LIFE INSURANCE AND PENSION CONTRACTS II: THE LIFE CYCLE MODEL WITH RECURSIVE UTILITY

BY

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ABSTRACT

We analyze optimal consumption and pension insurance during the life time of a consumer using the life cycle model, when the consumer has recursive utility. The relationship between substitution of consumption and risk aversion is highlighted, and clarified by the introduction of this type of preferences. We illustrate how recursive utility can be used to explain the empirical consumption puzzle for aggregates. This indicates a plausible choice for the parameters of the utility function, relevant for the consumer in the life cycle model. Optimal life insurance is considered, as well as the portfolio choice problem related to optimal exposures in risky securities. A major finding is that it is optimal for the typical insurance buyer to smooth adverse shocks to the financial market, unlike what is implied by the conventional model. This has implications for what type of contracts the life and pension insurance industry should offer.

KEYWORDS

Optimal pension insurance, optimal life insurance, recursive utility, defined benefit, defined contribution, consumption puzzle, investment problem.

1. INTRODUCTION

For the standard life cycle model with additive and separable utility, risk aversion and intertemporal elasticity of substitution in consumption sometimes play conflicting roles in life and pension insurance. We propose to look at a wider class of utility functions, namely recursive utility, to sort out some of these problems. Here, we obtain several new insights of relevance for life and pension insurance, thereby extending the early works of Yaari (1965), Hakansson (1969), Samuelson (1969), Merton (1969, 1971), Fisher (1973) and Cox and Huang (1989) among others.

The present paper is a companion paper to Aase (2015), extending the analysis to recursive utility. In the present paper, we focus on issues where the

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conventional model has its most obvious weaknesses, and we illustrate throughout by using market data.

The paper based on the conventional model demonstrates that when there is consumption in several periods in a world with a perfect credit market with no financial risk, this model seems to work well. This is also what the standard literature takes for granted. In a one period problem with financial uncertainty and consumption only on the last time point, the so-called *timeless* situation, the standard model is still known to give reasonable results. A pure life insurance contract may, perhaps, be considered in this category. When there is consumption in several periods (at least two) and there is also financial risk, we have a so-called *temporal* problem (see e.g., Mossin (1969) and Kreps (1988)). In such situations induced preferences violate the substitution axiom, so the von Neumann–Morgenstern expected utility (Eu) theory does not have axiomatic underpinnings. As it turns out, for such problems the conventional model does not fit historical data all that well when aggregated over agents. Our alternative model gives a better description of aggregated data.

With a typical customer of this kind in the life cycle model, we find the optimal consumption of a consumer, the optimal pension of a pension holder and the optimal amount of life insurance, all in closed form solutions, and discuss how these differ from the corresponding quantities in the expected utility model. A main finding is that a typical recursive utility customer finds it optimal to smooth consumption shocks stemming from adverse shocks to the financial market. The associated optimal investment strategy explains aggregate data, where the myopic counterpart fails. Our analysis reveals how the *mutuality principle* in a dynamic framework, which hinges on expected utility, is no longer valid with recursive utility.

As is well known (see e.g., Epstein and Zin (1989), Duffie and Epstein (1992a, b), Duffie and Skiadas (1994), Weil (1989), Kreps and Porteus (1978)), a major advantage with recursive utility is that it disentangles intertemporal substitution from risk aversion. In the context of the life cycle model, this specification of preferences tells us how and where these different properties of an individual consumer, or insurance customer, influence the optimal consumption.

A major, theoretical paper analyzing optimal consumption and portfolio selection is Schroder and Skiadas (1999), in which they discuss a variety of specifications, while we focus on the Kreps–Porteus specification only. From a formal point of view, we add to the theory demonstrating how recursive utility can alternatively be analyzed by employing the stochastic maximum principle, which works well for the version of recursive utility which gives the most unambiguous disentangling of risk aversion from consumption substitution. Second, we focus on the smoothing property of optimal consumption, and pension, and the implications for the insurance industry. Third, we illustrate numerically by using market data.

The paper is organized as follows: In Section 2 we specify the financial market, which the agent takes as given in the life cycle model. Here we introduce the uncertainty in our model, the wealth dynamics of an agent, and other variables and concepts that we use later.

In Section 3, we give a brief introduction to recursive utility along the lines of Duffie and Epstein (1992a, b). In Section 4, we formulate the first order conditions for optimal consumption using the stochastic maximum principle. In Section 4.1, we present the complete solution to the problem of finding the optimal consumption path for an agent with recursive utility, in closed form.

In Section 5, we briefly discuss the classical consumption puzzles in light of the findings in the previous section, and look at some empirical regularities. In Section 6 we take a small "detour", where we explain how to obtain the equity premium and the equilibrium interest rate using our approach.

In Section 7 we return to the life cycle model, and find the optimal pension for a typical insurance customer, in closed form. In Section 8 we find the optimal life insurance contract with recursive utility, and compare to the corresponding results of the conventional model. In Section 9 we present the solution to the optimal portfolio choice problem, and Section 10 summarizes. The Appendix contains some of the more technical proofs/material.

2. The financial market

We consider a consumer/insurance customer who has access to a securities market, as well as a credit market and pension and life insurance contracts. The securities market can be described by the vector $\nu_t = \mu_t - r_t \cdot 1_N$ of expected returns of N risky securities in excess of the risk-less instantaneous return r_t, 1_N is an N-vector of 1's. σ_t is an $N \times N$ matrix of diffusion coefficients of the risky asset prices, normalized by the asset process, so that $\sigma_t \sigma_t'$ is the instantaneous covariance matrix for asset returns. Both ν_t and σ_t are assumed to adapted stochastic processes. Here, N is the dimension of an underlying Brownian motion B as well.

We assume that the cumulative return process R_t^n is an ergodic process for each n, $n = 1, 2, \ldots, N$, where $dX_t^n = X_t^n dR_t^n$, and X_t^n is the cum dividend price process of the nth risky asset.

Underlying is a probability space $(\Omega, \mathcal{F}, P)$ and an increasing information filtration $\mathcal{F}_t$ generated by the d-dimensional Brownian motion, and satisfying the "usual" conditions. Each price process X_t^n is a continuous stochastic process, and we suppose that $\sigma^{(0)} = 0$, so that $r_t := \mu_0(t)$ is the risk-free interest rate, the return on a zeroth asset, also a stochastic process. T is the finite horizon of the economy. Our insurance customer lives a random time T_x from age x on, the time where an insurance contract has been issued. The support of T_x is $[0, \tau]$ where $\tau < T$. The state price deflator, or stochastic discount factor, $\pi(t)$ is given by

$$\pi_t = \xi_t e^{-\int_0^t r_s\, ds}, \tag{1}$$

where the "density" process ξ has the representation

$$\xi_t = \exp\Big(- \int_0^t \eta_s' \cdot dB_s - \frac{1}{2} \int_0^t \eta_s' \cdot \eta_s ds \Big). \tag{2}$$

Here $\eta(t)$ is the market-price-of-risk for the discounted price process $X_t e^{-\int_0^t r_s ds}$, defined by

$$\sigma(\omega, t)\eta(\omega, t) = \nu(\omega, t), \quad (\omega, t) \in \Omega \times [0, T], \tag{3}$$

where the nth component of ν_t equals $(\mu_n(t) - r_t)$, the excess rate of return on security n, $n = 1, 2, \ldots, N$. From Ito's lemma, it follows from (2) that

$$d\xi_t = -\xi_t \; \eta_t' \cdot dB_t, \tag{4}$$

and from (1) it follows that

$$d\pi_t = -r_t \pi_t dt - \pi_t \eta_t' dB_t, \tag{5}$$

gives the dynamics of the state price. We assume that Novikov's condition is satisfied, so that the density ξ_t is a martingale.

Abstracting from life insurance for the moment, the agent is represented by an endowment process e (income) and a utility function $U : L_+ \to R$, where

$$L = \left\{ c : c_t \text{ is } \mathcal{F}_t\text{-measurable, and } E\left(\int_0^{T_x} c_t^2 \, dt \right) < \infty \right\}.$$

L_+, the positive cone of L, is the set of consumption rate processes.

The form of the function U is specified in the section on recursive utility coming next. The remaining life time T_x of the x year old insurance customer is assumed independent of the risky securities X. The information filtration $\mathcal{F}_t$ is enlarged to account for events like $T_x > t$.

In general is utility U defined over the space C of consumption pairs (c, z), where c is an adapted nonnegative consumption-rate process with $\int_0^{T_x} c_t dt < \infty$ almost surely and z is an $\mathcal{F}_{T_x}$-measurable nonnegative random variable describing terminal consumption, later to be interpreted as the amount of life insurance.

A trading strategy $\theta = (\theta^0, \ldots, \theta^N)$ signifies the number of shares of the different N+1 given assets. It is supposed to satisfy some regularity conditions that we do not need to specify here. Following Duffie (2001), Ch 9, p. 206, given initial wealth $w_0 > 0$, we say that (c, z, θ) is budget-feasible, denoted $(c, z, \theta) \in \Lambda(w_0)$, if (c, z) is a consumption choice in C and θ is an admissible trading strategy satisfying

$$\theta_t \cdot X_t = w_0 + \int_0^t \theta_s dX_s - \int_0^t c_s \, ds \geq 0, \quad t \in [0, T_x], \tag{6}$$

and

$$\theta_{T_x} \cdot X_{T_x} \geq z. \tag{7}$$

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The first restriction (6) is that the current market value $\theta_t \cdot X_t$ of the trading strategy is nonnegative and equal to its initial value w_0, plus any gains from security trade, less the cumulative consumption to date. The second restriction (7) is that the terminal portfolio value is sufficient to cover the terminal consumption. We now have the problem, for each initial wealth w_0

$$\sup_{(c,z,\theta)\in\Lambda(w_0)} U(c,z). \tag{8}$$

The type of situation described above is known as a *temporal* problem of choice. In such a situation, it is far from clear that the time additive and separable form of utility is the natural representation of preferences (an early reference is Mossin (1969). See also Kreps (1988)).

It is convenient to represent trading strategies in terms of fractions $\varphi_t' = (\varphi_t^{(1)}, \varphi_t^{(2)}, \dots, \varphi_t^{(N)})$ of total wealth held in the risky securities. That is, for a given trading strategy θ, we let

$$\varphi_t^n := \frac{\theta_t^n X_t^n}{\theta_t \cdot X_t}, \quad \theta_t \cdot X_t \neq 0,$$

with $\varphi_t^n = 0$ if $\theta_t \cdot X_t = 0$.

Given a consumption process c and an adapted process φ, our problem is to find a nonnegative wealth process W_t satisfying

$$dW_t = \big(W_t(\varphi_t' \cdot \nu_t + r_t) - c_t\big)dt + W_t\varphi_t' \cdot \sigma_t dB_t, \quad W_0 = w. \tag{9}$$

In order for W to remain nonnegative, an admissible control (c, φ) has the property that $\varphi_t = 0$ and $W_t = c_t = 0$ for t larger than the stopping time $\inf\{s : W_s = 0\}$.

The above formulation of our problem invites the use of dynamic programming. This problem can alternatively be formulated as a constrained optimization problem, where the agent maximizes his/her utility subject to a budget constraint (see e.g., Pliska (1986), Cox and Huang (1989)). This is made use of in the Part I paper (Aase, 2015). This invites the use of Kuhn–Tucker, as demonstrated in Section 4, as well as in the Appendix. With ordinary time additive, expected utility this method works well and actually gives a more general solution technique for this problem than that of dynamic programming. The solutions coincide under an assumption about complete markets.

3. Recursive utility

We now introduce recursive utility. An important property with this type of utility representation is that uncertainty is "dated" by the time of its resolution, and the individual regards uncertainties resolving at different times as being different. The consumer can thus prefer late resolution of uncertainty to early, or vice versa, which naturally makes this theory somewhat complex, but at the same time more realistic.

We use the framework established by Duffie and Epstein (1992a,b) and Duffie and Skiadas (1994) which elaborates the foundational work by Kreps and Porteus (1978) of recursive utility in dynamic models. This approach leads to the separation of risk aversion from the elasticity of intertemporal substitution in consumption, within a time-consistent model framework.

The recursive utility $U : L \rightarrow \mathbb{R}$ is defined as follows by two primitive functions: $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $A : \mathbb{R} \rightarrow \mathbb{R}$.

The function $f(c_t, V_t)$ represents a felicity index at time t, and A is associated with a measure of absolute risk aversion (of the Arrow–Pratt type) for the agent. In addition to current consumption c_t, the function f also depends on future utility V_t time t, a stochastic process with volatility $\tilde{\sigma}_V(t)$ at time t.

Focusing on optimal consumption, let τ be the planners horizon. The utility process V for a given consumption process c, satisfying $V_\tau = 0$, is given by the representation

$$V_t = E_t\left\{ \int_t^\tau \big(f(c_s, V_s) - \frac{1}{2} A(V_s)\, \tilde{\sigma}_V(s)'\tilde{\sigma}_V(s)\big) ds \right\}, \quad t \in [0, \tau]. \tag{10}$$

If, for each consumption process c_t, there is a well-defined utility process V, the stochastic differential utility U is defined by $U(c) = V_0$, the initial utility. The pair (f, A) generating V is called an aggregator.

One may think of A as associated with a concave function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that $A(v) = -\frac{h''(v)}{h'(v)}$, where h is twice continuously differentiable. U is monotonic and risk averse if $A(\cdot) \geq 0$ and f is jointly concave and increasing in consumption. Then, the term $\frac{1}{2} A(V_s)\, \tilde{\sigma}_V(s)'\tilde{\sigma}_V(s)$ becomes the Arrow–Pratt approximation to the certainty equivalent.

The equation (10) is a quadratic backward stochastic differential equation (BSDE), and existence and uniqueness of solutions to such equations is in general far from granted. These topics have been dealt with in the original paper Duffie and Epstein (1992b), and are also part of contemporary research in applied mathematics, see e.g., Øksendal and Sulem (2014), or Peng (1990). For the particular BSDE (10) existence and uniqueness follows from Duffie and Lions (1992).

The representation (10) is motivated from the corresponding discrete time model of Epstein and Zin (1989). The common starting point for recursive utility is that utility at time t is given by $V_t = g(c_t, m(V_{t+1}))$ for some function g, where m is a certainty equivalent at time t. If h is a von Neumann–Morgenstern index, then $m(V) = h^{-1}(E[h(V)])$. The certainty equivalent m is then assumed to satisfy some smoothness properties. With Brownian information V_t is an Ito process, and based on this Duffie and Epstein (1992b) demonstrate how (10) is justified.

Stochastic differential utility disentangles intertemporal substitution from risk aversion: In the case of deterministic consumption, $\tilde{\sigma}_V(t) = 0$ for all t. The risk aversion A is then irrelevant since it multiplies a zero variance. Thus certainty preferences, including the willingness to substitute consumption across time, are determined by f alone. Only risk attitudes are affected by changes in A for f fixed. In particular, if

$$\tilde{A}(\cdot) \geq A(\cdot),$$

where U and $\tilde{U}$ are utility functions corresponding to (f, A) and $(f, \tilde{A})$ respectively, then $\tilde{U}$ is more risk averse than U in the sense that any consumption process c rejected by U in favor of some deterministic process $\bar{c}$ would also be rejected by $\tilde{U}$.

We work with the so-called Kreps–Porteus specification, which corresponds to the aggregator (f, A) with the constant elasticity of substitution (CES) form

$$f(c, v) = \frac{\delta}{1-\rho} \frac{c^{(1-\rho)} - v^{(1-\rho)}}{v^{-\rho}} \quad \text{and} \quad A(v) = \frac{\gamma}{v}. \tag{11}$$

The parameter $\delta \geq 0$ is the agent's impatience rate, $\rho \geq 0$, $\rho \neq 1$ is the time preference and $\gamma \geq 0$, $\gamma \neq 1$, is the relative risk aversion. The parameter $\psi = 1/\rho$ is the elasticity of intertemporal substitution in consumption, referred to later as the EIS-parameter. The higher the value of ρ, the higher the agent's aversion to consumption fluctuations *across time* in a deterministic world. The higher the value of γ, the more aversion the agent has to consumption fluctuations in the *next period*, due to the different possible states of the world that can occur. Clearly, these two properties of an individual's preferences are different. In the conventional model $\gamma = \rho$.

Let us now consider the following ordinally equivalent specification. When the aggregator (f, A) is given corresponding to the utility function U, there exists a strictly increasing and smooth function $g(\cdot)$ such that the ordinally equivalent $\hat{U} = g \circ U$ has the aggregator $(\hat{f}, \hat{A})$ where

$$\hat{f}(c, v) = ((1-\gamma)v)^{-\frac{\gamma}{1-\gamma}} f(c, ((1-\gamma)v)^{\frac{1}{1-\gamma}}), \qquad \hat{A} = 0. \tag{12}$$

The current connection is

$$\hat{U} = \frac{1}{1-\gamma} U^{1-\gamma}.$$

This specification we sometimes find convenient to work with, in particular in connection with life insurance. Here, $\hat{f}$ has the CES form

$$\hat{f}(c, v) = \frac{\delta}{1-\rho} \frac{c^{(1-\rho)} - ((1-\gamma)v)^{\frac{1-\rho}{1-\gamma}}}{((1-\gamma)v)^{\frac{\gamma-\rho}{1-\gamma}}}, \qquad \hat{A}(v) = 0. \tag{13}$$

It is emphasized in the above references that the reduction to a normalized aggregator $(\hat{f}, 0)$ does not mean that intertemporal utility is risk neutral, or that we have lost the ability to separate risk aversion from consumption substitution.

The normalized version is used to prove existence and uniqueness of the solution to the BSDE, see Duffie and Epstein (1992b) and Duffie and Lions (1992).

The standard additive and separable utility has aggregator

$$f_1(c, v) = u(c) - \delta v, \qquad A_1 = 0, \tag{14}$$

in this framework (an ordinally equivalent representation). Clearly, the agent with the conventional utility is not risk neutral even if $A_1 = 0$.

In the next section, we demonstrate how to analyze problems involving recursive utility using the stochastic maximum principle. Both version (11) and (13) can be analyzed by our method. The remaining analysis will be valid for both versions. When it comes to solving for the optimal consumption path, or the optimal life insurance contract, we use version (13). Identical results can be obtained by our method from using the ordinally equivalent representation (11) instead.

The analysis is seemingly more technically involved once we depart from the separable and additive expected utility. Despite of this we demonstrate below that simple, intuitive and transparent results still emerge using this more general framework.

4. The first order conditions of optimal consumption

The typical consumer's problem is to solve the problem (8). Instead of solving this problem directly, we solve an equivalent one. As is well known (e.g., Cox and Huang (1989) or Pliska (1986)), in a complete market the dynamic program (8), (9) has the same solution as the following simpler, yet more general problem

$$\sup_{c \in L} U(c),$$

subject to

$$E\Big\{ \int_0^\tau c_t \pi_t dt \Big\} \le E\Big\{ \int_0^\tau e_t \pi_t dt \Big\},$$

where e is the consumer's given endowment process. Here, we focus on the terminal condition $V_\tau = 0$. Mortality will be taken up in Section 7.

Now define $V_t^c := V_t$ and $Z(t) := \tilde{\sigma}_V(t)$. The pair $(V_t^c, Z(t))$ is then the solution of the BSDE

$$\begin{cases} dV_t = -\tilde{f}(t, c_t, V_t, Z(t))\, dt + Z(t)\, dB_t \\ V_\tau = 0, \end{cases} \tag{15}$$

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where

$$\tilde{f}(t, c_t, V_t, Z(t)) = \begin{cases} f(c_t, V_t) - \frac{1}{2} A(V_t)\, Z(t)' Z(t), & \text{if version (11) is used;} \\ \hat{f}(c_t, V_t), & \text{if version (13) is used;} \end{cases} \tag{16}$$

Notice that (15) covers both the versions (11) and (13). Important is here that the volatility $Z(t) := \tilde{\sigma}_V(t)$ is part of the preference specification.

Existence and uniqueness of solutions of the BSDE (15) is treated in the general literature on this subject. For a reference, see Theorem 2.5 in Øksendal and Sulem (2014), or Hu and Peng (1995). For the aggregator of the Kreps and Porteus type, a Lipschitz condition in the above references related to the drift term of the BSDE is not satisfied, however, existence and uniqueness has then been proven in Duffie and Lions (1992) for diffusion processes.

For a multiplier $\alpha > 0$, define the Lagrangian of the optimization problem by

$$\mathcal{L}(c; \alpha) = U(c) - \alpha\, E\Big(\int_0^\tau \pi_t (c_t - e_t) dt \Big).$$

Suppose for each $\alpha > 0$ we can find an optimal c_t^α such that

$$\sup_c \mathcal{L}(c; \alpha) = \mathcal{L}(c^\alpha; \alpha), \tag{17}$$

without constraints. Next, suppose we can find an α_0 such that

$$E\Big(\int_0^\tau \pi_t (c_t^{\alpha_0} - e_t) dt \Big) = 0. \tag{18}$$

Then

$$c^* := c^{\alpha_0},$$

is optimal for the original constrained problem. In view of this, we first maximize $\mathcal{L}(c; \alpha)$ over all c without constraints, for given $\alpha > 0$.

Here, we utilize the stochastic maximum principle (see Pontryagin (1959)), Bismut (1978), Kushner (1972), Bensoussan (1983), Peng (1990), and Øksendal and Sulem (2014)): We are given the backward stochastic differential equation (BSDE) (15). The objective function is the Lagrangian $\mathcal{L}(c; \alpha)$ defined above.

The Hamiltonian for this problem is

$$H(t, c, v, z, y) = y_t\, \tilde{f}(t, c_t, v_t, z_t) - \alpha\, \pi_t (c_t - e_t), \tag{19}$$

where y_t is the adjoint variable.

Sufficient conditions for the existence of a unique optimal solution to the stochastic maximum principle can be found in the literature, see e.g., Theorem 3.1 in Øksendal and Sulem (2014). Hu and Peng (1995) also study existence and uniqueness of the solution to coupled FBSDE. A unique solution exists in the present case provided this also holds for the BSDE (10).

The adjoint equation is

$$\begin{cases} dY_t = Y_t\big(\frac{\partial \tilde{f}}{\partial v}(t, c_t, V_t, Z(t))\,dt + \frac{\partial \tilde{f}}{\partial z}(t, c_t, V_t, Z(t))\,dB_t\big), \\ Y_0 = 1. \end{cases} \tag{20}$$

where we use the notation $Z(t) = \tilde{\sigma}_V(t)$ as explained, and z as the generic variable. If c is optimal, we therefore have

$$Y_t = \exp\Big(\int_0^t \Big\{\frac{\partial \tilde{f}}{\partial v}(s, c_s, V_s, Z(s)) - \frac{1}{2}\Big(\frac{\partial \tilde{f}}{\partial z}(s, c_s, V_s, Z(s))\Big)^2\Big\}\,ds + \int_0^t \frac{\partial \tilde{f}}{\partial z}(s, c_s, V_s, Z(s))\,dB(s)\Big) \quad a.s., \tag{21}$$

Maximizing the Hamiltonian with respect to c gives the first order condition

$$y\,\frac{\partial \tilde{f}}{\partial c}(t, c, v, z) - \alpha\,\pi = 0,$$

or

$$\alpha\,\pi_t = Y(t)\frac{\partial \tilde{f}}{\partial c}(t, c_t, V(t), Z(t)) \quad \text{a.s. for all } t \in [0, T]. \tag{22}$$

Notice that the state price deflator π_t at time t depends, through the adjoint variable Y_t, on the entire, optimal paths $(c_s, V_s, Z_s))$ for $0 \leq s \leq t$, which means that the economy may not display the usual Markovian structure. One of the strengths of the stochastic maximum principle is that the Hamiltonian is allowed to depend on the state.

When $\gamma = \rho$ then $Y_t = e^{-\delta t}$ for the aggregator (14) of the conventional model, so the state price deflator is a Markov process, and dynamic programming is appropriate. If $\gamma \neq \rho$ on the other hand, we use the stochastic maximum principle in the continuous-time model of this paper.

4.1. The optimal consumption

We now solve for the version (13) of recursive utility. In the Appendix, we present the technical details. From the equations (22) and (16), we have the following first order conditions

$$\alpha\pi_t = Y_t\,\hat{f}_c(c_t^{\alpha}, V_t), \tag{23}$$

since $\tilde{f}_c = \hat{f}_c$ for this version. For simplicity of notation, we write $\hat{f} = f$ from now on. For this version, we obtain that

$$V_t = E_t\Big(\int_t^{\tau} f(c_s^{\alpha}, V_s)ds\Big), \tag{24}$$

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which implies that

$$\begin{cases} dV_t = -f(c_t^\alpha, V_t)\,dt + Z(t)\,dB_t, \\ V_\tau = 0, \end{cases} \tag{25}$$

and

$$Y_t = \exp\Big(\int_0^t f_v(c_s^\alpha, V_s)ds\Big). \tag{26}$$

Here, the dynamic representation for Y_t can be written

$$dY_t = Y_t\, f_v(c_t^\alpha, V_t)\,dt, \tag{27}$$

and we notice that for this version Y is a process of bounded variation, which is not the case for the other version. From (13) we find c_t, using that

$$f_c(c, v) := \frac{\partial f(c, v)}{\partial c} = \delta\, \frac{c^{-\rho}}{\big((1-\gamma)v\big)^{\frac{\gamma-\rho}{1-\gamma}}}.$$

Solving for c_t, the theory guarantees the existence of a V_t and a $\tilde{\sigma}_V(t)$ such that

$$c_t^\alpha = \Big(\frac{\alpha\pi_t((1-\gamma)\,V_t)^{\frac{\gamma-\rho}{1-\gamma}}}{\delta\, Y_t}\Big)^{-\frac{1}{\rho}}. \tag{28}$$

In the Appendix, it is shown that the optimal consumption has the dynamics

$$\frac{dc_t^\alpha}{c_t^\alpha} = \mu_c(t)\,dt + \sigma_c(t)\,dB_t, \tag{29}$$

where

$$\mu_c(t) = \frac{1}{\rho}\,(r_t - \delta) + \frac{1}{2\rho}\left(1 + \frac{1}{\rho}\right)\eta_t'\eta_t - \frac{(\gamma-\rho)}{\rho^2}\,\eta_t'\sigma_V(t)$$
$$+\frac{1}{2}\frac{(\gamma-\rho)\gamma(1-\rho)}{\rho^2}\sigma_V'(t)\sigma_V(t), \tag{30}$$

and

$$\sigma_c(t) = \frac{1}{\rho}\Big(\eta_t + (\rho-\gamma)\sigma_V(t)\Big). \tag{31}$$

Here, $\sigma_V(t)$ and V_t exist as a solution to the system of forward/backward stochastic differential equations explained in the Appendix.

By Ito's formula, it follows that

$$c_t^\alpha = c_0\, e^{\int_0^t(\mu_c(s)-\frac{1}{2}\sigma_c(s)'\sigma_c(s))ds+\int_0^t \sigma_c(s)dB_s}, \tag{32}$$

where $\mu_c(t)$ and $\sigma_c(t)$ are as determined above.

From (28) and the fact that the recursive utility function we work with is homogeneous of degree one, there is a one-to-one correspondence between c_0

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and α. Given a suitable integrability condition (see the Appendix), there is a unique c_0 that satisfies the budget constraint with equality with corresponding α_0. Under these assumptions we have a complete characterization of the optimal consumption in terms of the primitives of the model, by The Saddle Point Theorem.

Since the agent takes the market as given, it is of interest to study how shocks to the state price π affect the optimal consumption. To this end, it is convenient to rewrite the expression for the optimal consumption in terms of the state price. Using the dynamics of π in (5), we can write (32) as

$$c_t^* = c_0 \pi_t^{-\frac{1}{\rho}} e^{\int_0^t (-\frac{\delta}{\rho} + \frac{1}{2\rho}(\gamma-\rho)(1-\gamma)\sigma_V'(s)\sigma_V(s))ds + \frac{1}{\rho}(\rho-\gamma)\int_0^t \sigma_V(s)\,dB_s}. \tag{33}$$

In terms of the density process ξ the expression is

$$c_t^* = c_0 \xi_t^{-\frac{1}{\rho}} e^{\int_0^t (\frac{1}{\rho}(r_s-\delta) + \frac{1}{2\rho}(\gamma-\rho)(1-\gamma)\sigma_V'(s)\sigma_V(s))ds + \frac{1}{\rho}(\rho-\gamma)\int_0^t \sigma_V(s)\,dB_s}, \tag{34}$$

where

$$\pi_t = e^{-\int_0^t r_s ds} \xi_t = e^{-\int_0^t r_s ds} e^{-\int_0^t \eta_s dB_s - \frac{1}{2}\int_0^t \eta_s' \eta_s ds}. \tag{35}$$

It can be shown that the same consumption dynamics as given in (29)–(34) also result for the specification (11) of recursive utility, which then leads to the same expression for the optimal consumption. For example, the expected optimal consumption at time t as of time zero given by

$$E(c_t^*) = c_0 E\left\{e^{\int_0^t (\mu_c(s) ds}\right\}. \tag{36}$$

When $\rho = \gamma$, or $\gamma = 1/\psi$, the optimal consumption dynamics for the conventional model results. As a direct comparison with (33) and (34) the conventional model gives

$$c_t^* = c_0 \pi_t^{-\frac{1}{\rho}} e^{-\frac{\delta}{\rho} t} = c_0 \xi_t^{-\frac{1}{\rho}} e^{\int_0^t \frac{1}{\rho}(r_s - \delta) ds} \quad (\text{when } \rho = \gamma). \tag{37}$$

Comparing with the corresponding expressions for the conventional model ($\rho = \gamma$) we notice several important differences. Recall that the state price reflects what the consumer is willing to pay for an extra unit of consumption. In particular, with the conventional model in mind, it has been convenient to think of π_t as high in "times of crises" and low in "good times". Start with the optimal consumption given in (32) with $\sigma_c(t)$ as in (31), and consider for example a "shock" to the economy via the state price π_t. It is natural to think of this as stemming from a shock to the term $\int_0^t \eta_s dB_s$ via the process B. Assuming η positive, this lowers the state price, and seen in isolation, increases optimal consumption (see (33) and (35)). This is as for the conventional model. However, a shock from B has also an effect on the last factor in (31). Assuming σ_V positive, the direction of this shock depends on the sign of $(\rho - \gamma)$. When the individual prefers early resolution of uncertainty too late ($\gamma > \rho$), this shock has the opposite effect on c_t^*. As a consequence, the individual wants to dampen shocks to

TABLE 1

KEY US-DATA FOR THE TIME PERIOD 1889–1978. CONTINUOUS-TIME COMPOUNDING. $\hat{\kappa}_{M,c} = 0.4033$.

	Expectation	Standard dev.	Covariances
Consumption Growth	1.81%	3.55%	$\hat{\sigma}_{cM} = 0.002268$
Return S&P-500	6.78%	15.84%	$\hat{\sigma}_{Mb} = 0.001477$
Government Bills	0.80%	5.74%	$\hat{\sigma}_{cb} = -0.000149$
Equity Premium	5.98%	15.95%	

the economy. More precisely, the *optimal* strategy for the consumer is to smooth consumption provided the agent prefers early resolution of uncertainty to late.

A shock to the interest rate (in isolation) has the same effect on the recursive consumer as the conventional one. We summarize our findings in the following theorem.

Theorem 1. *Assume the preferences are such that σ_V is positive, and the market-price-of-risk η is positive. The individual with recursive utility will then prefer to smooth market shocks provided the consumer prefers early resolution of uncertainty to late ($\gamma > \rho$).*

When the expected utility consumer just follows is the wake of others as in a flock of sheep, the contents of the *mutuality principle*, the recursive individual displays a more sophisticated behavior under market uncertainty. It may depend on whether the individual has preference for early, or late resolution of uncertainty. With these two possibilities the mutuality principle does not necessarily hold for recursive utility. This is of importance for pension insurance. Some of the conventional wisdom has to be rewritten in presence of recursive utility.

The investment strategy that attains the optimal consumption of the agent is presented in Section 9. The recursive agent does not behave myopically, in contrast looks at several periods at the time. When times are good he consumes less than the myopic agent, invests more for the future, and can hence enjoy higher consumption when times are bad than the expected utility maximizer.

5. SOME EMPIRICAL IMPLICATIONS OF THE RECURSIVE MODEL

In society aggregate consumption is observed to be smooth, with a relatively high growth rate, see e.g., Table 1, where the summary statistics of the data used in the Mehra and Prescott (1985) paper is presented[1]. By $\sigma_{cM}(t)$ we mean the instantaneous covariance rate between the return on the index S&P-500 and the consumption growth rate, in the model a measurable, ergodic process. Similarly, $\sigma_{Mb}(t)$ and $\sigma_{cb}(t)$ are the corresponding covariance rates between the index M and government bills b and between aggregate consumption c and Government bills, respectively[2]. $\kappa_{M,c}(t)$ is the associated instantaneous correlation coefficient.

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The *first* major problem with the conventional life cycle model is to explain the smooth path of aggregate consumption observed in society. That is, we now imagine that the consumer plays the role of the representative agent, who consumes the aggregate consumption.[3] Observe that the volatility in (31) can be made arbitrarily small i.e., when the market-price-of-risk $\eta_t \approx (\gamma - \rho)\sigma_V(t)$, then $\sigma_c(t) \approx 0$. In contrast, when $\sigma_c(t) = \eta_t/\gamma$ for $\gamma = \rho$ as in the expected utility model, it follows that only the first term on the right-hand side is present. For the estimated value of η_t, this requires an unreasonably large value of γ to match the low estimate of the consumption volatility. In the recursive model, this is seen to be very different.

The *second* major problem with the conventional model is to explain the relatively large estimate of the growth rate of aggregate consumption in society for plausible values of the parameters. For the estimated value of η_t, and the large value of γ required to match the low estimated volatility, this leads to a low value of the impatience rate δ in (30) when $\gamma = \rho$ to match the estimate of the consumption growth rate.

With the additional terms in (30) this is different. Here, ρ takes the role that γ plays in the two first terms in the conventional model, but unlike for that model, ρ needs not be large to fit the US-consumption and stock market data of Mehra and Prescott (1985), summarized in Table 1, due to the last two terms in (30).

As an example, consider a situation where $\gamma > \rho$. By inspection of the forth term on the right-hand side in (30), depending on the values of $\sigma_V(t)$, a large consumption growth rate is possible when $\rho < 1$. The third term puts a limit on how much larger than ρ the risk aversion γ can be in order to match the estimated value of the growth rate. Other combinations can of course be found, again depending on the term $\sigma_V(t)$.

Summarizing, an important part of the asset pricing and consumption puzzles is related to how aggregate consumption can be so smooth at such a relatively large growth rate as indicated by the data summarized in Table 1, and still conform to the conventional model. This is much better accounted for by the recursive model, as demonstrated by the expressions (30) and (31).

6. The equilibrium interest rate and the equity premium

In this paper, the agent takes the market as given in the life cycle model. However, one may also consider a "representative agent" equilibrium, where the agent takes the aggregate consumption as given (Lucas (1978)). Here, the objective is to determine equilibrium risk premiums and the equilibrium interest rate, where the agent on the margin just holds the market portfolio. This agent is, of course, not like the individual we have considered in the life cycle model. For once, this agent does not hold any insurance in equilibrium, nor does he or she hold any government bonds. But note, still such instruments can be prices in this model: The equilibrium price is the one where the agent is just indifferent to holding the instrument. The advantage of considering this construction here, is that the preferences of the customer should be close to the preferences of the representative agent in an equilibrium setting, or, at least the latter will serve as a guide to determine the preferences of the former. We should in general consider a model that aggregates to something reasonable. We now explore the consequences of the market clearing condition.

As a direct consequence of the above expressions for the growth rate of consumption and the volatility of consumption, when consumption is considered as aggregate consumption in society and the consumer is the representative agent, from (3) we get the following equilibrium equity premium

$$\varphi_t' \sigma_t \eta_t = \mu_W(t) - r_t,$$

where $\varphi_t' \sigma_t = \sigma_W'(t)$ is the volatility of the wealth portfolio. Here, we have used the representation of the wealth portfolio (9). Sometimes the market portfolio is considered a proxy for the wealth portfolio, (but this is not always a good assumption). This is market clearing in the market for risky securities. We just give a short illustration of what this might lead to.

Using (31) and rearranging, we get

$$\mu_W(t) - r_t = \rho \sigma_c'(t) \sigma_W(t) + (\gamma - \rho) \sigma_V'(t) \sigma_W(t). \tag{38}$$

Provided a representative agent equilibrium exists, Aase (2014) shows that in equilibrium $\sigma_W(t) = (1-\rho)\sigma_V(t) + \rho\sigma_c(t)$ for all t, for either version of recursive utility. This relationship determines the volatility of the wealth portfolio in terms of primitives of the model, which are preferences ($\sigma_V(t)$ and ρ) and aggregate consumption ($\sigma_c(t)$).

Using (38) and solving for $\sigma_V(t)$ in the above equilibrium relationship, this gives $\sigma_V(t) = \frac{1}{1-\rho}(\sigma_W(t) - \rho\sigma_c(t))$, and the following risk premium of the wealth portfolio

$$\mu_W(t) - r_t = \frac{\rho(1-\gamma)}{1-\rho} \sigma_c'(t) \sigma_W(t) + \frac{\gamma - \rho}{1-\rho} \sigma_W'(t) \sigma_W(t). \tag{39}$$

This formula can be extended to yield the equilibrium risk premium of any risky asset having volatility $\sigma_R(t)$. The result is

$$\mu_R(t) - r_t = \frac{\rho(1-\gamma)}{1-\rho} \sigma_c'(t) \sigma_R(t) + \frac{\gamma - \rho}{1-\rho} \sigma_W'(t) \sigma_R(t). \tag{40}$$

The first term on the right-hand side corresponds to the consumption based CAPM of Breeden (1979), while the second term corresponds to the market based CAPM of Mossin (1966), the latter only valid in a "timeless" setting, i.e., a one period model with consumption only on the terminal time, in its original derivation.

A formula for the equilibrium risk-free interest rate we now obtain as follows: We insert the market-price-of-risk η_t obtained from (31) in the expression for $\mu_c(t)$ in (30). This gives

$$\rho\mu_c(t) = r_t - \delta + \frac{1}{2}\left(1 + \frac{1}{\rho}\right)(\rho\sigma_c'(t) + (\gamma - \rho)\sigma_V'(t))(\rho\sigma_c(t) + (\gamma - \rho)\sigma_V(t)) + \frac{1}{\rho}(\rho - \gamma)(\rho\sigma_c'(t) + (\gamma - \rho)\sigma_V'(t))\sigma_V(t) + \frac{1}{2}\frac{\gamma}{\rho}(1 - \rho)(\gamma - \rho)\sigma_V'(t)\sigma_V(t).$$

From this expression we obtain the equilibrium risk-free interest rate in terms of $\sigma_V(t)$ as

$$r_t = \delta + \rho\,\mu_c(t) - \frac{1}{2}\rho(1 + \rho)\,\sigma_c'\sigma_c - \rho(\gamma - \rho)\sigma_c'(t)\sigma_V(t) - \frac{1}{2}(\gamma - \rho)(1 - \rho)\sigma_V'(t)\sigma_V(t). \tag{41}$$

The final step is to use the expression for $\sigma_V(t) = \frac{1}{1-\rho}(\sigma_W(t) - \rho\sigma_c(t))$ in this formula. The result is

$$r_t = \delta + \rho\mu_c(t) - \frac{1}{2}\,\frac{\rho(1 - \rho\gamma)}{1 - \rho}\,\sigma_c'(t)\sigma_c(t) + \frac{1}{2}\,\frac{\rho - \gamma}{1 - \rho}\,\sigma_W'(t)\sigma_W(t). \tag{42}$$

The present derivation is different from the ones in the literature, showing that the results (38) and (41) are robust.

In their seminal paper on the subject, Duffie and Epstein (1992a) derives the same expression (40) for the risk premium, based on dynamic programming. They have no expression for the equilibrium, real interest rate r_t. In their derivation, using the Bellman equation, the volatilities involved needed to be constants.

We see that when time preference can be separated from risk preferences, the former is contained in all the terms appearing in the conventional model, since only consumption related parameters occur in that framework. When the quantity $\sigma_V(t)$ enters, the relative risk aversion γ also appears.

Consider for example the three first terms on the right-hand side of r_t in (41). The two first terms are as in the classical Ramsey model, where there are no risky securities (see Ramsey (1928)). The third term is the precautionary savings term, still only depending on the individual's time preference. Risk aversion only appears in the last two terms, where also the wealth portfolio of risky securities enters.

Also the structure of the risk premium in (38) is noteworthy. The first term is the covariance rate between aggregate consumption and the wealth portfolio, in which case the time preference enters. Only when γ is different from ρ a second term appears, where the risk aversion also enters.

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Consider a calibration to the data summarized in Table 1. By fixing the impatience rate δ to some reasonable value, $\delta = 0.02$ say, one solution to the two equations (39) and (42) yields $\gamma = 2.11$ and $\rho = 0.74$. Here, we have assumed $\sigma_W(t) = 0.10$ and with an instantaneous correlation with the market portfolio M, $\kappa_{W,M} = 0.80$. The resulting preference parameters seem plausible, and many other reasonable combinations fit the equations as well.

In contrast, a similar calibration of the conventional Eu-model leads to the (unique) values $\gamma = 26$ and $\delta = -0.015$, none of which are very plausible.

7. Consequences of recursive utility for optimal consumption/pensions

7.1. Optimal consumption

We now return to the life cycle model with our typical insurance customer, having recursive utility with parameters that may take values as described in the previous section.

The consequences of recursive preferences are noteworthy for the life cycle model. Of particular importance is the separation between risk aversion and time preference. We have demonstrated that this model can be calibrated to give reasonable values for the preference parameters, which is not the case for the conventional model.

The recursive model thus opens up possibilities to discuss pension plans based on a model for behavior that (i) fits data when aggregated, (ii) has an axiomatic underpinning, and (iii) deviates from the simple and naive solution of the myopic model.

Having summarized important properties of optimal consumption in Theorem 1, we now take a look at what the life cycle model tells us about optimal pensions.

7.2. The optimal pension insurance contract

We now discuss optimal pensions in the life cycle model. For this, we will need the following standard actuarial concepts: Let the survival probability of an x-year old pension insurance customer be given by $P(T_x > t) = l_{x+t}/l_x$, where $l(x)$ is the decrement function. The single premium of an annuity paying one unit per unit of time is given by the formula

$$\bar{a}_x^{(r)} = \int_0^{\tau} e^{-rt} \frac{l_{x+t}}{l_x} dt, \tag{43}$$

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where r is the short term interest rate, and the single premium of a "temporary annuity" which terminates after time n is

$$\bar{a}^{(r)}_{x:\bar{n}|} = \int_0^n e^{-rt} \frac{l_{x+t}}{l_x} dt. \tag{44}$$

For simplicity of exposition, we assume there to be only one risky asset ($N = 1$), where σ_t, ν_t, r_t and $\sigma_V(t)$ are all deterministic and constant in time.

Recall the basic optimization problem of Section 4. With mortality included, the associated Lagrangian is

$$\mathcal{L}(c;\alpha) = U(c) - \alpha\, E\Big(\int_0^\tau \pi_t (c_t - e_t) P(T_x > t) dt\Big). \tag{45}$$

where

$$U(c) = V_0 = E\Big\{\int_0^\tau \big(f(c_s, V_s) - \frac{1}{2} A(V_s)\, \tilde{\sigma}_V(s)'\tilde{\sigma}_V(s)\big) P(T_x > t) ds\Big\}. \tag{46}$$

Consider the following income process e_t:

$$e_t = \begin{cases} y, & \text{if } t \leq n; \\ 0, & \text{if } t > n \end{cases} \tag{47}$$

where y is a constant, interpreted as the consumer's salary when working, and n is the time of retirement for an x-year old. Equality in the budget constraint can then be written

$$E\Big(\int_0^\tau (e_t - c_t^*)\pi_t P(T_x > t) dt\Big) = 0.$$

This is the "Principle of Equivalence" of Actuarial Science in the present setting. Here, $[0, \tau]$ is the support of the remaining life time T_x of an x-year old pension customer. The above condition can be written

$$\int_0^n \left(yE(\pi_t)\frac{l_{x+t}}{l_x} - E(c_t^*\pi_t)\frac{l_{x+t}}{l_x}\right) dt + \int_n^\tau (-1) E(c_t^*\pi_t)\frac{l_{x+t}}{l_x}\, dt = 0.$$

Now we use that c_t^* and π_t are both geometric Brownian motions under the assumptions of this section. Using (30) and (31) in (32) and the representation for π_t given in (5), we find that c_0 can be written

$$c_0 = y \frac{\bar{a}^{(r)}_{x:\bar{n}|}}{\bar{a}^{(\tilde{r})}_x},$$

where

$$\hat{r} = r - \frac{1}{\rho}(r-\delta) + \frac{1}{2}\frac{1}{\rho}\left(1-\frac{1}{\rho}\right)\eta'\eta + \frac{1}{\rho}\left(\frac{1}{\rho}-1\right)(\rho-\gamma)\,\eta\sigma_V$$
$$-\frac{1}{\rho}(\gamma-\rho)\Big(\frac{1}{\rho}(\gamma-\rho)+\frac{1}{2}(1-\gamma)\Big)\sigma_V^2. \tag{48}$$

By inspection of equation (28) for the optimal consumption and the representation given in (32), we notice that this also determines the Lagrange multiplier $\alpha_0 > 0$. From the results in Section 4.1, it follows that the optimal life time consumption ($t \in [0, n]$) and pension ($t \in [n, \tau]$) is

$$c_t^* = y\,\frac{\bar{a}^{(r)}_{x:\bar{n}|}}{\bar{a}^{(\hat{r})}_{x}}\exp\left\{\left(\frac{1}{\rho}(r-\delta)+\frac{1}{2\rho}\eta^2+\frac{1}{2\rho}(\gamma-\rho)(1-\gamma)\sigma_V^2\right)t\right.$$
$$\left.+\frac{1}{\rho}(\eta+(\rho-\gamma)\sigma_V)\,B_t\right\}, \tag{49}$$

provided the agent is alive at time t (otherwise $c_t^* = 0$), where the expressions for μ_c and σ_c given in (30) and (31) have been used.

The premium intensity p_t at time t while working is given by $p_t := y - c_t^*$, an $\mathcal{F}_t$-adapted process for ($t \in [0, n]$), provided the agent is alive at time t.

As can be seen, the optimal pension is being smoothened in the same manner as the optimal consumption in Section 4.1, summarized in Theorem 1. A positive shock to the economy via the term B_t increases the optimal pension benefits via the term ηB_t, which may be mitigated, or strengthened, by the term $(\rho-\gamma)\sigma_V B_t$, depending on the sign of $(\gamma-\rho)\sigma_V$. Similarly, a negative shock is dampened in the optimal pension provided $\gamma > \rho$, when $\sigma_V > 0$. This indicates that the pensioner in this model behaves considerably more sophisticated than the one modeled by expected utility. We summarize as follows:

Theorem 2. *Under the same assumptions as in Theorem 1, the individual with recursive utility will prefer a pension plan that smooths market shocks provided the consumer prefers early resolution of uncertainty to late ($\gamma > \rho$).*

This indicates that the pension customer may prefer a defined benefit pension plan to a defined contribution, or unit linked plan under these assumptions.

How can this optimal pension be implemented in the real world? When an insurance company takes over the responsibility of investing the pension fund on behalf of its customers, the company needs to find a strategy that works. This has been possible in the past, so it should also be possible in the future. Again the optimal investment strategy for each individual is presented in Section 9.

Clearly it involves time diversification, which requires a proper regulatory regime, and long term budgeting practices rather than the current short term (yearly), and adequate equity capital.

7.3. Comparative statics

In the present model, we have the possibility to investigate what happens to optimal consumption when conditions in the market for risky assets change. As an example, consider the *partial effect* on the expected value of optimal consumption at time t, as seen from time $t = 0$, of an increase in the market-price-of-risk. From this, it follows that the expected value as of time zero of the optimal consumption/pension at any time t in (36) is

$$E(c_t^*) = y\,\frac{\bar{a}^{(r)}_{x:\bar{n}|}}{\bar{a}^{(\hat{r})}_x}\exp\Big\{\frac{1}{\rho}\Big(r-\delta+\frac{1}{2}\Big(1+\frac{1}{\rho}\Big)\eta'\cdot\eta+\frac{1}{\rho}\,(\rho-\gamma)\,\eta\sigma_V + \frac{\gamma}{2\rho}\,(\gamma-\rho)(1-\rho)\sigma_V^2\Big)t\Big\}. \tag{50}$$

From (48), we see that the expected optimal consumption grows with time provided

$$r > \delta - \frac{1}{2}\left(1+\frac{1}{\rho}\right)\eta^2 - \frac{1}{\rho}(\rho-\gamma)\eta\sigma_V - \frac{\gamma}{2\rho}\,(\gamma-\rho)(1-\rho)\sigma_V^2. \tag{51}$$

The corresponding condition in the classical Ramsey model is $r > \delta$.

In order to compute the derivative of $E(c_t^*)$ in (50) with respect to the market-price-of-risk parameter η, we notice that η (when $N = 1$) appears both in the exponential function in the numerator, as well as in the adjusted interest rate $\hat{r}$ in the denominator. After some routine calculations, we then find the expression

$$\frac{\partial E(c_t^*)}{\partial\eta} = y\,\frac{\bar{a}^{(r)}_{x:\bar{n}|}}{\bar{a}^{(\hat{r})}_x}\Big(\exp\Big\{\frac{1}{\rho}\Big(r-\delta+\frac{1}{2}\left(1+\frac{1}{\rho}\right)\eta'\cdot\eta+\frac{1}{\rho}\,(\rho-\gamma)\,\eta\sigma_V + \frac{\gamma}{2\rho}\,(\gamma-\rho)(1-\rho)\sigma_V^2\Big)t\Big\}\Big)\cdot\Big(\Big(\eta\,\frac{1}{\rho}(1+\frac{1}{\rho})+\frac{1}{\rho^2}(\rho-\gamma)\sigma_V\Big)(t-\hat{t}_1)\Big), \tag{52}$$

where

$$\hat{t}_1 = \frac{\frac{1}{\rho}(\frac{1}{\rho}-1)(\eta+(\gamma-\rho)\sigma_V)}{\frac{1}{\rho}(1+\frac{1}{\rho})\eta+\frac{1}{\rho^2}(\rho-\gamma)\sigma_V}\,\hat{t}, \tag{53}$$

and $\hat{t}$ is determined by the equality

$$\int_0^\tau s\,\frac{l_{x+s}}{l_x}\,e^{-\hat{r}s}\,ds = \hat{t}\int_0^\tau \frac{l_{x+s}}{l_x}\,e^{-\hat{r}s}\,ds = \hat{t}\,\bar{a}^{(\hat{r})}_x,$$

by the first mean value theorem for integrals.

From this, we can investigate what effect an increase in the expected rate of return of the risky asset has on the expected optimal consumption. We consider four situations:

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1. Suppose $\hat{t}_1 > 0$ and the factor $(\eta \frac{1}{\rho}(1+\frac{1}{\rho}) + \frac{1}{\rho^2}(\rho - \gamma)\sigma_V) > 0$, the expected consumption would first decrease for $t < \hat{t}_1$, and then increase for $t > \hat{t}_1$. When $\gamma > \rho$ and $\rho < 1$ this holds. Since, σ_V may depend on the preference parameters, this situation may also occur when $\rho > \gamma$ and $\rho > 1$, depending on σ_V. This corresponds to a substitution effect.
2. If $\eta(1 + \rho) > (\gamma - \rho)\sigma_V > 0$, $\eta + (\gamma - \rho)\sigma_V > 0$ and $\rho > 1$, or if $\eta(1 + \rho) > (\gamma - \rho)\sigma_V > 0$, $\eta + (\gamma - \rho)\sigma_V < 0$ and $\rho < 1$, the expected consumption is an increasing function of η. Typically, this happens if $\rho > \gamma$ and $\rho > 1$. This corresponds to an income effect.
3. If $\eta(1 + \rho) < (\gamma - \rho)\sigma_V$ and $\frac{1}{\rho}(\frac{1}{\rho} - 1)(\eta + (\gamma - \rho)\sigma_V) < 0$, the expected consumption is an increasing function in η for $0 \leq t \leq \hat{t}_1$ and a decreasing function for $t > \hat{t}_1$. This can happen for $\gamma > \rho > 1$, but also for $\rho > \gamma$ and $\rho > 1$ depending on the values of σ_V.
4. If $\eta(1 + \rho) < (\gamma - \rho)\sigma_V$ and $\frac{1}{\rho}(\frac{1}{\rho} - 1)(\eta + (\gamma - \rho)\sigma_V) > 0$, the expected consumption is a decreasing function in η for all t. In this case, $\rho < 1$. This can happen when $\gamma > \rho$, but also when $1 > \rho > \gamma$, depending on the values of σ_V.

We illustrate with an example.

For the standard recursive model when $\sigma_V = (\sigma_W - \rho\sigma_c)/(1 - \rho)$ (representative agent), and $\gamma > \rho$ and $\rho < 1$, the first category above may apply and $E(c_t^*)$ as of time zero would first decrease for $t < \hat{t}_1$, and then increase for $t > \hat{t}_1$. If $\rho > 1$ on the other hand, then the income effect dominates.

This shows that the recursive model can give a variety of results. For instance an increase in the market-price-of-risk can lead to a decrease in expected consumption early in life, and an increase later when $t > \hat{t}_1$. The substitution effect then dominates the income effect for the consumer/insurance customer. The conventional model is only consistent with the income effect, and no substitution, since γ is larger than one for this model.

7.4. Consequences for the insurance industry

When stock market uncertainty is present, a main result about optimal pensions is summarized in Theorem 2. Insurance companies pay the pensions from funds, which in bad times are lower than in good times. Such companies have the possibility and ability, however, to take a long term view and "harvest" the equity premium in the financial markets by diversification across time.

Because of the equity premium puzzle, there has been expressed doubt whether the equity premium is "too large". This is an observed (estimated) quantity, and as the results of Sections 5 and 6 indicate, there may simply be nothing wrong with the "high" equity premium of the last century, nor with the "low" equilibrium interest rate, the "smooth" aggregate consumption and the "large" growth rate of the latter. It was just the map that did not fit the terrain.

With this perspective in mind, insurance companies could consider providing the type of pension and life insurance contracts that many people seem to prefer, namely that of smoothening life time consumption across both time and states of nature. Since individuals have a much shorter time perspective than the insurance industry, they are not equally well prepared to "time diversify" the way this industry can. Even so, the solution to the investment problem is presented in Section 9.

With regard to the part of the pensions paid by governments each year to the whole generation of people above a certain age, from the conventional model we conclude that it is not optimal to insure the entire society against crises and bad times on a year by year basis. If aggregate consumption in society is down in one particular year, everyone is in principle worse off, by the mutuality principle. However, this principle is typically related to a one-period setting, and as we have demonstrated, in the multi-period framework it hinges on the assumption of expected utility.

The dynamic, recursive model tells an additional story. It points towards more consumption smoothening and immunization to market swings when $\gamma > \rho$, and a strengthening of market movements when $\gamma < \rho$. How can society help implement the former? Also governments sometimes chose to take a long term view and average across time as well as over the states of nature. However, a discussion of this topic leads us into fiscal policy and macroeconomics, which is beyond the scope of this presentation.

8. LIFE INSURANCE

We now turn to life insurance in the recursive model. Since life insurance has many of the characteristics of an ordinary insurance contract, one would conjecture that risk aversion is the more prominent property for this type of contracts, while consumption substitution is more essential for pensions. We now address this distinction.

Recursive utility is now a function $U : L_+ \times L_+ \to \mathbb{R}$. The problem can be formulated as follows:

$$\sup_{z,c \geq 0} U(c, z),$$

subject to

$$E\left(\pi_{T_x} W(T_x)\right) \geq E\left\{\pi_{T_x} z\right\},$$

where $W(t)$ is the consumer's net saving at time t given by

$$W_t = (\pi_t)^{-1} \int_0^t \pi_s (e_s - c_s) ds. \tag{54}$$

This budget constraint says that the present value of the terminal wealth is sufficient to cover the amount of life insurance. In life and pension insurance this constraint is in expectation, meaning pooling over the population. It is this element that gives the individual the benefit of using the life and pension insurance market to save for longevity. Without such a market, the budget constraint would instead be an (a.s.) inequality between the corresponding random variables. Clearly the above constraint is less strict, hence gives at least as large life time consumption, including life insurance, as without insurance available.

We proceed as before and assume first a fixed horizon τ in the initial specification of recursive utility. Then future utility is given by

$$V_t = E\Big(\int_t^{\tau} f(c_s, V_s)ds + u(z) \Big),$$

assuming u is a bequest utility function (Schroder and Skiadas (1999) treat terminal utility), where z is the amount of life insurance payable at time of death of the insured. As for the conventional model, this quantity is a random variable. Here, the assumption is that the agent is alive at time t. Recursive utility is now given by $U(c, z) = V_0$.

The Lagrangian of the problem is

$$\mathcal{L}(c, z; \alpha) = U(c, z) - \alpha E\left[\pi_{T_x} z - \int_0^{\tau} \pi_t (e_t - c_t) \frac{l_{x+t}}{l_x} dt \right].$$

Using directional derivatives (see the appendix), the first order condition in c is

$$\nabla_c \mathcal{L}(c, z; \alpha; \tilde{c}) = 0, \quad \forall \tilde{c} \in L_+,$$

which is equivalent to

$$\alpha \pi_t = Y_t \frac{\partial f}{\partial c}(c_t, V_t) \quad a.s. \quad \text{for all } t \in [0, \tau],$$

independent of the horizon τ, and of mortality, since the survival probability simply cancels. This leads to the optimal consumption/pension for any $\alpha > 0$

$$c_t = \Big(\frac{\alpha \pi_t ((1-\gamma) V_t)^{\frac{\gamma - \rho}{1-\gamma}}}{\delta \, Y_t} \Big)^{-\frac{1}{\rho}}.$$

Likewise, the first order condition in the amount of life insurance z is

$$\nabla_z \mathcal{L}(c, z; \alpha; \tilde{z}) = 0, \quad \forall \tilde{z} \in L_+,$$

which is equivalent to

$$E\Big\{ \Big(Y_{T_x} \frac{\partial u(z)}{\partial z} - \alpha \pi_{T_x} \Big) \tilde{z} \Big\} = 0, \quad \forall \tilde{z} \in L_+. \tag{55}$$

Here, z and $\tilde{z}$ are $\mathcal{F} \vee \sigma(T_x)$ - measurable. It is at this stage we extend to the horizon T_x. For (55) to hold true, it follows that

$$z = u'^{-1}\Big(\frac{\alpha \pi_{T_x}}{Y_{T_x}} \Big), \tag{56}$$

assuming the derivative of the bequest utility function u' is invertible.

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As an illustration suppose $u(z) = \frac{1}{1-\theta} z^{1-\theta}$, so that θ is the relative risk aversion of the bequest utility function. Then, the optimal amount of life insurance is

$$z = \left(\frac{\alpha \pi_{T_x}}{Y_{T_x}}\right)^{-\frac{1}{\theta}}. \tag{57}$$

Comparing this with the corresponding expression z^{cm} for the conventional model, which is

$$z^{cm} = \left(\frac{\alpha \pi_{T_x}}{Y_{T_x}}\right)^{-\frac{1}{\gamma}} \quad \text{(conventional model)},$$

where $Y_{T_x} = e^{-\delta T_x}$, we notice that this is quite analogous, except for a more complicated formula for the adjoint variable Y in the recursive model.

In both models, *risk aversion* is seen to be the essential property for the optimal amount of life insurance, not consumption substitution. Recall, in the conventional model there is only one parameter (with two distinct interpretations). This is not to say that the time preference ρ does not matter for the recursive specification (Y_{T_x} depends on both γ and ρ), but ρ does not affect the state price deflator π_t at the terminal time, which is the important issue here.

As with pensions, the multiplier α is determined from equality in the budget constraint. Thus we consider the equation

$$E\left[\pi_{T_x} z - \int_0^{\tau} \pi_t (e_t - c_t) \frac{l_{x+t}}{l_x} dt\right] = 0.$$

With a constant income of y up to the time n of retirement, and a pension thereafter as the basis for determining the endowment process e, we obtain the equation

$$\alpha^{-\frac{1}{\theta}} E\left\{\int_0^{\tau} \exp\left(-\left(r\left(1-\frac{1}{\theta}\right)t - \frac{1}{2}\eta'\eta\frac{1}{\theta}\left(1-\frac{1}{\theta}\right)t - \eta\left(1-\frac{1}{\theta}\right)B_t\right.\right.\right.$$
$$\left.\left.\left. + \frac{1}{\theta}\int_0^t f_v(c_u, V_u)du\right)\right) f_x(t)dt\right\} + \alpha^{-\frac{1}{\rho}} \bar{a}_x^{(\hat{r})} = y\bar{a}_{x:\bar{n}|}^{(r)}, \tag{58}$$

where $\hat{r}$ is as given in (48), equation (26) for Y has been used, and $f_x(t)$ is the probability density function of T_x. The formula for $f_v(c_t, V_t)$ is given in (A.18) in the appendix, we have used the constant relative risk aversion (CRRA) bequest function $u'(z) = z^{-\theta}$, and made the common assumption that $l_{x+\tau} = 0$. This determines the multiplier α_0. It is at this point that pooling takes place in the contract. In this situation, the optimal consumption ($t \in [0, n]$) and pension ($t \in (n, \tau)$) is given by

$$c_t^* = \alpha_0^{-\frac{1}{\rho}} \exp\left\{\left(\frac{1}{\rho}(r-\delta) + \frac{1}{2\rho}\eta^2 + \frac{1}{2\rho}(\gamma-\rho)(1-\gamma)\sigma_V^2\right)t\right.$$
$$\left. + \frac{1}{\rho}(\eta + (\rho-\gamma)\sigma_V)B_t\right\}, \tag{59}$$

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provided the agent is alive at time t, and the optimal amount of life insurance at time T_x of death of the insured is

$$z^* = \left(\frac{\alpha_0 \pi_{T_x}}{Y_{T_x}}\right)^{-\frac{1}{\theta}}. \tag{60}$$

The premium intensity paid while working is $p_t = y - c_t^*$, which is naturally larger than without life insurance included.

When $\rho = \gamma$, the equation (58) for α_0 simplifies to

$$\alpha^{-\frac{1}{\theta}}\big(1 - r_1 \bar{a}_x^{(r_1)}\big) + \alpha^{-\frac{1}{\rho}} \bar{a}_x^{(\hat{r})} = y \bar{a}_{x:\bar{n}|}^{(r)}, \tag{61}$$

where

$$r_1 := r - \frac{1}{\theta}(r - \delta) + \frac{1}{2}\eta' \eta \frac{1}{\theta}\left(1 - \frac{1}{\theta}\right),$$

which is the same result as obtained in the conventional model. Also, the optimal amount of life insurance is determined jointly, through the constant α_0, with the optimal consumption/pension.

9. Portfolio choice with recursive utility

We now address the optimal investment strategy that the recursive utility consumer will use in order to obtain the optimal consumption.

Consider an agent with recursive utility who takes the market introduced in Section 2 as given. In this setting, we now analyze optimal portfolio choice. We then have the following result

Theorem 3. *The optimal portfolio fractions in the risky assets are given by*

$$\varphi(t) = \frac{1-\rho}{\gamma - \rho}(\sigma_t \sigma_t')^{-1} \nu_t - \frac{\rho(1-\gamma)}{\gamma - \rho}(\sigma_t \sigma_t')^{-1}(\sigma_t \sigma_c^*(t)),$$

assuming $\gamma \neq \rho$. Here, $\sigma_{c^*}(t)$ is the volatility of the optimal consumption growth rate of the individual.

Proof. First, we recall the dynamics of the optimal consumption for the individual investor under consideration. The volatility $\sigma_{c^*}(t)$ has been shown in (31) of Section 4.1 to be

$$\sigma_{c^*}(t) = \frac{1}{\rho}\Big(\eta_t + (\rho - \gamma)\sigma_V(t)\Big),$$

where $\sigma_t \eta_t = \nu_t$ is the market-price-of-risk given in Section 2. Also, the volatility of utility is given by

$$\sigma_V(t) = \frac{1}{1-\rho}(\sigma_W(t) - \rho \sigma_{c^*}),$$

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as shown in Aase (2014), where $\sigma_W(t)$ is the volatility of the agent's wealth portfolio. The dynamics of the wealth is given in (9) of Section 2, implying that $\sigma_W(t) = \sigma_t' \varphi_t$ (see Section 6). This leads to a single equation for φ_t, and the solution is given by the above formula. ■

The optimal fractions with recursive utility depend on both risk aversion and time preference as well as the volatility σ_{c^*} of the optimal consumption c_t^* of the agent. This latter quantity is usually not directly observable for an individual. However, for institutions this matter is different.

In each period, the consumer both consumes and invests for future consumption. Compared to the expected utility consumer, the recursive agent consumes less in good times, and then invests more for future consumption, and vice versa in bad times. This is how this consumer can average consumption across time in a more efficient manner than the conventional theory predicts. The recursive utility maximizer considers more than one period at the time which allows for a smoother consumption path. Here, the expected utility agent is just myopic.

Based on the conventional, pure demand theory of this paper, by assuming a relative risk aversion of around two, the optimal fraction in equity is 119% follows from the standard formula $\varphi = \frac{1}{\gamma}(\sigma_t \sigma_t')^{-1} \nu_t$ (see Mossin (1968), Merton (1971), Samuelson (1969)), using the summary statistics of Table 1, and assuming one single risky asset, the index itself. In contrast, depending upon estimates, the typical household holds between 6% to 20% in equity. Conditional on participating in the stock market, this number increases to about 40% in financial assets. Recent estimates are close to 60%, including indirect holdings via pension funds invested in the stock market. In the above application, this formula reduces to $\varphi = \frac{1}{\gamma}(\sigma_R \sigma_R')^{-1}(\mu_R - r)$. Notice that here $\gamma = \rho$.

One could object to this that the conventional model is consistent with a value for γ around 26 only. Using this value instead, the optimal fraction in equity is down to around 9%, which in isolation seems reasonable enough. However, such a high value for the relative risk aversion is considered implausible, as discussed before.

As an illustration of the general formula, consider the standard situation with one risky and one risk-free asset, interpreting the S&P-500 index as the risky security, and employ the data of Table 1. The recursive model explains an average of 14 per cent in risky securities for the following parameter values $\gamma = 2.6$ and $\rho = 0.96$. Given participation in the stock market, when $\gamma = 2.5$ and $\rho = 0.74$, then $\varphi = 0.40$. If $\varphi = 0.60$, this can correspond to $\gamma = 2.0$ and $\rho = 0.7$, etc., a potential resolution of this puzzle.

In addition to the insurance industry, other interesting applications would be to management of funds that invests public wealth to the benefits of the citizens of a country, or the members of a society large enough for an estimate of the volatility of the consumption growth rate of the group to be available. One such example is the Norwegian Government Pension Fund Global (formerly the Norwegian Petroleum Fund).

10. Summary

Expected utility in a temporary setting is demonstrated to fail capitally on most accounts. Already Mossin (1969) indicated this, but then there were few alternatives to extending expected utility to several periods. Today this is different, and there are alternatives, the most promising one being recursive utility, the topic of this paper.

For the version of recursive utility that we consider, the agent is not myopic, and consumes less and invests more in good times than the expected utility agent, thereby having more to consume in bad times. The same conclusions carry over to pension insurance. This behavior explains aggregate data. Not so for the Eu-model, where several serious puzzles remain.

The agent in the life cycle model both consumes and invests in every period, yet the optimal investment strategy in the Eu-theory does not depend on the optimal consumption. This changes with the recursive model. Also here the optimal investment policy explains aggregate data, where puzzles abound in the Eu-version.

Consequences of all this for pension insurance are pointed out in the paper. A section on life insurance is also included.[4]

Notes

1. There are of course newer data sets, and for other countries than the US, but they all retain these basic features. The data is adjusted from discrete-time to continuous-time compounding.

2. These quantities are "estimated" directly from the original data obtained from R. Mehra, assuming stationarity, and estimates are denoted by $\hat{\sigma}_{cM}$, etc.

3. Such a construction cannot serve as a typical pension insurance customer, but is included here to get a feeling for the preference relation used.

4. I am happy to acknowledge the contribution of the referees for the end result of this paper. Any remaining errors, or opaque formulations, are my responsibility.

5. The topic of such systems constitutes an active research field in parts of applied mathematics today, see e.g., Øksendal and Sulem (2013).

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APPENDIX A

In this Appendix, we present the analysis for the version (13). Starting with the first order condition (22), which is true for both versions, we get immediately that

$$\alpha \pi_t = Y_t \hat{f}_c(c_t, V_t). \tag{A.1}$$

The dynamics of V and Y can be found from (15) and (20) recalling that $A(v) = \gamma / v$ in these equations must be replaced by $\hat{A}(v) = 0$ for the ordinally equivalent version, and $\hat{f}$ replaces f. We now write f for $\hat{f}$. From this we obtain that

$$V_t = E_t\Big(\int_t^{\tau} f(c_s, V_s) ds \Big), \tag{A.2}$$

and

$$Y_t = \exp\Big(\int_0^t f_v(c_s, V_s) ds \Big). \tag{A.3}$$

Here, the dynamic representation for Y_t can be written

$$dY_t = Y_t \, f_v(c_t, V_t) \, dt, \tag{A.4}$$

and we notice that for this version Y is a process of bounded variation. From (A.1) we find c_t, using that

$$f_c(c, v) := \frac{\partial f(c, v)}{\partial c} = \delta \, \frac{c^{-\rho}}{\big((1-\gamma)v\big)^{\frac{\gamma-\rho}{1-\gamma}}}.$$

Solving for c_t, the theory guarantees the existence of a V_t and a $\tilde{\sigma}_V(t)$ such that

$$c_t = \Big(\frac{\alpha \pi_t ((1-\gamma) V_t)^{\frac{\gamma-\rho}{1-\gamma}}}{\delta \, Y_t} \Big)^{-\frac{1}{\rho}}. \tag{A.5}$$

Recall that the standard additive and separable utility has aggregator

$$\tilde{f}(c, v) = u(c) - \delta v, \qquad \tilde{A} = 0.$$

in this framework. When $\gamma = \rho$ then $Y_t = e^{-\delta t}$ and since α is arbitrary, the expression for the optimal consumption in (A.5) then becomes the same as for the conventional model, which is

$$c_t^* = (\alpha e^{\delta t} \pi_t)^{-\frac{1}{\gamma}} \quad \text{a.s.}, \ t \geq 0. \tag{A.6}$$

For this model, the state price deflator is a Markov process, and dynamic programming is an appropriate technique to apply.

From the equations (A.2) and (A.4) we get, when substituting in from (A.5)

$$dY_t = h(t, Y_t, V_t) \, dt; \quad 0 \leq t \leq \tau, \tag{A.7}$$

$$Y_0 = 1 \quad \text{(forward SDE)}, \tag{A.8}$$

and

$$dV_t = -g(t, Y_t, V_t) \, dt + \tilde{\sigma}_V(t) \, dB_t; \quad 0 \leq t \leq \tau, \tag{A.9}$$

$$V_\tau = 0 \quad \text{(backward SDE)}, \tag{A.10}$$

where

$$h(t, Y_t, V_t) = Y_t f_v\Big(\Big(\frac{\alpha\pi_t((1-\gamma)V_t)^{\frac{(1-\rho)}{(1-\gamma)}-1}}{\delta\, Y_t}\Big)^{-\frac{1}{\rho}}, V_t\Big), \tag{A.11}$$

and

$$g(t, Y_t, V_t) = f\Big(\Big(\frac{\alpha\pi_t((1-\gamma)V_t)^{\frac{(1-\rho)}{(1-\gamma)}-1}}{\delta\, Y_t}\Big)^{-\frac{1}{\rho}}, V_t\Big). \tag{A.12}$$

The system (A.7)–(A.10) is a coupled system of forward-backward stochastic differential equations[5]. In equations (A.9) and (A.10), both V *and* $\tilde{\sigma}_V(t)$ are determined.

A.1. Directional derivatives

As a comment, the above first order conditions can alternatively be found using directional derivatives. Recall that the Lagrangian for the problem is given by

$$\mathcal{L}(c;\alpha) = U(c) - \alpha E\Big(\int_0^\tau \pi_t(c_t - e_t)dt\Big).$$

In order to find, for any $\alpha > 0$, the first order condition for the representative consumer's problem, several approaches are possible. Duffie and Epstein (1992a) use dynamic programming. In the above, we used the stochastic maximum principle and forward/backward stochastic differential equations. Here, we use Kuhn–Tucker and variational calculus. The latter two methods are both very general, and in particular neither relies on a Markov structure.

In maximizing the Lagrangian of the problem, we calculate the directional derivative $(\bigtriangledown U(c))(h)$ where $\bigtriangledown U(c)$ is the gradient of U at c. The directional, or Gateau derivative is defined as $(\bigtriangledown U(c))(h) = \lim_{\alpha\to 0}\frac{U(c+\alpha h)-U(c)}{\alpha}$.

Since U is continuously differentiable, this gradient is a linear and continuous functional, and thus, by the Riesz representation theorem, it is given by an inner product. By dominated convergence this utility gradient is

$$(\bigtriangledown U(c))(h_t) = E\Big(\int_0^\tau Y_t \frac{\partial f}{\partial c}(c_t, V_t)\, h_t\, dt\Big). \tag{A.13}$$

where

$$Y_t = \exp\Big(\int_0^t \frac{\partial f}{\partial v}(c_s, V_s)\, ds\Big) \quad a.s., \tag{A.14}$$

see Duffie and Skiadas (1994). The first order condition is that the directional derivative of the Lagrangian is zero at the optimal c_t, called c_t^α, in all directions $h \in L$:

$$\bigtriangledown\mathcal{L}(c^\alpha, \alpha; h) = 0 \quad \text{for all} \quad h \in L.$$

This is equivalent to

$$E\Big\{\int_0^\tau \Big(Y_t \frac{\partial f}{\partial c}(c_t^\alpha, V_t) - \alpha\pi_t\Big)h_t\, dt\Big\} = 0 \quad \text{for all} \quad h \in L. \tag{A.15}$$

The result is that for the Riesz-representation of the gradient of U to be equal to the state price deflator π_t it is necessary and sufficient that

$$\alpha\pi_t = Y_t\, f_c(c_t^\alpha, V_t), \quad \text{a.s. for all } t \in [0, T], \tag{A.16}$$

the same as (A.1) obtained by the stochastic maximum principle.

A.2. The dynamics of the optimal consumption

Finally, we find the dynamics of the optimal consumption c_t^* of equation (A.5). We use the notation for the volatility function $\tilde{\sigma}_V(t)$

$$\tilde{\sigma}_V(t) = (1-\gamma)\, V_t \sigma_V(t),$$

where $\sigma_V(t)$ is a vector of volatilities of the (normalized) growth rate of V_t. The quantity $\sigma_V(t)$ is assumed to be an ergodic stochastic process. This transformation accomplishes two things. First, the model for the stochastic utility is moved to percentage form, which is innocuous. Second, the multiplication of utility by $(1-\gamma)$ secures that the sign of σ_V is positive, since V and $(1-\gamma)$ have the same sign. This only requires $V \neq 0$.

The consumer takes the stock market as given, where the state price deflator π is assumed to satisfy

$$d\pi_t = -r_t \pi_t dt - \pi_t \eta_t' dB_t, \tag{A.17}$$

where η_t is the market-price-of-risk, an ergodic stochastic process, and r_t is the risk free interest rate, also a stochastic process. We now use Ito's lemma for the variables π_t, V_t and Y_t in order to find the optimal consumption. We use the notation $c = c(\pi_t, V_t, Y_t) := c_t^{\alpha}$, where the latter is given in (A.5), for $c : R^3 \to R$, a real and smooth function. First, we need to compute the following partial derivatives.

$$\frac{\partial c(\pi_t, V_t, Y_t)}{\partial \pi} = -\frac{1}{\rho}\left(\frac{c_t^{\alpha}}{\pi_t}\right), \quad \frac{\partial c(\pi_t, V_t, Y_t)}{\partial v} = -\frac{1}{\rho}\left(\frac{\gamma-\rho}{1-\gamma}\right)\left(\frac{c_t^{\alpha}}{V_t}\right),$$

$$\frac{\partial c(\pi_t, V_t, Y_t)}{\partial y} = \frac{1}{\rho}\left(\frac{c_t^{\alpha}}{Y_t}\right), \quad \frac{\partial^2 c(\pi_t, V_t, Y_t)}{\partial \pi^2} = \frac{1}{\rho}\left(\frac{1}{\rho}+1\right)\left(\frac{c_t^{\alpha}}{\pi_t^2}\right),$$

$$\frac{\partial^2 c(\pi_t, V_t, Y_t)}{\partial \pi \partial v} = \frac{\gamma-\rho}{\rho^2(1-\gamma)}\left(\frac{c_t^{\alpha}}{\pi_t V_t}\right), \quad \frac{\partial^2 c(\pi_t, V_t, Y_t)}{\partial v^2} = \frac{c_t^{\alpha}\,(\rho-\gamma)\,\gamma\,(\rho-1)}{V_t^2\,\rho^2\,(1-\gamma)^2}.$$

Since, Y_t is of bounded variation, we get the following from the multidimensional version of Ito's lemma

$$dc_t^{\alpha} = \frac{\partial c}{\partial \pi} d\pi_t + \frac{\partial c}{\partial v} dV_t + \frac{\partial c}{\partial y} dY_t + \frac{1}{2}\frac{\partial^2 c}{\partial \pi^2} d\pi_t^2 + \frac{\partial^2 c}{\partial \pi \partial v} d\pi_t dV_t + \frac{1}{2}\frac{\partial^2 c}{\partial v^2} dV_t^2.$$

Using the representations for the variables π_t given in (A.17), for V_t given in (A.9), (A.10) and (A.12) and for Y_t given in (A.7), (A.8) and (A.11), and the following partial derivative

$$f_v(c, v) := \frac{\partial f(c, v)}{\partial v} = \frac{\delta}{1-\rho}\left(c^{1-\rho}\big((1-\gamma)v\big)^{-\frac{1-\rho}{1-\gamma}}(\rho-\gamma) + (\gamma-1)\right), \tag{A.18}$$

this altogether gives

$$\frac{dc_t^{\alpha}}{c_t^{\alpha}} = \mu_c(t)\, dt + \sigma_c(t)\, dB_t, \tag{A.19}$$

where

$$\begin{aligned} \mu_c(t) = {} & \frac{1}{\rho}(r_t - \delta) + \frac{1}{2}\frac{1}{\rho}\left(1+\frac{1}{\rho}\right)\eta_t'\eta_t - \frac{(\gamma-\rho)}{\rho^2}\,\eta_t'\,\sigma_V(t) \\ & + \frac{1}{2}\frac{(\gamma-\rho)\gamma(1-\rho)}{\rho^2}\sigma_V'(t)\sigma_V(t), \end{aligned} \tag{A.20}$$

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and

$$\sigma_c(t) = \frac{1}{\rho}\Big(\eta_t + (\rho - \gamma)\sigma_V(t)\Big). \tag{A.21}$$

Here, $\psi = 1/\rho$, and σ_V(t) is as defined above. This gives the optimal consumption path c_t^α in (32) of Section 4.1 for any $\alpha > 0$, and is also the basis for the optimal pensions and the optimal amount of life insurance, satisfying the budget constraint of the agent, in this paper.

Finally, we address the existence issue of an optimal consumption: The utility function we work with satisfies $U(\lambda c) = \lambda U(c)$ for any $\lambda > 0$. As a consequence, there is a one-to-one correspondence between c_0 and α. To make things simple, suppose c_t^α is a geometric Brownian motion. Then

$$E\bigg(\int_0^\tau \pi_t c_0\, e^{\int_0^t (\mu_c(s) - \frac{1}{2}\sigma_c(s)'\sigma_c(s))ds + \int_0^t \sigma_c(s)dB_s} dt\bigg)$$
$$= c_0 E\bigg(\int_0^\tau \pi_t\, e^{\int_0^t (\mu_c(s) - \frac{1}{2}\sigma_c(s)'\sigma_c(s))ds + \int_0^t \sigma_c(s)dB_s} dt\bigg) = E\bigg(\int_0^\tau \pi_t\, e_t\, dt\bigg) < \infty. \tag{A.22}$$

The first term on the left-hand side is finite by Schwartz' inequality, since the variance of a geometric Brownian motion is finite. The first equality follows since $c_0 \in \mathbb{R}$ is a real constant. By assumption $e \in L_+$ so the last inequality follows. The second equality now follows by choice of c_0, so that the budget constraint holds with equality. For more general cases, see Schroder and Skiadas (1999) and Duffie and Lions (1992).