HEREDITARILY ANTISYMMETRIC OPERATOR ALGEBRAS

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Abstract   We introduce a notion of 'hereditarily antisymmetric' operator algebras and prove a structure theorem for them in finite dimensions. We also characterize those operator algebras in finite dimensions which can be made upper triangular and prove matrix analogs of the theorems of Dilworth and Mirsky for finite posets. Some partial results are obtained in the infinite dimensional case.

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In this paper an *operator algebra* is a linear subspace of some $\mathcal{B}(\mathcal{H})$ which is stable under products, where $\mathcal{H}$ is a complex Hilbert space. It is *unital* if it contains the identity operator I. Self-adjointness is not assumed. Indeed, an operator algebra $\mathcal{A}$ is said to be *antisymmetric* if $\mathcal{A} \cap \mathcal{A}^*$ is $\{0\}$ or $\mathbb{C} \cdot I$. This means that a nonunital antisymmetric operator algebra contains no nonzero self-adjoint operators, and the only self-adjoint operators a unital one contains are the real multiples of the identity operator (Proposition 3.1). Antisymmetric operator algebras were introduced in [9] and have attracted occasional attention, e.g., [5, 6].

Following [10], we regard linear subspaces $\mathcal{V}$ of $M_n = M_n(\mathbb{C}) \cong \mathcal{B}(\mathbb{C}^n)$ as matrix or 'quantum' analogs of relations on finite sets. In this picture, the quantum analog of a preorder relation, i.e., a relation which is reflexive and transitive, is a unital operator algebra. This intuition behind this idea, and its relation to the physics of finite state quantum systems, is discussed in §1.

Partially ordered sets are preordered sets which satisfy the extra condition of antisymmetry ($a \leqslant b$ and $b \leqslant a$ implies $a = b$). On the face of it, a natural matrix version of this condition might be for the operator algebra to be antisymmetric in the sense defined above. This suggests that antisymmetric operator algebras should be rather special compared to general operator algebras, in something like the way that posets are special compared to general preorders. But that does not seem to be the case. Of course, just by counting dimensions, it is easy to see that an operator algebra in M_n cannot be antisymmetric if its dimension is at least $\frac{n^2}{2} + 1$ (Proposition 3.5). But operator algebras of lower dimension than this typically should not nontrivially intersect the real linear

space of self-adjoint matrices. In other words, the operator algebras of dimension less than $\frac{n^2}{2} + 1$ which are *not* antisymmetric are exceptional, not the other way around.

In contrast, I will show in this paper that operator algebras which enjoy a sort of 'hereditary' antisymmetry condition are unusually well behaved. In particular, the main result, Theorem 5.9, establishes that in finite dimensions they must have a very special structure. A key result on the way, Theorem 5.1, characterizes those operator algebras in M_n which can be put in upper triangular form. I propose that hereditarily antisymmetric operator algebras are the right notion of 'quantum poset'.

One aspect of the special structure of these algebras is that they can always be decomposed into a diagonal subalgebra (relative to a suitable basis) and a nilpotent ideal (Corollary 3.9 plus Theorem 5.9). The intuition for this decomposition could be that the diagonal subalgebra represents the 'equality' part of the quantum partial order and the nilpotent ideal represents the 'strict inequality' part. Moreover, every nilpotent operator algebra is hereditarily antisymmetric (Theorem 4.3(v) $\Rightarrow$ (i) plus Corollary 4.4), so we are led to the view that nilpotent operator algebras are 'quantum' strict orders. In support of this idea, I prove matrix analogs of two basic theorems about finite posets, Dilworth's theorem (in the nontrivial direction) and Mirsky's theorem, for nilpotent operator algebras in M_n (Theorems 6.5 and 6.8). The quantum Mirsky theorem has essentially the same proof as its classical analog, while the quantum Dilworth theorem has a very nonclassical proof.

Trivially, any subalgebra of an antisymmetric operator algebra must itself be antisymmetric. So requiring $\mathcal{A}$ and all of its subalgebras to be antisymmetric is no different from merely requiring $\mathcal{A}$ to be antisymmetric. Whereas imposing this condition on quotients of $\mathcal{A}$ does not even make sense, as the concept of antisymmetry depends on the representation in M_n and one loses this when passing to quotients. These are not the kinds of hereditary conditions we want. Rather, there is another natural kind of 'subobject' and 'quotient' besides ordinary subalgebras and quotient algebras, and requiring them to be antisymmetric becomes a nontrivial condition (Definition 2.1).

In infinite dimensions the prospect of a general structure theory is limited by the possibility that there could be bounded operators with no nontrivial invariant subspaces. However, assuming a positive solution to the transitive algebra problem, we can at least show that any hereditarily antisymmetric dual operator algebra can be put in upper triangular form, in the infinite dimensional sense of being contained in the nest algebra for some maximal nest (Theorem 8.2). The same technique applied to a single operator shows that if the invariant subspace problem for Hilbert space operators has a positive solution, then every bounded operator can be put in upper triangular form, in the same sense (Theorem 8.4).

A word about notation. The operator algebra M_n comes equipped with a natural involution, namely the Hermitian transpose operation. However, we will sometimes want to work with matrices relative to some nonorthogonal basis of $\mathbb{C}^n$, in which case the Hilbert space adjoint of an operator is not expressed by the Hermitian transpose of its matrix. In these cases I will write $\widetilde{M}_n$ for the unital algebra of $n \times n$ complex matrices without any distinguished involution. Thus, results about $\widetilde{M}_n$ will hold for the matrix representations of linear operators on $\mathbb{C}^n$ relative to any, possibly nonorthogonal, basis of $\mathbb{C}^n$.

TABLE 1. Analogy between relations on a finite set X and linear subspaces of $M_n = M_n(\mathbb{C})$.

$R \subseteq X \times X$	$\mathcal{V} \subseteq M_n$
Reflexive	$I \in \mathcal{V}$
Symmetric	$\mathcal{V} = \mathcal{V}^*$
Transitive	$\mathcal{V}^2 \subseteq \mathcal{V}$

In a slight abuse of notation, if $A \in B(\mathcal{H})$ and P is the orthogonal projection onto a closed subspace $\mathcal{E} \subseteq \mathcal{H}$, I will often identify PAP with an operator in $B(\mathcal{E})$.

Unless qualified as 'orthogonal', the word 'projection' will always mean 'possibly nonorthogonal projection', i.e., a bounded operator P satisfying $P = P^2$ but not necessarily $P = P^*$.

Throughout this paper the scalar field is complex. The standard basis of $\mathbb{C}^n$ will be denoted (e_i).

1. 'Quantum' preorders

The idea that unital operator algebras are 'quantum' preorders arises from the more general idea that operator spaces in finite dimensions — that is, linear subspaces $\mathcal{V}$ of M_n — are 'quantum' analogs of relations on a set with n elements. The usual conditions of reflexivity, symmetry, and transitivity of a relation correspond to $\mathcal{V}$ being unital, self-adjoint, and an algebra (Table 1). This is seen in the fact that the subspace $\text{span}\{E_{ij} : (i, j) \in R\}$ of M_n induced by a relation R on the set $\{1, \ldots, n\}$ satisfies one of these algebraic conditions if and only if R satisfies the corresponding relational condition. (E_{ij} is the matrix with a 1 in the (i, j) entry and 0's elsewhere.) Thus unital self-adjoint subalgebras of M_n are regarded as 'quantum equivalence relations' (reflexive, symmetric, transitive), operator systems as 'quantum graphs' (reflexive, symmetric), and general unital subalgebras as 'quantum preordered sets' (reflexive, transitive). The idea of operator systems as quantum graphs has been particularly fruitful; see, e.g., [4, 11–13].

The adjective 'quantum' is justified in the latter case by the fact that the confusability graph in classical error correction becomes a confusability operator system in quantum error correction. That is, where classically we use a graph to catalog which pairs of states might be indistinguishable after passage through a noisy channel, we would in the setting of quantum mechanics use an operator system to carry this information. This is explained in detail in [4, 13].

The idea that unital operator algebras are 'quantum' preorders has a similar physical justification. To see this, first consider the classical setting of a finite state system with phase space $S = \{s_1, \ldots, s_n\}$. Suppose we have a family of classical channels represented by (left) stochastic matrices $P^\lambda = (p_{ij}^\lambda)$ which can be applied to the system. Here each P^λ represents a probabilistic transformation of S, with p_{ij}^λ being the probability of the state s_j transitioning to the state s_i.

The relation 's_j has a nonzero probability of transitioning to s_i under some P^λ' merely describes a directed graph on the vertex set S. But the relation 'there is a sequence of channels $P^{\lambda_1}, \ldots, P^{\lambda_m}$ under which s_j has a nonzero probability of transitioning

to s_i' is transitive: if there is some sequence of channels which takes s_k to s_j with nonzero probability and another sequence of channels which takes s_j to s_i with nonzero probability, then the concatenation of the two sequences takes s_k to s_i with nonzero probability. If we include the identity channel as the $m = 0$ case, then this relation is also reflexive, i.e., it is a preorder.

This framework could describe an experimental scenario where we have some family of classical channels which we are able to apply to a finite state system, and the preorder $s_i \preceq s_j$ reflects our ability to convert the state s_j into the state s_i, with nonzero probability, by sequentially applying some of the channels which are available to us. It is not a partial order because it might be possible to transition from s_j to s_i and then back to s_j. However, there may also be 'invariant' subsets of S with the property that once the state of the system lies in such a subset it cannot escape. These would correspond to *lower* subsets for the preorder $\preceq$, i.e., subsets $S_0 \subseteq S$ with the property that $j \in S_0$ and $i \preceq j$ imply $i \in S_0$.

Now consider the analogous quantum setup. The pure states of a finite quantum system are represented as unit vectors in $\mathbb{C}^n$ and the mixed states by positive unit trace matrices in M_n. A quantum channel is a completely positive trace preserving map $\Phi : M_n \to M_n$, taking mixed states to mixed states. We can always express Φ in the form $\Phi(A) = \sum K_i A K_i^*$ where the *Kraus matrices* K_i satisfy $\sum K_i^* K_i = I$.

Given some available set of quantum channels Φ^λ, we can ask the same question as in the classical case: for which unit vectors v and w is there a nonzero probability of transitioning from v to w after the application of some sequence of Φ^λ's? Where in the classical setting this information was represented by a preorder on the set of states, in the quantum setting it will be represented by the unital algebra $\mathcal{A}$ generated by the Kraus matrices of the available channels Φ^λ. (The Kraus matrices are not uniquely determined by the Φ^λ, but their linear span is, and hence so is the unital algebra they generate.) Namely, there is a nonzero probability of transitioning from v to w after the application of some sequence of Φ^λ's if and only if we have $\langle Av, w \rangle \neq 0$ for some $A \in \mathcal{A}$. More generally, if v and w are unit vectors in $\mathbb{C}^n \otimes \mathbb{C}^k$ for some k, representing pure states of some composite system, then it is possible to transition from v to w if and only if $\langle (A \otimes I_k)v, w \rangle \neq 0$ for some $A \in \mathcal{A}$. In fact, this property characterizes $\mathcal{A}$: it is the unique linear subspace of M_n for which this is true [13, Proposition 6.2].

In the classical setting we also had invariant subsets from which one could not escape; the analogous quantum notion would be a linear subspace of $\mathbb{C}^n$ which is invariant for every operator in $\mathcal{A}$.

To summarize, in finite state quantum systems unital operator algebras play a role analogous to that played by preorders in finite state classical systems. Thus unital operator algebras are 'quantum' preorders, in the same way that operator systems are 'quantum' graphs [13].

2. Hereditary antisymmetry

Moving back to the general idea that one can think of linear subspaces of M_n as being somehow analogous to relations on a set with n elements: in this picture $\mathbb{C}^n$ plays the

role of an n element set on which a relation is defined, and the linear subspace $\mathcal{V} \subseteq M_n$ specifies that relation by the condition that two unit vectors v and w in $\mathbb{C}^n$ are related if $\langle Av, w \rangle \neq 0$ for some $A \in \mathcal{V}$. From this point of view, the natural notion of a 'subobject' of $\mathcal{V}$ is its compression to some subspace of $\mathbb{C}^n$ (cf. [13, §4]). Now if $\mathcal{V}$ is an algebra, then its compressions $P\mathcal{V}P$, for $P \in M_n$ an orthogonal projection onto a subspace of $\mathbb{C}^n$, generally are not algebras. However, in some cases, they are.

An *invariant subspace* for an operator algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a closed subspace $\mathcal{E}$ of $\mathcal{H}$ with the property that $A(\mathcal{E}) \subseteq \mathcal{E}$ for all $A \in \mathcal{A}$ (what happens in $\mathcal{E}$ stays in $\mathcal{E}$). In this paper a *co-invariant subspace* will be a subspace whose orthocomplement is invariant for $\mathcal{A}$, or equivalently a subspace which is invariant for $\mathcal{A}^*$. Finally, a *semi-invariant subspace* is the orthogonal difference of two invariant subspaces, i.e., a closed subspace of the form $\mathcal{E}_1 \ominus \mathcal{E}_2 = \mathcal{E}_1 \cap \mathcal{E}_2^\perp$ where $\mathcal{E}_1$ and $\mathcal{E}_2$ are invariant and $\mathcal{E}_2 \subseteq \mathcal{E}_1$. Equivalently, $\mathcal{E} \subseteq \mathcal{H}$ is semi-invariant if there is an orthogonal decomposition $\mathcal{H} = \mathcal{F}_1 \oplus \mathcal{E} \oplus \mathcal{F}_2$ such that $\mathcal{F}_1 \oplus \mathcal{E}$ is invariant and $\mathcal{E} \oplus \mathcal{F}_2$ is co-invariant (Figure 1). According to [1, Theorem 2.16], a closed subspace $\mathcal{E}$ is semi-invariant for $\mathcal{A}$ if and only if the map $A \mapsto PAP$ is a homomorphism from $\mathcal{A}$ into $\mathcal{B}(\mathcal{E})$, i.e., $PABP = (PAP)(PBP)$ for all $A, B \in \mathcal{A}$. Here P is the orthogonal projection onto $\mathcal{E}$. In particular, if $\mathcal{E}$ is semi-invariant for $\mathcal{A}$ then $P\mathcal{A}P \subseteq \mathcal{B}(\mathcal{E})$ is still an operator algebra (and it is unital if $\mathcal{A}$ was).

In infinite dimensions we will mainly be interested in weak* closed operator algebras, necessitating some small modifications in the next definition. I will defer discussion of this aspect to §7. The rest of the present section deals only with the finite dimensional setting.

Definition 2.1. Let $\mathcal{A} \subseteq B(\mathbb{C}^n) \cong M_n$ be an operator algebra and let $P \in M_n$ be the orthogonal projection onto a subspace $\mathcal{E} \subseteq \mathbb{C}^n$. Then $P\mathcal{A}P$ is

(i) a *subobject* of $\mathcal{A}$ if $\mathcal{E}$ is invariant for $\mathcal{A}$;

(ii) a *quotient* of $\mathcal{A}$ if $\mathcal{E}$ is co-invariant for $\mathcal{A}$;

(iii) a *subquotient* of $\mathcal{A}$ if $\mathcal{E}$ is semi-invariant for $\mathcal{A}$.

$\mathcal{A}$ is *hereditarily antisymmetric* if every subquotient of $\mathcal{A}$ is antisymmetric.

There should be no confusion with the term 'quotient' because in this paper the word will always be used in the above sense and never in the more general sense of 'quotient by an ideal'.

By the comment made just above, subobjects, quotients, and subquotients are always operator algebras. The intuition for why our subobjects are rightly thought of as subobjects was explained above. For quotients, the idea is that if $\mathcal{E}$ is invariant then the action of $\mathcal{A}$ on $\mathbb{C}^n$ descends to an action on $\mathbb{C}^n/\mathcal{E} \cong \mathcal{E}^\perp$.

For the definition of hereditary antisymmetry in infinite dimensions, see Definition 7.1.

Note that $\mathbb{C}^n$ is a semi-invariant subspace for any $\mathcal{A} \subseteq M_n$, so that $\mathcal{A} = I\mathcal{A}I$ is always a subobject and a quotient of itself.

The next proposition is basic.

Proposition 2.2. *Let $\mathcal{A} \subseteq M_n$ be an operator algebra. Then any subquotient of a subquotient of $\mathcal{A}$ is a subquotient of $\mathcal{A}$.*

Proof. Let $P\mathcal{A}P$ be a subquotient of $\mathcal{A}$, where P is the orthogonal projection onto a semi-invariant subspace $\mathcal{E}$ for $\mathcal{A}$. Say that $\mathcal{E} = \mathcal{E}_1 \ominus \mathcal{E}_2$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are invariant subspaces for $\mathcal{A}$. Then let $QP\mathcal{A}PQ = Q\mathcal{A}Q$ be a subquotient of $P\mathcal{A}P$, where Q is the orthogonal projection onto a semi-invariant subspace $\mathcal{F}$ for $P\mathcal{A}P$. Say that $\mathcal{F} = \mathcal{F}_1 \ominus \mathcal{F}_2$, where $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{E}$ are invariant subspaces for $P\mathcal{A}P$. We must show that $\mathcal{F}$ is semi-invariant for $\mathcal{A}$; this will imply that $Q\mathcal{A}Q$ is a subquotient of $\mathcal{A}$.

I claim that $\mathcal{E}_2 + \mathcal{F}_1$ is invariant for $\mathcal{A}$. To see this, let $v \in \mathcal{E}_2$, $w \in \mathcal{F}_1$, and $A \in \mathcal{A}$. Then $Aw \in \mathcal{E}_1$ (since $\mathcal{F}_1 \subseteq \mathcal{E}_1$) and $PAw = (PAP)w \in \mathcal{F}_1$ (since $\mathcal{F}_1$ is invariant for $P\mathcal{A}P$). This shows that $Aw \in \mathcal{E}_1 \ominus (\mathcal{E} \ominus \mathcal{F}_1) = \mathcal{E}_2 + \mathcal{F}_1$, and therefore also $A(v + w) = Av + Aw \in \mathcal{E}_2 + \mathcal{F}_1$. So $\mathcal{E}_2 + \mathcal{F}_1$ is invariant, as claimed. By the same reasoning, $\mathcal{E}_2 + \mathcal{F}_2$ is invariant for $\mathcal{A}$, and thus $\mathcal{F} = (\mathcal{E}_2 + \mathcal{F}_1) \ominus (\mathcal{E}_2 + \mathcal{F}_2)$ is semi-invariant for $\mathcal{A}$. This is what we needed to show. $\qquad\square$

Corollary 2.3. *Any subquotient of a hereditarily antisymmetric operator algebra in M_n is hereditarily antisymmetric.*

The following class of examples might give some intuition for the nature of subobjects, quotients, and subquotients.

Example 2.4. Let $\preceq$ be a preorder on the set $\{1, \ldots, n\}$ and define

$$\mathcal{A}_{\preceq} = \text{span}\{E_{ij} : i \preceq j\} \subseteq M_n,$$

where E_{ij} is the matrix with a 1 in the (i, j) entry and 0's elsewhere. This is an algebra because $E_{ij}E_{jk} = E_{ik}$ and $i \preceq j, j \preceq k \Rightarrow i \preceq k$. It is unital because $I = E_{11} + \cdots + E_{nn}$.

Suppose $\mathcal{E} \subseteq \mathbb{C}^n$ is an invariant subspace for $\mathcal{A}_{\preceq}$. For any $v \in \mathcal{E}$ and $1 \leqslant i \leqslant n$ we must have $E_{ii}v \in \mathcal{E}$, and this shows that $\mathcal{E}$ must be the span of some subset of the standard basis $\{e_1, \ldots, e_n\}$. Thus, invariant subspaces for $\mathcal{A}_{\preceq}$ correspond to certain subsets of $\{1, \ldots, n\}$. A moment's thought shows that the subsets of $\{1, \ldots, n\}$ which correspond to invariant subspaces are precisely the lower sets, i.e., sets X which satisfy $i \preceq j \in X \Rightarrow i \in X$, while the subsets which correspond to orthocomplements of invariant subspaces are precisely the upper sets satisfying the opposite condition. The semi-invariant subspaces for $\mathcal{A}_{\preceq}$ are therefore the subspaces of the form $\text{span}\{e_i : i \in X\}$, where $X \subseteq \{1, \ldots, n\}$ is the difference of two lower sets. This is equivalent to saying that X is convex, i.e., $i \preceq j \preceq k$ and $i, k \in X$ $\Rightarrow j \in X$.

Thus, the subobjects of $\mathcal{A}$ correspond to lower subsets of $\{1, \ldots, n\}$ under $\preceq$, the quotients correspond to upper subsets, and the subquotients correspond to convex subsets.

In this example the ordinary subalgebras of $\mathcal{A}_{\preceq}$ which contain all the diagonal matrices correspond to preorders on $\{1, \ldots, n\}$ which are weaker than $\preceq$ (cf. Theorem 4.8).

Sometimes a modified version of a semi-invariant subspace, which is not orthogonal to $\mathcal{E}_2$ (in the notation from the beginning of this section), is more natural. In those situations the following notion can be helpful.

Definition 2.5. Let $\mathcal{A} \subseteq M_n$ be an operator algebra and let $\mathcal{E} = \mathcal{E}_1 \ominus \mathcal{E}_2$ be a semi-invariant subspace for $\mathcal{A}$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are invariant subspaces. A *companion subspace* of $\mathcal{E}$

(relative to the expression of $\mathcal{E}$ as $\mathcal{E}_1 \ominus \mathcal{E}_2$, but I will take this as understood) is any complementary subspace $\mathcal{F}$ of $\mathcal{E}_2$ in $\mathcal{E}_1$. That is, $\mathcal{F}$ is any linear subspace of $\mathcal{E}_1$ satisfying $\mathcal{E}_2 + \mathcal{F} = \mathcal{E}_1$ and $\mathcal{E}_2 \cap \mathcal{F} = \{0\}$. The *natural projection* onto $\mathcal{F}$ is the (nonorthogonal) projection onto $\mathcal{F}$ whose kernel is $\mathcal{E}_2 \oplus \mathcal{E}_1^{\perp}$.

Proposition 2.6. *Let $\mathcal{E} = \mathcal{E}_1 \ominus \mathcal{E}_2$ be a semi-invariant subspace for an operator algebra $\mathcal{A} \subseteq M_n$ and let $\mathcal{F}$ be a companion subspace of $\mathcal{E}$. Let P be the orthogonal projection onto $\mathcal{E}$, let $P_0 : \mathcal{F} \to \mathcal{E}$ be its restriction to $\mathcal{F}$, and let $Q \in M_n$ be the natural projection onto $\mathcal{F}$. Then $\Phi : T \mapsto P_0 T P_0^{-1}$ defines an isomorphism between $Q\mathcal{A}Q \subseteq B(\mathcal{F})$ and $P\mathcal{A}P \subseteq B(\mathcal{E})$.*

Proof. I claim that $\Phi(QAQ) = PAP$ for all $A \in \mathcal{A}$; this will show that Φ maps $Q\mathcal{A}Q$ bijectively onto $P\mathcal{A}P$. The fact that Φ is an algebra homomorphism is straightforward.

To prove the claim, observe first that for any $v \in \mathcal{E}_1$ we have $Pv, Qv \in v + \mathcal{E}_2$ since $\ker(P) \cap \mathcal{E}_1 = \ker(Q) \cap \mathcal{E}_1 = \mathcal{E}_2$; likewise for $P_0 v$ when $v \in \mathcal{F}$ and $P_0^{-1} v$ when $v \in \mathcal{E}$. So fixing $A \in \mathcal{A}$ and $v \in \mathcal{E}$, we have $QP_0^{-1}v = P_0^{-1}v \in v + \mathcal{E}_2$, and then $AQP_0^{-1}v \in Av + \mathcal{E}_2$, and then $QAQP_0^{-1}v \in Av + \mathcal{E}_2$ and $P_0 QAQP_0^{-1}v \in Av + \mathcal{E}_2$. But also $PAPv = PAv \in Av + \mathcal{E}_2$, so that

$$PAPv - \Phi(QAQ)v \in \mathcal{E}_2 \cap \mathcal{E} = \{0\}.$$

Since v was arbitrary, this shows that $\Phi(QAQ) = PAP$. $\qquad\square$

3. Basic facts

Before we discuss hereditarily antisymmetric operator algebras, it will be helpful to collect some basic facts about operator algebras, both antisymmetric and not. These results mostly pertain to the finite dimensional setting which will be our primary focus.

The first result, however, applies to both the finite and infinite dimensional settings. It provides a simple alternative characterization of antisymmetry. This was [9, Proposition 1 (i, iv)], but I include the proof for the sake of completeness.

Proposition 3.1. *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be an operator algebra. Then $\mathcal{A}$ is antisymmetric if and only if every self-adjoint element of $\mathcal{A}$ is a scalar multiple of the identity. If $\mathcal{A}$ is weak* closed, then it is antisymmetric if and only if it contains no orthogonal projections besides 0 and (possibly) I.*

Proof. The forward implication in the first assertion is trivial. For the reverse implication, suppose $\mathcal{A}$ is not antisymmetric. Then there is some $A \in \mathcal{A} \cap \mathcal{A}^*$ which is not a scalar multiple of the identity. Since both A and its adjoint belong to $\mathcal{A}$, both its real and imaginary parts $\text{Re}(A) = \frac{1}{2}(A + A^*)$ and $\text{Im}(A) = \frac{1}{2i}(A - A^*)$ also belong to $\mathcal{A}$, and (since $A = \text{Re}(A) + i \cdot \text{Im}(A)$) they cannot both be scalar multiples of the identity. So $\mathcal{A}$ contains a self-adjoint operator which is not a scalar multiple of the identity.

The forward implication in the second assertion is also trivial. For the reverse implication, suppose $\mathcal{A}$ is not antisymmetric. Then it contains a nonscalar self-adjoint operator A by the first part of the proposition. Since $\mathcal{A}$ is weak* closed, it then contains the von Neumann algebra generated by A, and hence it must contain a nontrivial orthogonal projection. $\qquad\square$

Next, we show that antisymmetric algebras can always be unitized. This works in infinite dimensions too.

Proposition 3.2. *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a nonunital operator algebra. If $\mathcal{A}$ is antisymmetric then so is its unitization $\mathcal{A} + \mathbb{C} \cdot I$.*

Proof. Suppose $\mathcal{A} + \mathbb{C} \cdot I$ is not antisymmetric. Then according to Proposition 3.1 there exists a self-adjoint operator of the form $A + aI$ with $A \in \mathcal{A}$ nonzero. Then $A + aI = A^* + \bar{a}I$ implies that $A = A^* + bI$, where $b = \bar{a} - a$. But then $A^2 = A^*A + bA \in \mathcal{A}$, and hence $A^*A \in \mathcal{A}$. Thus, $\mathcal{A}$ contains a nonzero self-adjoint operator, which means that it cannot be antisymmetric either. $\square$

Proposition 3.1 can be strengthened in the finite dimensional setting. We need the following lemma, which is probably well known.

Lemma 3.3. *Every subalgebra of $l_n^\infty = l^\infty(\{1, \ldots, n\})$ is self-adjoint.*

Proof. Let $\mathcal{A}$ be a subalgebra of l_n^∞, let $f \in \mathcal{A}$, and let $X = \text{Ran}(f) \cup \{0\}$. For each nonzero $a \in X$, find a complex polynomial p satisfying $p(a) = 1$ and $p(b) = 0$ for all other $b \in X$. Then p has no constant term, so $f_a = p \circ f \in \mathcal{A}$. This shows that we can write $f = \sum_{a \in X \setminus \{0\}} a f_a$ where each f_a belongs to $\mathcal{A}$ and takes only the values 0 and 1. Thus $\bar{f} = \sum \bar{a} f_a$ also belongs to $\mathcal{A}$. We conclude that $\mathcal{A}$ is self-adjoint. $\square$

The situation for $l^\infty = l^\infty(\mathbb{N})$ could not be more different; see Proposition 7.9.

It is standard that the self-adjoint unital subalgebras of l_n^∞ correspond to the equivalence relations on $\{1, \ldots, n\}$, by associating an equivalence relation $\sim$ to the set of functions in l_n^∞ which are constant on each block of $\sim$. Any nonunital self-adjoint subalgebra is, for some equivalence relation $\sim$, the set of all functions which are constant on each block of $\sim$ and which vanish on some specified block.

Proposition 3.4. *Let $\mathcal{A} \subseteq M_n$ be an operator algebra. The following are equivalent:*

(i) *$\mathcal{A}$ is antisymmetric;*

(ii) *every self-adjoint element of $\mathcal{A}$ is a scalar multiple of the identity;*

(iii) *there are no orthogonal projections in $\mathcal{A}$ besides 0 and (possibly) I;*

(iv) *every unitary element of $\mathcal{A} + \mathbb{C} \cdot I$ is a scalar multiple of the identity;*

(v) *every normal element of $\mathcal{A}$ is a scalar multiple of the identity.*

Proof. The equivalence of (i), (ii), and (iii) was shown in Proposition 3.1.

For (i) $\Rightarrow$ (v), suppose $\mathcal{A}$ contains a normal operator A which is not a scalar multiple of the identity. Working in an orthonormal basis which diagonalizes A, the operator algebra generated by A constitutes a subalgebra of the diagonal subalgebra of M_n, which can be identified with l_n^∞. We infer from Lemma 3.3 that A^* also belongs to this subalgebra, and hence that $A^* \in \mathcal{A}$. So $A \in \mathcal{A} \cap \mathcal{A}^*$, showing that $\mathcal{A}$ is not antisymmetric.

For (v) $\Rightarrow$ (iv), suppose there is a nonscalar unitary $U = A + \alpha I$ with $A \in \mathcal{A}$. Then A is a nonscalar normal operator in $\mathcal{A}$.

For (iv) $\Rightarrow$ (ii), suppose there is a self-adjoint operator A in $\mathcal{A}$ which is not a scalar multiple of the identity. Then $e^{iAt} \in \mathcal{A} + \mathbb{C} \cdot I$ for every $t \in \mathbb{R}$; one can infer this from

Lemma 3.3 or simply consider the power series expansion of e^{iAt}. This is a one-parameter unitary group, and since $A = \lim_{t\to 0} \frac{1}{it}(e^{iAt} - I)$, the operators e^{iAt} cannot all be scalar multiples of the identity. (Alternatively, one can deduce the implication (iv) $\Rightarrow$ (ii) by applying the comment made after Lemma 3.3 to the algebra generated by A in an orthonormal basis which diagonalizes A.) $\qquad\square$

The equivalence with conditions (iv) and (v) fails in infinite dimensions; see Example 7.5.

Next, we note a simple dimensional restriction on antisymmetric subalgebras of M_n.

Proposition 3.5. *Suppose an operator algebra $\mathcal{A} \subseteq M_n$ has dimension at least $\frac{n^2}{2} + 1$. Then $\mathcal{A}$ is not antisymmetric.*

Proof. M_n has real dimension $2n^2$, and the set of self-adjoint $n \times n$ matrices with zero trace is a real linear subspace of M_n of real dimension $n^2 - 1$. Then any complex linear subspace of M_n whose complex dimension is at least $\frac{n^2}{2} + 1$ will have real dimension at least $n^2 + 2$ and hence must nontrivially intersect the space of self-adjoint matrices with zero trace. Thus, no operator algebra of dimension at least $\frac{n^2}{2} + 1$ can be antisymmetric. $\qquad\square$

Putting operator algebras in upper triangular form, i.e., finding a basis with respect to which the matrix of every element of the algebra is upper triangular, will be a recurring theme in this paper. The next result is basic. In order for the notion of 'upper triangular' to be meaningful, we need not merely a basis but an *ordered* basis in which the basis vectors appear in a specified order.

Proposition 3.6. *Let $\mathcal{A} \subseteq M_n$ be an operator algebra and suppose there is a (possibly nonorthogonal) ordered basis of $\mathbb{C}^n$ with respect to which the matrix of every element of $\mathcal{A}$ is upper triangular. Then there is an ordered orthonormal basis with the same property. If in addition the matrix with respect to the first basis of every element of $\mathcal{A}$ is constant on its main diagonal, then the same will be true of the matrix with respect to the second basis of every element of $\mathcal{A}$.*

Proof. Suppose $(v_1, \ldots, v_n)$ is an ordered basis with respect to which the matrix of every element of $\mathcal{A}$ is upper triangular. This means that $Av_i \in \text{span}\{v_1, \ldots, v_i\}$ for all $A \in \mathcal{A}$ and $1 \leqslant i \leqslant n$, or, to put it differently, for every i the subspace $\text{span}\{v_1, \ldots, v_i\}$ is invariant for $\mathcal{A}$. Applying the Gram–Schmidt orthonormalization procedure to this basis produces an ordered orthonormal basis $(w_1, \ldots, w_n)$ with the property that $\text{span}\{w_1, \ldots, w_i\} = \text{span}\{v_1, \ldots, v_i\}$ for all $1 \leqslant i \leqslant n$. So each $\text{span}\{w_1, \ldots, w_i\}$ is invariant for $\mathcal{A}$, and this shows that every element of $\mathcal{A}$ has an upper triangular matrix for the (w_i) basis.

The second assertion follows from the fact that the main diagonal entries of an upper triangular matrix are precisely its eigenvalues. So if the matrix of $A \in \mathcal{A}$ with respect to the (v_i) basis is upper triangular and constant on its main diagonal, then A has only one eigenvalue and so its matrix with respect to the (w_i) basis will also be constant on its main diagonal. $\qquad\square$

The final theorem of this section shows that in finite dimensions, operator algebras typically contain plenty of nonorthogonal projections (with the exceptions described in

Theorem 4.3). As stated, it applies to block diagonal matrices in which each block is upper triangular and constant on its main diagonal — the type just treated in Proposition 3.6. Pictorially, these are matrices of the form shown in Figure 2. Jordan matrices are particular examples of matrices of this type. Therefore I will call any matrix of the above form 'Jordanesque'. Let us record this in a definition.

Definition 3.7. A block diagonal matrix in $\widetilde{M}_n$ in which each block is upper triangular and constant on its main diagonal is *Jordanesque*. If the sizes of the blocks are $n_1, \ldots, n_k$ (so that $n_1 + \cdots + n_k = n$), then we may also say the matrix is $(n_1, \ldots, n_k)$-*Jordanesque*.

Any linear operator on $\mathbb{C}^n$ has a matrix in Jordan form relative to some ordered basis of $\mathbb{C}^n$. But this basis need not be orthogonal, so it is important that the preceding definition applies to $\widetilde{M}_n$, which has no preferred involution.

It is straightforward to check that the sum and product of any two $(n_1, \ldots, n_k)$-Jordanesque matrices are again $(n_1, \ldots, n_k)$-Jordanesque. Thus the set of all $(n_1, \ldots, n_k)$-Jordanesque matrices is a unital operator algebra.

The crux of the next proof is the simple fact that if $A \in \widetilde{M}_n$ is strictly upper triangular (i.e., upper triangular and zero on the main diagonal) then $A^n = 0$.

Theorem 3.8. *Let $A \in \widetilde{M}_n$ be Jordanesque and let λ be one of its diagonal entries (i.e., one of its eigenvalues). Assume $\lambda \neq 0$. Then the algebra generated by A contains the diagonal matrix whose (i, i) entry is 1 if $a_{ii} = \lambda$ and is 0 otherwise, where $A = (a_{ij})$.*

Proof. Let $\mathcal{A}$ be the algebra generated by A. For any nonzero eigenvalue μ besides λ, the matrix $A^2 - \mu A \in \mathcal{A}$ has zero diagonal on every block where A has diagonal entries μ, but its diagonal entries on every block where A has diagonal entries λ are nonzero. The restriction of $A^2 - \mu A$ to any of the former blocks is strictly upper triangular, so that $(A^2 - \mu A)^n$ is zero on all of these blocks. Thus, if $\mu_1, \ldots, \mu_k$ are the nonzero eigenvalues of A besides λ, then $(A^2 - \mu_1 A)^n \cdots (A^2 - \mu_k A)^n \in \mathcal{A}$ is nonzero only on blocks where A has diagonal entries λ, and on those blocks its diagonal entries all take the nonzero value $\lambda' = \lambda^{kn}(\lambda - \mu_1)^n \cdots (\lambda - \mu_k)^n$.

Multiplying by $\frac{1}{\lambda'}$ and restricting to the nonzero blocks, we reduce to the case where A is upper triangular and its diagonal entries are all 1. We must now show that $I \in \mathcal{A}$. Write $A = I + A_0$, where A_0 is strictly upper triangular. For any $B \in \mathcal{A}$ we have $A_0 B = AB - B \in \mathcal{A}$. So inductively $A_0 + A_0^2$, $A_0^2 + A_0^3, \ldots, A_0^{n-1} + A_0^n = A_0^{n-1}$ all belong to $\mathcal{A}$. Then the alternating sum

$$(I + A_0) - (A_0 + A_0^2) + \cdots \pm (A_0^{n-2} + A_0^{n-1}) \mp A_0^{n-1} = I$$

belongs to $\mathcal{A}$, as desired.   □

Corollary 3.9. *If $\mathcal{A} \subseteq M_n$ is an operator algebra all of whose elements are Jordanesque, then we have $\mathcal{A} = \mathcal{A}_{\mathrm{diag}} + \mathcal{A}_{\mathrm{nil}}$, where $\mathcal{A}_{\mathrm{diag}}$ is the algebra of diagonal matrices in $\mathcal{A}$ and $\mathcal{A}_{\mathrm{nil}}$ is the ideal of strictly upper triangular (i.e., nilpotent) matrices in $\mathcal{A}$.*

Proof. Let $A \in \mathcal{A}$. Then applying Theorem 3.8 to each nonzero eigenvalue of A and taking a linear combination, we get that the diagonal matrix B whose diagonal entries

agree with those of A belongs to $\mathcal{A}$. Thus, every matrix in $\mathcal{A}$ can be decomposed into the sum of a diagonal matrix and a strictly upper triangular matrix, both of which belong to $\mathcal{A}$. This shows that $\mathcal{A} = \mathcal{A}_{\text{diag}} + \mathcal{A}_{\text{nil}}$. $\qquad\square$

Since any linear operator on $\mathbb{C}^n$ can be put in Jordan form by a suitable choice of (not necessarily orthogonal) basis, the preceding results apply to any linear operator on $\mathbb{C}^n$ and the algebra it generates.

4. Two basic examples

There are two prototypical examples of hereditarily antisymmetric operator algebras in M_n. We will see in Theorem 5.9 that any hereditarily antisymmetric operator algebra in M_n is a sort of combination of these two types.

Example 4.1. Let $n \in \mathbb{N}$ and let $\mathcal{T}_n \subseteq M_n$ be the set of upper triangular matrices which are constant on the main diagonal (Figure 3). This is a unital operator algebra, and it is clearly antisymmetric.

In this example we are working in the standard basis of $\mathbb{C}^n$, which is orthonormal. (That is, $\mathcal{T}_n$ is defined in terms of M_n, not $\widetilde{M}_n$.) But the same definition could be made relative to any ordered vector space basis of $\mathbb{C}^n$: let $(v_1, \ldots, v_n)$ be any ordered basis of $\mathbb{C}^n$ and let $\mathcal{A}$ be the set of operators whose matrices relative to this basis are upper triangular and constant on the main diagonal. However, as a consequence of Proposition 3.6, this construction is no more general than the orthonormal case. That result implies that this new algebra $\mathcal{A}$ will be identical to the algebra $\mathcal{T}_n$ with respect to some ordered orthonormal basis. So we have one example of this type for each dimension n, and allowing nonorthonormal bases does not expand the class of examples.

Proposition 4.2. *For each $n \in \mathbb{N}$, the operator algebra $\mathcal{T}_n$ of Example 4.1 is hereditarily antisymmetric, and every operator algebra that properly contains it is not antisymmetric. In particular, it is a maximal hereditarily antisymmetric operator algebra.*

Proof. In order to check the first assertion, we must identify the semi-invariant subspaces for $\mathcal{T}_n$. But $\mathcal{T}_n$ contains the shift matrix (Figure 4), whose only invariant subspaces are $\{0\}$, $\text{span}\{e_1\}$, $\text{span}\{e_1, e_2\}, \ldots, \mathbb{C}^n$. So these are the only possible invariant subspaces for $\mathcal{T}_n$. Conversely, it is easy to see that each of these subspaces is invariant for any operator in $\mathcal{T}_n$. So these are precisely the invariant subspaces for $\mathcal{T}_n$, and the nonzero semi-invariant subspaces are therefore those of the form $\text{span}\{e_i, \ldots, e_j\}$ for $i \leqslant j$. The compression of $\mathcal{T}_n$ to any such subspace is simply a lower dimensional version of $\mathcal{T}_n$, and hence is antisymmetric. So $\mathcal{T}_n$ is hereditarily antisymmetric.

Next, we show that $\mathcal{T}_n$ is maximal antisymmetric. Suppose $\mathcal{A} \subseteq M_n$ is an operator algebra that properly contains $\mathcal{T}_n$ and let $A \in \mathcal{A} \setminus \mathcal{T}_n$. There are two cases to consider. First, suppose A is upper triangular but not constant on the main diagonal. By subtracting an element of $\mathcal{T}_n$, we may assume that A has no entries above the main diagonal, i.e., it is a diagonal matrix. But then it is normal but not a scalar multiple of the identity, which shows that $\mathcal{A}$ is not antisymmetric by Proposition 3.4.

In the second case A has some nonzero entry a_{ij} with $i > j$. Choose such a pair (i, j) with $i - j$ maximal. Then AU^{i-j} is upper triangular, its $(0, 0)$ entry is 0, and its (i, i) entry is a_{ij}. (The matrix U was introduced in the first part of this proof.) So we reduce to the first case. This completes the proof that $\mathcal{T}_n$ is maximal antisymmetric.

Since there is no larger antisymmetric algebra in M_n, there is certainly no larger hereditarily antisymmetric algebra in M_n. Thus $\mathcal{T}_n$ is also maximal hereditarily antisymmetric. $\qquad\square$

The operator algebras $\mathcal{T}_n$ are quite special. They can be characterized in several alternative ways. Recall that Burnside's theorem on matrix algebras states that any operator algebra properly contained in M_n has a nontrivial invariant subspace.

Theorem 4.3. *Let $\mathcal{A} \subseteq M_n$ be an operator algebra. The following are equivalent:*

(i) *$\mathcal{A}$ is contained in the algebra $\mathcal{T}_n$ relative to some ordered orthonormal basis;*

(ii) *every similar operator algebra (i.e., every $S\mathcal{A}S^{-1}$ for $S \in M_n$ invertible) is antisymmetric;*

(iii) *$\mathcal{A}$ is antisymmetric with respect to any inner product on $\mathbb{C}^n$;*

(iv) *$\mathcal{A}$ contains no projections besides 0 and (possibly) I;*

(v) *every operator in $\mathcal{A}$ has exactly one eigenvalue.*

Proof. (i) $\Rightarrow$ (ii): Let $S \in M_n$ be invertible; we must show that $S\mathcal{T}_nS^{-1}$ is antisymmetric. Now $S\mathcal{T}_nS^{-1}$ consists of the operators whose matrices relative to the (Se_i) basis are upper triangular and constant on the main diagonal. According to Proposition 3.6, this means that $S\mathcal{T}_nS^{-1}$ is just another $\mathcal{T}_n$ with respect to some other orthonormal basis. This shows that there are no nonscalar self-adjoint operators in $S\mathcal{T}_nS^{-1}$.

(ii) $\Rightarrow$ (iii): Assume that every operator algebra similar to $\mathcal{A}$ is antisymmetric and let $\{\cdot, \cdot\}$ be a new inner product on $\mathbb{C}^n$. Then there is an invertible positive operator T with the property that $\langle Tv, w \rangle = \{v, w\}$ for all $v, w \in \mathbb{C}^n$. Let $S = T^{-1/2}$. Denoting the adjoint operation relative to $\{\cdot, \cdot\}$ by $A^\dagger$, we then have, for any v and w,

$$\langle TA^\dagger v, w \rangle = \{A^\dagger v, w\} = \{v, Aw\} = \langle Tv, Aw \rangle = \langle A^*Tv, w \rangle.$$

Thus

$$A^\dagger = T^{-1}A^*T = S^2A^*S^{-2} = S(S^{-1}AS)^*S^{-1},$$

and hence $S^{-1}A^\dagger S = (S^{-1}AS)^*$. So $A^\dagger = A \in \mathcal{A}$ implies $S^{-1}AS = (S^{-1}AS)^*$, which by hypothesis implies that $S^{-1}AS$ is a scalar multiple of the identity, which implies that A is a scalar multiple of the identity. Thus by Proposition 3.1 $\mathcal{A}$ is antisymmetric relative to $\{\cdot, \cdot\}$.

(iii) $\Rightarrow$ (iv): Suppose $\mathcal{A}$ contains a projection P which is neither 0 nor I. Find bases of $\ker(P)$ and $\operatorname{ran}(P)$; together these form a basis for $\mathbb{C}^n$ which, if taken to be orthonormal, makes P an orthogonal projection. So relative to the inner product which makes the chosen basis orthonormal, $\mathcal{A}$ contains a nonscalar self-adjoint operator, i.e., it is not antisymmetric.

(iv) $\Rightarrow$ (v): If $\mathcal{A}$ contains an operator with more than one eigenvalue, then it contains a nonscalar projection by Theorem 3.8.

$$\begin{array}{cc} & \begin{array}{ccc} \mathcal{F}_1 & \mathcal{E} & \mathcal{F}_2 \end{array} \\ \begin{array}{c} \mathcal{F}_1 \\ \mathcal{E} \\ \mathcal{F}_2 \end{array} & \left[\begin{array}{ccc} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{array} \right] \end{array}$$

Figure 1. $\mathcal{F}_1 \oplus \mathcal{E}$ is invariant and $\mathcal{E} \oplus \mathcal{F}_2$ is co-invariant.
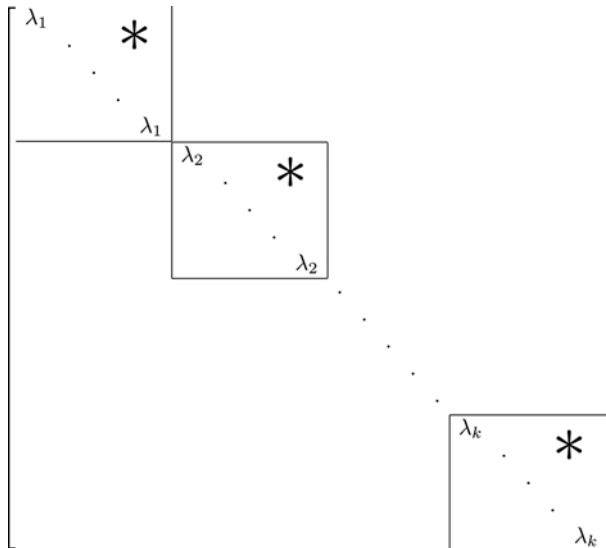

Figure 2. A Jordanesque matrix.

(v) $\Rightarrow$ (i): Suppose every operator in $\mathcal{A}$ has exactly one eigenvalue. We prove that there is an ordered orthonormal basis relative to which every operator in $\mathcal{A}$ is upper triangular. The fact that every operator in $\mathcal{A}$ has exactly one eigenvalue then implies that these upper triangular matrices are all constant on the main diagonal.

The proof goes by induction on n, where $\mathcal{A} \subseteq M_n$. The assertion is trivial for $n = 1$. For $n > 1$, let $\mathcal{A}$ be such an algebra and observe that every subquotient of $\mathcal{A}$ has the same property that every operator in it has only one eigenvalue. (If $\mathcal{H} = \mathcal{F}_1 \oplus \mathcal{E} \oplus \mathcal{F}_2$ is an orthogonal decomposition such that $\mathcal{F}_1$ and $\mathcal{F}_1 \oplus \mathcal{E}$ are invariant for A, then the eigenvalues of A are the eigenvalues of its compressions to $\mathcal{E}$, $\mathcal{F}_1$, and $\mathcal{F}_2$; see Figure 1.) By Burnside's theorem, $\mathcal{A}$ has a nontrivial invariant subspace $\mathcal{E}$. Then the induction hypothesis applies to the compressions of $\mathcal{A}$ to $\mathcal{E}$ and $\mathcal{E}^\perp$, so we can find ordered orthonormal bases of $\mathcal{E}$ and $\mathcal{E}^\perp$ such that the matrix of the compression of any operator in $\mathcal{A}$ to either $\mathcal{E}$ or $\mathcal{E}^\perp$ is upper triangular. Then the matrix of any operator in $\mathcal{A}$ has the same property relative to the concatenation of these two bases. This proves (i). $\square$

The proof of (v) $\Rightarrow$ (i) appears in a more general form in Theorem 5.1.

$$\begin{bmatrix} \lambda & & & & * \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \lambda \end{bmatrix}$$

Figure 3. Upper triangular and constant on the main diagonal.

$$U = \begin{bmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 0 & & & & 0 \end{bmatrix}$$

Figure 4. The shift matrix.

Since any subquotient of $\mathcal{A}$ would inherit property (v), the following corollary is immediate.

Corollary 4.4. *Every subalgebra of $\mathcal{T}_n$ is hereditarily antisymmetric.*

Now let us turn to the second prototypical class of examples.

Definition 4.5. Let $\mathbf{v} = \{v_1, \ldots, v_n\}$ be a basis for $\mathbb{C}^n$ and define $\mathcal{D}_{\mathbf{v}} \subseteq M_n$ to be the set of operators for which each v_i is an eigenvalue. Equivalently, these are the operators whose matrix for the $\mathbf{v}$ basis is diagonal.

If the $\mathbf{v}$ basis is orthogonal, then $\mathcal{D}_{\mathbf{v}}$ clearly contains many self-adjoint operators. In order to make $\mathcal{D}_{\mathbf{v}}$ antisymmetric we need some amount of nonorthogonality. Define the *nonorthogonality graph associated with* $\mathbf{v}$ to have vertices $\{1, \ldots, n\}$ and an edge between i and j if $\langle v_i, v_j \rangle \neq 0$. Say that $\mathbf{v}$ is *anti-orthogonal* if, for any $1 \leqslant i < j \leqslant n$ and $X \subseteq \{1, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, n\}$ (i.e., $\{1, \ldots, n\}$ with i and j omitted), we have $\langle P v_i, v_j \rangle \neq 0$, where P is the orthogonal projection onto the orthocomplement of $\mathrm{span}\{v_k : k \in X\}$. (In particular, taking $X = \emptyset$ yields $\langle v_i, v_j \rangle \neq 0$ for all i and j, i.e., the nonorthogonality graph is complete.)

Theorem 4.6. *Let $\mathbf{v}$ be a basis for $\mathbb{C}^n$, $n > 1$. Then $\mathcal{D}_{\mathbf{v}}$ is antisymmetric if and only if the nonorthogonality graph associated with $\mathbf{v}$ is connected. It is hereditarily antisymmetric if and only if $\mathbf{v}$ is anti-orthogonal.*

Proof. Suppose the nonorthogonality graph associated with $\mathbf{v}$ is disconnected. Then there is a nonempty proper subset $X \subset \{1, \ldots, n\}$ such that $\langle v_i, v_j \rangle = 0$ for any $i \in X$ and $j \notin X$. The operator A which satisfies $A v_i = v_i$ for all $i \in X$ and $A v_j = 0$ for all $j \notin X$ is then a nonscalar orthogonal projection which belongs to $\mathcal{D}_{\mathbf{v}}$. So $\mathcal{D}_{\mathbf{v}}$ is not antisymmetric.

Conversely, suppose the nonorthogonality graph associated with $\mathbf{v}$ is connected. Let $A \in \mathcal{D}_{\mathbf{v}}$ and suppose $A = A^*$. Write $Av_i = \lambda_i v_i$. Then for any i we have

$$\lambda_i \|v_i\|^2 = \langle Av_i, v_i \rangle = \langle v_i, Av_i \rangle = \bar{\lambda}_i \|v_i\|^2,$$

so each λ_i is real. For any i and j we then have

$$\lambda_i \langle v_i, v_j \rangle = \langle Av_i, v_j \rangle = \langle v_i, Av_j \rangle = \lambda_j \langle v_i, v_j \rangle,$$

so that $\langle v_i, v_j \rangle \neq 0$ implies $\lambda_i = \lambda_j$. Since the nonorthogonality graph is connected, this makes every λ_i take the same value λ, so that $A = \lambda I$. We have shown that the only self-adjoint elements of $\mathcal{D}_{\mathbf{v}}$ are scalar multiples of the identity, so $\mathcal{D}_{\mathbf{v}}$ is antisymmetric.

Now suppose $\mathbf{v}$ is anti-orthogonal. It is easy to see that the invariant subspaces for $\mathcal{D}_{\mathbf{v}}$ are precisely the subspaces of the form $\mathrm{span}\{v_i : i \in X\}$ for some $X \subseteq \{1, \ldots, n\}$. Thus the semi-invariant subspaces are all the orthogonal differences of subspaces of this form. The structure of the compression of $\mathcal{D}_{\mathbf{v}}$ to the subspace $\mathrm{span}\{v_i : i \in X\} \ominus \mathrm{span}\{v_i : i \in Y\}$, with $X \subset Y$, is clear because $\mathrm{span}\{v_i : i \in Y \setminus X\}$ is a companion subspace on which $\mathcal{D}_{\mathbf{v}}$ acts diagonally. Thus, the compression of $\mathcal{D}_{\mathbf{v}}$ to any semi-invariant subspace will be a lower dimensional version of $\mathcal{D}_{\mathbf{v}}$, and by anti-orthogonality for a basis whose nonorthogonality graph is complete. So the compression is antisymmetric by the first part of the theorem. We have shown that $\mathcal{D}_{\mathbf{v}}$ is hereditarily antisymmetric.

Conversely, suppose $\mathbf{v}$ is not anti-orthogonal. Then there exist i, j, and $X \subseteq \{1, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, n\}$ such that $\langle Pv_i, v_j \rangle = 0$, where P is the orthogonal projection onto the orthocomplement of $\mathrm{span}\{v_k : k \in X\}$. Then the orthogonal difference $\mathcal{E}$ between $\mathrm{span}\{v_k : k \in X \cup \{i, j\}\}$ and $\mathrm{span}\{v_k : k \in X\}$ is a two-dimensional semi-invariant subspace which has $\mathrm{span}\{v_i, v_j\}$ as a companion subspace. The compression of $\mathcal{D}_{\mathbf{v}}$ to this subspace is then just the set of 2×2 matrices which are diagonal with respect to the basis $\{Pv_i, Pv_j\}$ of $\mathbb{C}^2$, which is orthogonal. So this compression is not antisymmetric, and therefore $\mathcal{D}_{\mathbf{v}}$ is not hereditarily antisymmetric. $\qquad\square$

This result shows that there are antisymmetric operator algebras which are not hereditarily antisymmetric.

Example 4.7. Let $\mathbf{v} = \{v_1, v_2, v_3\}$ be a basis for $\mathbb{C}^3$ with the property that $\langle v_1, v_2 \rangle$ and $\langle v_2, v_3 \rangle$ are nonzero but $\langle v_1, v_3 \rangle = 0$. Then $\mathbf{v}$ has a connected nonorthogonality graph but is not anti-orthogonal, so $\mathcal{D}_{\mathbf{v}}$ is antisymmetric but not hereditarily antisymmetric.

If $\mathbf{v}$ is anti-orthogonal, then $\mathcal{D}_{\mathbf{v}}$ is not only hereditarily antisymmetric but also maximal for this property. In order to prove this, we first need to characterize the operator algebras which contain $\mathcal{D}_{\mathbf{v}}$. Fortunately, this characterization is easy and explicit. The construction was already seen in Example 2.4. Given any basis $\mathbf{v}$ of $\mathbb{C}^n$, let E_{ij} be the operator whose matrix for the $\mathbf{v}$ basis has a 1 in the (i, j) entry and 0's elsewhere. (So $E_{ij}v_j = v_i$ and $E_{ij}v_k = 0$ for $k \neq j$.)

Theorem 4.8. *Let $\mathbf{v} = \{v_1, \ldots, v_n\}$ be a basis for $\mathbb{C}^n$. For any preorder $\preceq$ on $\{1, \ldots, n\}$, the set*

$$\mathcal{A}_{\preceq} = \mathrm{span}\{E_{ij} : i \preceq j\}$$

is an operator algebra which contains $\mathcal{D}_{\mathbf{v}} = \mathrm{span}\{E_{11}, \ldots, E_{nn}\}$. *Conversely, every operator algebra in M_n which contains $\mathcal{D}_{\mathbf{v}}$ has this form.*

Proof. The fact that $\mathcal{A}_{\preceq}$ is an operator algebra is a simple verification. For the second assertion, let $\mathcal{A} \subseteq M_n$ be an operator algebra which contains $\mathcal{D}_{\mathbf{v}}$. For any $A \in \mathcal{A}$ we can write $A = \sum a_{ij} E_{ij}$, where $A = (a_{ij})$. Since $a_{ij} E_{ij} = E_{ii} A E_{jj} \in \mathcal{A}$, it follows that $\mathcal{A}$ equals $\mathrm{span}\{E_{ij} : (i, j) \in X\}$ for some subset X of $\{1, \ldots, n\}^2$, i.e., some relation on $\{1, \ldots, n\}$. Since each E_{ii} belongs to $\mathcal{D}_{\mathbf{v}} \subseteq \mathcal{A}$, the pairs (i, i) all belong to X, i.e., X is reflexive. It is transitive because $E_{ij} E_{jk} = E_{ik}$. Therefore it is a preorder. $\qquad\square$

Theorem 4.9. *Suppose $\mathbf{v} = \{v_1, \ldots, v_n\}$ is an anti-orthogonal basis for $\mathbb{C}^n$. Then $\mathcal{D}_{\mathbf{v}}$ is a maximal hereditarily antisymmetric operator algebra.*

Proof. Let $\preceq$ be a preorder on $\{1, \ldots, n\}$ and let $\mathcal{A}_{\preceq}$ be as in Theorem 4.8. If $\mathcal{A}_{\preceq}$ properly contains $\mathcal{D}_{\mathbf{v}}$ then there must be at least one pair of distinct elements which are comparable. Thus, there must exist $1 \leqslant j \leqslant n$ for which $X = \{i : i \preceq j\}$ contains at least one number besides j. Let $\mathcal{E} = \mathrm{span}\{e_i : i \in X\}$; this is an invariant subspace for $\mathcal{A}_{\preceq}$.

Find a nonzero vector $v \in \mathcal{E}$ which is orthogonal to $\mathrm{span}\{e_i : i \in X \setminus \{j\}\}$. Write $v = \sum_{i \in X} a_i e_i$ and consider the operator $A = \sum_{i \in X} a_i E_{ij} \in \mathcal{A}_{\preceq}$, where E_{ij} is as above. We have $A e_i = 0$ for all $i \in X \setminus \{j\}$ and $A v = a_j v$. Thus the compression of A to $\mathcal{E}$ is a nonzero scalar multiple of the orthogonal projection onto $\mathrm{span}\{v\}$, and this shows that $\mathcal{A}_{\preceq}$ is not hereditarily antisymmetric. Since every operator algebra properly containing $\mathcal{D}_{\mathbf{v}}$ has this form, the latter must be maximal hereditarily antisymmetric. $\qquad\square$

The algebras $\mathcal{T}_n$ were maximal hereditarily antisymmetric merely by virtue of being maximal antisymmetric (Proposition 4.2). This is not the case for the algebras $\mathcal{D}_{\mathbf{v}}$.

Proposition 4.10. *Let $\mathbf{v} = \{v_1, \ldots, v_n\}$ be a basis of $\mathbb{C}^n$ whose nonorthogonality graph is connected and suppose $n > 2$. Then $\mathcal{D}_{\mathbf{v}}$ is not maximal antisymmetric.*

Proof. First suppose $\langle v_i, v_j \rangle = 0$ for some i and j. Then $\mathcal{A} = \mathcal{D}_{\mathbf{v}} + \mathbb{C} \cdot E_{ij}$ is an algebra which properly contains $\mathcal{D}_{\mathbf{v}}$. Let $A \in \mathcal{A}$ be self-adjoint. Writing $A = \lambda_1 E_{11} + \cdots + \lambda_n E_{nn} + \alpha E_{ij}$, we have

$$\langle A v_i, v_j \rangle = \lambda_i \langle v_i, v_j \rangle = 0$$

and

$$\langle v_i, A v_j \rangle = \langle v_i, \lambda_j v_j + \alpha v_i \rangle = \bar{\alpha} \|v_i\|^2,$$

and equating these expressions yields $\alpha = 0$. So $A \in \mathcal{D}_{\mathbf{v}}$, which we know is antisymmetric by Theorem 4.6, and therefore A must be a scalar multiple of the identity. So in this case $\mathcal{D}_{\mathbf{v}}$ is not maximal antisymmetric.

Otherwise, if no two basis vectors are orthogonal, normalize the basis vectors so that $\|v_i\| = 1$ for all i and then fix i and j which minimize $|\langle v_i, v_j \rangle|$. Again let $\mathcal{A} = \mathcal{D}_{\mathbf{v}} + \mathbb{C} \cdot E_{ij}$. Suppose $A \in \mathcal{A}$ is self-adjoint, and again write $A = \lambda_1 E_{11} + \cdots + \lambda_n E_{nn} + \alpha E_{ij}$. Assume $\alpha \neq 0$, aiming for a contradiction.

By self-adjointness we have

$$\lambda_k = \langle A v_k, v_k \rangle = \langle v_k, A v_k \rangle = \bar{\lambda}_k$$

for any $k \neq j$, showing that λ_k is real for all $k \neq j$. Then for any $k, l \neq j$

$$\lambda_k \langle v_k, v_l \rangle = \langle A v_k, v_l \rangle = \langle v_k, A v_l \rangle = \lambda_l \langle v_k, v_l \rangle,$$

and since every $\langle v_k, v_l \rangle$ is nonzero this shows that $\lambda_k = \lambda_l$. So there is a single value λ such that $\lambda_k = \lambda$ for all $k \neq j$.

We therefore have

$$\lambda \langle v_k, v_j \rangle = \langle A v_k, v_j \rangle = \langle v_k, A v_j \rangle = \bar{\lambda}_j \langle v_k, v_j \rangle + \bar{\alpha} \langle v_k, v_i \rangle$$

for all $k \neq j$. That is,

$$(\lambda - \bar{\lambda}_j) \langle v_k, v_j \rangle = \bar{\alpha} \langle v_k, v_i \rangle,$$

or, more briefly, $\langle v_k, v_i \rangle = \beta \langle v_k, v_j \rangle$, where $\beta = \frac{\lambda - \bar{\lambda}_j}{\bar{\alpha}}$. With $k = i$, this yields

$$1 = |\beta| |\langle v_i, v_j \rangle|,$$

and then minimality of $|\langle v_i, v_j \rangle|$ yields

$$|\langle v_k, v_i \rangle| = |\beta| |\langle v_k, v_j \rangle| \geqslant |\beta| |\langle v_i, v_j \rangle| = 1$$

for all k not equal to i or j. (There is at least one such k; this is where we use $n > 2$.) But this is absurd since $\|v_k\| = \|v_i\| = 1$ and the two vectors are linearly independent. The contradiction shows that we must have $\alpha = 0$, and thus A belongs to $\mathcal{D}_\mathbf{v}$ and hence it must be a scalar multiple of the identity by Theorem 4.6. This completes the proof that $\mathcal{A}$ is antisymmetric, so we conclude that $\mathcal{D}_\mathbf{v}$ can always be strictly enlarged to an algebra which is still antisymmetric. □

If $n = 2$ then $\mathcal{D}_\mathbf{v}$ is maximal antisymmetric for any nonorthogonal basis $\mathbf{v} = \{v_1, v_2\}$ of $\mathbb{C}^2$. This is because its dimension is 2, and we know from Proposition 3.5 that no operator algebra contained in M_2 whose dimension is at least 3 can be antisymmetric.

One can ask, for which preorders $\preceq$ is the algebra $\mathcal{A}_{\preceq}$ antisymmetric? It seems natural to conjecture that in order for this to be the case, $\preceq$ would have to be a genuine partial order. However, this is not correct.

Example 4.11. Let $\mathbf{v} = \{v_1, v_2, v_3, v_4\}$ be a basis for $\mathbb{C}^4$ satisfying $\langle v_i, v_j \rangle \neq 0$ for all i and j, and assume the additional condition that no nonzero vector in $\mathcal{E} = \mathrm{span}\{v_1, v_2\}$ is orthogonal to every vector in $\mathcal{F} = \mathrm{span}\{v_3, v_4\}$, i.e., we have $\mathcal{E} \cap \mathcal{F}^\perp = \{0\}$. Define $\mathcal{A}$ to be $\mathcal{D}_\mathbf{v} + \mathrm{span}\{E_{12}, E_{21}\}$. This is the algebra $\mathcal{A}_{\preceq}$ from Theorem 4.8 for the preorder which sets $1 < 2$ and $2 < 1$, with no other comparability.

$\mathcal{A}$ is antisymmetric. To see this, suppose $A = \lambda_1 E_{11} + \lambda_2 E_{22} + \lambda_3 E_{33} + \lambda_4 E_{44} + \alpha E_{12} + \beta E_{21}$ is self-adjoint. As usual, the computation

$$\lambda_4 \|v_4\|^2 = \langle A v_4, v_4 \rangle = \langle v_4, A v_4 \rangle = \bar{\lambda}_4 \|v_4\|^2$$

implies that λ_4 is real and then

$$\lambda_3 \langle v_3, v_4 \rangle = \langle A v_3, v_4 \rangle = \langle v_3, A v_4 \rangle = \lambda_4 \langle v_3, v_4 \rangle$$

shows that $\lambda_3 = \lambda_4$. Let $B = A - \lambda_4 I$; we must show that $B = 0$.

But this is easy because B is self-adjoint and therefore $\mathrm{ran}(B) = \ker(B)^\perp$. The kernel of B contains $\mathcal{F}$, so this shows that its range is contained in $\mathcal{F}^\perp$. But it is also contained in $\mathcal{E}$. The hypothesis that $\mathcal{E} \cap \mathcal{F}^\perp = \{0\}$ then implies that $B = 0$.

At the same time, $\preceq$ can be a partial order without $\mathcal{A}_{\preceq}$ being antisymmetric. For instance, if $\preceq$ is a linear order then $\mathcal{A}_{\preceq}$ consists of the upper triangular operators for some ordering of the $\mathbf{v}$ basis, and it cannot be antisymmetric because its dimension is too large (Proposition 3.5).

5. Finite dimensional structure analysis

We are ready to discuss the general structure of hereditarily antisymmetric operator algebras in M_n. We start with the fact that they can always be upper triangularized. This will be an immediate corollary of the following theorem, which characterizes the operator algebras in finite dimensions which can be upper triangularized.

Recall that a subquotient of an operator algebra $\mathcal{A} \subseteq M_n$ is an operator algebra of the form $P\mathcal{A}P \subseteq B(\mathcal{E})$, where P is the orthogonal projection onto a semi-invariant subspace $\mathcal{E}$. Say that this subquotient is *full* if it equals $B(\mathcal{E})$.

Theorem 5.1. *Let $\mathcal{A} \subseteq M_n$ be an operator algebra. The following are equivalent:*

(i) *there is an ordered orthonormal basis of $\mathbb{C}^n$ with respect to which every operator in $\mathcal{A}$ has an upper triangular matrix;*

(ii) *there is an ordered vector space basis of $\mathbb{C}^n$ with respect to which every operator in $\mathcal{A}$ has an upper triangular matrix;*

(iii) *$\mathcal{A}$ has no full subquotients of dimension greater than* 1.

Proof. The equivalence of (i) and (ii) was Proposition 3.6. The proof of (iii) $\Rightarrow$ (i) goes by induction on n. It is trivial for $n = 1$. For $n > 1$, assume $\mathcal{A}$ has no full subquotients of dimension greater than 1. It must be properly contained in M_n as otherwise it would be a full subquotient of itself. Therefore Burnside's theorem yields that it has a nontrivial invariant subspace $\mathcal{E}$. Both $\mathcal{E}$ and $\mathcal{E}^{\perp}$ are then semi-invariant, and letting P and Q be the orthogonal projections onto these two subspaces, Proposition 2.2 yields that both $P\mathcal{A}P$ and $Q\mathcal{A}Q$ satisfy condition (iii), so inductively, we can find ordered orthonormal bases of $\mathcal{E}$ and $\mathcal{E}^{\perp}$ with respect to which every operator in $P\mathcal{A}P$ and $Q\mathcal{A}Q$ has an upper triangular matrix. The concatenation of these two bases is then an ordered orthonormal basis of $\mathbb{C}^n$ with respect to which every operator in $\mathcal{A}$ has an upper triangular matrix.

For (i) $\Rightarrow$ (iii), suppose there is an ordered orthonormal basis $\{f_1, \ldots, f_n\}$ of $\mathbb{C}^n$ with respect to which every operator in $\mathcal{A}$ has an upper triangular matrix. To reach a contradiction, assume that $\mathcal{A}$ has a full subquotient of dimension greater than 1. Consider first the case that the corresponding semi-invariant subspace $\mathcal{E}$ is actually invariant.

Let $v, w \in \mathcal{E}$ be linearly independent. Without loss of generality we can assume that there is an index $1 \leqslant k \leqslant n$ with the property that $\langle v, f_k \rangle \neq 0$ and $\langle v, f_{k'} \rangle = \langle w, f_{k'} \rangle = 0$ for all $k' > k$. Now since $\mathcal{E}$ is invariant and $P\mathcal{A}P$ is full, there are operators $A, B \in \mathcal{A}$ satisfying $Av = v$, $Aw = 0$ and $Bv = 0$, $Bw = v$. Since A is upper triangular for the (f_i) basis and v has no components past k, the condition $Av = v$ implies that the (k, k) entry of A is 1. Since w also has no nonzero components past k, $Aw = 0$ then implies that $\langle w, f_k \rangle = 0$. But now B being upper triangular implies that $\langle Bw, f_k \rangle = 0$, which makes $Bw = v$ impossible. This contradiction shows that the case where $\mathcal{E}$ is invariant cannot happen.

Now consider the general case where $\mathcal{E}$ is merely semi-invariant. Let $\mathcal{F} \subseteq \mathcal{E}^\perp$ be an invariant subspace such that $\mathcal{F} \oplus \mathcal{E}$ is also invariant and let Q be the orthogonal projection onto $\mathcal{F}^\perp$. Then the vectors $Qf_1, \ldots, Qf_n$ span $\mathcal{F}^\perp$, so by removing each Qf_i which is a linear combination of $Qf_1, \ldots, Qf_{i-1}$ we get an ordered basis (w_j) of $\mathcal{F}^\perp$. Now any $A \in M_n$ which is upper triangular for (f_i) and for which $\mathcal{F}$ is invariant will satisfy $A(Qf_i - f_i) \in \mathcal{F}$ since $Qf_i - f_i \in \mathcal{F}$, and hence $QA(Qf_i - f_i) = 0$. Thus

$$QAQ(Qf_i) = QA(Qf_i - f_i) + QAf_i = QAf_i \in \mathrm{span}\{Qf_1, \ldots, Qf_i\},$$

i.e., QAQ will be upper triangular for the (w_j) basis of $\mathcal{F}^\perp$. We have shown that there is an ordered vector space basis of $\mathcal{F}^\perp$ with respect to which every operator in QAQ has an upper triangular matrix, and (invoking Proposition 3.6) this reduces us to the first case because QAQ still has PAP as a subquotient and $\mathcal{E}$ is invariant for QAQ. This completes the proof. $\qquad\square$

Several characterizations of upper triangularizability of algebras of matrices can be found in [8], and the implication (iii) $\Rightarrow$ (i) in the preceding theorem can be inferred from [8, Lemma 1.1.4]. However, as far as I know this implication has not been explicitly stated anywhere, and the reverse implication (i) $\Rightarrow$ (iii) seems to be entirely new; compare Example 8.5, which shows that it fails in infinite dimensions. (Note that the 'quotients' of [8] are what I am calling 'subquotients'.)

Corollary 5.2. *Let $\mathcal{A} \subseteq M_n$ be hereditarily antisymmetric. Then there is an ordered orthonormal basis of $\mathbb{C}^n$ with respect to which every operator in $\mathcal{A}$ has an upper triangular matrix.*

Mere antisymmetry is not a sufficient hypothesis to ensure this conclusion. We already saw, in Example 4.11, an antisymmetric operator algebra in M_4 which had an invariant subspace $\mathcal{E}$ of dimension 2 such that the corresponding subobject was full. Thus, according to Theorem 5.1 this algebra cannot be made upper triangular.

Corollary 5.3. *The antisymmetric operator algebra of Example 4.11 cannot be upper triangularized.*

Let us also note that $\mathcal{A}$ being upper triangular does not prevent the existence of an orthogonal projection P onto a subspace $\mathcal{E}$ of dimension greater than 1 such that $PAP \cong B(\mathcal{E})$. For instance, according to [11, Proposition 2.2], the algebra $\mathcal{D}$ of diagonal matrices in M_{k^2+k-1} satisfies $P\mathcal{D}P \cong M_k$ for some orthogonal projection P whose range has dimension k. Thus, the obstruction to upper triangularizing is not the existence of a k-clique, in the terminology of [11], but the existence of a k-clique which lives on a semi-invariant subspace.

Our structure theorem for hereditarily antisymmetric operator algebras in M_n will state that every such algebra is contained in the algebra of all Jordanesque matrices relative to some basis of $\mathbb{C}^n$. Let us introduce a notation for this algebra.

Definition 5.4. Let a *block ordered basis* of $\mathbb{C}^n$ be an ordered basis $(v_1, \ldots, v_n)$ together with a choice of $n_1, \ldots, n_k > 0$ satisfying $n_1 + \cdots + n_k = n$. Given a block ordered

basis $\mathbf{v}$, define $\mathcal{J}_{\mathbf{v}}$ to be the set of all operators on $\mathbb{C}^n$ whose matrices for $\mathbf{v}$ are $(n_1, \ldots, n_k)$-Jordanesque.

It will often be convenient to indicate the blocks explicitly in the listing of basis vectors, i.e., as $(v_1^1, \ldots, v_{n_1}^1, \ldots, v_1^k, \ldots, v_{n_k}^k)$. As a matter of normalization, we can assume that for each $1 \leqslant j \leqslant k$, the set $\{v_1^j, \ldots, v_{n_j}^j\}$ is orthonormal. Applying Proposition 3.6 to the span of each of these sets shows that the set of all Jordanesque matrices for any given basis equals the set of all Jordanesque matrices for a basis which is normalized in this way.

Depending on the nature of the block ordered basis $\mathbf{v}$, the algebra $\mathcal{J}_{\mathbf{v}}$ may or may not be hereditarily antisymmetric. The invariant subspaces for $\mathcal{J}_{\mathbf{v}}$ are precisely the subspaces of the form $\mathrm{span}\{v_1^1, \ldots, v_{m_1}^1, \ldots, v_1^k, \ldots, v_{m_k}^k\}$, where $0 \leqslant m_j \leqslant n_j$ for each j. We allow $m_j = 0$ to accommodate the possibility that no vectors from the jth block appear. Call this subspace the $(m_1, \ldots, m_k)$ *subspace.* The condition we need to ensure hereditary antisymmetry is the following.

Definition 5.5. Say that a block ordered basis $\mathbf{v} = (v_1^1, \ldots, v_{n_1}^1, \ldots, v_1^k, \ldots, v_{n_k}^k)$ is *normalized* if for each j the set $\{v_1^j, \ldots, v_{n_j}^j\}$ is orthonormal. Say that it is *suitably nonorthogonal* if, for every $0 \leqslant m_1 \leqslant n_1, \ldots, 0 \leqslant m_k \leqslant n_k$ and every $1 \leqslant j < j' \leqslant k$ such that $m_j < n_j$ and $m_{j'} < n_{j'}$, we have

$$\langle Pv_{m_j+1}^j, v_{m_{j'}+1}^{j'} \rangle \neq 0, \qquad (*)$$

where P is the orthogonal projection onto the orthocomplement of the $(m_1, \ldots, m_k)$ subspace.

Thus, in order to be suitably nonorthogonal, the basis must be orthonormal in each block and a finite number of additional conditions $(*)$ must be satisfied. For instance, taking $m_1 = \cdots = m_k = 0$ in $(*)$ shows that we require $\langle v_1^j, v_1^{j'} \rangle \neq 0$ for all $1 \leqslant j < j' \leqslant k$. Letting exactly one m_j be nonzero, it is not too hard to see that $(*)$ reduces to the requirement that $\langle v_i^j, v_1^{j'} \rangle \neq 0$ for all $j \neq j'$ and $1 \leqslant i \leqslant n_j$ (since the $(m_1, \ldots, m_k)$ subspace in this case is just $\mathrm{span}\{v_1^j, \ldots, v_{i-1}^j\}$, to which v_i^j is already orthogonal). If more than one of the m_j are nonzero then the condition becomes more complicated, but it should be generically satisfied: the bases for which $Pv_{m_j+1}^j$ and $Pv_{m_{j'}+1}^{j'}$ are orthogonal are exceptional.

Say that an operator algebra $\mathcal{A} \subseteq \mathcal{J}_{\mathbf{v}}$ *distinguishes blocks* if for any $1 \leqslant j < j' \leqslant k$, there is an operator in $\mathcal{A}$ whose matrix has different values on the main diagonals of the j and j' blocks.

Theorem 5.6. *Let* $\mathbf{v} = \{v_1^1, \ldots, v_{n_1}^1, \ldots, v_1^k, \ldots, v_{n_k}^k\}$ *be a normalized block ordered basis of* $\mathbb{C}^n$. *If* $\mathbf{v}$ *is suitably nonorthogonal then* $\mathcal{J}_{\mathbf{v}}$ *is hereditarily antisymmetric. If* $\mathbf{v}$ *is not suitably nonorthogonal then* $\mathcal{J}_{\mathbf{v}}$ *is not hereditarily antisymmetric, nor is any subalgebra of* $\mathcal{J}_{\mathbf{v}}$ *which distinguishes blocks.*

Proof. Suppose $\mathbf{v}$ is suitably nonorthogonal. We must show that if $\mathcal{E}$ is semi-invariant for $\mathcal{J}_{\mathbf{v}}$ and $PAP \in B(\mathcal{E})$ is self-adjoint, where P is the orthogonal projection onto $\mathcal{E}$ and $A \in \mathcal{J}_{\mathbf{v}}$, then PAP is a scalar multiple of P.

Fix $\mathcal{E}$ and A such that PAP is self-adjoint. Now $\mathcal{E} = \mathcal{E}_1 \ominus \mathcal{E}_2$ where $\mathcal{E}_1$ is the $(s_1, \ldots, s_k)$ subspace and $\mathcal{E}_2$ is the $(r_1, \ldots, r_k)$ subspace, for some $0 \leqslant r_1 \leqslant s_1 \leqslant n_1, \ldots, 0 \leqslant r_k \leqslant s_k \leqslant n_k$. The basis vectors $v_{r_1+1}^1, \ldots, v_{s_1}^1, \ldots, v_{r_k+1}^k, \ldots, v_{s_k}^k$ need not belong to $\mathcal{E}$, but they span a companion subspace $\mathcal{F}$ and their orthogonal projections into $\mathcal{E}$ constitute a basis for $\mathcal{E}$. With respect to this basis PAP is Jordanesque. This follows from the fact that QAQ is Jordanesque, where Q is the natural projection onto $\mathcal{F}$, together with Proposition 2.6.

For each j, consider the orthogonal projection P_j onto $\mathrm{span}\{Pv_{r_j+1}^j, \ldots, Pv_{s_j}^j\}$. The matrix of $P_jAP_j = P_j(PAP)P_j$ for this basis is upper triangular and constant on the main diagonal, but it is also self-adjoint, and thus it follows from Propositions 3.6 and 4.2 that P_jAP_j is a scalar multiple of P_j.

Since this was true for all j, it follows that PAP is diagonal for the basis of $\mathcal{E}$ obtained by projecting the standard basis of $\mathcal{F}$ into $\mathcal{E}$. Say $P_jAP_j = \lambda_j P_j$, i.e., λ_j is the main diagonal entry of A on the jth block of $\mathbf{v}$. Fix distinct indices j, j' for which $r_j < s_j$ and $r_{j'} < s_{j'}$. Suitable nonorthogonality of $\mathbf{v}$ then implies that $Pv_{r_j+1}^j$ and $Pv_{r_{j'}+1}^{j'}$ are not orthogonal. Since these are eigenvectors belonging to the eigenvalues λ_j and $\lambda_{j'}$, the usual computation (as in the proof of Theorem 4.6, for example) then shows that self-adjointness of PAP implies $\lambda_j = \lambda_{j'}$. As j and j' were arbitrary (among the j's represented in $\mathcal{E}$), we conclude that PAP is a scalar multiple of P, as desired. This completes the proof of the first assertion of the theorem.

For the second assertion, suppose $\mathbf{v}$ is not suitably nonorthogonal; we must show that every subalgebra $\mathcal{A}$ of $\mathcal{J}_{\mathbf{v}}$ which distinguishes blocks is not hereditarily antisymmetric. (Obviously, this includes $\mathcal{J}_{\mathbf{v}}$ itself.) By Proposition 3.2, we can restrict attention to unital subalgebras. By Corollary 3.9, every such algebra contains the algebra $\mathcal{A}_{\mathrm{diag}}$ of operators whose matrix for the $\mathbf{v}$ basis is diagonal, with constant main diagonal entries on each block. So we must find such a matrix whose compression to some semi-invariant subspace (for $\mathcal{A}$) is nonscalar and self-adjoint.

Since $\mathbf{v}$ fails to be suitably nonorthogonal, there exist $m_1, \ldots, m_k$ and $1 \leqslant j < j' \leqslant k$ such that $Pv_{m_j+1}^j$ and $Pv_{m_{j'}+1}^{j'}$ are orthogonal, where P is the orthogonal projection onto the orthocomplement of the $(m_1, \ldots, m_k)$ subspace. Let $m'_l = m_l$ when $l \notin \{j, j'\}$, $m'_j = m_j + 1$, and $m'_{j'} = m_{j'} + 1$ and consider the orthogonal difference of the $(m'_1, \ldots, m'_k)$ subspace and the $(m_1, \ldots, m_k)$ subspace. These subspaces are invariant for $\mathcal{J}_{\mathbf{v}}$, and hence also for $\mathcal{A}$, so their orthogonal difference is semi-invariant for $\mathcal{A}$. It is spanned by the orthogonal vectors $Pv_{m_j+1}^j$ and $Pv_{m_{j'}+1}^{j'}$, and there is an operator in $\mathcal{A}_{\mathrm{diag}}$ whose j and j' eigenvalues are distinct and real, and thus whose compression to $\mathcal{E}$ is nonscalar and self-adjoint. We conclude that $\mathcal{A}$ is not hereditarily antisymmetric. $\square$

Theorem 5.6 tells us that if $\mathcal{J}_{\mathbf{v}}$ fails to be hereditarily antisymmetric then so does every subalgebra which distinguishes blocks. It does not tell us that if $\mathcal{J}_{\mathbf{v}}$ is hereditarily antisymmetric then the same is true of any subalgebra which distinguishes blocks. In fact, this can fail.

Example 5.7. Let $\mathbf{v} = \{v_1^1, v_2^1, v_1^2, v_2^2\}$ be a normalized suitably nonorthogonal basis of $\mathbb{C}^4$ satisfying $\langle v_2^1, v_2^2 \rangle = 0$. For bases of this size, suitable nonorthogonality amounts to the sets $\{v_1^1, v_2^1\}$ and $\{v_1^2, v_2^2\}$ both being orthonormal, v_1^1 not being orthogonal to either v_1^2 or v_2^2, v_1^2 not being orthogonal to either v_1^1 or v_2^1, and Pv_2^1 and Pv_2^2 not being orthogonal, where P is the orthogonal projection onto the orthocomplement of $\mathrm{span}\{v_1^1, v_1^2\}$. This is compatible with v_2^1 and v_2^2 being orthogonal.

According to Theorem 5.6, $\mathcal{J}_{\mathbf{v}}$ is hereditarily antisymmetric. However, the elements of $\mathcal{J}_{\mathbf{v}}$ which are diagonal for $\mathbf{v}$ has $\mathrm{span}\{v_2^1, v_2^2\}$ as an invariant subspace, and the elements of $\mathcal{J}_{\mathbf{v}}$ which are diagonal for $\mathbf{v}$ compress to all operators on this subspace which are diagonal for the orthonormal basis $\{v_2^1, v_2^2\}$. So these elements constitute a subalgebra of $\mathcal{J}_{\mathbf{v}}$ which distinguishes blocks but is not hereditarily antisymmetric.

Of course, if $\mathcal{J}_{\mathbf{v}}$ is hereditarily antisymmetric, then in particular it is antisymmetric, and hence so is any subalgebra. So every subalgebra is antisymmetric, but not necessarily hereditarily antisymmetric.

Before proving our structure theorem for hereditarily antisymmetric operator algebras, we need one more easy lemma.

Lemma 5.8. *Let $\mathcal{A} \subseteq \widetilde{M}_n$ be an operator algebra all of whose elements are upper triangular. Then there exists an operator $A \in \mathcal{A}$ all of whose main diagonal entries are real and with the property that $a_{ii} = a_{jj}$ implies $b_{ii} = b_{jj}$ for all $B \in \mathcal{A}$, where $A = (a_{ij})$ and $B = (b_{ij})$.*

Proof. The main diagonal entries of the matrices in $\mathcal{A}$ constitute a subalgebra of l_n^∞. By Lemma 3.3 and the comment following it, there is an equivalence relation on $\{1, \ldots, n\}$ such that this subalgebra consists of all the functions which are constant on each block (if $\mathcal{A}$ is unital) or all the functions which are constant on each block and vanish on some specified block (if it is not). In either case, we can find a function which takes a different real value on each block, and then take A to be a matrix in $\mathcal{A}$ whose main diagonal entries are this function. $\qquad \square$

Theorem 5.9. *Let $\mathcal{A} \subseteq M_n$ be a hereditarily antisymmetric operator algebra. Then there is a normalized suitably nonorthogonal block ordered basis $(v_1^1, \ldots, v_{n_1}^1, \ldots, v_1^k, \ldots, v_{n_k}^k)$ of $\mathbb{C}^n$ with respect to which the matrix of every element of $\mathcal{A}$ is Jordanesque.*

Proof. We will prove that there are numbers $n_1 + \cdots + n_k = n$ and a block ordered basis $(v_1^1, \ldots, v_{n_1}^1, \ldots, v_1^k, \ldots, v_{n_k}^k)$ with respect to which the matrix of every element of $\mathcal{A}$ is Jordanesque and such that $\mathcal{A}$ distinguishes blocks. After orthonormalizing each block, we get a basis which must be suitably nonorthogonal by the second part of Theorem 5.6.

We can assume that $\mathcal{A}$ is unital by Proposition 3.2. The proof goes by induction on n. First, according to Corollary 5.2 we can find an ordered orthonormal basis $\mathbf{w} = (w_1, \ldots, w_n)$ of $\mathbb{C}^n$ with respect to which the matrix of every operator in $\mathcal{A}$ is upper triangular. Then $\mathcal{E}_0 = \mathrm{span}\{w_1, \ldots, w_{n-1}\}$ is an invariant subspace, so we can inductively assume that $\mathcal{E}_0$ has a block ordered basis $\mathbf{v}_0$ with respect to which the compression of $\mathcal{A}$ to $\mathcal{E}_0$ distinguishes blocks and every element of which is Jordanesque.

Fix $A \in \mathcal{A}$ as in Lemma 5.8, relative to the $\mathbf{w}$ basis. Now the (n, n) entry of A for the $\mathbf{w}$ basis may equal at least one of its other main diagonal entries. If λ is this bottom right

entry of A, then this means that the entries of A in the main diagonal of exactly one of the blocks of the $\mathbf{v}_0$ basis are equal to λ. The other possibility is that λ is distinct from all other main diagonal entries (i.e., eigenvalues) of A.

In any case, since distinct blocks of $\mathbf{v}_0$ can be interchanged without consequence, we can assume that $\mathbf{v}_0 = (v_1^1, \ldots, v_{n_1}^1, \ldots, v_1^k, \ldots, v_{n_k-1}^k)$ where $n_1 + \cdots + n_k = n$ and the main diagonal entries of A in the final block are all λ. The case where λ is distinct from the other eigenvalues of A is accommodated by allowing the possibility that $n_k = 1$.

Next, let $\mathcal{F}$ be the span of $\mathbf{v}_1 = (v_1^1, \ldots, v_{n_1}^1, \ldots, v_1^{k-1}, \ldots, v_{n_{k-1}}^{k-1}, w_n)$ (omitting the kth block) and let P be the natural projection onto $\mathcal{F}$ with kernel $\text{span}\{v_1^k, \ldots, v_{n_k-1}^k\}$. Then $PAP \in B(\mathcal{F})$ has an upper triangular matrix for the $\mathbf{v}_1$ basis, and its bottom right entry is λ, and no other main diagonal entry takes this value. Thus λ is an eigenvalue of PAP. Let $v_{n_k}^k \in \mathcal{F}$ be an eigenvector for this eigenvalue, so that $PAv_{n_k}^k = \lambda v_{n_k}^k$. This implies that $Av_{n_k}^k \in \text{span}\{v_1^k, \ldots, v_{n_k}^k\}$, so the matrix of A is Jordanesque with respect to the basis $\mathbf{v} = (v_1^1, \ldots, v_{n_1}^1, \ldots, v_1^k, \ldots, v_{n_k}^k)$.

We must show that the matrix of every $B \in \mathcal{A}$ is Jordanesque with respect to the $\mathbf{v}$ basis. That is, the matrix of B with respect to this basis must be zero in all but the last n_k entries of its final column. (We already have, inductively, that the upper left $(n-1) \times (n-1)$ corner of this matrix is $(n_1, \ldots, n_{k-1})$-Jordanesque.)

Suppose this fails, and find $B \in \mathcal{A}$ whose final column $Bv_{n_k}^k$ has a nonzero v_i^j component for some $j < k$ but such that this and all other operators in $\mathcal{A}$ have zero components in the $v_{i+1}^j, \ldots, v_{n_j}^j, \ldots, v_1^{k-1}, \ldots, v_{n_{k-1}}^{k-1}$ entries. That is, v_i^j is the highest index where Jordanesqueness of some operator in $\mathcal{A}$ fails. Then $\mathcal{E}_1 = \text{span}\{v_1^1, \ldots, v_{n_1}^1, \ldots, v_1^j, \ldots, v_i^j, v_1^k, \ldots, v_{n_k}^k\}$ (i.e., $\mathbf{v}$ with all entries between v_i^j and v_1^k omitted) and $\mathcal{E}_2 = \text{span}\{v_1^1, \ldots, v_{n_1}^1, \ldots, v_1^j, \ldots, v_{i-1}^j, v_1^k, \ldots, v_{n_k-1}^k\}$ (i.e., the same list but also omitting v_i^j and $v_{n_k}^k$) are both invariant for $\mathcal{A}$, and so their orthogonal difference $\mathcal{E} = \mathcal{E}_1 \ominus \mathcal{E}_2$ is semi-invariant. This subspace is two-dimensional and $\text{span}\{v_i^j, v_{n_k}^k\}$ is a companion subspace. Let Q be the natural projection onto $\text{span}\{v_i^j, v_{n_k}^k\}$. Then QAQ is diagonal, with distinct real main diagonal entries, for the $\{v_i^j, v_{n_k}^k\}$ basis; QIQ is diagonal for the same basis with diagonal entries 1 and 1; and QBQ has a nonzero entry in the $(1, 2)$ corner. So QAQ has dimension at least 3, which by Proposition 2.6 means that the compression of $\mathcal{A}$ to $\mathcal{E}$ has dimension at least 3, and it is therefore not antisymmetric by Proposition 3.5. This contradicts hereditary antisymmetry of $\mathcal{A}$, and we conclude that the matrix of every operator in $\mathcal{A}$ for the $\mathbf{v}$ basis must be Jordanesque. $\qquad\square$

Thus, Corollary 3.9 applies to any hereditarily antisymmetric operator algebra in M_n.

The hypothesis of Theorem 5.1(iii) does not suffice to imply the conclusion that $\mathcal{A}$ can be made Jordanesque. For example, the set of all upper triangular matrices in M_n has no full subquotients of dimension greater than 1 (by Theorem 5.1, or by inspection), yet it cannot be put in Jordanesque form: its dimension is greater than the dimension of any $\mathcal{J}_\mathbf{v}$ in M_n.

We have a rather explicit characterization of the maximal hereditarily antisymmetric operator algebras.

Corollary 5.10. *The maximal hereditarily antisymmetric subalgebras of M_n are precisely the algebras $\mathcal{J}_\mathbf{v}$ for $\mathbf{v}$ a normalized suitably nonorthogonal block ordered basis of $\mathbb{C}^n$.*

This follows from Theorems 5.9 and 5.6: the former shows that every hereditarily antisymmetric algebra is contained in such an algebra and the latter shows that every such algebra is hereditarily antisymmetric.

6. 'Quantum' posets

I am proposing that unital hereditarily antisymmetric operator algebras may be regarded as 'quantum posets'. The qualifier 'quantum' is justified both by the direct physical interpretation discussed in §1 and on the grounds of kinship with other objects, such as quantum graphs and quantum metrics, which have similar physical interpretations [4, 7].

As I mentioned at the beginning of the paper, the nilpotent part of a hereditarily antisymmetric operator algebra (see Corollary 3.9 plus Theorem 5.9) plays the role of a strict order, i.e., it is the quantum version of $\prec$ rather than $\preceq$. Since every *nilpotent operator algebra* — every operator algebra consisting solely of nilpotent matrices — is hereditarily antisymmetric, I am also proposing that, in finite dimensions, nilpotent operator algebras are 'strict quantum orders'.

In order to give this proposal substance, we need some nontrivial results about operator algebras which are analogous to known results about posets. In this section I will prove operator algebraic analogs of the theorems of Mirsky (the maximal length of a chain equals the minimal size of a decomposition into antichains) and Dilworth (the maximal width of an antichain equals the minimal size of a decomposition into chains) for posets. However, the quantum Dilworth theorem only holds in the classically nontrivial direction.

First, we need to identify operator algebraic analogs of chains and antichains.

Definition 6.1. A *quantum antichain* for a nilpotent operator algebra $\mathcal{A} \subset M_n$ is a nonzero semi-invariant subspace $\mathcal{E} \subseteq \mathbb{C}^n$ whose corresponding subquotient $P\mathcal{A}P$ equals $\{0\}$. Its *width* is the dimension of $\mathcal{E}$. A *quantum chain* for $\mathcal{A}$ is a sequence of nonzero vectors $\mathcal{C} = (v_1, \ldots, v_k)$ in $\mathbb{C}^n$ with $v_{i+1} \in \mathcal{A}v_i$ for $1 \leqslant i < k$. Its *length* is k.

In the definition of quantum chains, we want $\mathcal{A}$ not to contain any nonzero projections, i.e., to be nilpotent, in order to express the idea that chains are strictly descending, not merely descending. Note that since $\mathcal{A}$ is an algebra, we actually get $v_j \in \mathcal{A}v_i$ whenever $i < j$.

It is convenient to know that in the definition of quantum antichains, semi-invariance of $\mathcal{E}$ is automatic.

Proposition 6.2. *Let $\mathcal{A} \subset M_n$ be a nilpotent operator algebra, let $\mathcal{E}$ be a subspace of $\mathbb{C}^n$, and let P be the orthogonal projection onto $\mathcal{E}$. If $P\mathcal{A}P = \{0\}$ then $\mathcal{E}$ is a quantum antichain for $\mathcal{A}$.*

Proof. This can be verified directly, but the quick way to see it is to invoke [1, Theorem 2.16], which states that $\mathcal{E}$ is semi-invariant if and only if the map $A \mapsto P A P$ is a homomorphism from $\mathcal{A}$ to $B(\mathcal{E})$. If $P\mathcal{A}P = \{0\}$ then this map must be the zero homomorphism. $\square$

Next, we need matrix versions of the idea of partitioning a poset into chains or antichains.

Definition 6.3. Let $\mathcal{A} \subset M_n$ be a nilpotent operator algebra.

(a) A *partition into quantum antichains* is an orthogonal decomposition $\mathbb{C}^n = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k$ where each $\mathcal{E}_i$ is a quantum antichain. Its *size* is k. It is *ordered* if $\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_i$ is invariant for each $1 \leqslant i \leqslant k$.

(b) A family of quantum chains $\mathcal{C}_1, \ldots, \mathcal{C}_k$ *spans* $\mathbb{C}^n$ if the span of $\bigcup_{i=1}^k \mathcal{C}_i$ equals $\mathbb{C}^n$. A *partition into quantum chains* is a family of quantum chains for which this union constitutes a basis for $\mathbb{C}^n$. Its *size* is k.

Intuitively, a partition into quantum antichains is ordered if $\mathcal{E}_i$ is 'below' $\mathcal{E}_j$ whenever $i < j$. Classically, given two disjoint antichains $\mathcal{C}_1$ and $\mathcal{C}_2$ in a poset, we can always perform a swap: let $\mathcal{C}_1'$ be the set of $x \in \mathcal{C}_1$ which lie above some element of $\mathcal{C}_2$, let $\mathcal{C}_2'$ be the set of $y \in \mathcal{C}_2$ which lie below some element of $\mathcal{C}_1$, and replace $\mathcal{C}_1$ and $\mathcal{C}_2$ with the antichains $(\mathcal{C}_1 \setminus \mathcal{C}_1') \cup \mathcal{C}_2'$ and $(\mathcal{C}_2 \setminus \mathcal{C}_2') \cup \mathcal{C}_1'$. Then no element of the first new antichain lies above any element of the second new antichain. Using this trick repeatedly, any partition into antichains can classically be converted into an ordered partition without changing its size. However, nothing like this is true in the quantum setting; compare Theorem 6.5 and Example 6.6.

Any family of quantum chains which spans $\mathbb{C}^n$ can be turned into a partition by removing selected elements. This follows from the fact that any subset of a quantum chain is a quantum chain — this is a consequence of the earlier comment that $v_j \in \mathcal{A} v_i$ whenever $i < j$ — so we can simply remove excess elements until there is no linear dependence. I record this fact in the following.

Proposition 6.4. *Let $\mathcal{A} \subset M_n$ be a nilpotent operator algebra and let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ be a family of quantum chains which spans $\mathbb{C}^n$. Then there is a partition into quantum chains $\mathcal{C}_1', \ldots, \mathcal{C}_{k'}'$ with $k' \leqslant k$ and each $\mathcal{C}_i'$ contained in some $\mathcal{C}_j$.*

(We might have $k' < k$ if some quantum chains disappear entirely in the pruning process.)

Now we can prove the 'quantum' Mirsky's theorem. Define $\mathcal{A}^i(\mathbb{C}^n)$ to be the span of $\{A_i \cdots A_1 v : A_1, \ldots, A_i \in \mathcal{A}, v \in \mathbb{C}^n\}$ and set $\mathcal{A}^0(\mathbb{C}^n) = \mathbb{C}^n$. If k is the smallest value for which $\mathcal{A}^k = \{0\}$, then the orthogonal differences

$$\mathcal{A}^{k-1}(\mathbb{C}^n) \oplus (\mathcal{A}^{k-2}(\mathbb{C}^n) \ominus \mathcal{A}^{k-1}(\mathbb{C}^n)) \oplus \cdots \oplus (\mathcal{A}^0(\mathbb{C}^n) \ominus \mathcal{A}^1(\mathbb{C}^n))$$

form an ordered partition into quantum antichains. I will call this the *top down partition*. (There is also a *bottom up* partition $\mathcal{E}_1 \oplus (\mathcal{E}_2 \ominus \mathcal{E}_1) \oplus \cdots \oplus (\mathcal{E}_k \ominus \mathcal{E}_{k-1})$, where $\mathcal{E}_1 = \{v \in \mathbb{C}^n : Av = 0 \text{ for all } A \in \mathcal{A}\}$ and inductively $\mathcal{E}_{i+1} = \{v \in \mathbb{C}^n : Av \in \mathcal{E}_i \text{ for all } A \in \mathcal{A}\}$. But we will not need this.)

Theorem 6.5. *Let $\mathcal{A} \subset M_n$ be a nilpotent operator algebra. Then the maximal length of a quantum chain equals the minimal size of an ordered partition into quantum antichains.*

Proof. Fix an ordered partition $\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k$ of $\mathbb{C}^n$ into quantum antichains, with k minimal. Given any quantum chain $(v_1, \ldots, v_l)$, we have $v_1 \in \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k = \mathbb{C}^n$, and inductively, since $v_{j+1} \in \mathcal{A} v_j$ for all j and $\mathcal{A}(\mathcal{E}_j) \subseteq \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_{j-1}$ for all j, we have $v_j \in \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_{k+1-j}$. This implies that the length of the quantum chain is at most k. We have shown that no quantum chain has length greater than k.

For the converse, let r be the smallest value for which $\mathcal{A}^r = \{0\}$ and let $\mathcal{A}^{r-1}(\mathbb{C}^n) \oplus (\mathcal{A}^{r-2}(\mathbb{C}^n) \ominus \mathcal{A}^{r-1}(\mathbb{C}^n)) \oplus \cdots \oplus (\mathcal{A}^0(\mathbb{C}^n) \ominus \mathcal{A}^1(\mathbb{C}^n))$ be the top down partition into quantum antichains. Its size is r. Since $\mathcal{A}^{r-1} \neq \{0\}$, there exists $v \in \mathbb{C}^n$ and $A_1, \ldots, A_{r-1} \in \mathcal{A}$ such that $A_{r-1} \cdots A_1 v \neq 0$. Thus $(v, A_1 v, \ldots, A_{r-1} \cdots A_1 v)$ is a quantum chain of length r. This shows that the maximal length of a quantum chain is at least as large as the minimal size of an ordered partition into quantum antichains. $\square$

Incidentally, this proof shows that the top down partition has minimal size among all ordered partitions. As essentially the same proof would work with the bottom up partition, it too has minimal size.

The proof of Theorem 6.5 corresponds to an easy proof of the classical theorem of Mirsky. Given any finite poset, let $\mathcal{C}_1$ be the set of maximal elements, let $\mathcal{C}_2$ be the set of new maximal elements after $\mathcal{C}_1$ is removed, and so on. This yields a partition into antichains such that every element of $\mathcal{C}_{i+1}$ lies under some element of $\mathcal{C}_i$, and one can then build a chain whose length is the size of this partition by starting with any element of the bottommost antichain and working up.

I mentioned earlier that in the quantum setting, arbitrary partitions into quantum antichains cannot necessarily be converted into ordered partitions. In fact, Theorem 6.5 fails for unordered partitions.

Example 6.6. Let $\mathcal{A}$ be the subalgebra of M_8 generated by the matrices $E_{31} + E_{61}$, $E_{23} + E_{73} - E_{26} + E_{46}$, and $E_{57} + E_{84} - E_{52} + E_{82}$. Thus, it is the linear span of these matrices together with the matrices $E_{41} + E_{71}$, $E_{51} + E_{81}$, and $E_{56} + E_{83}$. This is a nilpotent algebra, and $\mathcal{E}_1 = \mathrm{span}\{e_1, e_2\}$, $\mathcal{E}_2 = \mathrm{span}\{e_3, e_4, e_5\}$, and $\mathcal{E}_3 = \mathrm{span}\{e_6, e_7, e_8\}$ are all quantum antichains for $\mathcal{A}$. Thus $\mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \mathcal{E}_3$ is a partition of $\mathbb{C}^8$ into three quantum antichains. But there is a quantum chain of length 4, namely $(e_1, e_3 + e_6, e_4 + e_7, e_5 + e_8)$.

In some sense, the 'dual' result to Mirsky's theorem is Dilworth's theorem, which states that the minimal size of a partition into chains equals the maximal width of an antichain. Surprisingly, the quantum version of the trivial direction of this result fails.

Example 6.7. Let $\mathcal{A} = \mathrm{span}\{E_{14}, E_{24}, E_{34}\} \subset M_4$. This is a nilpotent operator algebra. It has a three-dimensional quantum antichain, namely $\mathrm{span}\{e_1, e_2, e_3\}$, but it also has a partition into the two quantum chains (e_4, e_1) and $(e_4 + e_3, e_2)$.

The harder direction of Dilworth's theorem does hold in the matrix setting, however. None of the usual proofs successfully transfers to the matrix setting, but there is a fairly easy linear algebra proof.

Theorem 6.8. *Let $\mathcal{A} \subset M_n$ be a nilpotent operator algebra. Then the minimal size of a partition of $\mathbb{C}^n$ into quantum chains is no larger than the maximal width of a quantum antichain.*

Proof. Let k be the largest value such that $\mathcal{A}^k$ is nonzero and let $\mathcal{E}_i = \mathcal{A}^i(\mathbb{C}^n) \ominus \mathcal{A}^{i+1}(\mathbb{C}^n)$ for $0 \leqslant i \leqslant k$, so that $\mathcal{E}_k \oplus \cdots \oplus \mathcal{E}_0$ is the top down partition. It will be convenient to have the indices descend in this way. Let d be the largest of the dimensions of the $\mathcal{E}_i$. We will find d quantum chains $(v_1, A_1^1 v_1, \ldots, A_k^1 \cdots A_1^1 v_1), \ldots, (v_d, A_1^d v_d, \ldots, A_k^d \cdots A_1^d v_d)$ which span $\mathbb{C}^n$. By Proposition 6.4, this is enough.

Let d_i be the dimension of $\mathcal{E}_i$, for $0 \leqslant i \leqslant k$. The goal will be to ensure that the vectors $v_1, \ldots, v_{d_0}$ orthogonally project to a basis of $\mathcal{E}_0$ and, for $1 \leqslant i \leqslant k$, the vectors $A_i^1 \cdots A_1^1 v_1, \ldots, A_i^{d_i} \cdots A_1^{d_i} v_{d_i}$ (the $(i+1)$st vectors in the first d_i quantum chains, which live in $\mathcal{E}_k \oplus \cdots \oplus \mathcal{E}_i$) orthogonally project to a basis of $\mathcal{E}_i$. This will suffice because it immediately implies that the terminal vectors in all the chains span $\mathcal{E}_k$, and then inductively that the last i vectors in all the chains span $\mathcal{E}_k \oplus \cdots \oplus \mathcal{E}_{k+1-i}$ for each i. Thus all the vectors in all the chains span $\mathbb{C}^n$.

The construction will be recursive, so that we choose the first i vectors in each chain before choosing any of the $(i+1)$st vectors. Those $(i+1)$st vectors will themselves be chosen sequentially. The key point is that if the vectors $A_i^1 \cdots A_1^1 v_1, \ldots, A_i^{d_i} \cdots A_1^{d_i} v_{d_i}$ orthogonally project to a basis of $\mathcal{E}_i$, then the same will be true of the vectors $\tilde{A}_i^1 \cdots \tilde{A}_1^1 \tilde{v}^1, \ldots, \tilde{A}_i^{d_i} \cdots \tilde{A}_1^{d_i} \tilde{v}^{d_i}$ for any $\tilde{A}_s^r$ sufficiently close to A_s^r and $\tilde{v}^r$ sufficiently close to v_r. This means that previous choices can be modified without affecting the fact that their projections in $\mathcal{E}_j$ span $\mathcal{E}_j$ for $j < i$, provided the modifications are sufficiently small.

We can start by letting $v_1, \ldots, v_{d_0}$ be a basis of $\mathcal{E}_0$ and setting $v_i = 0$ for $i > d_0$. Having chosen the first i vectors in each of the chains, we aim to choose the $(i+1)$st vectors sequentially, ensuring that for each $j \leqslant d_i$ the $(i+1)$st elements of the first j chains project to a linearly independent set in $\mathcal{E}_i$. When choosing the $(i+1)$st element of the jth chain, i.e., when choosing the operator A_i^j and possibly making small modifications to v_j and to $A_1^j, \ldots, A_{i-1}^j$, we just have to ensure that the projection of $A_i^j \cdots A_1^j v_j$ into $\mathcal{E}_i$ does not lie in a certain subspace, namely the span of the projections of the vectors $A_i^{j'} \cdots A_1^{j'} v_{j'}$ into $\mathcal{E}_i$ for $j' < j$. Call this span $\mathcal{F}$. To do this, find $w \in \mathbb{C}^n$ and $B_1, \ldots, B_i \in \mathcal{A}$ such that the projection of $B_i \cdots B_1 w$ into $\mathcal{E}_i$ does not lie in $\mathcal{F}$. This can be done because $\mathcal{A}^i(\mathbb{C}^n) = \mathcal{E}_k \oplus \cdots \oplus \mathcal{E}_i$. It will then suffice to show that we can find arbitrarily small values of t such that the projection of

$$t B_i (A_{i-1}^j + t B_{i-1}) \cdots (A_1^j + t B_1)(v_j + t w)$$

into $\mathcal{E}_i$ does not lie in $\mathcal{F}$. Then we can define A_i^j to be $t B_i$ and replace v_j with $v_j + t w$, A_1^j with $A_1^j + t B_1$, etc., and if t is small enough, this will not affect the spanning property at previous stages.

Now if the projection of $t B_i (A_{i-1}^j + t B_{i-1}) \cdots (A_1^j + t B_1)(v_j + t w)$ into $\mathcal{E}_i$ lies in $\mathcal{F}$ for all sufficiently small t, then all of its derivatives at $t = 0$ must lie in $\mathcal{F}$. But the $(i+1)$st derivative is $(i+1) B_i \cdots B_1 w$, which does not lie in $\mathcal{F}$, so this is impossible. Thus we are able to find arbitrarily small values of t which have the desired property. $\qquad\square$

7. Infinite dimensional examples

Now we turn to the infinite dimensional setting. In infinite dimensions it is natural to consider operator algebras which are weak* closed in $B(\mathcal{H})$. These are called *dual operator algebras*. In order to stay within this category, we must slightly modify the definitions of subobject, quotient, and subquotient used in finite dimensions.

Definition 7.1. Let $\mathcal{A} \subseteq B(\mathcal{H})$ be a dual operator algebra and let $P \in B(\mathcal{H})$ be the orthogonal projection onto a closed subspace $\mathcal{E} \subseteq \mathcal{H}$. Then $\overline{P\mathcal{A}P}^{wk*}$ is

  (i) a *subobject* of $\mathcal{A}$ if $\mathcal{E}$ is invariant for $\mathcal{A}$;

  (ii) a *quotient* of $\mathcal{A}$ if $\mathcal{E}$ is co-invariant for $\mathcal{A}$;

  (iii) a *subquotient* of $\mathcal{A}$ if $\mathcal{E}$ is semi-invariant for $\mathcal{A}$.

$\mathcal{A}$ is *hereditarily antisymmetric* if every subquotient of $\mathcal{A}$ is antisymmetric.

Of course, this definition reduces to Definition 2.1 in the finite dimensional setting, where weak* considerations become vacuous.

(The compression $P\mathcal{A}P$ is not automatically weak* closed. For example, let (x_n) be a dense sequence in the open unit disk $\mathbb{D}$ and define $x_0 = 0$ and $x_{-k} = \frac{1}{k}$ for all $k \in \mathbb{N}$. Then the set $\mathcal{A}$ of sequences (a_n) in $l^\infty(\mathbb{Z})$ with the property that $a_0 = 0$ and the map $x_n \mapsto a_n$ extends to a bounded analytic function on $\mathbb{D}$ is a weak* closed algebra — see the proof of Proposition 7.9 — which by the Schwarz lemma satisfies $|a_{-k}| \leqslant \frac{1}{k} \sup |a_n|$. The compression of $\mathcal{A}$ to $l^\infty(-\mathbb{N})$ is therefore contained in c_0 and contains all finite sequences, so it is not weak* closed. In this example $l^\infty(-\mathbb{N})$ is both invariant and co-invariant. The algebra $\mathcal{A}$ is not unital, but its unitization has the same property that its compression to $l^\infty(-\mathbb{N})$ is not weak* closed.)

The basic results from §2 go through without significant modification in infinite dimensions.

Proposition 7.2. *Let $\mathcal{A} \subseteq B(\mathcal{H})$ be a dual operator algebra. Then any subquotient of a subquotient of $\mathcal{A}$ is a subquotient of $\mathcal{A}$.*

This works because an operator algebra and its weak* closure have the same invariant subspaces, and hence the same semi-invariant subspaces, and because the compression of any weak* convergent net is weak* convergent.

Corollary 7.3. *Any subquotient of a hereditarily antisymmetric dual operator algebra is hereditarily antisymmetric.*

In infinite dimensions we generalize Definition 2.5 by taking a companion subspace $\mathcal{F}$ to be any topological complement of $\mathcal{E}_2$ in $\mathcal{E}_1$. Thus, it is a closed subspace of $\mathcal{E}_1$ satisfying $\mathcal{E}_2 + \mathcal{F} = \mathcal{E}_1$ and $\mathcal{E}_2 \cap \mathcal{F} = \{0\}$.

Proposition 7.4. *Let $\mathcal{E} = \mathcal{E}_1 \ominus \mathcal{E}_2$ be a semi-invariant subspace for a dual operator algebra $\mathcal{A} \subseteq B(\mathcal{H})$ and let $\mathcal{F}$ be a companion subspace of $\mathcal{E}$. Let P be the orthogonal projection onto $\mathcal{E}$, let $P_0 : \mathcal{F} \to \mathcal{E}$ be its restriction to $\mathcal{F}$, and let $Q \in M_n$ be the natural projection*

onto $\mathcal{F}$. Then $\Phi : T \mapsto P_0 T P_0^{-1}$ defines an isomorphism between $\overline{Q \mathcal{A} Q}^{wk*} \subseteq B(\mathcal{F})$ and $\overline{P \mathcal{A} P}^{wk*} \subseteq B(\mathcal{E})$.

The proof of this result requires the one additional observation that P_0 is invertible with bounded inverse by the Banach isomorphism theorem.

Propositions 3.1 and 3.2 were already stated for possibly infinite dimensional operator algebras. The infinite dimensional analog of Proposition 3.4 fails, however.

Example 7.5. Let $\mathcal{H} = l^2(\mathbb{Z})$ and let $\mathcal{A}$ be the set of operators in $\mathcal{B}(\mathcal{H})$ whose matrices relative to the standard basis of $l^2(\mathbb{Z})$ are upper triangular and constant on every diagonal. That is, the operators in $\mathcal{A}$ satisfy $\langle A e_j, e_i \rangle = 0$ when $i > j$ and $\langle A e_j, e_i \rangle = \langle A e_{j+1}, e_{i+1} \rangle$ for all $i, j \in \mathbb{Z}$.

This is a unital dual operator algebra, and it is antisymmetric because if $A \in \mathcal{A}$ is self-adjoint then $\langle A e_j, e_i \rangle = 0$ for all $i > j$ implies $\langle A e_j, e_i \rangle = 0$ for all $i < j$, i.e., A is diagonal and hence a scalar multiple of I. But $\mathcal{A}$ contains the bilateral shift operator $U : e_i \mapsto e_{i-1}$, which is normal and even unitary but not a scalar multiple of the identity. (Indeed, $\mathcal{A}$ is the unital dual operator algebra generated by U.)

Let us look at some infinite dimensional examples of hereditarily antisymmetric operator algebras. First we generalize the algebras $\mathcal{T}_n$ of Example 4.1 to infinite dimensions. There are a variety of ways to do this.

Example 7.6. Given a totally ordered set $(X, \preceq)$, define $\mathcal{T}_X \subset B(l^2(X))$ to be the set of operators whose matrix relative to the standard basis $\{e_x : x \in X\}$ is upper triangular and constant on the main diagonal. That is, $\langle A e_y, e_x \rangle = 0$ whenever $y \prec x$ and $\langle A e_x, e_x \rangle = \langle A e_y, e_y \rangle$ for all $x, y \in X$.

Proposition 7.7. *For any totally ordered set $(X, \preceq)$ the algebra $\mathcal{T}_X$ is unital, weak* closed, and hereditarily antisymmetric.*

Proof. Closure under weak* limits is seen by examining matrix entries. Hereditary antisymmetry follows from identifying the semi-invariant subspaces of $\mathcal{T}_X$ as those of the form $\text{span}\{e_x : x \in I\}$ where I is an interval in X. The compression of $\mathcal{T}_X$ to such a subspace would simply be $\mathcal{T}_I$, which is still antisymmetric (and already weak* closed). $\square$

The simplest cases are $X = \mathbb{Z}, \mathbb{N}$, and $-\mathbb{N}$ (the negative integers or, equivalently, $\mathbb{N}$ with the order reversed). $\mathcal{T}_{-\mathbb{N}}$ could equivalently be defined to be the set of bounded operators on $\mathbb{N}$ whose matrix for the standard basis is lower triangular and constant on the main diagonal. Using the opposite order on $\mathbb{N}$ effectively interchanges upper and lower triangular matrices. Note that $\mathcal{T}_{\mathbb{N}}$ and $\mathcal{T}_{-\mathbb{N}}$ are different: the former has a minimal invariant subspace which is the range of the operator E_{12}, while the latter has no minimal invariant subspace.

Proposition 7.8. *$\mathcal{T}_{\mathbb{Z}}, \mathcal{T}_{\mathbb{N}}$, and $\mathcal{T}_{-\mathbb{N}}$ are maximal hereditarily antisymmetric algebras.*

Proof. They are hereditarily antisymmetric by Proposition 7.7. For maximality, Let $\mathcal{A}$ be an operator algebra which properly contains one of these algebras. There are four

cases. First, if the matrix of some operator $A \in \mathcal{A}$ has a nonzero entry a_{ij} with $i \geqslant j + 2$, then $E_{i-1,i}, E_{j,i-1} \in \mathcal{A}$ and $E_{i-1,i} A E_{j,i-1}$ is a nonzero multiple of $E_{i-1,i-1}$. So $\mathcal{A}$ is not even antisymmetric. Second, if some $A \in \mathcal{A}$ has no nonzero matrix entries more than one diagonal below the main diagonal, but adjacent nonzero entries $a_{i+1,i}, a_{i,i-1}$ on the first subdiagonal, then A^2 has a nonzero $(i + 1, i - 1)$ entry, reducing to the first case. Third, if no operator in $\mathcal{A}$ has nonzero matrix entries more than one diagonal below the main diagonal, but some $A \in \mathcal{A}$ has a nonzero entry $a_{i+1,i}$, then both $a_{i+2,i+1}$ and $a_{i,i-1}$ must be zero (if they both exist; in the $\mathbb{N}$ or $-\mathbb{N}$ settings, one of these entries could be out of range). Moreover, those same entries must be zero for any $B \in \mathcal{A}$, as otherwise a linear combination of A and B would put us in the second case. It follows that $\mathrm{span}\{\ldots, e_{i-1}, e_i, e_{i+1}\}$ and $\mathrm{span}\{\ldots, e_{i-1}\}$ are both invariant, and so their orthogonal difference $\mathrm{span}\{e_i, e_{i+1}\}$ is semi-invariant. The compression of $\mathcal{A}$ to this two-dimensional subspace contains the identity matrix, the matrix $E_{i,i+1}$, and the compression of A, which is not upper triangular. So it is at least three-dimensional and therefore not antisymmetric by Proposition 3.5.

In the final case, every operator in $\mathcal{A}$ is upper triangular but $\mathcal{A}$ includes an operator A whose main diagonal entries are not constant. By subtracting a strictly upper triangular operator, we can assume that A is diagonal. Say $\langle Ae_i, e_i \rangle \neq \langle Ae_{i+1}, e_{i+1} \rangle$. Then $\mathrm{span}\{e_i, e_{i+1}\}$ is semi-invariant, and the compression of A to this subspace is diagonal but not a scalar multiple of the identity. So it is a nonscalar normal operator, and this shows that the compression of $\mathcal{A}$ is not antisymmetric by Proposition 3.4. We have shown that no operator algebra which properly contains $\mathcal{T}_{\mathbb{Z}}$, $\mathcal{T}_{\mathbb{N}}$, or $\mathcal{T}_{-\mathbb{N}}$ is hereditarily antisymmetric. □

It was easier to show that the algebras $\mathcal{T}_n$ are maximal hereditarily antisymmetric because these algebras were even maximal antisymmetric, which is an easier condition to check. However, that fact relied on Proposition 3.4, which no longer holds in infinite dimensions. In fact $\mathcal{T}_X$ is never maximal antisymmetric if X is infinite.

Proposition 7.9. *l^{∞} contains an infinite dimensional, weak* closed, unital, antisymmetric subalgebra.*

Proof. Let (x_n) be a dense sequence in the open unit disk $\mathbb{D}$. Define $\mathcal{A} \subset l^{\infty}$ to be the set of sequences (a_n) with the property that the map $x_n \mapsto a_n$ extends to a bounded analytic function on $\mathbb{D}$. This is clearly an infinite dimensional unital algebra, and it is antisymmetric because any analytic function which takes real values on a dense subset of $\mathbb{D}$ must be constant. For weak* closure, by the Krein–Smulian theorem it suffices to check closure under bounded pointwise convergence; since the predual of l^{∞} is separable, it suffices to consider bounded pointwise convergent sequences. If (f_n) is a sequence of analytic functions on $\mathbb{D}$ whose restrictions to the set $\{x_n\}$ are uniformly bounded and converge pointwise, then by continuity the f_n must be uniformly bounded, and Vitali's theorem (a consequence of Montel's theorem) then implies that this sequence converges uniformly on compact sets to a bounded analytic function on $\mathbb{D}$. So the pointwise limit of the restrictions to the set $\{x_n\}$ still belongs to $\mathcal{A}$. □

Proposition 7.10. *If $(X, \preceq)$ is any infinite totally ordered set, then $\mathcal{T}_X$ is properly contained in another weak* closed antisymmetric algebra.*

Proof. Fix a surjection $\phi : X \to \mathbb{N}$, let $\mathcal{A}$ be as in Proposition 7.9, and define $\mathcal{B} \subset B(l^2(X))$ to be the set of all operators whose matrix relative to the standard basis is upper triangular and whose main diagonal entries equal $f \circ \phi$ for some $f \in \mathcal{A}$. One straightforwardly checks that $\mathcal{B}$ is a weak* closed antisymmetric algebra. $\square$

If X is not discretely ordered, worse things can happen.

Proposition 7.11. *$\mathcal{T}_\mathbb{Q}$ is properly contained in another weak* closed hereditarily antisymmetric algebra.*

Proof. First, write $\mathbb{Q}$ as the disjoint union of a sequence of subsets X_n each of which is dense in $\mathbb{Q}$. (For instance, X_n could be the set of all rationals which, when written in lowest terms, have a denominator whose smallest prime factor is the nth prime, including $\mathbb{Z}$ in X_1, say.) Define $\phi : \mathbb{Q} \to \mathbb{N}$ by setting $\phi(x) = n$ when $x \in X_n$ and let $\mathcal{B}$ be as in the proof of Proposition 7.10, for this ϕ.

In this case, $\mathcal{B}$ is hereditarily antisymmetric because the semi-invariant subspaces are precisely the subspaces of the form $\mathrm{span}\{e_x : x \in I\}$ for some interval I in $\mathbb{Q}$, and the compression of $\mathcal{B}$ to any such subspace, if I contains more than a single point, consists of upper triangular operators whose diagonal entries take all the values of some function in $\mathcal{A}$. This uses the fact that every interval in $\mathbb{Q}$ of positive length contains points from every X_n. Since $\mathcal{A}$ is antisymmetric, the diagonal entries of any such compression, if nonconstant, cannot all be real, showing that the only self-adjoint operators in the compression of $\mathcal{B}$ are scalar multiples of the identity. We have shown that $\mathcal{B}$ is hereditarily antisymmetric. $\square$

We may also consider continuous analogs of $\mathcal{T}_X$.

Definition 7.12. For each $\epsilon > 0$, define $\mathcal{T}_\epsilon^0$ to be the set of operators $A \in B(L^2(\mathbb{R}))$ which satisfy $\langle Af, g \rangle = 0$ whenever f is supported on $(-\infty, a + \epsilon]$ and g is supported on $[a, \infty)$, for some $a \in \mathbb{R}$. Equivalently, A takes $L^2((-\infty, a + \epsilon]) \subset L^2(\mathbb{R})$ into $L^2((-\infty, a])$ for each a. This is a dual operator algebra. Let $\mathcal{T}_\epsilon$ be the unitization of $\mathcal{T}_\epsilon^0$.

In regard to the next result, note that any invariant subspace for the union of a chain $\{A_\lambda\}$ of operator algebras is invariant for each A_λ. So the same is true of semi-invariant subspaces, and this means that if each A_λ is hereditarily antisymmetric then $\bigcup A_\lambda$ cannot contain any operators which compress to a nonscalar self-adjoint operator on some semi-invariant subspace. However, its weak* closure might.

Proposition 7.13. *For each $\epsilon > 0$ the algebra $\mathcal{T}_\epsilon$ is hereditarily antisymmetric, but the weak* closure of the union $\bigcup_{\epsilon > 0} \mathcal{T}_\epsilon$ is not even antisymmetric.*

Proof. The nontrivial invariant subspaces for $\mathcal{T}_\epsilon^0$ (or $\mathcal{T}_\epsilon$) are precisely the subspaces of the form $L^2((-\infty, a] \cup X) \subset L^2(\mathbb{R})$, where $a \in \mathbb{R}$ and X is a measurable subset of $[a, a + \epsilon]$. It follows that every semi-invariant subspace has the form $L^2(Y)$ for some measurable $Y \subseteq \mathbb{R}$ (which is the difference of two sets of the preceding form, but we do not need this).

Let Y be any measurable subset of $\mathbb{R}$ and let P be the orthogonal projection of $L^2(\mathbb{R})$ onto $L^2(Y)$. Then for any $A \in \mathcal{T}_\epsilon^0$, the compression PAP satisfies $\langle PAPf, g \rangle = \langle Af, g \rangle = 0$ whenever f is supported on $(-\infty, a+\epsilon) \cap Y$ and g is supported on $[a, \infty) \cap Y$, for some a. Thus the same is true of any operator in $\overline{P\mathcal{T}_\epsilon^0 P}^{wk*}$. So any self-adjoint operator B in this set must satisfy $\langle Bf, g \rangle = \langle Bg, f \rangle = 0$ for all such f and g. In particular, if f is supported on $[a, a+\epsilon)$ then $\langle Bf, g \rangle = 0$ if g is either supported on $[a, \infty)$ or on $(-\infty, a+\epsilon)$, which implies that $Bf = 0$. This implies that $B = 0$. This shows that every subquotient of $\mathcal{T}_\epsilon^0$ is antisymmetric, i.e., $\mathcal{T}_\epsilon^0$ is hereditarily antisymmetric. Hereditary antisymmetry of $\mathcal{T}_\epsilon$ follows from Proposition 3.2.

For every $f \in L^\infty(\mathbb{R})$ and $r > 0$, the union $\bigcup_{\epsilon > 0} \mathcal{T}_\epsilon$ contains the operator $(Ag)(x) = f(x)g(x+r)$ which shifts everything in $L^2(\mathbb{R})$ left by r and then multiplies by f. As $r \to 0$, these operators converge weak* to the operator of multiplication by f. So every multiplication operator belongs to $\overline{\bigcup_{\epsilon > 0} \mathcal{T}_\epsilon}^{wk*}$, and thus this algebra is not antisymmetric. □

This example shows that the restriction to weak* closed algebras is important. We can have an algebra whose compression to any semi-invariant subspace contains no nonscalar self-adjoint operators, but whose weak* closure does not have this property (indeed, whose weak* closure itself contains nonscalar self-adjoint operators).

The phenomenon exhibited in Proposition 7.13 unfortunately limits our ability to reduce the analysis of arbitrary hereditarily antisymmetric dual operator algebras to the analysis of maximal hereditarily antisymmetric dual operator algebras.

There is also a natural generalization of Example 4.5 to infinite dimensions.

Example 7.14. Let $\mathcal{P} \subset B(\mathcal{H})$ be a maximal family of commuting (nonorthogonal) projections. Then its commutant

$$\mathcal{P}' = \{A \in B(H) : AP = PA \text{ for all } P \in \mathcal{P}\}$$

is a unital dual operator algebra.

The following special case is important enough to merit a mention.

Proposition 7.15. *Suppose (v_n) is a Schauder basis of the Hilbert space $\mathcal{H}$. Then the projections P_n satisfying $P_n v_n = v_n$ and $P_n v_k = 0$ for $k \neq n$ are uniformly bounded and commute. There is exactly one maximal family of commuting projections which contains the set $\{P_n\}$, and it consists of precisely those operators P_X, for some $X \subseteq \mathbb{N}$, which are bounded and satisfy $P_X v_n = v_n$ if $n \in X$ and $P_X v_n = 0$ if $n \notin X$.*

The first assertion is a standard fact about Schauder bases, and the second is just the easy observation that any projection that commutes with every P_n must have the stated form.

I will call the maximal family of commuting projections identified in Proposition 7.15 *the family of projections associated with the Schauder basis (v_n).*

Proposition 7.16. *Let $\mathcal{P} \subset B(\mathcal{H})$ be a maximal family of commuting projections. Then $\mathcal{P}'$ is antisymmetric if and only if $\mathcal{P}$ contains no orthogonal projections besides 0 and I.*

Proof. By maximality, there are no projections in $\mathcal{P}'$ other than those in $\mathcal{P}$. The result therefore follows from Proposition 3.1. $\qquad\square$

In order to characterize hereditary antisymmetry, we need a slightly stronger hypothesis.

Theorem 7.17. *Let $\mathcal{P} \subset B(\mathcal{H})$ be a maximal family of commuting projections which is uniformly bounded. Then $\mathcal{P}'$ is hereditarily antisymmetric if and only if whenever $P, Q, R \in \mathcal{P}$ satisfy $PQ = PR = QR = 0$ but $P, Q \neq 0$, the orthogonal projections of $\mathrm{ran}(P)$ and $\mathrm{ran}(Q)$ into $\mathrm{ran}(R)^{\perp}$ are not mutually orthogonal.*

Proof. Suppose $P, Q, R \in \mathcal{P}$ have the stated properties. Let $\tilde{R}$ be the orthogonal projection onto $\mathrm{ran}(R)^{\perp}$. Since $\mathrm{ran}(R)^{\perp}$ is co-invariant and therefore semi-invariant, the map $A \mapsto \tilde{R}A\tilde{R}$ is a homomorphism, and thus $\tilde{P} = \tilde{R}P\tilde{R}$ and $\tilde{Q} = \tilde{R}Q\tilde{R}$ are commuting projections. This shows that $\tilde{R}\mathcal{P}'\tilde{R}$ contains a pair of projections whose product is zero and whose ranges are orthogonal, and from this we get that the range of $\tilde{P} + \tilde{Q}$ is semi-invariant (cf. Proposition 2.2) and the compressions of P and Q to this subspace are nonzero orthogonal projections which sum to the identity. So $\mathcal{P}'$ has a subquotient which contains a nonscalar orthogonal projection, i.e., it is not hereditarily antisymmetric.

For the converse, suppose $\mathcal{P}'$ is not hereditarily antisymmetric. The set $\mathcal{U} = \{2P - I : P \in \mathcal{P}\}$ is an abelian group under operator product, so, using the uniform boundedness hypothesis, a theorem of Day and Dixmier [2, 3] implies that it can be conjugated to a family of unitaries. That is, there exists an invertible $S \in B(\mathcal{H})$ such that $S^{-1}(2P - I)S = 2S^{-1}PS - I$ is unitary for all $P \in \mathcal{P}$. But the spectrum of each of these operators is contained in $\{1, -1\}$, so each of these unitaries is self-adjoint and has the form $2\tilde{P} - I$ for some orthogonal projection $\tilde{P} = S^{-1}PS$. Thus the set $S^{-1}\mathcal{P}S$ is a maximal commuting family of orthogonal projections, i.e., it is the set of projections in the maximal abelian von Neumann algebra $(S^{-1}\mathcal{P}S)'$.

Now $(S^{-1}\mathcal{P}S)' = S^{-1}\mathcal{P}'S$, so the invariant subspaces for $\mathcal{P}'$ are precisely the subspaces of the form $S(\mathcal{E})$ where $\mathcal{E}$ is invariant for $(S^{-1}\mathcal{P}S)'$. But the invariant subspaces for the latter are just the ranges of the projections in $S^{-1}\mathcal{P}S$. Thus we have shown that the invariant subspaces for $\mathcal{P}$ are precisely the ranges of the projections in $\mathcal{P}$. The semi-invariant subspaces are therefore the orthogonal differences between ranges of projections in $\mathcal{P}$.

Since $\mathcal{P}'$ is not hereditarily antisymmetric, there is a semi-invariant subspace $\mathcal{E}$ such that the compression of $\mathcal{P}'$ to $\mathcal{E}$ is not antisymmetric. Say $\mathcal{E} = \mathrm{ran}(R_1) \ominus \mathrm{ran}(R_2)$ for some projections $R_1, R_2 \in \mathcal{P}$ with $\mathrm{ran}(R_2) \subset \mathrm{ran}(R_1)$.

Now $R_0 = R_1 - R_2$ is the natural projection onto a companion subspace of $\mathcal{E}$, and $R_0\mathcal{P}'R_0 = \mathcal{P}'R_0$ is weak* closed, so by Proposition 7.4, so is the compression of $\mathcal{P}'$ to $\mathcal{E}$.

Since this compression is not antisymmetric, it therefore contains nonscalar orthogonal projections P and Q whose sum is the orthogonal projection onto $\mathcal{E}$. These correspond via Proposition 7.4 to projections $\tilde{P}, \tilde{Q} \in \mathcal{P}R_0 \subseteq \mathcal{P}$ whose ranges orthogonally project onto orthogonal subspaces of $\mathrm{ran}(R_2)^{\perp}$. $\qquad\square$

If $\mathcal{P}$ is the family of projections associated with some Schauder basis, as in Proposition 7.15, and the basis is actually Riesz, then we are in the setting of

Theorem 7.17. In this case, at least, the operator algebra $\mathcal{P}'$ is maximal hereditarily antisymmetric.

Theorem 7.18. *Suppose $\mathcal{P}$ is the family of projections associated with a Riesz basis (v_n). If $\mathcal{P}'$ is hereditarily antisymmetric then it is maximal hereditarily antisymmetric.*

Proof. Let $\mathcal{A}$ be a dual operator algebra which properly contains $\mathcal{P}'$. Then there exists $k \in \mathbb{N}$ such that v_k is not an eigenvector for some operator in $\mathcal{A}$. Let $\{P_n\}$ be the projections from Proposition 7.15 and let X be the set of $n \in \mathbb{N}$ such that $P_n(Av_k) \neq 0$ for some $A \in \mathcal{A}$. Observe that for any such n we have $0 \neq P_n A P_k \in \mathcal{A}$, and hence $E_{nk} \in \mathcal{A}$, where as before E_{ij} is the operator which takes v_j to v_i and annihilates all other $v_{j'}$.

Now $\mathcal{E} = \mathrm{span}\{v_n : n \in X\}$ is an invariant subspace for $\mathcal{A}$ which contains v_k and at least one other v_n. Working in $\mathcal{E}$, find a nonzero vector v which is orthogonal to $\mathrm{span}\{v_n : n \in X \setminus \{k\}\}$ and write $v = \sum_{n \in X} a_n v_n$. Then consider the operator $A = \sum_{n \in X} a_n E_{nk}$; considered as an operator in $B(\mathcal{E})$, this is a nonzero scalar multiple of the orthogonal projection onto $\mathrm{span}\{v\}$, and it belongs to $\mathcal{A}$ because each E_{nk} belongs to $\mathcal{A}$ and the partial sums are uniformly bounded. This shows that $\mathcal{A}$ is not hereditarily antisymmetric. $\qquad\square$

8. Infinite dimensional structure analysis

The transitive algebra problem asks whether any dual operator algebra that is properly contained in $B(\mathcal{H})$ must have a nontrivial invariant subspace. Without knowing this to be the case, there is little we can say about the structure of such algebras. However, assuming the problem has a positive answer, we easily get an infinite dimensional analog of Theorem 5.1. As in the finite dimensional case, say that a subquotient of a dual operator algebra corresponding to a semi-invariant subspace $\mathcal{E}$ is *full* if it equals $B(\mathcal{E})$. We also need an infinite dimensional version of upper triangularity.

Definition 8.1. A *nest* in a Hilbert space $\mathcal{H}$ is a chain of closed subspaces, i.e., a family of closed subspaces which is totally ordered by inclusion. It is *maximal* if it is not properly contained in any other nest. An algebra $\mathcal{A} \subseteq B(\mathcal{H})$ is *upper triangular* for a nest if each subspace in the nest is invariant for $\mathcal{A}$.

Note that in finite dimensions a maximal nest simply looks like a nested sequence of subspaces, one of each possible dimension, and being upper triangular with respect to a maximal nest is the same as being upper triangular with respect to some orthonormal basis. In infinite dimensions, a nest is maximal if and only if it contains $\{0\}$ and $\mathcal{H}$, it is complete (closed under arbitrary joins and meets), and whenever $\mathcal{E}_1$ and $\mathcal{E}_2$ are distinct subspaces in the nest, with $\mathcal{E}_2 \subset \mathcal{E}_1$ and no other subspace in the nest intermediate between them, then $\mathcal{E}_1$ has codimension 1 in $\mathcal{E}_2$.

Theorem 8.2. *Let $\mathcal{A} \subseteq B(\mathcal{H})$ be a dual operator algebra with no full subquotients of dimension greater than 1. If the transitive algebra problem has a positive solution, then there is a maximal nest in $\mathcal{H}$ with respect to which $\mathcal{A}$ is upper triangular.*

Proof. Use Zorn's lemma to find a maximal chain of invariant subspaces. We must show that it is a maximal nest. It clearly contains $\{0\}$ and $\mathcal{H}$ and is complete. Thus let $\mathcal{E}_1$

and $\mathcal{E}_2$ be distinct subspaces in the chain and suppose $\mathcal{E}_2 \subset \mathcal{E}_1$ and there is no strictly intermediate subspace between them in the chain. Then $\mathcal{E}_1 \ominus \mathcal{E}_2$ is semi-invariant, and if its dimension were greater than 1 then by hypothesis the corresponding subquotient of $\mathcal{A}$ could not be full. A positive solution to the transitive algebra problem would then imply the existence of a proper closed subspace $\mathcal{F}$ of $\mathcal{E}_1 \ominus \mathcal{E}_2$ which is invariant for the compression of $\mathcal{A}$. But then $\mathcal{E}_2 \oplus \mathcal{F}$ would be an invariant subspace strictly intermediate between $\mathcal{E}_1$ and $\mathcal{E}_2$, contradicting maximality of the chain. We conclude that there is a maximal nest which consists only of invariant subspaces for $\mathcal{A}$, i.e., with respect to which $\mathcal{A}$ is upper triangular. $\qquad\square$

Like the implication (iii) $\Rightarrow$ (i) of Theorem 5.1, this theorem can be inferred from standard results; see [8, Lemma 7.1.11]. But its explicit statement is perhaps new.

Corollary 8.3. *Let $\mathcal{A} \subseteq B(\mathcal{H})$ be a hereditarily antisymmetric dual operator algebra. If the transitive algebra problem has a positive solution, then there is a maximal nest in $\mathcal{H}$ with respect to which $\mathcal{A}$ is upper triangular.*

The same technique can be applied to a single bounded operator. In this case the hypothesis we need is a positive solution to the invariant subspace problem. The following theorem is proven in the same way as Theorem 8.2 and hence also follows easily from ideas in [8]. Experts would surely consider it to be 'known', but I have not seen it explicitly written anywhere.

Theorem 8.4. *Let A be a bounded operator on an infinite dimensional Hilbert space. Assume that the invariant subspace problem for Hilbert space operators has a positive solution. Then there is a maximal nest in $\mathcal{H}$ with respect to which A is upper triangular.*

It seems, worthwhile to state this theorem explicitly, both to make the result available to nonexperts, and because it yields a reduction in the negative direction: in order to answer the invariant subspace problem negatively, we do not need to find a bounded operator for which every nonzero vector is cyclic, we only need to find a bounded operator which cannot be made upper triangular.

In contrast to the finite dimensional case (Theorem 5.1), the converse direction in Theorem 8.2 is false.

Example 8.5. Working on $l^2(\mathbb{N})$, let $A_1 = \mathrm{diag}(1, 0, 1, 0, \ldots)$, $A_2 = \mathrm{diag}(0, 1, 0, 1, \ldots)$, $A_3 = U \cdot \mathrm{diag}(2, 0, 2, 0, \ldots)$, and $A_4 = U \cdot \mathrm{diag}(0, 1, 0, 1, \ldots)$, where U is the backward shift operator. Consider the vectors

$$v = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{4} \\ 0 \\ \vdots \end{bmatrix} \qquad w = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{4} \\ \vdots \end{bmatrix}.$$

Then $A_1 v = v$, $A_1 w = 0$, $A_2 v = 0$, $A_2 w = w$, $A_3 v = w$, $A_3 w = 0$, $A_4 v = 0$, and $A_4 w = v$. Thus the algebra generated by the A_i has $\mathcal{E} = \text{span}\{v, w\}$ as an invariant subspace, and its compression to $\mathcal{E}$ equals $B(\mathcal{E}) \cong M_2$. So it has a full subquotient of dimension greater than 1, yet it is evidently upper triangular for the standard orthonormal basis of $l^2(\mathbb{N})$.

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