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Structure of small-amplitude quasiparallel magnetohydrodynamic shock waves in plasmas with anisotropic viscosity and thermal conductivity

M. S. RUDERMAN†

Departament of Mathematical and Computational Sciences, University of St Andrews, St Andrews, Fife KY16 9SS, Scotland, UK

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Shock waves in plasmas with strongly anisotropic viscosity and thermal conductivity are considered. The analysis is restricted to the case where the plasma beta is less than unity. The set of two equations that governs propagation of small-amplitude MHD waves at small angles with respect to the unperturbed magnetic field in such plasmas is derived. A qualitative analysis of this set of equations is carried out. It is shown that the shock structure is described by a solution that is a separatrix connecting two stationary points: a stable node and a saddle. This solution describes the structure of a fast quasiparallel shock wave, and it only exists when the ratio of the magnetic field component, perpendicular to the direction of shock-wave propagation after and before the shock is smaller than a critical value. This critical value is a function of the plasma beta. The structures of shock waves are calculated numerically for different values of the shock amplitude and the ratio of the coefficients of viscosity and thermal conductivity.

1. Introduction

Structures of small-amplitude quasiparallel MHD shock waves have been studied extensively. In order to describe these structures, the Cohen–Kurlsrud–Burgers (CKB) equation has been used (see e.g. Ruderman 1989; Kennel *et al.* 1990; Wu and Kennel 1992; Wu 1995). The CKB equation governs the propagation of small-amplitude nonlinear waves at small angles with respect to the unperturbed magnetic field in plasmas with isotropic viscosity and electrical conductivity. However, the solar coronal plasma is an example of a plasma with strongly anisotropic viscosity, electrical conductivity and thermal conductivity (see e.g. Priest 1982; Hollweg 1985; Ruderman *et al.* 1996). In the solar coronal plasma viscosity can be described by the first term of Braginskii's tensorial expression, which is at least five orders of magnitude larger than the other terms. The term that describes the heat flux along the magnetic field in Braginskii's expression for the heat flux is seven orders of magnetude larger than the other terms, which can consequently be neglected. Finite resistivity and the Hall effect can be neglected in comparison with viscosity

† Permanent address: Institute for Problems in Mechanics, Russian Academy of Sciences, 101 Vernadski Prospekt, 117526 Moscow, Russia.

and thermal conductivity (see e.g. the discussion in Ruderman *et al.* 1996). As a result, we arrive at a model of a plasma with strongly anisotropic viscosity and thermal conductivity.

The aim of the present paper is to study the structure of small-amplitude quasiparallel MHD shock waves in plasmas with strongly anisotropic viscosity and thermal conductivity. The paper is organized as follows. In the next section we present the set of basic equations and describe the main assumptions. In Sec. 3 we derive the set of two governing equations for small-amplitude quasiparallel waves. In Sec. 4 we give a qualitative analysis of the set of governing equations. In Sec. 5 an analytical solution describing the shock-wave structure for particular values of the parameters is presented and the results of numerical calculations of the shock-wave structures are given. Section 6 contains conclusions.

2. Basic equations

In accordance with the discussion in the previous section, the only dissipative processes taken into account are viscosity and thermal conductivity. We consider strongly magnetized plasmas where viscosity can be described by the first term of Braginskii's tensorial expression, so that the viscosity tensor $\hat{\boldsymbol{\pi}}$ takes the form (see Braginskii 1965)

$$\hat{\boldsymbol{\pi}} = \eta_0(\mathbf{b} \otimes \mathbf{b} - \tfrac{1}{3}\hat{\mathbf{I}})[3\mathbf{b} \cdot (\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{v} - \boldsymbol{\nabla} \cdot \mathbf{v}]\,, \tag{1}$$

where $\mathbf{b} = \mathbf{B}/B$ is the unit vector in the direction of the magnetic field and η_0 is the dynamic coefficient of viscosity. The thermal conductivity in the direction of the magnetic field strongly dominates the thermal conductivity in the directions perpendicular to the magnetic field. As a result, the expression for the heat flux can be written as

$$\mathbf{q} = -\kappa_\parallel \mathbf{b}(\mathbf{b} \cdot \boldsymbol{\nabla} T)\,, \tag{2}$$

where T is the temperature and $\kappa_\parallel$ is the coefficient of thermal conductivity. In what follows, we assume that the quantities η_0 and $\kappa_\parallel$ are constant.

We adopt Cartesian coordinates $(x,\ y,\ z)$ and consider one-dimensional plasma motions that depend on x only. With the use of (1) and (2), we write the set of MHD equations in the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0\,, \tag{3}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial \tilde{P}}{\partial x} + \frac{\eta_0}{\rho}\frac{\partial}{\partial x}\left[b_x^2\left(3b_x\mathbf{b}\cdot\frac{\partial \mathbf{v}}{\partial x} - \frac{\partial u}{\partial x}\right)\right], \tag{4}$$

$$\frac{\partial \mathbf{v}_\perp}{\partial t} + u\frac{\partial \mathbf{v}_\perp}{\partial x} = \frac{B_0}{\mu\rho}\frac{\partial \mathbf{B}_\perp}{\partial x} + \frac{\eta_0}{\rho}\frac{\partial}{\partial x}\left[b_x\mathbf{b}_\perp\left(3b_x\mathbf{b}\cdot\frac{\partial \mathbf{v}}{\partial x} - \frac{\partial u}{\partial x}\right)\right], \tag{5}$$

$$\frac{\partial \mathbf{B}_\perp}{\partial t} = \frac{\partial}{\partial x}(B_0\mathbf{v}_\perp - u\mathbf{B}_\perp)\,, \tag{6}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + (\gamma-1)T\frac{\partial u}{\partial x} = \frac{(\gamma-1)\kappa_\parallel}{R\rho}\frac{\partial}{\partial x}\left(b_x^2\frac{\partial T}{\partial x}\right) + \frac{(\gamma-1)\eta_0}{3R\rho}\left(3b_x\mathbf{b}\cdot\frac{\partial \mathbf{v}}{\partial x} - \frac{\partial u}{\partial x}\right)^2, \tag{7}$$

$$p = R\rho T\,. \tag{8}$$

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Here ρ is the density, p the pressure and u the x component of the velocity. B_0 is the x component of the magnetic field, which is constant in accordance with the induction equation and the equation of solenoidality. $\mathbf{v}_\perp$ and $\mathbf{B}_\perp$ are the components of the velocity and magnetic field perpendicular to the x axis. R and γ are the gas constant and the adiabatic exponent. The total pressure modified by viscosity is given by

$$\tilde{P} = p + \frac{B^2}{2\mu} + \frac{\eta_0}{3}\left(3b_x\mathbf{b}\cdot\frac{\partial\mathbf{v}}{\partial x} - \frac{\partial u}{\partial x}\right). \tag{9}$$

The set of equations (3)–(9) will be used in what follows in order to describe the structure of small-amplitude quasiparallel shock waves.

3. Derivation of governing equations for small-amplitude quasiparallel waves

We consider one-dimensional small-amplitude perturbations propagating in the positive direction of the x axis. All unperturbed quantities are constant. The plasma is assumed to be unperturbed at infinity, so that all quantites tend to their unperturbed values as $x \to \infty$. The term 'quasiparallel' means that the angle between the direction of wave propagation and the unperturbed magnetic field is small. This implies that the perpendicular component of the unperturbed magnetic field, $\mathbf{B}_{\perp 0}$, is small in comparison with the parallel component B_0: $|\mathbf{B}_{\perp 0}| \ll B_0$. In what follows, the subscript '0' indicates an unperturbed quantity.

We introduce the dimensionless characteristic amplitude of perturbations $\epsilon \ll 1$ and use the singular perturbation method in order to derive the governing equations for small-amplitude perturbations. In accordance with this method, we introduce the running variable $\xi = x - Vt$, where V is a constant velocity to be determined. In addition, we have to introduce the so-called 'slow' time. In the case of ideal plasmas the corresponding governing equation was derived by Cohen and Kurlsrud (1974). This equation contains a nonlinear term that is cubic with respect to the wave amplitude ϵ. As a result, the effect of nonlinearity becomes important on a timescale of the order of ϵ^{-2} multiplied by the wave period. We assume that the same is true in the case of viscous thermal conductive plasmas considered in the present paper and introduce the 'slow' time $\tau = \epsilon^2 t$. Then we rewrite (3)–(7) and (9) as

$$\epsilon^2\frac{\partial\rho}{\partial\tau} - V\frac{\partial\rho}{\partial\xi} + \frac{\partial(\rho u)}{\partial\xi} = 0\,, \tag{10}$$

$$\epsilon^2\frac{\partial u}{\partial\tau} - V\frac{\partial u}{\partial\xi} + u\frac{\partial u}{\partial\xi} = -\frac{1}{\rho}\frac{\partial\tilde{P}}{\partial\xi} + \frac{\eta_0}{\rho}\frac{\partial}{\partial\xi}\left[b_x^2\left(3b_x\mathbf{b}\cdot\frac{\partial\mathbf{v}}{\partial\xi} - \frac{\partial u}{\partial\xi}\right)\right], \tag{11}$$

$$\epsilon^2\frac{\partial\mathbf{v}_\perp}{\partial\tau} - V\frac{\partial\mathbf{v}_\perp}{\partial\xi} + u\frac{\partial\mathbf{v}_\perp}{\partial\xi} = \frac{B_0}{\mu\rho}\frac{\partial\mathbf{B}_\perp}{\partial\xi} + \frac{\eta_0}{\rho}\frac{\partial}{\partial\xi}\left[b_x\mathbf{b}_\perp\left(3b_x\mathbf{b}\cdot\frac{\partial\mathbf{v}}{\partial\xi} - \frac{\partial u}{\partial\xi}\right)\right], \tag{12}$$

$$\epsilon^2\frac{\partial\mathbf{B}_\perp}{\partial\tau} - V\frac{\partial\mathbf{B}_\perp}{\partial\xi} = \frac{\partial}{\partial\xi}(B_0\mathbf{v}_\perp - u\mathbf{B}_\perp), \tag{13}$$

$$\epsilon^2\frac{\partial T}{\partial\tau} - V\frac{\partial T}{\partial\xi} + u\frac{\partial T}{\partial\xi} + (\gamma-1)T\frac{\partial u}{\partial\xi} = \frac{(\gamma-1)\kappa_\parallel}{R\rho}\frac{\partial}{\partial\xi}\left(b_x^2\frac{\partial T}{\partial\xi}\right) + \frac{(\gamma-1)\eta_0}{3R\rho}\left(3b_x\mathbf{b}\cdot\frac{\partial\mathbf{v}}{\partial\xi} - \frac{\partial u}{\partial\xi}\right)^2, \tag{14}$$

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$$\tilde{P} = p + \frac{B^2}{2\mu} + \frac{\eta_0}{3}\left(3b_x \mathbf{b}\cdot\frac{\partial \mathbf{v}}{\partial \xi} - \frac{\partial u}{\partial \xi}\right). \tag{15}$$

We now look for the solution to the set of equations (8) and (10)–(15) in the form of expansions in series with respect to ϵ. For $\mathbf{v}_\perp$ and $\mathbf{B}_\perp$, these expansions take the form

$$\mathbf{v}_\perp = \epsilon \mathbf{v}_\perp^{(1)} + \epsilon^3 \mathbf{v}_\perp^{(3)} + \dots , \tag{16}$$

$$\mathbf{B}_\perp = \epsilon \mathbf{B}_\perp^{(1)} + \epsilon^3 \mathbf{B}_\perp^{(3)} + \dots \quad . \tag{17}$$

For all other quantities, we write the expansions as

$$f = f_0 + \epsilon^2 f^{(2)} + \dots , \tag{18}$$

where f_0 represents an unperturbed quantity. For consistency, we should write terms of order ϵ^2 in the expansions (16) and (17). However, we should then find that these terms are proportional to the first terms in the expansions (16) and (17), and consequently could be included in these first terms. Therefore, for the sake of mathematical simplicity, we drop these terms of order ϵ^2 from the very beginning.

Now, in order to derive governing equations for small-amplitude waves, we calculate the successive approximations with respect to ϵ. In the first-order approximation we collect terms of order ϵ in (10)–(15). Such terms are only present in (12) and (13), so that the equations of the first approximation are

$$V\frac{\partial \mathbf{v}_\perp^{(1)}}{\partial \xi} + \frac{B_0}{\mu\rho_0}\frac{\partial \mathbf{B}_\perp^{(1)}}{\partial \xi} = 0 , \tag{19}$$

$$V\frac{\partial \mathbf{B}_\perp^{(1)}}{\partial \xi} + B_0\frac{\partial \mathbf{v}_\perp^{(1)}}{\partial \xi} = 0 . \tag{20}$$

It follows from these equations that

$$\mathbf{v}_\perp^{(1)} = -\frac{V}{B_0}(\mathbf{B}_\perp^{(1)} - \mathbf{B}_{\perp 0}^{(1)}) , \qquad V = \frac{B_0}{(\mu\rho_0)^{1/2}} , \tag{21}$$

where $\mathbf{B}_{\perp 0}^{(1)} = \epsilon^{-1}\mathbf{B}_{\perp 0}$. When deriving (21) we have taken into account that $\mathbf{v}_\perp \to 0$ and $\mathbf{B}_\perp \to \mathbf{B}_{\perp 0}$ as $\xi \to \infty$.

In the second order approximation we collect terms of the order ϵ^2. Such terms are present in (8), (10), (11), (14), and (15), while they are absent in (12) and (13). As a result we have

$$V\frac{\partial \rho^{(2)}}{\partial \xi} - \rho_0\frac{\partial u^{(2)}}{\partial \xi} = 0 , \tag{22}$$

$$V\frac{\partial u^{(2)}}{\partial \xi} = \frac{1}{\rho_0}\frac{\partial \tilde{P}^{(2)}}{\partial \xi} - \frac{3\nu}{B_0}\frac{\partial}{\partial \xi}\left(\mathbf{B}_\perp^{(1)}\cdot\frac{\partial \mathbf{v}_\perp^{(1)}}{\partial \xi} + \tfrac{2}{3}B_0\frac{\partial u^{(2)}}{\partial \xi}\right), \tag{23}$$

$$(\gamma - 1)T_0\frac{\partial u^{(2)}}{\partial \xi} - V\frac{\partial T^{(2)}}{\partial \xi} = \chi\frac{\partial^2 T^{(2)}}{\partial \xi^2}, \tag{24}$$

$$\tilde{P}^{(2)} = p^{(2)} + \frac{1}{2\mu}(\mathbf{B}_\perp^{(1)})^2 + \frac{\rho_0\nu}{B_0}\left(\mathbf{B}_\perp^{(1)}\cdot\frac{\partial \mathbf{v}_\perp^{(1)}}{\partial \xi} + \tfrac{2}{3}B_0\frac{\partial u^{(2)}}{\partial \xi}\right), \tag{25}$$

$$p^{(2)} = R(\rho_0 T^{(2)} + T_0\rho^{(2)}) , \tag{26}$$

where

$$\nu = \frac{\eta_0}{\rho_0}, \qquad \chi = \frac{(\gamma - 1)\kappa_\parallel}{\rho_0 R}. \tag{27}$$

It follows from (21)–(26) that

$$\rho^{(2)} = \frac{\rho_0}{V} u^{(2)}, \tag{28}$$

and $u^{(2)}$ satisfies the equation

$$\begin{aligned}\tfrac{4}{3}\nu\chi \frac{\partial^2 u^{(2)}}{\partial \xi^2} &+ \left(\tfrac{4}{3}\nu V + \chi \frac{\gamma V^2 - c_S^2}{\gamma V}\right) \frac{\partial u^{(2)}}{\partial \xi} + (V^2 - c_S^2) u^{(2)} \\ &= \frac{\nu\chi V}{B_0^2} \frac{\partial^2 (\mathbf{B}_\perp^{(1)})^2}{\partial \xi^2} + \frac{(2\nu + \chi)V^2}{2B_0^2} \frac{\partial (\mathbf{B}_\perp^{(1)})^2}{\partial \xi} + \frac{V^3}{2B_0^2}[(\mathbf{B}_\perp^{(1)})^2 - (\mathbf{B}_{\perp 0}^{(1)})^2]. \end{aligned} \tag{29}$$

In the third-order approximation we collect terms of order ϵ^3 in (12) and (13) and obtain

$$\begin{aligned} V \frac{\partial \mathbf{v}_\perp^{(3)}}{\partial \xi} + \frac{B_0}{\mu \rho_0} \frac{\partial \mathbf{B}_\perp^{(3)}}{\partial \xi} &= \frac{\partial \mathbf{v}_\perp^{(1)}}{\partial \tau} + u^{(2)} \frac{\partial \mathbf{v}_\perp^{(1)}}{\partial \xi} + \frac{B_0}{\mu \rho_0^2} \rho^{(2)} \frac{\partial \mathbf{B}_\perp^{(1)}}{\partial \xi} \\ &\quad - \frac{\nu}{B_0^2} \frac{\partial}{\partial \xi} \left[\mathbf{B}_\perp^{(1)} \left(3\mathbf{B}_\perp^{(1)} \cdot \frac{\partial \mathbf{v}_\perp^{(1)}}{\partial \xi} + 2B_0 \frac{\partial u^{(2)}}{\partial \xi} \right) \right], \end{aligned} \tag{30}$$

$$V \frac{\partial \mathbf{B}_\perp^{(3)}}{\partial \xi} + B_0 \frac{\partial \mathbf{v}_\perp^{(3)}}{\partial \xi} = \frac{\partial \mathbf{B}_\perp^{(1)}}{\partial \tau} + \frac{\partial}{\partial \xi}(u^{(2)} \mathbf{B}_\perp^{(1)}). \tag{31}$$

The left-hand sides of (30) and (31) coincide with the left-hand sides of (19) and (20). The set of equations (19) and (20) considered as linear homogeneous algebraic equations with respect to $\partial \mathbf{v}_\perp^{(1)}/\partial \xi$ and $\partial \mathbf{B}_\perp^{(1)}/\partial \xi$ posseses a non-trivial solution. This implies that the set of equations (30) and (31) considered as linear inhomogeneous algebraic equations with respect to $\partial \mathbf{v}_\perp^{(3)}/\partial \xi$ and $\partial \mathbf{B}_\perp^{(3)}/\partial \xi$ is compatible only if the right-hand sides of (30) and (31) satisfy the compatibility condition. In order to derive this compatibility condition, we multiply (31) by V/B_0 and extract the result from (30). As a result, we obtain the equation for variables of the first and second approximations, which, with the use of (21) and (28) can be reduced to

$$\frac{\partial \mathbf{B}_\perp^{(1)}}{\partial \tau} + \frac{1}{2} \frac{\partial}{\partial \xi}(u^{(2)} \mathbf{B}_\perp^{(1)}) - \frac{3\nu}{4B_0^2} \frac{\partial}{\partial \xi} \mathbf{B}_\perp^{(1)} \left(\frac{\partial (\mathbf{B}_\perp^{(1)})^2}{\partial \xi} - \frac{4B_0^2}{3V} \frac{\partial u^{(2)}}{\partial \xi} \right) = 0. \tag{32}$$

We now return to the variables t and x and use the approximate equalitites $u \approx \epsilon^2 u^{(2)}$ and $\mathbf{B}_\perp \approx \epsilon \mathbf{B}_\perp^{(1)}$ to arrive at

$$\begin{aligned}\tfrac{4}{3}\nu\chi \frac{\partial^2 u}{\partial x^2} &+ \left(\tfrac{4}{3}\nu V + \chi \frac{\gamma V^2 - c_S^2}{\gamma V}\right) \frac{\partial u}{\partial x} + (V^2 - c_S^2) u \\ &= \frac{\nu\chi V}{B_0^2} \frac{\partial^2 (\mathbf{B}_\perp)^2}{\partial x^2} + \frac{(2\nu + \chi)V^2}{2B_0^2} \frac{\partial (\mathbf{B}_\perp)^2}{\partial x} + \frac{V^3}{2B_0^2}[(\mathbf{B}_\perp)^2 - (\mathbf{B}_{\perp 0})^2], \end{aligned} \tag{33}$$

$$\frac{\partial \mathbf{B}_\perp}{\partial t} + V \frac{\partial \mathbf{B}_\perp}{\partial x} + \frac{1}{2} \frac{\partial}{\partial x}(u \mathbf{B}_\perp) - \frac{3\nu}{4B_0^2} \frac{\partial}{\partial x} \mathbf{B}_\perp \left(\frac{\partial (\mathbf{B}_\perp)^2}{\partial x} - \frac{4B_0^2}{3V} \frac{\partial u}{\partial x} \right) = 0. \tag{34}$$

This set of equations describes small-amplitude quasiparallel waves in plasmas with strongly anisotropic viscosity and thermal conductivity under the assumption that

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the perturbations of all quantities except $\mathbf{B}_\perp$ vanish as x tends to infinity and $\mathbf{B}_\perp \to \mathbf{B}_{\perp 0}$ as $x \to \infty$. When $\nu = \chi = 0$, it is straightforward to show that the set of equations (33) and (34) is reduced to the Cohen–Kulsrud equation (Cohen and Kulsrud 1974).

In the next section the set of equations (33) and (34) is used in order to study the structure of small-amplitude quasiparallel shock waves.

4. Structure of shock waves: qualitative analysis

We look for solutions to the set of equations (33) and (34) that describe structures of shock waves. They are solutions in the form of travelling waves with permanent shapes moving in the positive direction of the x axis with velocities of shock waves. We write the velocity of a shock wave as $V(1+\epsilon^2 C)$, so that solutions describing the structures of shock waves depend on $\theta = l^{-1}[x - V(1+\epsilon^2 C)t]$, where $l = V^{-1}(\nu^2+\chi^2)^{1/2}$ is the dissipative length. Then (33) and (34) are reduced to

$$\tfrac{8}{3}\bar{\nu}\bar{\chi}\frac{d^2U}{d\theta^2} + [\tfrac{8}{3}\bar{\nu} + 2\bar{\chi}(1-\beta\gamma^{-1})]\frac{dU}{d\theta} + 2(1-\beta)U = 2\bar{\nu}\bar{\chi}\frac{d^2h^2}{d\theta^2} + (2\bar{\nu}+\bar{\chi})\frac{dh^2}{d\theta} + h^2 - h_0^2\,, \tag{35}$$

$$\left(2C - U - 2\bar{\nu}\frac{dU}{d\theta} + \tfrac{3}{2}\bar{\nu}\frac{dh^2}{d\theta}\right)\mathbf{h} = 2C\mathbf{h}_0\,, \tag{36}$$

where we use the notation

$$\beta = \frac{c_S^2}{V^2}\,, \qquad \bar{\nu} = \frac{\nu}{Vl}\,, \qquad \bar{\chi} = \frac{\chi}{Vl}\,, \qquad U = \frac{u}{\epsilon^2 V}\,, \qquad \mathbf{h} = \frac{\mathbf{B}_\perp}{\epsilon B_0}. \tag{37}$$

Note that $\bar{\nu}^2 + \bar{\chi}^2 = 1$.

Let us first consider the particular case where $C\mathbf{h}_0 = 0$. Then it follows from (36) that

$$2\bar{\nu}\frac{dU}{d\theta} + U = 2C + \tfrac{3}{2}\bar{\nu}\frac{dh^2}{d\theta}. \tag{38}$$

A solution to the set of equations (35) and (38) that describes the structure of a shock wave has to satisfy the condition that $h \to h_0$, $U \to 0$ as $\theta \to \infty$ and $h \to h_1$, $U \to U_1$ as $\theta \to -\infty$, where h_1 and U_1 are constants. However, (35) and (38) constitute a set of linear homogeneous differential equations with constant coefficients with respect to h^2 and U. This set of differential equations does not possess a non-trivial solution that tends to constant quantities as $\theta \to \pm\infty$. Hence in the case under consideration there are no solutions to the set of equations (35) and (36) that describe the structures of shock waves. In particular, there is no solution that describes the structure of the switch-on shock wave where $h \to 0$ as $\theta \to \infty$.

We now proceed to the general case where $C\mathbf{h}_0 \neq 0$. In this case, in particular $\mathbf{B}_{\perp 0} \neq 0$. Up to now, ϵ has been an arbitrary small constant. In what follows we determine it exactly and take $\epsilon = |\mathbf{B}_{\perp 0}|/B_0$, so that $h_0 = 1$. Then it is straightforward to see that ϵ is approximately equal to the angle between the equilibrium magnetic field and the direction of wave propagation. We choose a coordinate system such that $\mathbf{h}_0$ is in the positive direction of the y axis. Then it follows from (36) that $h_z \equiv 0$ and h_y does not change sign because it cannot take the value zero. This, in particular, implies that there are no solutions to the set of equations (35) and (36) describing the structure of intermediate shock waves. Since $h_y \to 1$ as $\theta \to \infty$, this quantity is positive everywhere, so that $h_y = h$ and we can rewrite (36) as

$$2\bar{\nu}\frac{dU}{d\theta} + U = 3\bar{\nu}h\frac{dh}{d\theta} + 2C\frac{h-1}{h}. \tag{39}$$

We substitute (39) into (35) to arrive at

$$\tfrac{2}{3}[4\bar{\nu} + \bar{\chi}(1 - 3\beta\gamma^{-1})]\frac{dU}{d\theta} + 2(1-\beta)U = 2\left[2\bar{\nu}h + \bar{\chi}\left(h - \frac{4C}{3h^2}\right)\right]\frac{dh}{d\theta} + h^2 - 1\,. \tag{40}$$

It is straightforward to reduce (39) and (40) to a set of two ordinary differential equations of first order:

$$\kappa(h)\frac{dh}{d\theta} = -3\bar{\nu}\gamma h(h-1)[4C(1-\beta) - h(h+1)] - \zeta h^2\left(U - 2C\frac{h-1}{h}\right), \tag{41}$$

$$\begin{aligned}\kappa(h)\frac{dU}{d\theta} = {} & -\tfrac{9}{2}\bar{\nu}\gamma h^2(h-1)[4C(1-\beta) - h(h+1)] \\ & -\tfrac{1}{2}[3\zeta h^3 + \bar{\nu}^{-1}\kappa(h)]\left(U - 2C\frac{h-1}{h}\right),\end{aligned} \tag{42}$$

where

$$\kappa(h) = \bar{\nu}\bar{\chi}[8\gamma C - 3(\gamma + 3\beta)h^3]\,, \qquad \zeta = 2\gamma\bar{\nu}(1-3\beta) - \bar{\chi}(\gamma - 3\beta)\,. \tag{43}$$

There are three stationary points of the set of equations (41) and (42), given by $h = 1$, $U = 0$ and $h = h_{1,2}$, $U = U_{1,2}$, where $h_{1,2}$ are the roots of the equation

$$h^2 + h - 4(1-\beta)C = 0\,, \tag{44}$$

and

$$U_{1,2} = 2C(1 - h_{1,2}^{-1})\,. \tag{45}$$

These three positions are the stationary points of the set of equations (41) and (42) if $\kappa(1) \neq 0$ and $\kappa(h_{1,2}) \neq 0$.

When $\zeta \neq 0$, the right-hand sides of (41) and (42) are also simultaneously equal to zero when $h = h_3$ and $U = U_3$, where h_3 and U_3 are given by

$$\kappa(h_3) = 0\,, \tag{46a}$$

$$U_3 = \frac{3(h_3 - 1)}{4\gamma\zeta}\{4\gamma^2\bar{\nu}(h_3+1) - h_3^2(\gamma + 3\beta)[4\gamma\bar{\nu} + \bar{\chi}(\gamma - 3\beta)]\}\,. \tag{46b}$$

However, the point (h_3, U_3) is not a stationary point, since $\kappa(h)$ is also a coefficient of the derivatives on the left-hand sides of (41) and (42). Hence (h_3, U_3) is a singular point of the set of equations (41) and (42).

The solution to the set of equations (41) and (42) that describes the structure of a shock wave has to be a separatrix connecting two stationary points. Since $h > 0$, the whole separatrix has to be in the right half of the phase plane (h, U). Hence the necessary condition for the existence of the solution describing the structure of a shock wave is that equation (44) possesses a positive root. This condition takes the form $(1-\beta)C > 0$. In what follows, we denote the positive root of (44) by h_1. Then $h_2 = -(1 + h_1) < 0$, and the separatrix that describes the structure of a shock wave has to connect the stationary points $(1, 0)$ and (h_1, U_1).

In what follows, we restrict our analysis to the case $\beta < 1$, as in the solar corona.

Let us study the stationary point (h_1, U_1). First, we note that (h_1, U_1) is not a stationary point when $\kappa(h_1) = 0$. Then there is no solution that tends to (h_1, U_1) when $\theta \to -\infty$, and consequently there is no solution to the set of equations (41) and (42) describing the structure of a shock wave. Therefore we assume that $\kappa(h_1) \neq 0$ in what follows. We now introduce the perturbations of quantities h and U, and write them as

$$h = h_1 + \tilde{h}\,, \qquad U = U_1 + \tilde{U}\,. \tag{47}$$

We now substitute (47) into (41) and (42), and linearize the obtained equations with respect to $\tilde{h}$ and $\tilde{U}$. As a result, we arrive at

$$\kappa_1 \frac{d\tilde{h}}{d\theta} = a_{11}\tilde{h} + a_{12}\tilde{U}\,, \tag{48a}$$

$$\kappa_1 \frac{d\tilde{U}}{d\theta} = a_{21}\tilde{h} + a_{22}\tilde{U}\,, \tag{48b}$$

where $\kappa_1 = \kappa(h_1)$, and the coefficients a_{ij} are given by

$$a_{11} = 2[\gamma\bar{\nu}(3h_1^3 - 4C) - C\bar{\chi}(\gamma - 3\beta)]\,, \tag{49a}$$

$$a_{12} = -\zeta h_1^2\,, \tag{49b}$$

$$a_{21} = \gamma h_1^{-2}(3h_1^3 - 4C)(3h_1^3\bar{\nu} - 2C\bar{\chi})\,, \tag{49c}$$

$$a_{22} = -\gamma[3\bar{\nu}(1 - 3\beta)h_1^3 - \bar{\chi}(3h_1^3 - 4C)]\,. \tag{49d}$$

In deriving (49), we have used the relation

$$4C(1 - \beta) = h_1(h_1 + 1)\,. \tag{50}$$

The characteristic equation takes the form

$$\kappa_1^2\lambda^2 - \kappa_1\lambda(a_{11} + a_{22}) + a_{11}a_{22} - a_{12}a_{21} = 0\,, \tag{51}$$

where λ is the characteristic exponent. For the discriminant D of this equation, we obtain

$$D = \kappa_1^2\{\gamma\bar{\nu}[8C - 3(1 + 3\beta)h_1^3] + 3\bar{\chi}[\gamma h_1^3 - 2(\gamma - \beta)C]\}^2 + 12\kappa_1^2\bar{\nu}\bar{\chi}\beta\gamma(\gamma - 1)(4C - 3h_1^3)^2\,, \tag{52}$$

so that $D > 0$. This implies that the stationary point (h_1, U_1) is either a node or a saddle. In order to distinguish between these two possibilities, we calculate $\det(a_{ij})$. As a result, we have

$$\det(a_{ij}) \equiv a_{11}a_{22} - a_{12}a_{21} = -\tfrac{3}{2}\gamma\kappa_1 h_1(h_1 - 1)(2h_1 + 1)\,. \tag{53}$$

The quantity $\kappa(h_1)$ plays an important role in the analysis. It is straightforward to show that $\kappa_1 > 0$ when $h_1 < h_c$, and $\kappa_1 < 0$ when $h_1 > h_c$, where h_c is given by

$$h_c = \frac{\gamma + [\gamma^2 + 6\gamma(1 - \beta)(\gamma + 3\beta)]^{1/2}}{3(1 - \beta)(\gamma + 3\beta)}. \tag{54}$$

The dependence of h_c on β is shown in Fig. 1 for $\gamma = \frac{5}{3}$. The function $h_c(\beta)$ decreases monotonically for $\beta < \frac{1}{6}(3 - \gamma)$ and increases monotonically for $\beta > \frac{1}{6}(3 - \gamma)$. When $\beta = \beta_m = \frac{1}{6}(3 - \gamma)$, it takes its minimum value

$$h_c^m = \frac{2}{(3 + \gamma)^2}\{2\gamma + [2\gamma(\gamma^2 + 8\gamma + 9)]^{1/2}\}\,.$$

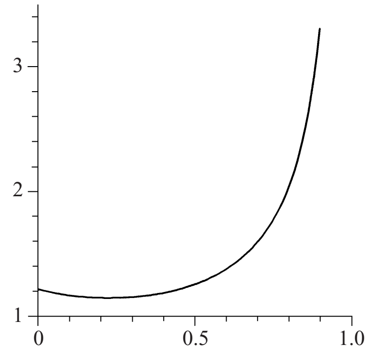

Figure 1. The dependence of h_c on β for $\gamma = \frac{5}{3}$.

It is straightforward to show that $h_c^m > 1$. For $\gamma = \frac{5}{3}$, we have $\beta_m = \frac{2}{9}$ and $h_c^m \approx 1.146$. For $\beta \ll 1$, as in the solar corona, $h_c = \frac{1}{3}(1 + 7^{1/2}) \approx 1.215$ for any $\gamma > 1$.

Let us now consider three cases:

(i) $0 < h_1 < 1$. In this case $\kappa_1 > 0$, so that $\det(a_{ij}) > 0$, and the stationary point (h_1, U_1) is a node. In addition, we get

$$a_{11} + a_{22} = -\frac{h_1}{2(1-\beta)}\{2\gamma\bar{\nu}[(1-3\beta)^2 h_1^2 + 2(1-h_1)(1+2h_1)] \\ +3\bar{\chi}[\gamma(1-\beta)(1-h_1)(1+2h_1) + \beta(\gamma-1)(1+h_1)]\} < 0\,. \quad (55)$$

This inequality implies that both roots of (51) are negative, and consequently the stationary point (h_1, U_1) is a stable node. Hence there is no integral curve that tends to (h_1, U_1) as $\theta \to -\infty$, i.e. no integral curve can start from (h_1, U_1), and consequently there is no solution to the set of equations (41) and (42) that describes the structure of a shock wave.

(ii) $1 < h_1 < h_c$. In this case $\det(a_{ij}) < 0$, and the point (h_1, U_1) is a saddle. Consequently there are exactly two integral curves starting from this point.

(iii) $h_1 > h_c$. In this case $\kappa_1 < 0$ and $h_1 > 1$, so that $\det(a_{ij}) > 0$, and the stationary point (h_1, U_1) is once again a node. The quantity $a_{11} + a_{22}$ can be written as

$$a_{11} + a_{22} = \frac{2C\bar{\nu}\bar{\chi}[12\beta\gamma(\gamma-1)\bar{\nu} + (\gamma-3\beta)^2\bar{\chi}] - \gamma\kappa_1[(1+3\beta)\bar{\nu} + \bar{\chi}]}{\bar{\nu}\bar{\chi}(\gamma+3\beta)} \\ > 0\,, \quad (56)$$

so that (h_1, U_1) is once again a stable node, and there is no solution describing the structure of a shock wave.

Let us now study the stationary point $(1, 0)$. Since the separatrix does not exist for $h_1 \leqslant 1$ and $h_1 \geqslant h_c$, we restrict our analysis to the case where $1 < h_1 < h_c$. In order to study the stationary point $(1, 0)$, we take $h = 1 + \tilde{h}$ and $U = \tilde{U}$, and linearize the set of equations (41) and (42) with respect to $\tilde{h}$ and $\tilde{U}$. As a result, we arrive at the same set of linear equations (48) with the coefficients a_{ij} given by the same expressions (49), but with 1 substituted for h_1. In particular, κ_1 is replaced

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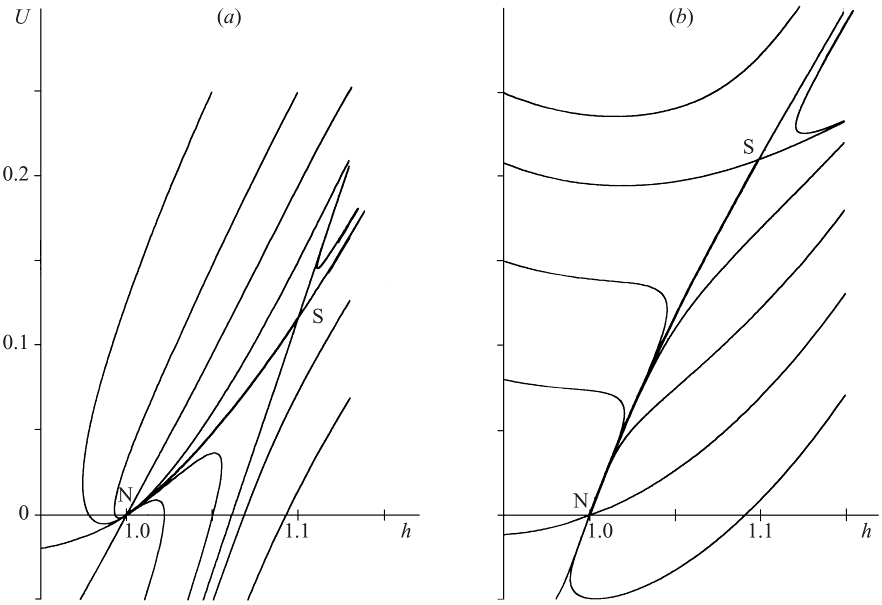

Figure 2. Integral curves of the set of equations (41) and (42) for $\bar{\nu} = \bar{\chi} = 2^{-1/2}$, $\gamma = \frac{5}{3}$, and $h_1 = 1.1$: (*a*) $\beta = 0.15$ and $\zeta > 0$; (*b*) $\beta = 0.5$ and $\zeta < 0$.

by $\kappa_0 = \kappa(1)$. Correspondingly, the expression for D is given by (52) with 1 and κ_0 substituted for h_1 and κ_1, so that $D > 0$. In addition, we have

$$\det(a_{ij}) = \tfrac{3}{2}\gamma\kappa_0(h_1 - 1)(h_1 + 2)\,, \tag{57}$$

$$a_{11} + a_{22} = -(1-\beta)^{-1}\big\{\gamma\bar{\nu}[(1-3\beta)^2 + 2(h_1 - 1)(h_1 + 2)] + \tfrac{3}{2}\bar{\chi}[(\gamma - \beta)(h_1 - 1)(h_1 + 2) + 2\beta(\gamma - 1)]\big\} < 0\,. \tag{58}$$

It is straightforward to obtain

$$\kappa_0 = (1-\beta)^{-1}[2\gamma h_1(h_1+1) - 3(1-\beta)(\gamma + 3\beta)] > (1-\beta)^{-1}[(1-3\beta)^2 + (\gamma - 1)(1 + 3\beta)] > 0\,. \tag{59}$$

The inequality $D > 0$ together with (57)–(59) implies that the stationary point $(1, 0)$ is a stable node.

Summarizing the analysis of the stationary points, we conclude that the separatrix starting from the stationary point (h_1, U_1) and ending at the stationary point $(1, 0)$ can exist if $1 < h_1 < h_c$. It is shown in the Appendix that this separatrix does exist. Hence the ratio of the perpendicular component of the magnetic field after and before the shock wave is h_1. The quantity $h_1 - 1$ can be considered as the dimensionless shock-wave amplitude. To avoid misunderstanding, we have to make the following point. In this paper we consider small-amplitude shock waves. However, this does not mean that the quantity $h_1 - 1$ is small. Instead, it means that perturbations of all quantities in the shock structure are of order $\epsilon \ll 1$. To ensure this, it is enough to impose the condition that $h_1|\mathbf{B}_{\perp 0}|$ be of order ϵB_0. Since $\epsilon = |\mathbf{B}_{\perp 0}|/B_0$, this condition is reduced to $h_1 \sim 1$. As we have already seen, the solution describing the shock structure only exists when $h_1 < h_c$. Since $h_c \sim 1$ for

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all $\beta < 1$ except for β very close to unity, the condition $h_1 \sim 1$ is automatically satisfied unless β is very close to unity.

Two illustrative examples of the picture of integral curves in the phase plane (h, U) are shown in Fig. 2, where (*a*) corresponds to $\zeta > 0$ and (*b*) to $\zeta < 0$. N and S indicate the (stable) node and the saddle. Since the component of the magnetic field perpendicular to the propagation direction increases across the shock wave and does not change sign, the separatrix describes the structure of a fast shock wave. Hence, in contrast to the CKB equation (see e.g. Wu 1995), the set of equations (33) and (34) describes neither the structure of a switch-on shock wave nor the structures of an intermediate shock wave.

5. Structure of shock waves: analytical solution and numerical examples

In this section we present an analytical solution for particular parameter values and the results of numerical calculations.

Let us consider the particular case $\zeta = 0$. Although this condition has no special physical meaning, it is important from a mathematical point of view. We shall see in what follows that an analytical expression for the separatrix can be obtained when $\zeta = 0$. The condition $\zeta = 0$ can be satisfied when either $\beta < \frac{1}{3}$ or $\beta > \frac{1}{3}\gamma$. If β is in one of these two intervals, this condition determines the (positive) ratio $\bar{\nu}/\bar{\chi}$. Since, by definition, $\bar{\nu}^2 + \bar{\chi}^2 = 1$, we then obtain

$$\bar{\nu} = \frac{|\gamma - 3\beta|}{[4\gamma^2(1 - 3\beta)^2 + (\gamma - 3\beta)^2]^{1/2}}, \tag{60a}$$

$$\bar{\chi} = \frac{2\gamma|1 - 3\beta|}{[4\gamma^2(1 - 3\beta)^2 + (\gamma - 3\beta)^2]^{1/2}}. \tag{60b}$$

When $\zeta = 0$, (41) reduces with the aid of (50) to

$$\kappa(h)\frac{dh}{d\theta} = -3\gamma\bar{\nu}h(h - 1)(h_1 - h)(h + h_1 + 1)\,, \tag{61}$$

so that the structure of a shock wave is described by

$$\theta = -\frac{1}{3\gamma\bar{\nu}}\int^h \frac{\kappa(\bar{h})\,d\bar{h}}{\bar{h}(\bar{h} - 1)(h_1 - \bar{h})(\bar{h} + h_1 + 1)}. \tag{62}$$

In principle, the integral on the right-hand side of (62) can be expressed in terms of elementary functions. However, this expression is very long and complicated, so we do not write it down here.

Let us now obtain an analytical expression for the separatrix in the phase plane. The set of equations (41) and (42) can be reduced to a single linear equation for the function U. In principle, the solution to this equation can be found in the form of a quadrature. However, the calculations are very lengthy, and, in addition, it is not so trivial to select the separatrix from the obtained one-parameter family of integral curves. Therefore we prefer to use a different approach and simply guess the general form of the solution corresponding to the separatrix. We look for the solution to the set of equations (41) and (42) in the form

$$U(h) = 2\alpha C\frac{h - 1}{h} + (1 - \alpha)\frac{h^2 - 1}{2(1 - \beta)}, \tag{63}$$

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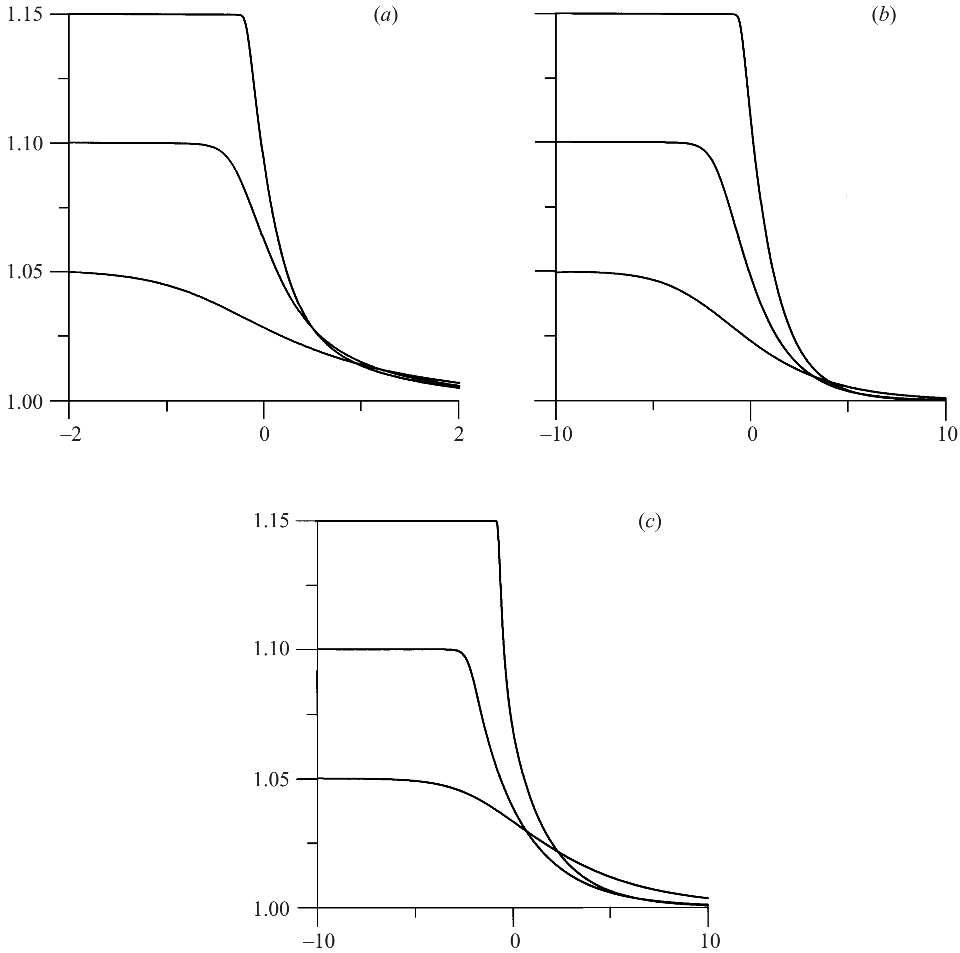

Figure 3. The dimensionless perpendicular component of the magnetic field h as a function of θ in the shock wave structure for $\gamma = \frac{5}{3}$ and $\beta = 0.1$: (a) $\bar{\nu}/\bar{\chi} = 0.1$; (b) 1; (c) 10. The lower, middle and upper curves in all three parts correspond to $h_1 = 1.05$, 1.1 and 1.15 respectively.

where α is a constant to be determined. It is straightforward to check that $U(1) = 0$ and $U(h_1) = U_1$, so that the curve determined by (63) connects the stationary points $(1, 0)$ and (h_1, U_1) for any value of α. We substitute (63) into (41) and (42) and find that (63) gives a solution when $\zeta = 0$ and α is given by

$$\alpha = -\frac{2\gamma(1 - 3\beta)}{\gamma - 9\beta + 3\gamma\beta + 9\beta^2}. \tag{64}$$

Hence in the case where $\zeta = 0$ the separatrix is determined by

$$U = F(h) \equiv \frac{(h-1)[3h(h+1)(\gamma - 3\beta) - 8\gamma C(1 - 3\beta)]}{2h(\gamma - 9\beta + 3\gamma\beta + 9\beta^2)}. \tag{65}$$

It can be shown that $F(h)$ increases monotonically both for $\beta < \frac{1}{3}$ and for $\beta > \frac{1}{3}\gamma$.

In Fig. 3 the behaviour of h in the shock wave structure is shown for $\beta = 0.1$, $\gamma = \frac{5}{3}$, and different values of $\bar{\nu}/\bar{\chi}$ and h_1. We can see that the thickness of the shock structure grows when h_1 is decreased. The characteristic thickness of the shock-wave structure is of the order of the dissipative length l when $\bar{\nu} \ll \bar{\chi}$, while it is a few times larger than l when $\bar{\nu} \gtrsim \bar{\chi}$. Hence the quantity l can be considered as a characteristic thickness of the shock structure for all values of the coefficients of viscosity and thermal conductivity and for all but very small (< 0.05) values of the dimensionless shock-wave amplitude $h_1 - 1$. The important property is that the thickness of the structure is independent of the angle ϵ between the equilibrium magnetic field and the direction of wave propagation. This result is in contrast to the corresponding result for the CKB equation, where the thickness of the shock-wave structure is proportional to $\epsilon^{-2}\nu_i$ (when there is no electrical resistivity or thermal conductivity), where ν_i is the coefficient of isotropic viscosity.

6. Conclusions

In this paper the structures of small-amplitude quasiparallel shock waves in plasmas with strongly anisotropic viscosity and thermal conductivity have been studied. A set of two equations for the parallel velocity and the perpendicular component of the magnetic field has been derived with the use of the singular perturbation method. This set of two equations describes the propagation of small-amplitude perturbations at small angles with respect to the equilibrium magnetic field and with phase velocities close to the Alfvén velocity. In particular, it describes the structures of small-amplitude quasiparallel MHD shock waves. The analysis of the shock-wave structures here has been restricted to the case where the plasma beta is smaller than unity. The plasma beta was introduced as the ratio of the sound speed to the Alfvén speed. The quantity h_1, which is the ratio of the values of the magnetic field component perpendicular to the direction of the wave propagation after and before a shock wave, plays an important role in the analysis. The quantity $h_1 - 1$ can be considered as the shock-wave amplitude. It has been shown that the derived set of two equations describes the structure of fast shock waves where $h_1 > 1$ under the condition that $h_1 < h_c$. The quantity h_c depends on the plasma beta, and tends to infinity when the plasma beta tends to unity. In contrast to the CKB equation, the derived set of two equations does not describe the structure of a switch-on shock wave or the structure of an intermediate shock wave.

The fact that the derived set of equations does not describe the structures of fast shock waves when $h_1 \geqslant h_c$, switch-on shock waves or intermediate shock waves implies that in these cases the strongly anisotropic viscosity and thermal conductivity alone cannot build-up the shock-wave structure. Other dissipative mechanisms, such as isotropic shear viscosity, isotropic thermal conductivity and/or finite electrical resistivity, have to be taken into account.

Note that the solution found in the present paper does not describe the structure of slow shocks either. However, this results from the assumption that $\beta < 1$. The set of equations (33) and (34) is obtained under the assumption that perturbations propagate with a velocity close to V. The velocities of fast and slow small-amplitude quasiparallel shocks are close to V and c_S respectively when $\beta < 1$, while they are close to c_S and V respectively when $\beta > 1$. Hence the set of equations (33) and (34) does not describe the structure of a slow shock when $\beta < 1$, while it does not describe the structure of a fast shock when $\beta > 1$.

The structures of shock waves have been calculated numerically for different values of the coefficients of the anisotropic viscosity and thermal conductivity and for different values of h_1. The numerical results show that for all but very small

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(< 0.05) values of the dimensionless shock-wave amplitude $h_1 - 1$, the thickness of the shock structure is of order $l = V^{-1}(\nu^2 + \chi^2)^{1/2}$, where ν is the kinematic coefficient of viscosity and χ is proportional to the coefficient of thermal conductivity (see (27)). The important property is that this thickness is independent of the angle ϵ between the equilibrium magnetic field and the direction of shock-wave propagation. This result is in contrast to the theory of small-amplitude quasiparallel shock waves in isotropic plasmas described by the CKB equation. The characteristic thickness of the structure of these shock waves is proportional to ϵ^{-2}.

Let us now apply the results obtained to some typical coronal conditions. We take the temperature $T_0 = 10^6$ K, the electron number density $n_e = 10^{15}$ m^{-3} and the magnetic field $B_0 = 1$ mT. Then, for the collisional times of electrons and ions, we obtain $\tau_e \approx 2 \times 10^{-2}$ s and $\tau_i \approx 1$ s. The electron and ion gyrofrequencies are $\omega_e \approx 2 \times 10^8$ s^{-1} and $\omega_i \approx 10^5$ s^{-1}, so that $\tau_e\omega_e \approx 4 \times 10^6$ and $\tau_i\omega_i \approx 10^5$. Braginskii (1965) gives the following approximate expressions for η_0 and $\kappa_\parallel$:

$$\eta_0 \approx k_B n_e T_0 \tau_i \,, \qquad \kappa_\parallel \approx 3k_B^2 m_e^{-1} n_e T_0 \tau_e \,, \tag{66}$$

where k_B is the Boltzmann constant (we assume that the temperatures of ions and electrons are equal and so are their number densities). With the use of (66), we obtain $\eta_0 \approx 1.4 \times 10^{-2}$ kg s^{-1} m^{-1} and $\kappa_\parallel \approx 1.2 \times 10^4$ kg m s$^{-3}K^{-1}$. Then $\nu \approx 10^{10}$ m^2 s^{-1} and $\chi \approx 4.3 \times 10^{11}$ m^2 s^{-1}. For the Alfvén velocity, we have $V \approx 2.2 \times 10^6$ m s^{-1}, and finally we obtain for the characteristic thickness of the shock wave structure $l \approx 2 \times 10^5$ m.

For the isotropic coefficient of viscosity, we get (see Braginskii 1965) $\nu_i \approx \nu(\tau_i\omega_i)^{-2} \approx 1$ m^2 s, and the characteristic thickness of the structure of a shock wave is of the order $\epsilon^{-2}\nu_i V^{-1} \approx 5 \times 10^{-6}\epsilon^{-2}$ m. Even for an extremely small shock-wave amplitude $\epsilon = 10^{-2}$, we obtain that the characteristic thickness of a shock wave is 5×10^{-2} m, i.e. six orders of magnitude smaller than the characteristic thickness of shock waves in plasmas with strongly anisotropic viscosity and thermal conductivity. Hence the use of the isotropic viscosity for the estimation of the characteristic thickness of shock-wave structures in the solar corona can lead to incorrect results.

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Appendix

In this appendix we proof that the separatrix connecting the two stationary points $(1, 0)$ and (h_1, U_1) of the set of equations (41) and (42) does exist under the conditions that $\beta < 1$ and $1 < h_1 < h_c$. We give only a sketch of the proof, omitting details.

Let us rewrite the set of equations (41) and (42) as

$$\frac{d\mathbf{W}}{d\theta} = \mathbf{H}(h, U) \,, \tag{A 1}$$

where $\mathbf{W} = (h, U)$ and the phase velocity vector $\mathbf{H} = (H_1, H_2)$ is determined by the right-hand sides of (41) and (42). We start our proof with a consideration of the curve in the phase plane determined by the condition $H_1 = 0$. It is straightforward

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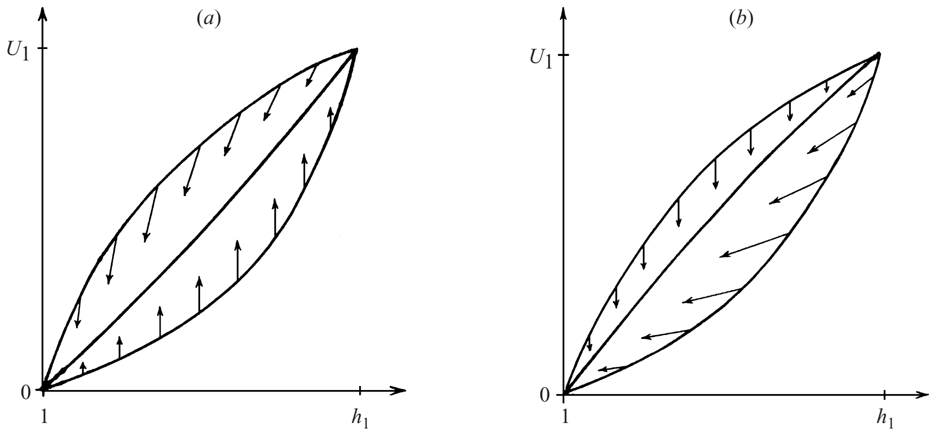

Figure 4. Reciprocal location of the separatrix and the curves $U = F(h)$ and $U = G(h)$ in the phase plane: (*a*) $\zeta > 0$; (*b*) $\zeta < 0$. The curve $U = F(h)$ is the upper curve in (*a*) and the lower curve in (*b*). The curve $U = G(h)$ is the lower curve in (*a*) and the upper curve in (*b*). In both parts the separatrix is shown by the middle curve. The arrows show the phase velocity vector **H**.

to show that this curve is given by

$$U = G(h) \equiv \frac{h-1}{\zeta h}\{3\gamma\bar{\nu}h(h+1) - 2C[4\gamma\bar{\nu} + \bar{\chi}(\gamma - 3\beta)]\} . \qquad \text{(A 2)}$$

It is easy to see that $G(1) = 0$ and $G(h_1) = U_1$, so that this curve connects the stationary points $(1, 0)$ and (h_1, U_1). When $U = G(h)$, we have

$$H_2 = \frac{3\gamma(h-1)(h_1 - h)(h + h_1 + 1)}{2\zeta h}, \qquad \text{(A 3)}$$

so that the phase velocity vector **H** is directed upwards when $\zeta > 0$ and downwards when $\zeta < 0$. The curve $U = G(h)$ is shown in Fig. 4(*a*) for $\zeta > 0$ (the lower curve) and in Fig. 4(*b*) for $\zeta < 0$ (the upper curve). The arrows show the vector **H**.

Let us now consider the curve $U = F(h)$, where $F(h)$ is given by (65). It is straightforward to find that

$$\zeta[F(h) - G(h)] = \frac{3(h-1)(h_1 - h)(h + h_1 + 1)[12\gamma\beta\bar{\nu}(\gamma - 1) + \bar{\chi}(\gamma - 3\beta)^2]}{2h(\gamma - 9\beta + 3\gamma\beta + 9\beta^2)} > 0. \qquad \text{(A 4)}$$

Hence $F(h) > G(h)$ for $\zeta > 0$, and $F(h) < G(h)$ for $\zeta < 0$. In Fig. 4(*a*), $U = F(h)$ is the upper curve, while in Fig. 4(*b*) it is the lower curve. We denote the region between the curves $U = F(h)$ and $U = G(h)$ by $\mathscr{D}$. It can be shown that at the curve $U = F(h)$ the following inequalities are valid for the components of the phase velocity vector:

$$H_1 < 0 , \qquad \zeta\left(\frac{H_2}{H_1} - \frac{dF}{dh}\right) > 0 . \qquad \text{(A 5)}$$

It follows from (A 5) that at the curve $U = F(h)$ the phase velocity vector is directed into the region $\mathscr{D}$. Since the phase velocity vector is directed into $\mathscr{D}$ at its boundary, no integral curve can leave $\mathscr{D}$.

Let $\mathbf{W} = \overline{\mathbf{W}}(\theta) \equiv (\bar{h}(\theta), \overline{U}(\theta))$ be the equation of the integral curve that leaves the saddle point (h_1, U_1) in the unstable direction and satisfies the condition $d\bar{h}/d\theta < 0$ at this point. In the vicinity of the point (h_1, U_1) there is an inverse function $\theta = \bar{\theta}(h)$, so that the equation of this curve can be written as $U = \overline{U}(h) \equiv \overline{U}(\bar{\theta}(h))$. It can be shown that at $h = h_1$ the inequality

$$\left(\frac{d\overline{U}}{dh} - \frac{dF}{dh}\right)\left(\frac{d\overline{U}}{dh} - \frac{dG}{dh}\right) < 0 \tag{A 6}$$

holds. This inequality implies that the curve $U = \overline{U}(h)$ lies in the region $\mathscr{D}$ for $0 < h_1 - h \ll 1$. Then, in accordance with the results of this appendix, the integral curve $\mathbf{W} = \overline{\mathbf{W}}(\theta)$ cannot leave the region $\mathscr{D}$. Therefore there are only two possibilities. The first is that the curve $\mathbf{W} = \overline{\mathbf{W}}(\theta)$ ends at the same stationary point (h_1, U_1). The second is that this curve ends at the stationary point $(1, 0)$. However, it can be shown that the two integral curves that come to the stationary point (h_1, U_1) from the stable directions lie outside the region $\mathscr{D}$ in the vicinity of this stationary point, so that the first possibility cannot be realized. Therefore the integral curve $\mathbf{W} = \overline{\mathbf{W}}(\theta)$ has to end at the stationary point $(1, 0)$. Hence the separatrix connecting the stationary points (h_1, U_1) and $(1, 0)$ does exist.

It is straightforward to see that $H_1 < 0$ inside the region $\mathscr{D}$, so that $dh/d\theta < 0$. This implies that h increases monotonically in the shock-wave structure when θ changes from $+\infty$ to $-\infty$.

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