{M}}[\mathbf{K}_0(\mathbf{x})]\|_F^2\} < +\infty. \tag{2.14}$$

(ii) *If $\|\mathbf{G}(\mathbf{x})\|_F \leqslant \beta_G < +\infty$ almost surely, in which β_G is a positive random variable independent of **x**, then $\|\mathbf{K}_0(\mathbf{x})\|_F \leqslant \beta_0 < +\infty$ almost surely, in which β_0 is the positive random variable, independent of **x**, such that $\beta_0 = \sqrt{n}\, e^{\beta_G}$.*

Proof For (i), we have $\|\exp_{\mathbb{M}}(\mathbf{G}(\mathbf{x}))\|_2 \leqslant e^{\|\mathbf{G}(\mathbf{x})\|_2}$. Since $[\mathbf{K}_0(\mathbf{x})] = \exp_{\mathbb{M}}(\mathbf{G}(\mathbf{x}))$ and since $\|\mathbf{K}_0(\mathbf{x})\|_F \leqslant \sqrt{n}\, \|\mathbf{K}_0(\mathbf{x})\|_2$, it can then be deduced that $\|\mathbf{K}_0(\mathbf{x})\|_F \leqslant \sqrt{n}\, e^{\|\mathbf{G}(\mathbf{x})\|_F}$ and then $E\{\|\mathbf{K}_0(\mathbf{x})\|_F^2\} \leqslant n\, E\{e^{2\|\mathbf{G}(\mathbf{x})\|_F}\}$. Since $[\mathbf{G}]$ is a second-order random field, the inequality (2.14) is deduced from equation (2.10). For (ii), the proof is directly deduced from the inequality (2.12). $\square$

2.2.2 *Square-type representation of random field $[\mathbf{K}_0]$*

Let $g \mapsto h(g;a)$ be a given function from $\mathbb{R}$ in $\mathbb{R}^+$, depending on one positive real parameter a. For all fixed a, it is assumed that:

(i) $h(.;a)$ is a strictly monotonically increasing function on $\mathbb{R}$, which means that $h(g;a) < h(g';a)$ if $-\infty < g < g' < +\infty$;

(ii) there are real numbers $0 < c_h < +\infty$ and $0 < c_a < +\infty$, such that, for all g in $\mathbb{R}$, we have $h(g;a) \leqslant c_a + c_h\, g^2$.

It should be noted that $c_a \geqslant h(0, a)$. In addition, from (i) it can be deduced that, for $g < 0$, we have $h(g; a) < h(0, a) \leqslant c_a < c_a + c_h g^2$. Therefore, the inequality given in (ii), which is true for $g < 0$, allows the behaviour of the increasing function $g \mapsto h(g; a)$ to be controlled for $g > 0$. Finally, the introduced hypotheses imply that, for all $a > 0$, $g \mapsto h(g; a)$ is a one-to-one mapping from $\mathbb{R}$ onto $\mathbb{R}^+$ and consequently, the reciprocity mapping, $v \mapsto h^{-1}(v; a)$, is a strictly monotonically increasing function from $\mathbb{R}^+$ onto $\mathbb{R}$.

The square-type representation of random field $[\mathbf{K}_0]$ belonging to $\mathscr{C}_n^+$ is then defined as follows. For all $\mathbf{x}$ in D,

$$[\mathbf{K}_0(\mathbf{x})] = \mathbb{L}([\mathbf{G}(\mathbf{x})]), \tag{2.15}$$

in which $\{[\mathbf{G}(\mathbf{x})], \mathbf{x} \in D\}$ is a random field belonging to $\mathscr{C}_n^{2,S}$ and where $[G] \mapsto \mathbb{L}([G])$ is a measurable mapping from $\mathbb{M}_n^S(\mathbb{R})$ into $\mathbb{M}_n^+(\mathbb{R})$ which is defined as follows. The matrix $[K_0] = \mathbb{L}([G]) \in \mathbb{M}_n^+(\mathbb{R})$ is written as

$$[K_0] = [L]^T [L], \tag{2.16}$$

in which $[L]$ is an upper triangular ($n \times n$) real matrix with positive diagonal, which is written as

$$[L] = \mathscr{L}([G]), \tag{2.17}$$

where $[G] \mapsto \mathscr{L}([G])$ is the measurable mapping from $\mathbb{M}_n^S(\mathbb{R})$ into $\mathbb{M}_n^U(\mathbb{R})$ defined by

$$[\mathscr{L}([G])]_{jj'} = [G]_{jj'}, \; 1 \leqslant j < j' \leqslant n, \tag{2.18}$$

$$[\mathscr{L}([G])]_{jj} = \sqrt{h([G]_{jj}; a_j)}, \; 1 \leqslant j \leqslant n, \tag{2.19}$$

in which $a_1, \ldots, a_n$ are positive real numbers.

If $[\mathbf{K}_0]$ is any random field given in $\mathscr{C}_n^+$, then there exists a unique random field $[\mathbf{G}]$ with values in the $\mathbb{M}_n^S(\mathbb{R})$ such that, for all $\mathbf{x}$ in D,

$$[\mathbf{G}(\mathbf{x})] = \mathbb{L}^{-1}([\mathbf{K}_0(\mathbf{x})]), \tag{2.20}$$

in which $\mathbb{L}^{-1}$ is the reciprocity function of $\mathbb{L}$, from $\mathbb{M}_n^+(\mathbb{R})$ into $\mathbb{M}_n^S(\mathbb{R})$, which is explicitly defined as follows. For all $1 \leqslant j \leqslant j' \leqslant n$,

$$[\mathbf{G}(\mathbf{x})]_{jj'} = [\mathscr{L}^{-1}([\mathbf{L}(\mathbf{x})])]_{jj'} \quad , \quad [\mathbf{G}(\mathbf{x})]_{j'j} = [\mathbf{G}(\mathbf{x})]_{jj'}. \tag{2.21}$$

in which $[L] \mapsto \mathscr{L}^{-1}([L])$ is the unique reciprocity mapping of $\mathscr{L}$ (due to the existence of $v \mapsto h^{-1}(v; a)$) defined on $\mathbb{M}_n^U(\mathbb{R})$, and where $[\mathbf{L}(\mathbf{x})]$ follows from the Cholesky factorisation of random matrix $[\mathbf{K}_0(\mathbf{x})] = [\mathbf{L}(\mathbf{x})]^T [\mathbf{L}(\mathbf{x})]$ (see equations (2.15) and (2.16)).

Example of function h

Let us give an example which shows that there exists at least one such a construction. For instance, we can choose $h = h^{\text{APM}}$, in which the function h^{APM} is defined in [57] as follows. Let be $s = \delta/\sqrt{n+1}$ in which δ is a parameter such that $0 < \delta < \sqrt{(n+1)/(n-1)}$ and which allows the statistical fluctuations level to be controlled. Let be $a_j = 1/(2s^2) + (1 - j)/2 > 0$ and $h^{\text{APM}}(g; a) = 2s^2 F_{\Gamma_a}^{-1}(F_W(g/s))$ with $F_W(\widetilde{w}) = \int_{-\infty}^{\widetilde{w}} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}w^2) \, dw$ and

$F_{\Gamma_a}^{-1}(u) = \gamma$ the reciprocal function such that $F_{\Gamma_a}(\gamma) = u$ with $F_{\Gamma_a}(\gamma) = \int_0^\gamma \frac{1}{\Gamma(a)} t^{a-1} e^{-t} dt$ and $\Gamma(a) = \int_0^{+\infty} t^{a-1} e^{-t} dt$. Then, for all $j = 1, \ldots, n$, it can be proven [57] that $g \mapsto h^{\mathrm{APM}}(g; a_j)$ is a strictly monotonically increasing function from $\mathbb{R}$ into $\mathbb{R}^+$ and there are positive real numbers c_h and c_{a_j} such that, for all g in $\mathbb{R}$, we have $h^{\mathrm{APM}}(g; a_j) \leqslant c_{a_j} + c_h g^2$. In addition, it can easily be seen that the reciprocity function is written as $h^{\mathrm{APM}-1}(v; a) = s F_W^{-1}(F_{\Gamma_a}(v/(2s^2)))$.

Proposition 2 *Let $[\mathbf{G}]$ be a random field belonging to $\mathscr{C}_n^{2,S}$ and let $[\mathbf{K}_0]$ be the random field belonging to $\mathscr{C}_n^+$ such that, for all $\mathbf{x}$ in D, $[\mathbf{K}_0(\mathbf{x})] = \mathbb{L}([\mathbf{G}(\mathbf{x})])$. We then have the following results:*

(i) *There exist two real numbers $0 < \gamma_0 < +\infty$ and $0 < \gamma_1 < +\infty$ such that, for all $\mathbf{x}$ in D, we have*

$$\|\mathbf{K}_0(\mathbf{x})\|_F \leqslant \gamma_0 + \gamma_1 \|\mathbf{G}(\mathbf{x})\|_F^2 \quad a.s, \tag{2.22}$$

$$E\{\|\mathbf{K}_0(\mathbf{x})\|_F\} < +\infty, \tag{2.23}$$

$$\textit{If } E\{\|\mathbf{G}(\mathbf{x})\|_F^4\} < +\infty, \textit{ then } E\{\|\mathbf{K}_0(\mathbf{x})\|_F^2\} < +\infty. \tag{2.24}$$

(ii) *If $\|\mathbf{G}(\mathbf{x})\|_F \leqslant \beta_G < +\infty$ a.s, in which β_G is a positive random variable independent of $\mathbf{x}$, we then have $\|\mathbf{K}_0(\mathbf{x})\|_F \leqslant \beta_0 < +\infty$ almost surely, in which β_0 is the positive random variable, independent of $\mathbf{x}$, such that $\beta_0 = \gamma_0 + \gamma_1 \beta_G^2$.*

Proof (i) Since $[\mathbf{K}_0(\mathbf{x})] = [\mathbf{L}(\mathbf{x})]^T [\mathbf{L}(\mathbf{x})]$, we have $\|\mathbf{K}_0(\mathbf{x})\|_F \leqslant \|\mathbf{L}(\mathbf{x})\|_F^2$ with $\|\mathbf{L}(\mathbf{x})\|_F^2 = \sum_j h([\mathbf{G}(\mathbf{x})]_{jj}; a_j) + \sum_{j<j'} [\mathbf{G}(\mathbf{x})]_{jj'}^2$. Therefore, $\|\mathbf{K}_0(\mathbf{x})\|_F \leqslant \sum_j c_{a_j} + \max\{c_h, 1/2\} \sum_{j,j'} [\mathbf{G}(\mathbf{x})]_{jj'}^2$ which yields inequality (2.22) with $\gamma_0 = \sum_j c_{a_j}$ and $\gamma_1 = \max\{c_h, 1/2\}$. Taking the mathematical expectation yields $E\{\|\mathbf{K}_0(\mathbf{x})\|_F\} \leqslant \gamma_0 + \gamma_1 E\{\|\mathbf{G}(\mathbf{x})\|_F^2\} < +\infty$ because $[\mathbf{G}]$ is a second-order random field. Taking the square and then the mathematical expectation of equation (2.22) yields equation (2.24).
(ii) The proof is directly deduced from equation (2.22). □

2.3 Representation of any random field $[\mathbf{G}]$ in $\mathscr{C}_n^{2,S}$

As previously explained, we are interested in the identification of the random field $[\mathbf{K}]$ which belongs to $\mathscr{C}_n^+$, by solving a statistical inverse problem related to a stochastic boundary value problem. Such an identification is carried out using the proposed representation of $[\mathbf{K}]$ (defined by equation (2.3)) as a function of the random field $[\mathbf{K}_0]$ which is written either as $[\mathbf{K}_0(\mathbf{x})] = \exp_{\mathbb{M}}([\mathbf{G}(\mathbf{x})])$ (see equation (2.9)) or as $[\mathbf{K}_0(\mathbf{x})] = [\mathscr{L}([\mathbf{G}(\mathbf{x})])]^T [\mathscr{L}([\mathbf{G}(\mathbf{x})])]$ (see equations (2.16) and (2.17)). In these two representations, $[\mathbf{G}]$ is any random field in $\mathscr{C}_n^{2,S}$, which has to be identified (instead of $[\mathbf{K}]$ in $\mathscr{C}_n^+$). Consequently, we have to construct a representation of any random field $[\mathbf{G}]$ in $\mathscr{C}_n^{2,S}$. Since any $[\mathbf{G}]$ in $\mathscr{C}_n^{2,S}$ is a second-order random field, a general representation of $[\mathbf{G}]$, adapted to its identification, is the polynomial chaos expansion for which the coefficients of the expansion constitutes a family of functions from D into $\mathbb{M}_n^S(\mathbb{R})$.

It is well-known that a direct identification of such a family of matrix-valued functions cannot easily be done. An adapted representation must be introduced consisting in

choosing a deterministic vector basis, then representing [**G**] on this deterministic vector basis and finally, performing a polynomial chaos expansion of the random coefficients on this deterministic vector basis. The representation is generally in high stochastic dimension. Consequently, the identification of a very large number of coefficients must be done. It is then interesting to use a statistical reduction and thus to choose, for the deterministic vector basis, the Karhunen–Loève vector basis. Such a vector basis is constituted of the family of the eigenfunctions of the compact covariance operator of random field [**G**].

2.3.1 *Covariance operator of random field* [**G**] *and eigenvalue problem*

Let $[G_0(\mathbf{x})] = E\{[\mathbf{G}(\mathbf{x})]\}$ be the mean function of [**G**], defined on D with values in $\mathbb{M}_n^S(\mathbb{R})$. The covariance function of random field [**G**] is the function $(\mathbf{x}, \mathbf{x}') \mapsto C_{\mathbf{G}}(\mathbf{x}, \mathbf{x}')$, defined on $D \times D$, with values in the space $\mathbb{M}_n^S(\mathbb{R}) \otimes \mathbb{M}_n^S(\mathbb{R})$ of fourth-order tensors, such that

$$C_{\mathbf{G}}(\mathbf{x}, \mathbf{x}') = E\{([\mathbf{G}(\mathbf{x})] - [G_0(\mathbf{x})]) \otimes ([\mathbf{G}(\mathbf{x}')] - [G_0(\mathbf{x}')])\}. \tag{2.25}$$

It is assumed that $[G_0]$ is a uniformly bounded function on D, i.e.,

$$\|G_0\|_{\infty} := \operatorname{ess\,sup}_{\mathbf{x} \in D} \|G_0(\mathbf{x})\|_F < +\infty, \tag{2.26}$$

and that function $C_{\mathbf{G}}$ is square integrable on $D \times D$. Consequently, the covariance operator $\text{Cov}_{\mathbf{G}}$, defined by the kernel $C_{\mathbf{G}}$, is a Hilbert–Schmidt, symmetric, positive operator in the Hilbert space $L^2(D, \mathbb{M}_n^S(\mathbb{R}))$ equipped with the inner product,

$$\ll [G_i], [G_j] \gg = \int_D \operatorname{tr}\{[G_i(\mathbf{x})]^T [G_j(\mathbf{x})]\} \, d\mathbf{x}, \tag{2.27}$$

and the associated norm $|||[G_i]||| = \ll [G_i], [G_i] \gg^{1/2}$. The eigenvalue problem related to the covariance operator $\text{Cov}_{\mathbf{G}}$ consists in finding the family $\{[G_i(\mathbf{x})], \mathbf{x} \in D\}_{i \geqslant 1}$ of the normalised eigenfunctions with values in $\mathbb{M}_n^S(\mathbb{R})$ and the associated positive eigenvalues $\{\sigma_i\}_{i \geqslant 1}$ with $\sigma_1 \geqslant \sigma_2 \geqslant \ldots \to 0$ and $\sum_{i=1}^{+\infty} \sigma_i^2 < +\infty$, such that

$$\int_D C_{\mathbf{G}}(\mathbf{x}, \mathbf{x}') : [G_i(\mathbf{x}')] \, d\mathbf{x}' = \sigma_i \, [G_i(\mathbf{x})] \,, \quad i \geqslant 1, \tag{2.28}$$

in which $\{C_{\mathbf{G}}(\mathbf{x}, \mathbf{x}') : [G_i(\mathbf{x}')]\}_{k\ell} = \sum_{k'\ell'} \{C_{\mathbf{G}}(\mathbf{x}, \mathbf{x}')\}_{k\ell k'\ell'} \{[G_i(\mathbf{x}')]\}_{k'\ell'}$. The normalised family $\{[G_i(\mathbf{x})], \mathbf{x} \in D\}_{i \geqslant 1}$ is a Hilbertian basis of $L^2(D, \mathbb{M}_n^S(\mathbb{R}))$ and consequently, $\ll [G_i], [G_j] \gg = \delta_{ij}$. It can easily be verified that

$$E\{\|\mathbf{G}(\mathbf{x})\|_F^2\} = \|G_0(\mathbf{x})\|_F^2 + \sum_{i=1}^{+\infty} \sigma_i \|G_i(\mathbf{x})\|_F^2 < +\infty \,, \quad \forall \mathbf{x} \in D, \tag{2.29}$$

$$\int_D E\{\|\mathbf{G}(\mathbf{x})\|_F^2\}\,d\mathbf{x} = |||G_0|||^2 + \sum_{i=1}^{+\infty} \sigma_i < +\infty. \tag{2.30}$$

We now give some properties which will be useful later. Since $C_\mathbf{G}$ is a covariance function, we have $\|C_\mathbf{G}(\mathbf{x},\mathbf{x}')\|^2 \leqslant \mathrm{tr}\{C_\mathbf{G}(\mathbf{x},\mathbf{x})\} \times \mathrm{tr}\{C_\mathbf{G}(\mathbf{x}',\mathbf{x}')\}$ in which $\|C_\mathbf{G}(\mathbf{x},\mathbf{x}')\|^2 = \sum_{k\ell k'\ell'}\{C_\mathbf{G}(\mathbf{x},\mathbf{x}')\}^2_{k\ell k'\ell'}$ and $\mathrm{tr}\{C_\mathbf{G}(\mathbf{x},\mathbf{x})\} = \sum_{k\ell}\{C_\mathbf{G}(\mathbf{x},\mathbf{x})\}_{k\ell k\ell} \geqslant 0$. We can then deduce the following Lemma.

Lemma 4 *If $\mathbf{x} \mapsto \mathrm{tr}\{C_\mathbf{G}(\mathbf{x},\mathbf{x})\}$ is integrable on D, then $C_\mathbf{G}$ is square integrable on $D \times D$. If $\mathbf{x} \mapsto \mathrm{tr}\{C_\mathbf{G}(\mathbf{x},\mathbf{x})\}$ is bounded on D, then $C_\mathbf{G}$ is bounded on $D \times D$ (and thus square integrable on $D \times D$) and the eigenfunctions $\mathbf{x} \mapsto [G_i(\mathbf{x})]$ are bounded functions from D into $\mathbb{M}_n^S(\mathbb{R})$. For $i \geqslant 1$, we then have $\|G_i\|_\infty = ess\ sup_{\mathbf{x} \in D}\|G_i(\mathbf{x})\|_F < +\infty$.*

Using additional assumptions about the kernel $C_\mathbf{G}$, more results concerning the decrease rate of eigenvalues $\{\sigma_i\}_{i \geqslant 1}$ can be obtained.

Lemma 5 *We have the following results for $d = 1$ and for $d \geqslant 2$.*

(a) *For $d = 1$, it is proven (see Theorem 9.1 of Chapter II of [72]) that, if for a given integer $\mu \geqslant 1$ and for all integer α such that $1 \leqslant \alpha \leqslant \mu$, the functions $(x,x') \mapsto \partial^\alpha C_\mathbf{G}(x,x')/\partial x'^\alpha$ are bounded functions on $D \times D$, then $\sum_{i=1}^{+\infty} \sigma_i^{\zeta+2/(2\mu+1)} < +\infty$ for any $\zeta > 0$.*

(b) *For $d \geqslant 2$ (finite integer), a useful result is given by Theorem 4 of [38]. Let us assume that D has a sufficiently smooth boundary ∂D. For $\mathbf{x}$ in D, let $\mathbf{h}$ be such that $\mathbf{x} + \mathbf{h}$ and $\mathbf{x} + 2\mathbf{h}$ belong to D. We then define the difference operator $\varDelta_\mathbf{h}$ such that, for all $\mathbf{x}'$ fixed in D, $\varDelta_\mathbf{h} C_\mathbf{G}(\mathbf{x},\mathbf{x}') = C_\mathbf{G}(\mathbf{x}+\mathbf{h},\mathbf{x}') - C_\mathbf{G}(\mathbf{x},\mathbf{x}')$ and $\varDelta_\mathbf{h}^2$ is defined as the second iterate of $\varDelta_\mathbf{h}$. For all $\mathbf{x}'$ fixed in D, for a given positive integer μ such that $\mu \geqslant 1$ and for a given real ζ such that $0 < \zeta \leqslant 1$, let be*

$$\|C_\mathbf{G}(\cdot,\mathbf{x}')\|_\mu = \sum_{|\boldsymbol{\alpha}| \leqslant \mu} \sup_{\mathbf{x} \in D}\|\mathscr{D}^{\boldsymbol{\alpha}} C_\mathbf{G}(\mathbf{x},\mathbf{x}')\| + \sum_{|\boldsymbol{\alpha}| = \mu} \sup_{\mathbf{x},\mathbf{h}} \frac{\|\varDelta_\mathbf{h}^2 \mathscr{D}^{\boldsymbol{\alpha}} C_\mathbf{G}(\mathbf{x},\mathbf{x}')\|}{\|\mathbf{h}\|^{\mu+\zeta}},$$

in which $\mathscr{D}^{\boldsymbol{\alpha}} C_\mathbf{G}(\mathbf{x},\mathbf{x}') = \partial^{|\boldsymbol{\alpha}|} C_\mathbf{G}(\mathbf{x},\mathbf{x}')/\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}$ with $|\boldsymbol{\alpha}| = \alpha_1 + \ldots + \alpha_d$. Therefore, if $C_\mathbf{G}$ is continuous on $D \times D$ such that $\sup_{\mathbf{x}' \in D}\|C_\mathbf{G}(\cdot,\mathbf{x}')\|_\mu < +\infty$, we then have the following decrease of eigenvalues: $\sigma_i = O(i^{-1-(\mu+\zeta)/d})$.

(c) *For $d \geqslant 2$ (finite integer), a similar result to (b) in Lemma 5 can be found in [45] under hypotheses weaker than those introduced above, but in practice, it seems much more difficult to verify these hypotheses for a given kernel $C_\mathbf{G}$.*

Comment about the eigenvalue problem

Any symmetric $(n \times n)$ real matrix $[G]$ can be represented by a real vector $\mathbf{w}$ of dimension $n_\mathbf{w} = n(n+1)/2$ such that $[G] = \mathscr{G}(\mathbf{w})$ in which $\mathscr{G}$ is the one-to-one linear mapping from $\mathbb{R}^{n_\mathbf{w}}$ in $\mathbb{M}_n^S(\mathbb{R})$, such that, for $1 \leqslant i \leqslant j \leqslant n$, one has $[G]_{ij} = [G]_{ji} = w_k$ with $k = i + j(j-1)/2$. Let $\mathscr{L}_{n_\mathbf{w}}^2$ be the set of all the $\mathbb{R}^{n_\mathbf{w}}$-valued second-order random fields $\{\mathbf{W}(\mathbf{x}), \mathbf{x} \in D\}$ defined on probability space $(\Theta, \mathscr{T}, \mathscr{P})$. Consequently, any random field

[**G**] in $\mathscr{C}_n^{2,S}$ can be written as

$$[\mathbf{G}] = \mathscr{G}(\mathbf{W}), \quad \mathbf{W} = \mathscr{G}^{-1}([\mathbf{G}]), \tag{2.31}$$

in which **W** is a random field in $\mathscr{L}_{n_w}^2$. The eigenvalue problem defined by equation (2.28) can be rewritten as

$$\int_D [C_{\mathbf{W}}(\mathbf{x}, \mathbf{x}')] \, \mathbf{w}^i(\mathbf{x}') \, d\mathbf{x}' = \sigma_i \, \mathbf{w}^i(\mathbf{x}) \,, \quad i \geqslant 1, \tag{2.32}$$

in which $[C_{\mathbf{W}}(\mathbf{x}, \mathbf{x}')] = E\{(\mathbf{W}(\mathbf{x}) - \mathbf{w}^0(\mathbf{x}))(\mathbf{W}(\mathbf{x}') - \mathbf{w}^0(\mathbf{x}'))^T\}$ and where $\mathbf{w}^0 = \mathscr{G}^{-1}([G_0])$. For $i \geqslant 1$, the eigenfunctions $[G_i]$ are then given by $[G_i] = \mathscr{G}(\mathbf{w}^i)$.

2.3.2 *Chaos representation of random field* [**G**]

Under the hypotheses introduced in Section 2.3.1, random field [**G**] admits the following Karhunen–Loève decomposition:

$$[\mathbf{G}(\mathbf{x})] = [G_0(\mathbf{x})] + \sum_{i=1}^{+\infty} \sqrt{\sigma_i} \, [G_i(\mathbf{x})] \, \eta_i, \tag{2.33}$$

in which $\{\eta_i\}_{i \geqslant 1}$ are uncorrelated random variables with zero mean and unit variance. The second-order random variables $\{\eta_i\}_{i \geqslant 1}$ are represented using the following polynomial chaos expansion:

$$\eta_i = \sum_{j=1}^{+\infty} y_i^j \, \Psi_j(\{\Xi_k\}_{k \in \mathbb{N}}), \tag{2.34}$$

in which $\{\Xi_k\}_{k \in \mathbb{N}}$ is a countable set of independent normalised Gaussian random variables and where $\{\Psi_j\}_{j \geqslant 1}$ is the polynomial chaos basis composed of normalised multivariate Hermite polynomials such that $E\{\Psi_j(\{\Xi_k\}_{k \in \mathbb{N}}) \, \Psi_{j'}(\{\Xi_k\}_{k \in \mathbb{N}})\} = \delta_{jj'}$. Since $E\{\eta_i \eta_{i'}\} = \delta_{ii'}$, it can be deduced that

$$\sum_{j=1}^{+\infty} y_i^j \, y_{i'}^j = \delta_{ii'}. \tag{2.35}$$

2.4 Random upper bound for random field [K]

Lemma 6 *If* $\sum_{i=1}^{+\infty} \sqrt{\sigma_i} \, \|G_i\|_\infty < +\infty$, *then for all* **x** *in* D,

$$\|\mathbf{G}(\mathbf{x})\|_F \leqslant \beta_G < +\infty \quad a.s, \tag{2.36}$$

in which β_G *is the second-order positive-valued random variable,*

$$\beta_G = \|G_0\|_\infty + \sum_{i=1}^{+\infty} \sqrt{\sigma_i} \, \|G_i\|_\infty \, |\eta_i|, \quad E\{\beta_G^2\} < +\infty. \tag{2.37}$$

Proof The expression of β_G defined in equation (2.37) is directly deduced from equation (2.33). The random variable β_G can be written as $\beta_G = \|G_0\|_\infty + \widehat{\beta}$. Clearly, if $E\{\widehat{\beta}^2\} = \sum_{i,i'} \sqrt{\sigma_i}\sqrt{\sigma_{i'}}\|G_i\|_\infty\|G_{i'}\|_\infty E\{|\eta_i|\,|\eta_{i'}|\} < +\infty$, then $E\{\beta_G^2\} < +\infty$. We have $E\{|\eta_i|\,|\eta_{i'}|\} \leqslant \sqrt{E\{\eta_i^2\}}\sqrt{E\{\eta_{i'}^2\}} = 1$ and thus $E\{\widehat{\beta}^2\} \leqslant (\sum_{i=1}^{+\infty}\sqrt{\sigma_i}\|G_i\|_\infty)^2$. $\qquad\square$

Remark 2.3 It should be noted that, if $\|G_i\|_\infty < c_\infty < +\infty$ for all $i \geqslant 1$ with c_∞ independent of i, we then have $\sum_{i=1}^{+\infty}\sqrt{\sigma_i}\|G_i\|_\infty < +\infty$ if:

(i) for $d = 1$, the hypothesis of (a) in Lemma 5 holds for $\mu = 2$. The proof is the following. We have $\sum_{i=1}^{+\infty}\sqrt{\sigma_i}\|G_i\|_\infty < c_\infty\sum_{i=1}^{+\infty}\sqrt{\sigma_i}$. Since $\sigma_i \to 0$ for $i \to +\infty$, there exists an integer $i_0 \geqslant 1$ such that, for all $i \geqslant i_0$, we have $\sigma_i < 1$. For $\mu = 2$, then there exits $0 < \zeta < 1/10$ such that $\sum_{i=i_0}^{+\infty}\sqrt{\sigma_i} \leqslant \sum_{i=i_0}^{+\infty}\sigma_i^{\zeta+2/5} < +\infty$, which yields $\sum_{i=1}^{+\infty}\sqrt{\sigma_i} < +\infty$, and consequently, $\sum_{i=1}^{+\infty}c_\infty\sqrt{\sigma_i} < +\infty$.

(ii) for $d \geqslant 2$ (finite integer), the hypothesis of (b) in Lemma 5 holds for $\mu = d$. The proof is then the following. We have $\sum_{i=1}^{+\infty}\sqrt{\sigma_i}\|G_i\|_\infty < c_\infty\sum_{i=1}^{+\infty}\sqrt{\sigma_i}$ and for $\mu = d$, $\sqrt{\sigma_i} = O(i^{-1-\zeta/(2d)})$ and consequently, for $0 < \zeta \leqslant 1$, $\sum_{i=1}^{+\infty}\sqrt{\sigma_i} < +\infty$.

The previous results allow the following proposition to be proven.

Proposition 3 *Let σ_i and $[G_i]$ be defined in Section 2.3.1. If $\sum_{i=1}^{+\infty}\sqrt{\sigma_i}\|G_i\|_\infty < +\infty$, then for all $\mathbf{x}$ in D,*

$$\|\mathbf{K}(\mathbf{x})\|_F \leqslant \beta < +\infty \quad a.s, \tag{2.38}$$

in which β is a positive-valued random variable independent of $\mathbf{x}$. Let β_G be the second-order positive-valued random variable defined by equation (2.37).

(i) *(Exponential-type representation) If $[\mathbf{K}(\mathbf{x})]$ is represented by equation (2.3) with equation (2.9),*

$$[\mathbf{K}(\mathbf{x})] = \frac{1}{1+\varepsilon}[\underline{L}(\mathbf{x})]^T\{\varepsilon[I_n] + \exp_{\mathbb{M}}([\mathbf{G}(\mathbf{x})])\}[\underline{L}(\mathbf{x})], \tag{2.39}$$

then, $\beta = \underline{k}_1\sqrt{n}(\varepsilon + e^{\beta_G})/(1+\varepsilon)$.

(ii) *(Square-type representation) If $[\mathbf{K}(\mathbf{x})]$ is represented by equation (2.3) with equations (2.16) and (2.17),*

$$[\mathbf{K}(\mathbf{x})] = \frac{1}{1+\varepsilon}[\underline{L}(\mathbf{x})]^T\{\varepsilon[I_n] + [\mathscr{L}([\mathbf{G}(\mathbf{x})])]^T[\mathscr{L}([\mathbf{G}(\mathbf{x})])]\}[\underline{L}(\mathbf{x})], \tag{2.40}$$

then, $\beta = \underline{k}_1(\sqrt{n}\,\varepsilon + \gamma_0 + \gamma_1\beta_G^2)/(1+\varepsilon)$ in which γ_0 and γ_1 are two positive and finite real numbers. In addition the random variable β is such that

$$E\{\beta\} < +\infty. \tag{2.41}$$

Proof Proposition 3 results from Lemma 2, Propositions 1 and 2, and Lemma 6. $\qquad\square$

2.5 Approximation of random field [G]

Taking into account equations (2.33) to (2.35), we introduce the approximation $[\mathbf{G}^{(m,N)}]$ of the random field $[\mathbf{G}]$ such that

$$[\mathbf{G}^{(m,N)}(\mathbf{x})] = [G_0(\mathbf{x})] + \sum_{i=1}^{m} \sqrt{\sigma_i}\,[G_i(\mathbf{x})]\,\eta_i, \tag{2.42}$$

$$\eta_i = \sum_{j=1}^{N} y_i^j\,\Psi_j(\boldsymbol{\Xi}), \tag{2.43}$$

in which the $\{\Psi_j\}_{j=1}^{N}$ only depends on a random vector $\boldsymbol{\Xi} = (\Xi_1, \ldots, \Xi_{N_g})$ of N_g independent normalised Gaussian random variables $\Xi_1, \ldots, \Xi_{N_g}$ defined on probability space $(\Theta, \mathcal{T}, \mathscr{P})$. The coefficients y_i^j are supposed to verify $\sum_{j=1}^{N} y_i^j y_{i'}^j = \delta_{ii'}$ which ensures that the random variables, $\{\eta_i\}_{i=1}^{m}$, are uncorrelated, centred random variables with unit variance, which means that $E\{\eta_i \eta_{i'}\} = \delta_{ii'}$. The relation between the coefficients can be rewritten as

$$[y]^T\,[y] = [I_m], \tag{2.44}$$

in which $[y] \in \mathbb{M}_{N,m}(\mathbb{R})$ is such that $[y]_{ji} = y_i^j$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant N$. Introducing the random vectors $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_m)$ and $\boldsymbol{\Psi}(\boldsymbol{\Xi}) = (\Psi_1(\boldsymbol{\Xi}), \ldots, \Psi_N(\boldsymbol{\Xi}))$, equation (2.43) can be rewritten as

$$\boldsymbol{\eta} = [y]^T\,\boldsymbol{\Psi}(\boldsymbol{\Xi}). \tag{2.45}$$

Equation (2.44) means that $[y]$ belongs to the compact Stiefel manifold.

$$\mathbb{V}_m(\mathbb{R}^N) = \left\{ [y] \in \mathbb{M}_{N,m}(\mathbb{R})\,;\,[y]^T\,[y] = [I_m] \right\}. \tag{2.46}$$

With the above hypotheses, the covariance operators of random fields $[\mathbf{G}]$ and $[\mathbf{G}^{(m,N)}]$ coincide on the subspace spanned by the finite family $\{[G_i]\}_{i=1}^{m}$.

3 Parameterisation of discretised random fields in the general class and identification strategy

Let us consider the approximation $\{[\mathbf{G}^{(m,N)}(\mathbf{x})], \mathbf{x} \in D\}$ of $\{[\mathbf{G}(\mathbf{x})], \mathbf{x} \in D\}$ defined by equations (2.42) to (2.44). The corresponding approximation $\{[\mathbf{K}^{(m,N)}(\mathbf{x})], \mathbf{x} \in D\}$ of random field $\{[\mathbf{K}(\mathbf{x})], \mathbf{x} \in D\}$ defined by equation (2.39) or by equation (2.40), is rewritten, for all $\mathbf{x}$ in D, as

$$[\mathbf{K}^{(m,N)}(\mathbf{x})] = \mathscr{K}^{(m,N)}(\mathbf{x}, \boldsymbol{\Xi}, [y]), \tag{3.1}$$

in which $(\mathbf{x}, \mathbf{y}, [y]) \mapsto \mathscr{K}^{(m,N)}(\mathbf{x}, \mathbf{y}, [y])$ is a mapping defined on $D \times \mathbb{R}^{N_g} \times \mathbb{V}_m(\mathbb{R}^N)$ with values in $\mathbb{M}_n^+(\mathbb{R})$.

- The first objective of this section is to construct a parameterised general class of random fields, $\{[\mathbf{K}^{(m,N)}(\mathbf{x})], \mathbf{x} \in D\}$, in introducing a minimal parameterisation of the compact Stiefel manifold $\mathbb{V}_m(\mathbb{R}^N)$ with an algorithm of complexity $O(Nm^2)$.

- The second objective will be the presentation of an identification strategy of an optimal random field $\{[\mathbf{K}^{(m,N)}(\mathbf{x})], \mathbf{x} \in D\}$ in the parameterised general class, using partial and limited experimental data.

3.1 Minimal parameterisation of the compact Stiefel manifold $\mathbb{V}_m(\mathbb{R}^N)$

Here, we introduce a particular minimal parameterisation of the compact Stiefel manifold $\mathbb{V}_m(\mathbb{R}^N)$ using matrix exponentials (see, e.g., [1]). The dimension of $\mathbb{V}_m(\mathbb{R}^N)$ being $v = mN - m(m+1)/2$, the parameterisation consists in introducing a surjective mapping from $\mathbb{R}^v$ onto $\mathbb{V}_m(\mathbb{R}^N)$. The construction is as follows.

(i) Let $[a]$ be given in $\mathbb{V}_m(\mathbb{R}^N)$ and let $[a_\perp] \in \mathbb{M}_{N,N-m}(\mathbb{R})$ be the orthogonal complement of $[a]$, which is such that $[a \ a_\perp]$ is in $\mathbb{O}(N)$. Consequently, we have,

$$[a]^T [a] = [I_m], \qquad [a_\perp]^T [a_\perp] = [I_{N-m}], \qquad [a_\perp]^T [a] = [0_{N-m,m}]. \qquad (3.2)$$

The columns of matrix $[a_\perp]$ can be chosen as the vectors of the ortho-normal basis of the null space of $[a]^T$. In practice [30], $[a_\perp]$ can be constructed using the QR factorisation of matrix $[a] = [Q_N] [R_m]$ and $[a_\perp]$ is then made up of the columns $j = m+1, \ldots, N$ of $[Q_N] \in \mathbb{O}(N)$.

(ii) Let $[A]$ be a skew-symmetric $(m \times m)$ real matrix which then depends on $m(m-1)/2$ parameters denoted by $z_1, \ldots, z_{m(m-1)/2}$ and such that, for $1 \leqslant i < j \leqslant m$, one has $[A]_{ij} = -[A]_{ji} = z_k$ with $k = i + (j-1)(j-2)/2$ and for $1 \leqslant i \leqslant m$, $[A]_{ii} = 0$.

(iii) Let $[B]$ be a $((N-m) \times m)$ real matrix which then depends on $(N-m)m$ parameters denoted by $z_{m(m-1)/2+1}, \ldots, z_{m(m-1)/2+(N-m)m}$ and such that, for $1 \leqslant i \leqslant N-m$ and $1 \leqslant j \leqslant m$, one has $[B]_{ij} = z_{m(m-1)/2+k}$ with $k = i + (j-1)(N-m)$. Introducing $\mathbf{z} = (z_1, \ldots, z_v)$ in $\mathbb{R}^v$, there is a one-to-one linear mapping $\mathscr{S}$ from $\mathbb{R}^v$ into $\mathbb{M}_m^{SS}(\mathbb{R}) \times \mathbb{M}_{N-m,m}(\mathbb{R})$ such that

$$\{[A], [B]\} = \mathscr{S}(\mathbf{z}), \qquad (3.3)$$

in which matrices $[A]$ and $[B]$ are defined in (i) and (ii) above as a function of $\mathbf{z}$.

Let $t > 0$ be a parameter which is assumed to be fixed. Then, for an arbitrary $[a]$ in $\mathbb{V}_m(\mathbb{R}^N)$, a first minimal parameterisation of the compact Stiefel manifold $\mathbb{V}_m(\mathbb{R}^N)$ can be defined by the mapping $\mathscr{M}_{[a]}$ from $\mathbb{R}^v$ onto $\mathbb{V}_m(\mathbb{R}^N)$ such that

$$[y] = \mathscr{M}_{[a]}(\mathbf{z}) := [a \ a_\perp] \left\{ \exp_{\mathbb{M}} \left(t \begin{bmatrix} A & -B^T \\ B & 0 \end{bmatrix} \right) \right\} [I_{N,m}]. \qquad (3.4)$$

In the context of the present development, we are interested in the case for which $N \geqslant m$ and possibly, in the case for which $N \gg m$ with N very large. The evaluation of the mapping $\mathscr{M}_{[a]}$ defined by equation (3.4) has a complexity $O(N^3)$.

We then propose to use a second form of minimal parameterisation of the compact Stiefel with a reduced computational complexity. This parameterisation, which is derived from the results presented in [20], is defined by the mapping $\mathscr{M}_{[a]}$ from $\mathbb{R}^v$ onto $\mathbb{V}_m(\mathbb{R}^N)$

such that

$$[y] = \mathscr{M}_{[a]}(\mathbf{z}) := [a \; Q] \left\{ \exp_{\mathbb{M}} \left(t \begin{bmatrix} A & -R^T \\ R & 0 \end{bmatrix} \right) \right\} [I_{2m,m}], \tag{3.5}$$

in which $[a \; Q] \in \mathbb{M}_{N,2m}(\mathbb{R})$. The matrix $[Q]$ is in $\mathbb{V}_m(\mathbb{R}^N)$ and $[R]$ is an upper triangular $(m \times m)$ real matrix. These two matrices are constructed using the QR factorisation of the matrix $[a_\perp] [B] \in \mathbb{M}_{N,m}(\mathbb{R})$,

$$[a_\perp] [B] = [Q] [R]. \tag{3.6}$$

The evaluation of the mapping $\mathscr{M}_{[a]}$ defined by equation (3.5) has a complexity $O(Nm^2)$.

3.2 Parameterised general class of random fields and parameterised random upper bound

Using equation (3.5), the parameterised general class of random field $\{[\mathbf{K}^{(m,N)}(\mathbf{x})], \mathbf{x} \in D\}$ is then defined as

$$\mathscr{K} := \{\mathscr{K}^{(m,N)}(\cdot, \boldsymbol{\Xi}, \mathscr{M}_{[a]}(\mathbf{z})) \, ; \, \mathbf{z} \in \mathbb{R}^v\}. \tag{3.7}$$

It should be noted that, for $\mathbf{z} = \mathbf{0}$, $[y] = \mathscr{M}_{[a]}(\mathbf{0}) = [a]$, which corresponds to the random field $[\mathbf{K}^{(m,N)}] = \mathscr{K}^{(m,N)}(\cdot, \boldsymbol{\Xi}, [a])$. The following proposition corresponds to Proposition 3 for the approximation $[\mathbf{K}^{(m,N)}]$ of random field $[\mathbf{K}]$.

Proposition 4 *The random field* $[\mathbf{K}^{(m,N)}] = \mathscr{K}^{(m,N)}(\cdot, \boldsymbol{\Xi}, \mathscr{M}_{[a]}(\mathbf{z}))$, $\mathbf{z} \in \mathbb{R}^v$, *is such that*

$$\|\mathbf{K}^{(m,N)}\|_F \leqslant \gamma(\boldsymbol{\Xi}, \mathbf{z}) < +\infty \tag{3.8}$$

almost surely and for all $\mathbf{z} \in \mathbb{R}^v$, *where* $\gamma : \mathbb{R}^{N_g} \times \mathbb{R}^v \to \mathbb{R}$ *is a measurable positive function. For the square-type representation of random fields, there exists a constant* $\overline{\gamma}$, *independent on N and* $\mathbf{z}$, *such that*

$$E\{\gamma(\boldsymbol{\Xi}, \mathbf{z})\} \leqslant \overline{\gamma} < +\infty \quad \text{for all } \mathbf{z} \in \mathbb{R}^v. \tag{3.9}$$

Moreover, if $\sum_{i=1}^{+\infty} \sqrt{\sigma_i} \|G_i\|_\infty < +\infty$, *with* σ_i *and* $[G_i]$ *defined in Section 2.3.1, (3.9) is satisfied for a constant* $\overline{\gamma}$ *independent of m.*

Proof Following the proof of Lemma 6, we obtain

$$\|\mathbf{G}^{(m,N)}(\mathbf{x})\|_F \leqslant \|G_0\|_\infty + \sum_{i=1}^{m} \sqrt{\sigma_i}\|G_i\|_\infty|\eta_i(\boldsymbol{\Xi}, \mathbf{z})| := \delta(\boldsymbol{\Xi}, \mathbf{z}), \tag{3.10}$$

with $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_m) = \mathscr{M}_{[a]}(\mathbf{z})^T \boldsymbol{\Psi}(\boldsymbol{\Xi})$. Using Proposition 3, we then obtain equation (3.8) with $\gamma = \underline{k}_1 \sqrt{n}(\varepsilon + e^\delta)/(1 + \varepsilon)$ for the exponential-type representation and $\gamma = \underline{k}_1(\sqrt{n}\,\varepsilon + \gamma_0 + \gamma_1 \delta^2)/(1 + \varepsilon)$ for the square-type representation. Using $E\{\eta_i^2\} = 1$, it can be shown that $E\{\delta^2\} \leqslant 2\|G_0\|_\infty^2 + 2(\sum_{i=1}^{m} \sqrt{\sigma_i}\|G_i\|_\infty)^2 := \overline{\zeta}_m$, where $\overline{\zeta}_m < +\infty$ is independent on N and $\mathbf{z}$. Therefore, for the square-type representation, we obtain equation (3.9) with $\overline{\gamma} = \underline{k}_1(\sqrt{n}\,\varepsilon + \gamma_0 + \gamma_1 \overline{\zeta}_m)/(1 + \varepsilon)$. Moreover, if $\sum_{i=1}^{+\infty} \sqrt{\sigma_i} \|G_i\|_\infty < +\infty$, then $\overline{\zeta}_m \leqslant \overline{\zeta}_\infty < +\infty$ and equation (3.9) holds for $\overline{\gamma} = \underline{k}_1(\sqrt{n}\,\varepsilon + \gamma_0 + \gamma_1 \overline{\zeta}_\infty)/(1 + \varepsilon)$. □

3.3 Brief description of the identification procedure

Let $\mathscr{B}$ be the non-linear mapping which, for any given random field $[\mathbf{K}] = \{[\mathbf{K}(\mathbf{x})], \mathbf{x} \in D\}$ introduced and studied in Section 2, associates a unique random observation vector $\mathbf{U}^{\mathrm{obs}} = \mathscr{B}([\mathbf{K}])$ with values in $\mathbb{R}^{m_{\mathrm{obs}}}$. The non-linear mapping $\mathscr{B}$ is constructed in solving the elliptic stochastic boundary value problem as explained in Section 4. We are then interested in identifying the random field $[\mathbf{K}]$ using partial and limited experimental dataset $\mathbf{u}^{\mathrm{exp},1}, \ldots, \mathbf{u}^{\mathrm{exp},v_{\mathrm{exp}}}$ in $\mathbb{R}^{m_{\mathrm{obs}}}$ (m_{obs} and v_{exp} are small). In high stochastic dimension (which is the assumption of the present paper), such a statistical inverse problem is an ill-posed problem if no additional available information is introduced. As explained in [59, 60], this difficulty can be circumvented (1) in introducing an algebraic prior model (APM), $[\mathbf{K}^{\mathrm{APM}}]$, of random field $[\mathbf{K}]$, which contains additional information and satisfying the required mathematical properties, and (2) in using an adapted identification procedure. Below, we summarise and we adapt this procedure to the new representations and their parameterisations of random field $[\mathbf{K}]$ that we propose in this paper. The steps of the identification procedure are the following:

Step 1

Introduction of a family $\{[\mathbf{K}^{\mathrm{APM}}(\mathbf{x}; \mathbf{w})], \mathbf{x} \in D\}$ of APMs for random field $[\mathbf{K}]$. This family depends on an unknown parameter $\mathbf{w}$ (for instance, $\mathbf{w}$ can be made up of the mean function, spatial correlation lengths, dispersion parameters controlling the statistical fluctuations, parameters controlling the shape of the tensor-valued correlation function, parameters controlling the symmetry class, etc). For fixed $\mathbf{w}$, the probability law and the generator of independent realisations of the APM are known. For example, for the modelling of groundwater Darcy flows, a typical choice for the APM would consist in a homogeneous and isotropic log-normal random field $[\mathbf{K}^{\mathrm{APM}}(\mathbf{x}; \mathbf{w})] = \exp(G^{\mathrm{APM}})[I_d]$ with G^{APM} a homogeneous real-valued Gaussian random field having a covariance function of the Matérn family. With this APM, $\mathbf{w}$ consists of four scalar parameters that are the mean value of G and the three parameters of the Matérn covariance. For other examples of APMs of non-Gaussian positive-definite matrix-valued random fields, we refer the reader to [57] for the anisotropic class, to [64] for the isotropic class, to [31] for bounded random fields in the anisotropic class, to [32] for random fields with any symmetry class (isotropic, cubic, transversal isotropic, tetragonal, trigonal, orthotropic), and finally, to [33] for a very general class of bounded random fields with any symmetry properties from the isotropic class to the anisotropic class.

Step 2

Identification of an optimal value $\mathbf{w}^{\mathrm{opt}}$ of parameter $\mathbf{w}$ using the experimental dataset, the family of stochastic solutions $\mathbf{U}^{\mathrm{obs}}(\mathbf{w}) = \mathscr{B}([\mathbf{K}^{\mathrm{APM}}(\cdot; \mathbf{w})])$ and a statistical inverse method such as the moment method, the least-square method or the maximum likelihood method [55, 59, 61, 66]). The optimal APM $\{[\mathbf{K}^{\mathrm{OAPM}}(\mathbf{x})], \mathbf{x} \in D\} := \{[\mathbf{K}^{\mathrm{APM}}(\mathbf{x}; \mathbf{w}^{\mathrm{opt}})], \mathbf{x} \in D\}$ is then obtained. Using the generator of realisations of the APM, v_{KL} independent realisations $[K^{(1)}], \ldots, [K^{(v_{\mathrm{KL}})}]$ of random field $[\mathbf{K}^{\mathrm{OAPM}}]$ can be generated with v_{KL} as large as it is desired without inducing a significant computational cost.

Step 3

Choice of a type of representation for random field $[\mathbf{K}]$ and, using equation (2.3) with equation (2.10) or with equation (2.20), the optimal APM $\{[\mathbf{G}^{\mathrm{OAPM}}(\mathbf{x})], \mathbf{x} \in D\}$ of random field $[\mathbf{G}]$ is deduced. For all $\mathbf{x} \in D$, $[\mathbf{G}^{\mathrm{OAPM}}(\mathbf{x})] = \log_{\mathbb{M}}([\mathbf{K}_0^{\mathrm{OAPM}}(\mathbf{x})])$ for the exponential-type representation and $[\mathbf{G}^{\mathrm{OAPM}}(\mathbf{x})] = \mathbb{L}^{-1}([\mathbf{K}_0^{\mathrm{OAPM}}(\mathbf{x})])$ for the square-type representation, with

$$[\mathbf{K}_0^{\mathrm{OAPM}}(\mathbf{x})] = (1 + \varepsilon)[\underline{L}(\mathbf{x})]^{-T} [\mathbf{K}^{\mathrm{OAPM}}(\mathbf{x})] [\underline{L}(\mathbf{x})] - \varepsilon[I_n]. \tag{3.11}$$

It is assumed that random field $[\mathbf{G}^{\mathrm{OAPM}}]$ belongs to $\mathscr{C}_n^{2,S}$. From the ν_{KL} independent realisations $[K^{(1)}],\ldots,[K^{(\nu_{\mathrm{KL}})}]$ of random field $[\mathbf{K}^{\mathrm{OAPM}}]$, it can be deduced the ν_{KL} independent realisations $[G^{(1)}],\ldots,[G^{(\nu_{\mathrm{KL}})}]$ of random field $[\mathbf{G}^{\mathrm{OAPM}}]$.

Step 4

Use of the ν_{KL} independent realisations $[G^{(1)}],\ldots,[G^{(\nu_{\mathrm{KL}})}]$ of random field $[\mathbf{G}^{\mathrm{OAPM}}]$ and use of the adapted statistical estimators for estimating the mean function $[G_0^{\mathrm{OAPM}}]$ and the covariance function $C_{\mathbf{G}^{\mathrm{OAPM}}}$ of random field $[\mathbf{G}^{\mathrm{OAPM}}]$ (see Section 2.3.1). Then, calculation of the first m eigenvalues $\sigma_1 \geqslant \ldots \geqslant \sigma_m$ and the corresponding eigenfunctions $[G_1],\ldots,[G_m]$ of the covariance operator $\mathrm{Cov}_{\mathbf{G}^{\mathrm{OAPM}}}$ defined by the kernel $C_{\mathbf{G}^{\mathrm{OAPM}}}$. For a given convergence tolerance with respect to m, equation (2.42) is used to construct independent realisations of the random vector $\boldsymbol{\eta}^{\mathrm{OAPM}} = (\eta_1^{\mathrm{OAPM}},\ldots,\eta_m^{\mathrm{OAPM}})$ such that

$$[\mathbf{G}^{\mathrm{OAPM}(m)}(\mathbf{x})] = [G_0^{\mathrm{OAPM}}(\mathbf{x})] + \sum_{i=1}^{m} \sqrt{\sigma_i}\, [G_i(\mathbf{x})]\, \eta_i^{\mathrm{OAPM}}. \tag{3.12}$$

For $i = 1,\ldots,m$, the ν_{KL} independent realisations $\eta_i^{(1)},\ldots,\eta_i^{(\nu_{\mathrm{KL}})}$ of the random variable η_i^{OAPM} are calculated by

$$\eta_i^{(\ell)} = \frac{1}{\sqrt{\sigma_i}} \ll [G^{(\ell)}] - [G_0^{\mathrm{OAPM}}], [G_i] \gg \quad, \quad \ell = 1,\ldots,\nu_{\mathrm{KL}}. \tag{3.13}$$

Step 5

For a given convergence tolerance with respect to N and N_g in the polynomial chaos expansion defined by equation (2.45), use of the methodology based on the maximum likelihood and the corresponding algorithms presented in [59] for estimating a value $[y_0] \in \mathbb{V}_m(\mathbb{R}^N)$ of $[y]$ such that $\boldsymbol{\eta}^{\mathrm{OAPM}} = [y_0]^T \boldsymbol{\Psi}(\boldsymbol{\Xi})$.

Let us examine the following particular case for which the APM $[\mathbf{K}^{\mathrm{APM}}]$ of random field $[\mathbf{K}]$ is defined by equation (2.3) with either equations (2.10) or (2.20), in which $[\mathbf{G}^{\mathrm{APM}}]$ is chosen as a second-order Gaussian random field indexed by D with values in $\mathbb{M}_n^S(\mathbb{R})$. Therefore, the components $\eta_1,\ldots,\eta_m$ of the random vector $\boldsymbol{\eta}$ defined by equation (2.45) are independent real-valued normalised Gaussian random variables. We then have $N_g = m$. Let us assume that, for $1 \leqslant j \leqslant m \leqslant N$, the indices j of the polynomial chaos are ordered such that $\Psi_j(\boldsymbol{\Xi}) = \Xi_j$. It can then be deduced that $[y_0] \in \mathbb{V}_m(\mathbb{R}^N)$ is such that $[y_0]_{ji} = \delta_{ij}$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant N$.

Step 6

With the maximum likelihood method, estimation of a value $\check{\mathbf{z}}$ of $\mathbf{z} \in \mathbb{R}^\nu$ for the parameterised general class $\mathscr{K}$ defined by equations (3.1) and (3.7), using the family of stochastic solutions $\mathbf{U}^{\mathrm{obs}}(\mathbf{z}) = \mathscr{B}(\mathscr{K}^{(m,N)}(\cdot, \boldsymbol{\Xi}, \mathscr{M}_{[y_0]}(\mathbf{z})))$ and the experimental dataset $\mathbf{u}^{\mathrm{exp},1}, \ldots, \mathbf{u}^{\mathrm{exp},\nu_{\mathrm{exp}}}$. Then, calculation of $[\check{y}] = \mathscr{M}_{[y_0]}(\check{\mathbf{z}})$.

Step 7

Construction of *a posterior* model for random field $[\mathbf{K}]$ using the Bayesian method. In such a framework, the coefficients $[y]$ of the polynomial chaos expansion $\boldsymbol{\eta} = [y]^T \boldsymbol{\Psi}(\boldsymbol{\Xi})$ (see equation (2.45)) are modelled by a random matrix $[\mathbf{Y}]$ (see [58]) as proposed in [60] and consequently, $\mathbf{z}$ is modelled by a $\mathbb{R}^\nu$-valued random variable $\mathbf{Z}$. For the prior model $[\mathbf{Y}^{\mathrm{prior}}]$ of $[\mathbf{Y}]$, here we propose

$$[\mathbf{Y}^{\mathrm{prior}}] = \mathscr{M}_{[\check{y}]}(\mathbf{Z}^{\mathrm{prior}}) \quad , \quad \mathbf{Z}^{\mathrm{prior}} \sim \text{centred Gaussian vector}, \qquad (3.14)$$

which guaranties that $[\mathbf{Y}^{\mathrm{prior}}]$ is a random matrix with values in $\mathbb{V}_m(\mathbb{R}^N)$ whose statistical fluctuations are centred around $[\check{y}]$ (the maximum likelihood estimator of the set of coefficients $[y]$ computed at step 6). The Bayesian update allows the posterior distribution of random vector $\mathbf{Z}^{\mathrm{post}}$ to be estimated using the stochastic solution $\mathbf{U}^{\mathrm{obs}} = \mathscr{B}(\mathscr{K}^{(m,N)}(\cdot, \boldsymbol{\Xi}, \mathscr{M}_{[\check{y}]}(\mathbf{Z}^{\mathrm{prior}})))$ and the experimental dataset $\mathbf{u}^{\mathrm{exp},1}, \ldots, \mathbf{u}^{\mathrm{exp},\nu_{\mathrm{exp}}}$.

Finally, it should be noted that once the probability distribution of $\mathbf{Z}^{\mathrm{post}}$ has been estimated by Step 7, ν_{KL} independent realisations can be calculated for the random field $[\mathbf{G}^{\mathrm{post}}(\mathbf{x})] = [G_0^{\mathrm{OAPM}}(\mathbf{x})] + \sum_{i=1}^m \sqrt{\sigma_i} [G_i(\mathbf{x})] \eta_i^{\mathrm{post}}$ in which $\boldsymbol{\eta}^{\mathrm{post}} = [\mathbf{Y}^{\mathrm{post}}]^T \boldsymbol{\Psi}(\boldsymbol{\Xi})$ and where $[\mathbf{Y}^{\mathrm{post}}] = \mathscr{M}_{[\check{y}]}(\mathbf{Z}^{\mathrm{post}})$. The identification procedure can then be restarted from Step 4 replacing $[\mathbf{G}^{\mathrm{OAPM}}]$ by $[\mathbf{G}^{\mathrm{post}}]$.

4 Solution of the stochastic elliptic boundary value problem

Let $D \subset \mathbb{R}^d$ be a bounded open domain with smooth boundary. The following elliptic stochastic partial differential equation is considered,

$$-\mathrm{div}([\mathbf{K}] \cdot \nabla U) = f \quad \text{a.e. in } D, \qquad (4.1)$$

with homogeneous Dirichlet boundary conditions (for the sake of simplicity). The random field $\{[\mathbf{K}(\mathbf{x})], \mathbf{x} \in D\}$ belongs to the parameterised general class of random fields,

$$\mathscr{K} = \{\mathscr{K}^{(m,N)}(\cdot, \boldsymbol{\Xi}, \mathscr{M}_{[a]}(\mathbf{z})) \, ; \, \mathbf{z} \in \mathbb{R}^\nu\},$$

introduced in Section 3.2. This stochastic boundary value problem has to be solved at steps 6 and 7 of the identification procedure described in Section 3.3, respectively considering $\mathbf{z}$ as a deterministic or a random parameter. We introduce the map

$$u : D \times \mathbb{R}^{N_g} \times \mathbb{R}^\nu \to \mathbb{R}$$

such that $U = u(\mathbf{x}, \boldsymbol{\Xi}, \mathbf{z})$ is the solution of the stochastic boundary value problem for $[\mathbf{K}(\mathbf{x})] = \mathscr{K}^{(m,N)}(\mathbf{x}, \boldsymbol{\Xi}, \mathscr{M}_{[a]}(\mathbf{z}))$. The aim here is to construct an explicit approximation of the map u for its efficient use in the identification procedure.

In this section, a suitable functional framework will first be introduced for the definition of the map u. Then, numerical methods based on Galerkin projections will be analysed for the approximation of this map. Different numerical approaches will be introduced depending on the type of representation of random fields (square-type or exponential-type) and depending on the properties of approximation spaces. For the two types of representation of random fields (exponential-type or square-type). Finally, we will briefly describe complexity reduction methods based on low-rank approximations that exploit the tensor structure of the high-dimensional map u and allows its approximate representation to be obtained in high dimension. That makes affordable the application of the identification procedure for high-dimensional germs Ξ (high N_g) and high-order representation of random fields (high v).

4.1 Analysis of the stochastic boundary value problem

We denote by $\Gamma = \Gamma_{N_g} \otimes \Gamma_v$ a product measure on $\mathbb{R}^\mu := \mathbb{R}^{N_g} \times \mathbb{R}^v$, where Γ_{N_g} is the probability measure of random variable Ξ and where Γ_v is a finite measure on $\mathbb{R}^v$. Up to a normalisation, Γ_v is considered as the probability measure of a random vector $\mathbf{Z}$. We denote by $[C] : D \times \mathbb{R}^{N_g} \times \mathbb{R}^v \to \mathbb{M}_n^+(\mathbb{R})$ the map defined by

$$[C](\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathscr{K}^{(m,N)}(\mathbf{x}, \mathbf{y}, \mathscr{M}_{[a]}(\mathbf{z})),$$

and such that $[\mathbf{K}(\mathbf{x})] = [C](\mathbf{x}, \Xi, \mathbf{Z})$ is a $\sigma(\Xi, \mathbf{Z})$-measurable random field. Sometimes, the random field $\{[C](\mathbf{x}, \Xi, \mathbf{Z}), \mathbf{x} \in D\}$ will be denoted by $\{[\mathbf{C}(\mathbf{x})], \mathbf{x} \in D\}$. For a measurable function $h : \mathbb{R}^{N_g} \times \mathbb{R}^v \to \mathbb{R}$, the mathematical expectation of h is defined by

$$E_\Gamma(h) = E\{h(\Xi, \mathbf{Z})\} = \int_{\mathbb{R}^{N_g} \times \mathbb{R}^v} h(\mathbf{y}, \mathbf{z})\, \Gamma(d\mathbf{y}, d\mathbf{z}).$$

Lemma 7 *Under the hypotheses of Proposition 4, there exists a constant α and a positive measurable function $\gamma : \mathbb{R}^{N_g} \times \mathbb{R}^v \to \mathbb{R}$ such that, for Γ-almost all $(\mathbf{y}, \mathbf{z})$ in $\mathbb{R}^{N_g} \times \mathbb{R}^v$, we have*

$$0 < \alpha \leqslant \operatorname*{ess\,inf}_{\mathbf{x} \in D} \inf_{\mathbf{h} \in \mathbb{R}^n \setminus \{0\}} \frac{<[C](\mathbf{x}, \mathbf{y}, \mathbf{z})\mathbf{h}, \mathbf{h}>_2}{\|\mathbf{h}\|_2^2}, \tag{4.2}$$

$$\operatorname*{ess\,sup}_{\mathbf{x} \in D} \sup_{\mathbf{h} \in \mathbb{R}^n \setminus \{0\}} \frac{<[C](\mathbf{x}, \mathbf{y}, \mathbf{z})\mathbf{h}, \mathbf{h}>_2}{\|\mathbf{h}\|_2^2} \leqslant \gamma(\mathbf{y}, \mathbf{z}) < \infty. \tag{4.3}$$

Moreover, for the class of random fields corresponding to the square-type representation, we have $E_\Gamma(\gamma) \leqslant \bar{\gamma} < +\infty$, that means $\gamma \in L_\Gamma^1(\mathbb{R}^\mu)$.

Proof Lemma 1(ii) gives the existence of the lower bound $\alpha = \underline{k}_\varepsilon$. Proposition 4 yields $\|[C](\mathbf{x}, \mathbf{y}, \mathbf{z})\|_2 \leqslant \|[C](\mathbf{x}, \mathbf{y}, \mathbf{z})\|_F \leqslant \gamma(\mathbf{y}, \mathbf{z}) < \infty$, with γ a measurable function defined on $\mathbb{R}^{N_g} \times \mathbb{R}^v$. For the square-type representation of random fields, property (3.9) implies $E_\Gamma(\gamma) = \int_{\mathbb{R}^v} E_{\Gamma_{N_g}}\{\gamma(\Xi, \mathbf{z})\}\Gamma_v(d\mathbf{z}) \leqslant \bar{\gamma}$. $\qquad\square$

We introduce the bilinear form $C(\cdot, \cdot; \mathbf{y}, \mathbf{z}) : H_0^1(D) \times H_0^1(D) \to \mathbb{R}$ defined by

$$C(u, v; \mathbf{y}, \mathbf{z}) = \int_D \nabla v \cdot [C](\cdot, \mathbf{y}, \mathbf{z}) \nabla u \, d\mathbf{x}. \tag{4.4}$$

Let's introduce $\| \cdot \|_{H_0^1} = (\int_D |\nabla(\cdot)|^2 \, d\mathbf{x})^{1/2}$ the norm on $H_0^1(D)$, and $\| \cdot \|_{H^{-1}}$ the norm on the continuous dual space $H^{-1}(D)$.

Strong-stochastic solution

Proposition 5 *Assume $f \in H^{-1}(D)$. Then, for Γ-almost all $(\mathbf{y}, \mathbf{z})$ in $\mathbb{R}^{N_g} \times \mathbb{R}^v$, there exists a unique $u(\cdot, \mathbf{y}, \mathbf{z}) \in H_0^1(D)$ such that*

$$C(u(\cdot, \mathbf{y}, \mathbf{z}), v; \mathbf{y}, \mathbf{z}) = f(v), \quad \text{for all } v \in H_0^1(D), \tag{4.5}$$

and

$$\|u(\cdot, \mathbf{y}, \mathbf{z})\|_{H_0^1} \leqslant \frac{1}{\alpha} \|f\|_{H^{-1}}. \tag{4.6}$$

Proof Lemma 7 ensures the continuity and coercivity of bilinear form $C(\cdot, \cdot; \mathbf{y}, \mathbf{z})$ for Γ-almost all $(\mathbf{y}, \mathbf{z})$ in $\mathbb{R}^{N_g} \times \mathbb{R}^v$. The proof follows from a direct application of the Lax–Milgram theorem. □

Weak-stochastic solution

Let be $L_\Gamma^2(\mathbb{R}^\mu) = L_{\Gamma_{N_g}}^2(\mathbb{R}^{N_g}) \otimes L_{\Gamma_v}^2(\mathbb{R}^v)$ and $X = H_0^1(D) \otimes L_\Gamma^2(\mathbb{R}^\mu)$. Then X is a Hilbert space for the inner product norm $\| \cdot \|_X$ defined by

$$\|v\|_X^2 = E_\Gamma \left(\int_D \|\nabla v\|_2^2 \, d\mathbf{x} \right).$$

We also introduce the spaces $X^{(\gamma^s \mathbf{C}^r)}$ $(s, r \in \mathbb{N})$ of functions $v : D \times \mathbb{R}^\mu \to \mathbb{R}$ with bounded norm

$$\|v\|_{X^{(\gamma^s \mathbf{C}^r)}} = \left\{ E_\Gamma \left(\gamma^s \int_D \nabla v \cdot [C]^r \nabla v \, d\mathbf{x} \right) \right\}^{1/2}.$$

Lemma 8 *We have $\alpha^{3/2} \|v\|_X \leqslant \alpha \|v\|_{X^{(\mathbf{C})}} \leqslant \|v\|_{X^{(\mathbf{C}^2)}} \leqslant \|v\|_{X^{(\gamma \mathbf{C})}} \leqslant \|v\|_{X^{(\gamma^2)}}$, and therefore*

$$X^{(\gamma^2)} \subset X^{(\gamma \mathbf{C})} \subset X^{(\mathbf{C}^2)} \subset X^{(\mathbf{C})} \subset X$$

with dense embeddings.

Proof The inequalities satisfied by the norms are easily deduced from the properties of $[C]$ (Lemma 7). Then, it can easily be proven that $X^{(\gamma^2)}$ is dense in X, which proves the density of other embeddings. □

Let's introduce the bilinear form $a : X \times X \to \mathbb{R}$ defined by

$$a(u,v) = E_\Gamma(C(u,v)) = \int_{\mathbb{R}^\mu} \left(\int_D \nabla v \cdot [C](\cdot, \mathbf{y}, \mathbf{z}) \nabla u \, d\mathbf{x} \right) \Gamma(d\mathbf{y}, d\mathbf{z}),$$

and the linear form F belonging to the continuous dual space X' of X, defined by

$$\langle F, v \rangle = E_\Gamma(f(v)) = \int_{\mathbb{R}^\mu} f(v(\cdot, \mathbf{y}, \mathbf{z})) \Gamma(d\mathbf{y}, d\mathbf{z}).$$

From Lemma 7, it can easily be deduced the following lemma.

Lemma 9 $a : X \times X \to \mathbb{R}$ *is a symmetric bilinear form such that:*

(i) *a is continuous from* $X^{(\mathbf{C}^2)} \times X$ *to* $\mathbb{R}$,

$$|a(u,v)| \leqslant \|u\|_{X^{(\mathbf{C}^2)}} \|v\|_X \quad \forall (u,v) \in X^{(\mathbf{C}^2)} \times X. \tag{4.7}$$

(ii) *a is continuous from* $X^{(\mathbf{C})} \times X^{(\mathbf{C})}$ *to* $\mathbb{R}$,

$$|a(u,v)| \leqslant \|u\|_{X^{(\mathbf{C})}} \|v\|_{X^{(\mathbf{C})}} \quad \forall (u,v) \in X^{(\mathbf{C})} \times X^{(\mathbf{C})}. \tag{4.8}$$

(iii) *a is coercive,*

$$a(v,v) \geqslant \alpha \|v\|_X^2 \quad \forall v \in X. \tag{4.9}$$

Now we introduce a weak form of the parameterised stochastic boundary value problem:

$$\begin{aligned} &\text{Find } u \in X \text{ such that} \\ &a(u,v) = F(v) \quad \forall v \in X. \end{aligned} \tag{4.10}$$

We have the following result.

Proposition 6 *There exists a unique solution* $u \in X$ *to problem* (4.10), *and* $\|u\|_X \leqslant \frac{1}{\alpha} \|F\|_{X'}$. *Moreover, u verifies* (4.5) Γ-*almost surely.*

Proof The existence and uniqueness of solution can be deduced from a general result obtained in [46]. Here we provide a short proof for completeness sake. We denote by $A : D(A) \subset X \to X'$ the linear operator defined by $Au = a(u, \cdot)$, $D(A)$ being the domain of A. We introduce the adjoint operator $A^* : D(A^*) \subset X \to X'$ defined by $\langle u, A^*v \rangle = \langle Au, v \rangle$ for all $u \in D(A)$ and $v \in D(A^*)$, with $D(A^*) = \{v \in X; \exists c > 0 \text{ such that } |\langle Au, v \rangle| \leqslant c\|u\|_X \text{ for all } u \in D(A)\}$. The continuity property (4.7) implies that A is continuous from $X^{(\mathbf{C}^2)}$ to X' and therefore, $D(A) \supset X^{(\mathbf{C}^2)}$ is dense in X (using Lemma 8). That means that A is densely defined. The coercivity property (4.9) implies that $\|Av\|_{X'} \geqslant \alpha\|v\|_X$ for all $v \in X$, which implies that A is injective and the range $R(A)$ of A is closed. It also implies that A^* is injective and $R(A^*)$ is closed, and by the Banach closed range theorem, we then have that A is surjective, which proves the existence of a unique solution. Then, using the coercivity of a, we simply obtain $\|u\|_X^2 \leqslant \frac{1}{\alpha} a(u,u) = \frac{1}{\alpha} F(u) \leqslant \frac{1}{\alpha} \|F\|_{X'} \|u\|_X$. $\square$

Proposition 7 *The solution u of problem* (4.10) *is such that:*

(i) $u \in X^{(\mathbf{C})}$,

(ii) $u \in X^{(\gamma\mathbf{C})}$ *if* $\gamma \in L^1_{\Gamma}(\mathbb{R}^\mu)$,

(iii) $u \in X^{(\gamma^2)}$ *if* $\gamma \in L^2_{\Gamma}(\mathbb{R}^\mu)$.

Proof Using equation (4.5) with $v = u$ and equation (4.6), we obtain $\int_D \nabla u \cdot [\mathbf{C}]\nabla u \, d\mathbf{x} = f(u) \leqslant \|f\|_{H^{-1}}\|u\|_{H^1_0} \leqslant \frac{1}{\alpha}\|f\|^2_{H^{-1}}$. Taking the expectation yields $\|u\|^2_{X^{(\mathbf{C})}} \leqslant \frac{1}{\alpha}\|f\|^2_{H^{-1}} < \infty$, which proves (i). If $\gamma \in L^1_{\Gamma}$, $\|u\|^2_{X^{(\gamma\mathbf{C})}} = E_{\Gamma}(\gamma \int_D \nabla u \cdot [\mathbf{C}]\nabla u \, d\mathbf{x}) \leqslant \frac{1}{\alpha}\|f\|^2_{H^{-1}}E_{\Gamma}(\gamma) < \infty$, which proves (ii). Finally, if $\gamma \in L^2_{\Gamma}$, we have $\|u\|^2_{X^{(\gamma^2)}} = E_{\Gamma}(\gamma^2\|u\|^2_{H^1_0}) \leqslant \frac{1}{\alpha^2}\|f\|^2_{H^{-1}}E_{\Gamma}(\gamma^2) < \infty$, which proves (iii). $\qquad\square$

4.2 Galerkin approximation

Galerkin methods are introduced for the approximation of the solution u of problem (4.10). Let $X_N \subset X$ be an approximation space such that $X_N := X_{q,p} = V_q \otimes W_p$ with $V_q \subset H^1_0(D)$ (e.g., a finite element approximation space) and $W_p \subset L^2_{\Gamma}(\mathbb{R}^\mu)$ (e.g., a polynomial chaos approximation space). The Galerkin approximation $u_N \in X_N$ of u is defined by

$$a(u_N, v_N) = F(v_N) \quad \forall v_N \in X_N. \tag{4.11}$$

Note that the coercivity property of bilinear form a on $X_N \times X_N$ (Lemma 9(iii)) ensures the existence and uniqueness of a solution u_N to problem (4.11). The convergence of Galerkin approximations is now analysed in different situations corresponding to the different types of representation of random fields (exponential- or square-type), to different underlying measures Γ, and to different choices of approximation spaces. We first analyse the case where $X_N \subset X^{(\gamma)}$ which leads us to a natural strategy for the definition of a convergent sequence of approximations. Then, we analyse the use of more general approximation spaces that do not necessarily verify $X_N \subset X^{(\gamma)}$.

4.2.1 Case $X_N \subset X^{(\gamma)}$

Proposition 8 *Assuming $X_N \subset X^{(\gamma)} \subset X^{(\mathbf{C})}$, the solution $u_N \in X_N$ of* (4.11) *verifies*

$$\|u - u_N\|_{X^{(\mathbf{C})}} \leqslant \inf_{v_N \in X_N} \|u - v_N\|_{X^{(\mathbf{C})}}, \tag{4.12}$$

$$\text{and} \quad \|u - u_N\|_X \leqslant \frac{1}{\sqrt{\alpha}} \inf_{v_N \in X_N} \|u - v_N\|_{X^{(\mathbf{C})}}. \tag{4.13}$$

Therefore, if $\{X_N\}_{N \in \mathbb{N}} \subset X^{(\gamma)}$ is a sequence of approximation spaces such that $\cup_{N \in \mathbb{N}} X_N$ is dense in X, then there exists a subsequence of Galerkin approximations $\{u_N\}_{N \in \mathbb{N}}$ which converges to u in the $X^{(\mathbf{C})}$-norm and in the X-norm.

Proof Using the Galerkin orthogonality property of u_N and the continuity of a (Lemma 9(ii)) yield

$$\|u - u_N\|^2_{X^{(\mathbf{C})}} = a(u - u_N, u - u_N) = a(u - u_N, u - v_N) \leqslant \|u - u_N\|_{X^{(\mathbf{C})}}\|u - v_N\|_{X^{(\mathbf{C})}},$$

for all $v_N \in X_N$. Inequality (4.12) is obtained by taking the infimum over all $v_N \in X_N$. Then, inequality (4.13) is obtained by using the coercivity of a (Lemma 9(iii)). □

Let us now analyse the condition $X_N \subset X^{(\gamma)}$ of Proposition 8, with $X_N = V_q \otimes W_p$. Due to the tensor product structure of $X_N = V_q \otimes W_p$, the condition $X_N \subset X^{(\gamma)}$ is equivalent to $\sqrt{\gamma}\, W_p \subset L^2_\Gamma(\mathbb{R}^\mu)$, that means $E_\Gamma(\varphi^2 \gamma) < +\infty$ for all $\varphi \in W_p$.

Use of weighted approximation spaces (for both types of representation of random fields)
A possible approximation strategy consists in choosing weighted approximation spaces $W_p = \{\gamma^{-1/2}\varphi ; \varphi \in \widetilde{W}_p\}$ with $\widetilde{W}_p \subset L^2_\Gamma(\mathbb{R}^\nu)$. For example, if the measure Γ has finite moments of any order, $\widetilde{W}_p$ can be chosen as a classical (piecewise) polynomial space with degree p, e.g.,

$$\widetilde{W}_p = \mathbb{P}_p(\mathbb{R}^{N_g}) \otimes \mathbb{P}_p(\mathbb{R}^\nu) = \text{span}\{\mathbf{y}^\alpha \mathbf{z}^\beta ; \alpha \in \mathbb{N}^{N_g}, \beta \in \mathbb{N}^\nu, |\alpha| \leqslant p, |\beta| \leqslant p\}.$$

Indeed, for all $\varphi = \gamma^{-1/2}\mathbf{y}^\alpha \mathbf{z}^\beta \in W_p$,

$$E_\Gamma(\varphi^2 \gamma) = E_\Gamma(\mathbf{y}^{2\alpha}\mathbf{z}^{2\beta}) = \int_{\mathbb{R}^\mu} \mathbf{y}^{2\alpha}\mathbf{z}^{2\beta} \Gamma(d\mathbf{y}, d\mathbf{z}) < +\infty,$$

This approximation strategy is adapted to both types of representation of random fields (exponential- or square-type) since it does not require any assumption on γ. However, the use of weighted polynomial approximation spaces W_p leads to non-classical computational treatments.

Use of classical approximation spaces (for square-type representation of random fields)
More classical approximation spaces can be used provided some conditions on γ.

Lemma 10 *If $\gamma \in L^r_\Gamma(\mathbb{R}^\mu)$ and $W_p \subset L^s_\Gamma(\mathbb{R}^\mu)$ with some $r \geqslant 1$ and $s \geqslant 2$ such that $\frac{2}{s} + \frac{1}{r} = 1$, then $X_N \subset X^{(\gamma)}$. In particular $X_N \subset X^{(\gamma)}$ if $\gamma \in L^1_\Gamma(\mathbb{R}^\mu)$ and $W_p \subset L^\infty_\Gamma(\mathbb{R}^\mu)$.*

Proof For $r > 1$, letting $s' = \frac{s}{2}$ such that $\frac{1}{s'} + \frac{1}{r} = 1$, we have

$$E_\Gamma(\varphi^2 \gamma) \leqslant \{E_\Gamma(\varphi^{2s'})\}^{1/s'} \{E_\Gamma(\gamma^r)\}^{1/r} = \{E_\Gamma(\varphi^s)\}^{2/s} \{E_\Gamma(\gamma^r)\}^{1/r} < \infty.$$

The proof for $r = 1$ and $s = \infty$ is straightforward. □

Lemma 10 allows us to analyse the approximation when using the square-type representation of random fields, for which $\gamma \in L^1_\Gamma(\mathbb{R}^\mu)$. Indeed, in this case, Lemma 10 implies that $X_N \subset X^{(\gamma)}$ if $W_p \subset L^\infty_\Gamma(\mathbb{R}^\mu)$. In particular, this condition is satisfied if Γ has a bounded support and W_p is a (possibly piecewise) polynomial space, e.g., $W_p = \mathbb{P}_p(\mathbb{R}^{N_g}) \otimes \mathbb{P}_p(\mathbb{R}^\nu)$. We note that $\Gamma = \Gamma_{N_g} \otimes \Gamma_\nu$ has a bounded support if (i) the random germ Ξ for the representation of random fields in the class $\mathcal{K} = \{\mathcal{K}^{(m,N)}(\cdot, \Xi, \mathcal{M}_{[a]}(\mathbf{z})) ; \mathbf{z} \in \mathbb{R}^\nu\}$ has a bounded support $supp(\Gamma_{N_g})$, and (ii) the support of the measure Γ_ν on the parameter space $\mathbb{R}^\nu$ is bounded. This latter condition implies that the map u only provides the

solution to the boundary value problems associated to a subset of random fields in $\mathcal{K}$ (those corresponding to parameters $\mathbf{z} \in supp(\Gamma_v)$). In other words, this first result shows that when using the square-type representation of random fields with a germ Ξ with bounded support, the approximation is possible using classical polynomial approximation spaces.

Note that for having $X_N \subset X^{(\gamma)}$, weaker conditions on approximation spaces could be obtained by looking further at the properties of γ and the measure Γ. In particular, if W_p is the tensor product of polynomial spaces with (total or partial) degree p, that means $W_p = \mathbb{P}_p(\mathbb{R}^{N_g}) \otimes \mathbb{P}_p(\mathbb{R}^v)$, then $X_N \subset X^{(\gamma)}$ if for all $\mathbf{y}^{\boldsymbol{\alpha}}\mathbf{z}^{\boldsymbol{\beta}} \in W_p$,

$$E_\Gamma(\mathbf{y}^{2\boldsymbol{\alpha}}\mathbf{z}^{2\boldsymbol{\beta}}\gamma(\mathbf{y},\mathbf{z})) = \int_{\mathbb{R}^\mu} \mathbf{y}^{2\boldsymbol{\alpha}}\mathbf{z}^{2\boldsymbol{\beta}}\gamma(\mathbf{y},\mathbf{z})\Gamma(d\mathbf{y},d\mathbf{z}) < +\infty, \tag{4.14}$$

which is a condition on γ and on measure Γ. The following result justifies the applicability of classical (piecewise) polynomial spaces when using the square-type representation of random fields.

Proposition 9 *For the square-type representation of random fields, if W_p is the tensor product of (possibly piecewise) polynomial spaces and if $\Gamma = \Gamma_{N_g} \otimes \Gamma_v$ is such that probability measures Γ_{N_g} and Γ_v admit moments of any order, then $X_N = V_q \otimes W_p \subset X^{(\gamma)}$.*

Proof Following the proof of Proposition 4, we obtain $\gamma \leqslant \underline{k}_1(\sqrt{n}\varepsilon + \gamma_0 + \gamma_1\delta^2)$, with δ defined by equation (3.10) and such that

$$\delta^2 \leqslant 2\|G_0\|_\infty^2 + 2\left(\sum_{i=1}^m \sqrt{\sigma_i}\|G_i\|_\infty|\eta_i|\right)^2 \leqslant g_0 + g_1\sum_{i=1}^m \eta_i^2,$$

with $g_0 = 2\|G_0\|_\infty^2$ and $g_1 = 2\sum_{i=1}^m \sigma_i\|G_i\|_\infty^2$. From equation (2.45), we have $\boldsymbol{\eta} = [y]^T\boldsymbol{\Psi}(\mathbf{y})$, where $\boldsymbol{\Psi}(\mathbf{y})$ is a vector of polynomials in $\mathbf{y}$ and $[y]^T[y] = [I_m]$. The latter condition implies $|y_i^j| \leqslant 1$ for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant N$. Therefore, $\eta_i^2 \leqslant (\sum_{j=1}^N y_i^j\Psi_j(\mathbf{y}))^2 \leqslant N\sum_{j=1}^N \Psi_j(\mathbf{y})^2$, and $\delta(\mathbf{y},\mathbf{z})^2 \leqslant g_0 + g_1mN\sum_{j=1}^N \Psi_j(\mathbf{y})^2 := Q(\mathbf{y})$, where $Q(\mathbf{y})$ is a polynomial in $\mathbf{y}$. If the measures Γ_{N_g} and Γ_v have finite moments of any order, then any (piecewise) polynomial function on $\mathbb{R}^\mu$ is in $L_\Gamma^1(\mathbb{R}^\mu)$. Therefore, if W_p is a space of (piecewise) polynomial functions on $\mathbb{R}^\mu$, then for all $\varphi \in W_p$, we have

$$E_\Gamma(\varphi(\mathbf{y},\mathbf{z})^2\gamma(\mathbf{y},\mathbf{z})) \lesssim E_\Gamma(\varphi(\mathbf{y},\mathbf{z})^2(1 + Q(\mathbf{y})) < +\infty,$$

from which we deduce that $V_q \otimes W_p \subset X^{(\gamma)}$. $\qquad\square$

Since measures with bounded support have finite moments of any order, Proposition 9 is consistent with the first conclusions of Lemma 10. Moreover, we have that for the square-type representation of random fields, if $\mathbf{Z}$ is chosen as a Gaussian random variable (that means Γ_v is a Gaussian measure) or a random variable with bounded support, then the classical Hermite polynomial chaos space associated with a Gaussian germ Ξ can be

used. Note that the use of a measure Γ_ν whose support is $\mathbb{R}^\nu$ allows to explore the whole class of random fields $\mathscr{K}$ with the single map u.

Remark 4.1 The case of log-normal random fields corresponds to a particular case of the exponential-type representation for which the random field [**G**] is Gaussian and represented using a degree one Hermite polynomial chaos expansion with a Gaussian germ Ξ. In this case, it can be proven that if W_p is the tensor product of (possibly piecewise) polynomial spaces and if Γ_ν admit moments of any order, then $X_N = V_q \otimes W_p \subset X^{(\gamma)}$. This justifies the use of polynomial approximation spaces when considering the particular case of log-normal random fields. However, this result does not extend to other random fields with exponential-type representation.

4.2.2 *Case of general approximation spaces X_N*

In order to handle both types of representation of random fields (exponential-type and square-type) with a measure Γ with a possibly unbounded support, we here propose and analyse an approximation strategy which consists in using truncated approximation spaces. We consider a family of approximation spaces $\{V_q\}_{q \geqslant 1}$ such that $\cup_q V_q$ is dense in $H_0^1(D)$, and a family $\{W_p\}_{p \geqslant 1}$ such that $\cup_p W_p$ is dense in $L_\Gamma^2(\mathbb{R}^\mu)$. Classical (possibly piecewise) polynomial spaces with degree p can be chosen for W_p. Let X_N be defined as $V_q \otimes W_p$. Then, for $\tau > 0$, we introduce the approximation space

$$W_p^\tau = \{I_{\gamma \leqslant \tau} \varphi \, ; \varphi \in W_p\},$$

where $I_{\gamma \leqslant \tau}(\mathbf{y}, \mathbf{z}) = 1$ if $\gamma(\mathbf{y}, \mathbf{z}) \leqslant \tau$ and 0 if $\gamma(\mathbf{y}, \mathbf{z}) > \tau$. It can be proven that $\cup_{\tau > 0} \cup_{p \geqslant 1} W_p^\tau$ is dense in $L_\Gamma^2(\mathbb{R}^\mu)$. Let X_N^τ be defined as $V_q \otimes W_p^\tau$. For $\psi = I_{\gamma \leqslant \tau} \varphi$ in W_p^τ, we have $E_\Gamma(\gamma \psi^2) \leqslant \tau E_\Gamma(\varphi^2) < \infty$, that means $X_N^\tau := V_q \otimes W_p^\tau \subset X^{(\gamma)}$ for all $\tau > 0$. The Galerkin approximation of u in X_N^τ is denoted by u_N^τ.

Proposition 10 *The Galerkin approximation $u_N^\tau \in X_N^\tau$ of u satisfies*

$$\|u - u_N^\tau\|_{X^{(\mathbf{C})}}^2 \leqslant \tau \inf_{v \in X_N} \|u - v\|_X^2 + \|u I_{\gamma > \tau}\|_{X^{(\mathbf{C})}}^2, \tag{4.15}$$

and there exists a sequence of approximation spaces $X_{N(\tau)}^\tau = V_{q(\tau)} \otimes W_{p(\tau)}^\tau$ such that

$$\|u - u_{N(\tau)}^\tau\|_{X^{(\mathbf{C})}} \to 0 \quad as \quad \tau \to \infty. \tag{4.16}$$

Proof We have $u \in X^{(\mathbf{C})}$ (Proposition 7) and $X_N^\tau \subset X^{(\gamma)} \subset X^{(\mathbf{C})}$. We first note that for all $v \in X_N^\tau$,

$$\begin{aligned}
\|(u - v)\|_{X^{(\mathbf{C})}}^2 &= \|(u - v)I_{\gamma \leqslant \tau}\|_{X^{(\mathbf{C})}}^2 + \|(u - v)I_{\gamma > \tau}\|_{X^{(\mathbf{C})}}^2 \\
&\leqslant \|(u - v)I_{\gamma \leqslant \tau}\|_{X^{(\gamma)}}^2 + \|u I_{\gamma > \tau}\|_{X^{(\mathbf{C})}}^2 \\
&\leqslant \tau \|(u - v)I_{\gamma \leqslant \tau}\|_X^2 + \|u I_{\gamma > \tau}\|_{X^{(\mathbf{C})}}^2.
\end{aligned}$$

Then, using Proposition 8 (equation (4.13)), we obtain

$$\|u - u_N^\tau\|_{X^{(\mathbf{C})}}^2 \leqslant \inf_{v \in X_N^\tau} \|u - v\|_{X^{(\mathbf{C})}}^2 \leqslant \tau \inf_{v \in X_N^\tau} \|(u - v)I_{\gamma \leqslant \tau}\|_X^2 + \|u I_{\gamma > \tau}\|_{X^{(\mathbf{C})}}^2$$

$$= \tau \inf_{v \in X_N} \|(u - v)I_{\gamma \leqslant \tau}\|_X^2 + \|u I_{\gamma > \tau}\|_{X^{(\mathbf{C})}}^2$$

$$\leqslant \tau \inf_{v \in X_N} \|u - v\|_X^2 + \|u I_{\gamma > \tau}\|_{X^{(\mathbf{C})}}^2$$

Since $\|u\|_{X^{(\mathbf{C})}}$ is bounded, the second term converges to 0 as $\tau \to \infty$. Also, provided that $\cup_N X_N$ is dense in X, we can define a sequence of spaces $X_{N(\tau)}$ such that

$$\tau \inf_{v \in X_{N(\tau)}} \|u - v\|_X^2 \to 0 \quad \text{as} \quad \tau \to \infty,$$

which ends the proof.   □

4.3 High-dimensional approximation using sparse or low-rank approximations

The approximation of the map $u \in X$ requires the introduction of adapted complexity reduction techniques. Indeed, when introducing approximation spaces $V_q \subset H_0^1(D)$ and $W_p = W_p^{\mathbf{y}} \otimes W_p^{\mathbf{z}} \subset L_{\Gamma_{N_g}}^2(\mathbb{R}^{N_g}) \otimes L_{\Gamma_\nu}^2(\mathbb{R}^\nu)$, the resulting approximation space $X_N = V_q \otimes W_p$ may have a very high dimension $dim(V_q) \times dim(W_p)$. For classical non adapted constructions of approximation spaces W_p, the dimension of W_p has typically an exponential (or factorial) increase with N_g and ν (e.g. with polynomial spaces $W_p^{\mathbf{y}} = \mathbb{P}_p(\mathbb{R}^{N_g})$ and $W_p^{\mathbf{z}} = \mathbb{P}_p(\mathbb{R}^\nu)$). Therefore, the use of standard approximation techniques would restrict the applicability of the identification procedure to a class of random fields with a germ Ξ of small dimension (small N_g) and a low order in the representation of random fields (small ν).

4.3.1 *Sparse tensor approximation*

A first possible way to reduce the complexity and address high-dimensional problems is to use adaptive sparse tensor approximation methods [11]. Supposing that $\{\phi_i = \otimes_{k=1}^\mu \phi_{i_k}^{(k)}; i \in \mathbb{N}^\mu\}$ constitutes a basis of W, it consists in constructing a sequence of approximations $u_M \in V_q \otimes W_M$, with $W_M = \text{span}\{\phi_i; i \in \Lambda_M\}$, where $\Lambda_M \subset \mathbb{N}^\mu$ is a subset of M indices constructed adaptively. For some classes of random fields (that are simpler than the one proposed in the present paper), algorithms have been proposed for the adaptive construction of the sequence of subsets Λ_M and some results have been obtained for the convergence of the corresponding sequence of approximations. In [12], convergence results have been obtained for different classes of random fields and in particular for two classes of random fields that are closely related to the ones proposed in the present paper, namely random fields that are obtained with the exponential or the square of a random field which admits an affine decomposition in terms of the parameters. Under suitable conditions on the convergence of the series expansion of this random field, the authors prove the convergence with a convergence rate independent on the dimension μ. Such results may be obtained for the class of random fields proposed in the present

paper under some conditions on the underlying second order random field [**G**]. This will be addressed in future work.

4.3.2 *Low-rank approximations*

Low-rank tensor approximation methods can be used in order to reduce the complexity of the approximation of functions in tensor product spaces [34]. They consist in approximating the solution in a subset $\mathscr{M} \subset X_N$ of low-rank tensors. Different tensor structures of space X_N can be exploited, such as $X_N = V_q \otimes W_p$ or $X_N = V_q \otimes W_p^{\mathbf{y}} \otimes W_p^{\mathbf{z}}$. Also, if the measures Γ_{N_g} and Γ_ν are tensor product of measures, then $W_p^{\mathbf{y}}$ and $W_p^{\mathbf{z}}$ can be chosen as tensor product approximation spaces $W_p^{\mathbf{y}} = \otimes_{k=1}^{N_g} W_p^{\mathbf{y},(k)}$ and $W_p^{\mathbf{z}} = \otimes_{k=1}^{\nu} W_p^{\mathbf{z},(k)}$. Low-rank approximation methods then consist in approximating the solution in a subset $\mathscr{M} \subset X_N$ of low-rank tensors, which is a subset of low dimension in the sense that $\mathscr{M}$ can be parameterized by a small number of parameters (small compared to the dimension of X_N). Different low-rank formats can be used, such as canonical format, Tucker format, Hierarchical Tucker format or more general tree-based tensor formats (see e.g. [34]). These tensor subsets can be formally written as

$$\mathscr{M} = \{v = F_{\mathscr{M}}(w_1, \ldots, w_\ell); w_1 \in \mathbb{R}^{r_1}, \ldots, w_\ell \in \mathbb{R}^{r_\ell}\},$$

where $F_{\mathscr{M}}$ is a multi-linear map with values in X_N and where the $w_k \in \mathbb{R}^{r_k}$ ($k = 1, \ldots, \ell$) are the parameters. The dimension of such a parameterisation is $\sum_{k=1}^{\ell} r_k$. As an example, $\mathscr{M}$ can be chosen as the set of rank-m canonical tensors in $X_N = V_q \otimes W_p^{\mathbf{y}} \otimes W_p^{\mathbf{z}}$, defined by $\mathscr{M} = \{v = \sum_{i=1}^{m} w_i^{\mathbf{x}} \otimes w_i^{\mathbf{y}} \otimes w_i^{\mathbf{z}}; w_i^{\mathbf{x}} \in V_q, w_i^{\mathbf{y}} \in W_p^{\mathbf{y}}, w_i^{\mathbf{z}} \in W_p^{\mathbf{z}}\}$.

4.3.3 *Algorithms for the low-rank approximation of $u_N \in X_N$*

The Galerkin approximation $u_N \in X_N$ of u is the unique minimiser of the strongly convex functional $J : X_N \to \mathbb{R}$ defined by $J(v) = \frac{1}{2}a(v,v) - F(v)$. A low-rank approximation $u_r \in \mathscr{M}_r$ of u_N can then be obtained by solving the optimisation problem

$$\min_{v \in \mathscr{M}} J(v) = \min_{w_1 \in \mathbb{R}^{r_1}, \ldots, w_\ell \in \mathbb{R}^{r_\ell}} J(F_{\mathscr{M}}(w_1, \ldots, w_\ell)).$$

This problem can be solved using alternated minimisation algorithms or other optimisation algorithms (see [34]). Also, greedy procedures using low-rank tensor subsets can be introduced in order to construct a sequence of approximations $\{u^k\}_k$ with $u^{k+1} = u^k + w^{k+1}$ and $w^{k+1} \in \mathscr{M}$ defined by $J(u^k + w^{k+1}) = \min_{v \in \mathscr{M}} J(u^k + v)$. We refer to [23] for the analysis of a larger class of greedy algorithms in the context of convex optimisation problems in tensor spaces, and to [51] for the practical implementation of some algorithms in the context of stochastic parametric partial differential equations.

Remark 4.2 [On the approximability of the solution using low-rank approximations] A very few results are available concerning the complexity reduction that can be achieved using low-rank approximations, in particular for the approximation of high-dimensional maps arising in the context of high-dimensional parametric stochastic equations. Of

course, results about sparse polynomial approximation methods (e.g., [12]) could be directly translated in the context of low-rank tensor methods (a sparse decomposition on a multi-dimensional polynomial basis being a low-rank decomposition) but this would not allow to quantify the additional reduction that could be achieved by low-rank methods. Providing *a priori* convergence results for low-rank approximations for the proposed class of random fields would deserve an independent analysis which is out of the scope of the present paper. However, we notice that some standard arguments could be used in order to provide *a priori* convergence results of low-rank approximations in the 2-order tensor space $X = H_0^1(D) \otimes L_\Gamma^2(\mathbb{R}^\mu) = L_\Gamma^2(\mathbb{R}^\mu; H_0^1(D))$. Indeed, under suitable regularity conditions on the samples of the random fields in the class (related to properties of the covariance structure of the algebraic prior), the set of solutions $\mathscr{U} = \{u(\cdot, \mathbf{y}, \mathbf{z}); (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^\mu\}$ could be proven to be a subset of $H_0^1(D)$ with rapidly convergent Kolmogorov n-width[1], therefore proving the existence of a good sequence of low-dimensional spaces in $H_0^1(D)$ for the approximation of the set of solutions $\mathscr{U}$. At least the same convergence rate (with respect to the rank) could therefore be expected for low-rank approximations that are optimal with respect to a L^2-norm in the parameter domain. These questions will be investigated in a future work.

4.4 Remarks about the identification procedure

The identification procedure has been presented in Section 3.3. In view of the numerical analysis presented above, the following remarks can be done. Different approximations of the map u should be constructed at the different steps of the identification procedure. Indeed, the quality of approximations of u clearly depends on the choice of measure Γ_ν on the parameters space $\mathbb{R}^\nu$. It is recalled that the parameterisation of the class $\mathscr{K} = \{\mathscr{K}^{(m,N)}(\cdot, \Xi, \mathscr{M}_{[a]}(\mathbf{z})); \mathbf{z} \in \mathbb{R}^\nu\}$ of random fields depends on the set of parameters $[a] = \mathscr{M}_{[a]}(0)$. The map u depends on the choice of $[a]$ but it is clearly independent on the choice of measure Γ_ν, provided that the support $supp(\Gamma_\nu) = \mathbb{R}^\nu$. However, the Galerkin approximation u_N being optimal with respect to a norm that depends on the measure Γ_ν, the quality of the approximation clearly depends on measure Γ_ν. For that reason, the choice of measure Γ_ν can be updated throughout the identification procedure in order to improve the approximation of u as explained at the end of Section 3.3. Steps 6 and 7 described in Section 3.3 can then be clarified as follows.

At *Step 6*, the parameterisation of $\mathscr{K}$ around $[a] = [y_0]$ is used and an approximation of the corresponding map u in X is constructed using low-rank tensor methods. The use of this explicit map allows a fast maximisation of the likelihood function to be performed, yielding an estimation $\check{\mathbf{z}}$ of $\mathbf{z}$ (or equivalently $[\check{y}]$ of $[y]$).

At *Step 7*, the map u constructed in Step 6 could be used for the subsequent Bayesian update. However, it is also possible to construct a new map that takes into account the result of the maximum likelihood estimation. Therefore, the parameterisation of $\mathscr{K}$

---

[1] For example, if the set of solutions $\mathscr{U}$ is a bounded subset of $H^2(D)$ (which can be obtained with a simple a posteriori analysis if the random fields have samples with C^1 regularity), then the Kolmogorov n-width can be proven to converge as $n^{-1/d}$ where d is the spatial dimension.

around $[a] = [\check{y}]$ is used and the boundary value problem is solved again in order to obtain an approximation of the corresponding map $u(\cdot, \cdot, \cdot; [\check{y}])$ in X. This map can then be efficiently used for solving the Bayesian update problem.

5 Conclusion

In this paper, we have presented new results allowing an unknown non-Gaussian positive matrix-valued random field to be identified through a stochastic elliptic boundary value problem, solving a statistical inverse problem. In order to propose a constructive and efficient methodology and analysis of this challenging problem in high stochastic dimension, a new general class of non-Gaussian positive-definite matrix-valued random fields, adapted to the statistical inverse problems in high stochastic dimension for their experimental identification, has been introduced. For this class of random fields, two types of representation are proposed: the exponential-type representation and the square-type representation. Their properties have been analysed. A parameterisation of discretised random fields belonging to this general class has been proposed and analysed for the two types of representation. Such parameterisation has been constructed using a polynomial chaos expansion with random coefficients and a minimal parameterisation of the compact Stiefel manifold related to these random coefficients. Using this parameterisation of the general class, a complete identification procedure has been proposed. Such a statistical inverse problem requires to solve the stochastic boundary value problem in high stochastic dimension with efficient and accurate algorithms. New results of the mathematical and numerical analyses of the parameterised stochastic elliptic boundary value problem have been presented. The numerical solution provides an explicit approximation of the application that maps the parameterised general class of random fields to the corresponding set of random solutions. Since the proposed general class of random fields possibly contains random fields which are not uniformly bounded, a particular mathematical analysis has been developed and dedicated approximation methods have been introduced. In order to obtain an efficient algorithm for constructing the approximation of this very high-dimensional map, we have described possible complexity reduction methods using sparse or low-rank approximation methods that exploit the tensor structure of the solution which results from the parameterisation of the general class of random fields. In this paper, we do not provide any result concerning the complexity reduction that can be achieved when using low-rank or sparse approximation methods for the proposed class of random fields. These results would require a fine analysis of the regularity and structures of the solution map. Some recent results are already available for sparse approximation methods and for some simpler classes of random fields (see [12]). This type of results should be extended to the present class of random fields. Concerning low-rank methods, a very few quantitative convergence results are available. Some recent results are provided in [52] for the approximation of functions with Sobolev-type regularity. For the present class of random fields, specific analyses are necessary to understand the structure of the solution map induced by the proposed parameterisation and to quantify the complexity reduction that could be achieved with low-rank methods. These challenging issues will be addressed in future works.

Acknowledgement

References

[1] ABSIL, P.-A., MAHONY, R. & SEPULCHRE, R. (2004) Riemannian geometry of Grassmann manifolds with a view on algorithmic computation. *Acta Applicandae Math.* **80**, 199–220.

[2] ARNST, M., GHANEM, R. & SOIZE, C. (2010) Identification of Bayesian posteriors for coefficients of chaos expansion. *J. Comput. Phys.* **229**(9), 3134–3154.

[3] BABUSKA, I., TEMPONE, R. & ZOURARIS, G. E. (2004) Galerkin finite element approximations of stochastic elliptic partial differential equations. *SIAM J. Numer. Anal.* **42**(2), 800–825.

[4] BABUSKA, I., TEMPONE, R. & ZOURARIS, G. E. (2005) Solving elliptic boundary value problems with uncertain coefficients by the finite element method: The stochastic formulation. *Comput. Methods Appl. Mech. Eng.* **194**(12–16), 1251–1294.

[5] BABUSKA, I., NOBILE, F. & TEMPONE, R. (2007) A stochastic collocation method for elliptic partial differential equations with random input data. *SIAM J. Numer. Anal.* **45**(3), 1005–1034.

[6] BABUSKA, I., NOBILE, F. & TEMPONE, R. (2010) A stochastic collocation method for elliptic partial differential equations with random input data. *SIAM Rev.* **52**(2), 317–355.

[7] BOYAVAL, S., LE BRIS, C., MADAY, Y., NGUYEN, N. C. & PATERA, A. T. (2009) A reduced basis approach for variational problems with stochastic parameters: Application to heat conduction with variable robin coefficient. *Comput. Methods Appl. Mech. Eng.* **198**(41–44), 3187–3206.

[8] CAMERON, R. H. & MARTIN, W. T. (1947) The orthogonal development of non-linear functionals in series of Fourier-Hermite functionals. *Ann. Math., Second Series* **48**(2), 385–392.

[9] CANCES, E., EHRLACHER, V. & LELIEVRE, T. (2011) Convergence of a greedy algorithm for high-dimensional convex nonlinear problems. *Math. Models Methods Appl. Sci.* **21**(12), 2433–2467.

[10] CHARRIER, J. (2012) Strong and weak error estimates for elliptic partial differential equations with random coefficients. *SIAM J. Numer. Anal.* **50**(1), 216–246.

[11] COHEN, A., DEVORE, R. & SCHWAB, C. (2010) Convergence rates of best N-term Galerkin approximations for a class of elliptic sPDEs. *Found. Comput. Math.* **10**(6), 615–646.

[12] COHEN, A., CHKIFA, A. & SCHWAB, CH. (2013) *Breaking the curse of dimensionality in sparse polynomial approximation of parametric PDEs*, in review, SAM Report 2013–25, Seminar for Applied Mathematics, ETH Zurich, Zurich.

[13] DAS, S., GHANEM, R. & SPALL, J. (2008) Asymptotic sampling distribution for polynomial chaos representation of data: A maximum-entropy and fisher information approach. *SIAM J. Sci. Comput.* **30**(5), 2207–2234.

[14] DAS, S., GHANEM, R. & FINETTE, S. (2009) Polynomial chaos representation of spatio-temporal random field from experimental measurements. *J. Comput. Phys.* **228**, 8726–8751.

[15] DEB, M., BABUSKA, I. & ODEN, J. T. (2001) Solution of stochastic partial differential equations using Galerkin finite element techniques. *Comput. Methods Appl. Mech. Eng.* **190**, 6359–6372.

[16] DE OLIVEIRA, V., KEDEM, B. & SHORT, D. A. (1997) Bayesian prediction of transformed gaussian random fields. *J. Am. Stat.AP Assoc.* **92**(440), 1422–1433.

[17] DESCELIERS, C., GHANEM, R. & SOIZE, C. (2006) Maximum likelihood estimation of stochastic chaos representations from experimental data. *Int. J. Numer. Methods Eng.* **66**(6), 978–1001.

[18] DOOSTAN, A., GHANEM, R. & RED-HORSE, J. (2007) Stochastic model reduction for chaos representations. *Comput. Methods Appl. Mech. Eng.* **196**(37–40), 3951–3966.

[19] DOOSTAN, A. & IACCARINO, G. (2009) A least-squares approximation of partial differential equations with high-dimensional random inputs. *J. Comput. Phys.* **228**(12), 4332–4345.

[20] EDELMAN, A., ARIAS, T. A. & SMITH, S. T. (1998) The geometry of algorithms with orthogonality constraints. *SIAM J. Matrix Anal. Appl.*, **20**(2), 303–353.

[21] ERNST, O. G., MUGLER, A., STARKLOFF, H. J. & ULLMANN, E. (2012) On the convergence of generalized polynomial chaos expansions. *ESAIM: Math. Modelling Numer. Anal.* **46**(2), 317–339.

[22] FALCO, A. & NOUY, A. (2011) A proper generalized decomposition for the solution of elliptic problems in abstract form by using a functional Eckart-Young approach. *J. Math. Anal. Appl.* **376**(2), 469–480.

[23] FALCO, A. & NOUY, A. (2012) Proper generalized decomposition for nonlinear convex problems in tensor Banach spaces. *Numer. Math.* **121**(3), 503–530.

[24] FRAUENFELDER, P., SCHWAB, C. & TODOR, R. A. (2005) Finite elements for elliptic problems with stochastic coefficients. *Comput. Methods Appl. Mech. Eng.* **194**(2–5), 205–228.

[25] GALVIS, J. & SARKIS, M. (2009) Approximating infinity-dimensional stochastic Darcy's equations without uniform ellipticity. *SIAM J. Numer. Anal.* **47**(5), 3624–3651.

[26] GHANEM, R. & SPANOS, P. D. (1991) *Stochastic Finite Elements: A spectral Approach*, Spinger-Verlag, New York (revised edition, Dover Publications, New York, 2003).

[27] GHANEM, R. & KRUGER, R. M. (1996) Numerical solution of spectral stochastic finite element systems. *Comput. Methods Appl. Mech. Eng.* **129**(3), 289–303.

[28] GHANEM, R. & DOOSTAN, R. (2006) Characterization of stochastic system parameters from experimental data: A Bayesian inference approach. *J. Comput. Phys.* **217**(1), 63–81.

[29] GITTELSON, C. J. (2010) Stochastic Galerkin discretization of the log-normal isotropic diffusion problem. *Math. Models Methods Appl. Sci.* **20**(2), 237–263.

[30] GOLUB, G. H. & VAN LOAN, C. F. (1996) *Matrix Computations*, 3rd ed., The Johns Hopkins University Press, Baltimore.

[31] GUILLEMINOT, J., NOSHADRAVAN, A., SOIZE, C. & GHANEM, R. (2011) A probabilistic model for bounded elasticity tensor random fields with application to polycrystalline microstructures. *Comput. Methods Appl. Mech. Eng.* **200**(17–20), 1637–1648.

[32] GUILLEMINOT, J. & SOIZE, C. (2011) Non-Gaussian positive-definite matrix-valued random fields with constrained eigenvalues: Application to random elasticity tensors with uncertain material symmetries. *Int. J. Numer. Methods Eng.* **88**(11), 1128–1151.

[33] GUILLEMINOT, J. & SOIZE, C. (2013) Stochastic model and generator for random fields with symmetry properties: Application to the mesoscopic modeling of elastic random media. *Multiscale Model. Simul. (A SIAM Interdisciplinary Journal)*, **11**(3), 840–870.

[34] HACKBUSCH, W. (2012) *Tensor Spaces and Numerical Tensor Calculus*, Volume 42 of Springer series in computational mathematics, Springer, Heidelberg.

[35] HRISTOPULOS, D. T. (2003) Spartan Gibbs random field models for geostatistical applications. *SIAM J. Sci. Comput.* **24**(6), 2125–2162.

[36] KAIPIO, J. & SOMERSALO, E. (2005) *Statistical and Computational Inverse Problems*, Springer-Verlag, New York.

[37] KHOROMSKIJ, B. N. & SCHWAB, C. (2011) Tensor-structured Galerkin approximation of parametric and stochastic elliptic PDEs. *SIAM J. Sci. Comput.* **33**(1), 364–385.

[38] KÜHN, T. (1987) Eigenvalues of integral operators with smooth positive definite kernels. *Archiv der Mathematik* **49**, 525–534.

[39] LE MAITRE, O. P., KNIO, O. M. & NAJM, H. N. (2004) Uncertainty propagation using Wiener-Haar expansions. *J. Comput. Phys.* **197**(1), 28–57.

[40] LE MAITRE, O. P. & KNIO, O. M. (2010) *Spectral Methods for Uncertainty Quantification with Applications to Computational Fluid Dynamics*, Springer, Heidelberg.

[41] LUCOR, D., SU, C. H. & KARNIADAKIS, G. E. (2004) Generalized polynomial chaos and random oscillators. *Int. J. Numer. Methods Eng.* **60**(3), 571–596.

[42] MARZOUK, Y. M. & NAJM, H. N. (2009) Dimensionality reduction and polynomial chaos acceleration of Bayesian inference in inverse problems. *J. Comput. Phys.* **228**(6), 1862–1902.

[43] MATTHIES, H. G. & KEESE, A. (2005) Galerkin methods for linear and nonlinear elliptic stochastic partial differential equations. *Comput. Methods Appl. Mech. Eng.* **194**(12–16), 1295–1331.

[44] MATTHIES, H. G. & ZANDER, E. (2012) Solving stochastic systems with low-rank tensor compression. *Linear Algebr. Appl.* **436**(10), 3819–3838.

[45] MENEGATTO, V. A. & OLIVEIRA, C. P.(2012) Eigenvalue and singular value estimates for integral operators: A unifying approach. *Mathematische Nachrichten* **285**(17–18), 2222–2232.

[46] MUGLER, A. & STARKLOFF, H.-J. (2011) On elliptic partial differential equations with random coefficients. *Studia Universitatis Babes-Bolyai – Series Mathematica* **56**(2), 473–487.

[47] NOBILE, F., TEMPONE, R. & WEBSTER, C. G. (2008) A sparse grid stochastic collocation method for partial differential equations with random input data. *SIAM J. Numer. Anal.* **46**(5), 2309–2345.

[48] NOUY, A. (2007) A generalized spectral decomposition technique to solve a class of linear stochastic partial differential equations. *Comput. Methods Appl. Mech. Eng.* **196**(45–48), 4521–4537.

[49] NOUY, A. (2008) Generalized spectral decomposition method for solving stochastic finite element equations: Invariant subspace problem and dedicated algorithms. *Comput. Methods Appl. Mech. Eng.* **197**(51–52), 4718–4736.

[50] NOUY, A. (2009) Recent developments in spectral stochastic methods for the numerical solution of stochastic partial differential equations. *Arch. Comput. Methods Eng.* **16**(3), 251–285.

[51] NOUY, A. (2010) Proper generalized decompositions and separated representations for the numerical solution of high dimensional stochastic problems. *Arch. Comput. Methods Eng.* **17**(4), 403–434.

[52] SCHNEIDER, R. & USCHMAJEW, A. (2013) Approximation rates for the hierarchical tensor format in periodic Sobolev spaces. *J. Complexity* **30**(2), 56–71.

[53] SCHWAB, C. & GITTELSON, C. (2011) Sparse tensor discretizations of high-dimensional parametric and stochastic PDEs. *Acta Numerica* **20**, 291–467.

[54] SCHWAB, C. & STUART, A. M. (2012) Sparse deterministic approximation of Bayesian inverse problems. *Inverse Problems* **28**(4) 045003.

[55] SERFLING, R. J. (1980) *Approximation Theorems of Mathematical Statistics*, John Wiley & Sons, New York.

[56] SOIZE, C. & GHANEM, R. (2004) Physical systems with random uncertainties: Chaos representation with arbitrary probability measure. *SIAM J. Sci. Comput.* **26**(2), 395–410.

[57] SOIZE, C. (2006) Non-Gaussian positive-definite matrix-valued random fields for elliptic stochastic partial differential operators. *Comput. Methods Appl. Mech. Eng.* **195**, 26–64.

[58] SOIZE, C. & GHANEM, R. (2009) Reduced chaos decomposition with random coefficients of vector-valued random variables and random fields. *Comput. Methods Appl. Mech. Eng.* **198**(21–26), 1926–1934.

[59] SOIZE, C. (2010) Identification of high-dimension polynomial chaos expansions with random coefficients for non-Gaussian tensor-valued random fields using partial and limited experimental data. *Comput. Methods Appl. Mech. Eng.* **199**(33–36), 2150–2164.

[60] SOIZE, C. (2011) A computational inverse method for identification of non-Gaussian random fields using the Bayesian approach in very high dimension. *Comput. Methods Appl. Mech. Eng.* **200**(45–46), 3083–3099.

[61] SPALL, J. C. (2003) *Introduction to Stochastic Search and Optimization*, John Wiley and Sons, Hoboken, New Jersey.

[62] STEFANOU, G., NOUY, A. & CLÉMENT, A. (2009) Identification of random shapes from images through polynomial chaos expansion of random level set functions. *Int. J. Numer. Methods Eng.* **79**(2), 127–155.

[63] STUART, A. M. (2010) Inverse problems: A bayesian perspective *Acta Numerica* **19**, 451–559.

[64] TA, Q. A., CLOUTEAU, D. & COTTEREAU, R. (2010) Modeling of random anisotropic elastic media and impact on wave propagation. *Eur. J. Comput. Mech.* **19**(1–3), 241–253.

[65] TODOR, R. A. & SCHWAB, C. (2007) Convergence rates for sparse chaos approximations of elliptic problems with stochastic coefficients. *IMA J. Numer. Anal.* **27**(2), 232–261.

[66] WALTER, E. & PRONZATO, L. (1997) *Identification of Parametric Models from Experimental Data*, Springer, Heidelberg.

[67] WAN, X. L. & KARNIADAKIS, G. E. (2006) Multi-element generalized polynomial chaos for arbitrary probability measures. *SIAM J. Sci. Comput.* **28**(3), 901–928.

[68] WAN, X. L. & KARNIADAKIS, G. E. (2009) Solving elliptic problems with non-Gaussian spatially-dependent random coefficients. *Comput. Methods Appl. Mech. Engrg.* **198**(21–26) 1985–1995.

[69] WIENER, N. (1938) The homogeneous chaos. *Am. J. Math.* **60**(1), 897–936.

[70] XIU, D. B. & KARNIADAKIS, G. E. (2002) Wiener-Askey polynomial chaos for stochastic differential equations. *SIAM J. Sci. Comput.* **24**(2), 619–644.

[71] ZABARAS, N. & GANAPATHYSUBRAMANIAN, B. (2008) A scalable framework for the solution of stochastic inverse problems using a sparse grid collocation approach. *J. Comput. Phys.* **227**(9), 4697–4735.

[72] ZABREYKO, P. P., KOSHELEV, A. I., KRASNOSEL'SKII, M. A., MIKHLIN, S. G., RAKOVSHCHIK, L. S. & YA. STET'SENKO, V. (1975) *Integral Equations – A Reference Text*, Noordhoff International Publishing, Leyden.