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Existence of positive solutions for a semipositone *p*-Laplacian problem

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We prove the existence of positive solutions to a semipositone p-Laplacian problem combining mountain pass arguments, comparison principles, regularity principles and *a priori* estimates.

Keywords: mountain pass theorem; semipositone problem; positive solutions; p-Laplacian; maximum principles; *a priori* estimates

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1. Introduction

In this paper we study the existence of *positive* weak solutions to the problem

$$\left.\begin{aligned} -\Delta_p u &= \lambda f(u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where $\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian operator, $p > 2$. Ω is an open smooth bounded domain in $\mathbb{R}^N$, $N > 2$. The function $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f(0) < 0$ (*semipositone*). We assume that there exist $q \in (p-1, Np/(N-p) - 1)$, $A > 0$, $B > 0$ such that

$$\left.\begin{aligned} A(u^q - 1) \leqslant f(u) &\leqslant B(u^q + 1) && \text{for } u > 0, \\ f(u) &= 0 && \text{for } u \leqslant -1. \end{aligned}\right\} \tag{1.2}$$

We also assume an Ambrosetti–Rabinowitz type of condition, namely that there exist $\theta > p$ and $M \in \mathbb{R}$ such that

$$uf(u) \geqslant \theta F(u) + M, \tag{1.3}$$

where

$$F(u) = \int_0^u f(s)\,\mathrm{d}s.$$

The assumption $f(0) < 0$ implies that $u = 0$ is not a subsolution to (1.1), making the finding of positive solutions rather challenging; this was pointed out in [6].

The aim of this paper is to prove the following result.

THEOREM 1.1. *There exists $\lambda^* > 0$ such that if $\lambda \in (0, \lambda^*)$, then the problem (1.1) has a positive weak solution $u_\lambda \in C^{1,\beta}(\bar{\Omega})$ for some $\beta \in (0, 1)$.*

Our results extend [1, theorem 1.1], where the case $p = 2$ was studied. Extending such a theorem to $p > 2$ is not straightforward due to the lack of regularity and linearity of Δ_p. Associated to (1.1) we have a functional, which will be defined in the next section. We show that this functional has a critical point of mountain pass type and, consequently, a weak solution of (1.1) for appropriate values of $\lambda > 0$. Finally, using order properties of $-\Delta_p$, we prove that by further restricting λ such a solution is actually positive. For recent results on semipositone problems the reader is referred to [2, 3].

2. Preliminary results

Let $W_0^{1,p}(\Omega)$ denote the Banach space of functions in $L^p(\Omega)$ with first-order partial derivatives in $L^p(\Omega)$ and vanishing on $\partial\Omega$. By a weak solution to (1.1) we mean an element $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \,\mathrm{d}x = \lambda \int_\Omega f(u)\phi \,\mathrm{d}x \tag{2.1}$$

for all $\phi \in W_0^{1,p}(\Omega)$. We denote by $\|\cdot\|_s$ the norm in the space $L^s(\Omega)$ and by $\|\cdot\|_{1.p}$ the norm in the Sobolev space $W_0^{1,p}(\Omega)$.

Associated to (1.1) we have the functional $J_\lambda \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$J_\lambda(u) := \int_\Omega \frac{|\nabla u(x)|^p}{p}\,\mathrm{d}x - \int_\Omega \lambda F(u(x))\,\mathrm{d}x, \tag{2.2}$$

where

$$F(s) := \int_0^s f(r)\,\mathrm{d}r.$$

It is well known that J_λ is a functional of class C^1 (see [7]) and that the critical points of the functional J_λ are the weak solutions of (1.1). The proof of theorem 1.1 consists of two main steps:

(i) the proof of existence of one solution via the mountain pass theorem,

(ii) the proof that for proper values of λ the solution is indeed positive.

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It follows from (1.2) that there exist positive real numbers A_1, B_1 such that

$$F(u) \leqslant B_1(|u|^{q+1} + 1) \quad \text{for all } u \in \mathbb{R} \tag{2.3}$$

and

$$F(u) \geqslant A_1(|u|^{q+1} - 1) \quad \text{for all } u \geqslant 0. \tag{2.4}$$

For simplicity of the notation, we define $r = 1/(q + 1 - p) > 0$. Let $\varphi \in W_0^{1,p}(\Omega)$ denote a positive differentiable function with $\|\varphi\|_{1,p} = 1$. Let us define the constant

$$c = (2p^{-1}A_1^{-1}\|\varphi\|_{q+1}^{-q-1})^r, \tag{2.5}$$

which will be used in the next lemma.

The next two lemmas prove that J_λ satisfies the geometric hypotheses of the mountain pass theorem.

Lemma 2.1. *There exists $\lambda_1 > 0$ such that if $\lambda \in (0, \lambda_1)$, then $J_\lambda(c\lambda^{-r}\varphi) \leqslant 0$.*

Proof. Let $s = c\lambda^{-r}$, with c and r as defined above. Hence, due to (2.4),

$$\begin{aligned} J_\lambda(s\varphi) &= \int_\Omega \left\{ \frac{|\nabla(s\varphi)|^p}{p} - \lambda F(s\varphi) \right\} \mathrm{d}x \\ &\leqslant \frac{s^p}{p} - \lambda A_1 \int_\Omega (s^{q+1}\varphi^{q+1} - 1)\,\mathrm{d}x \\ &= \frac{s^p}{p} - A_1 s^{q+1}\|\varphi\|_{q+1}^{q+1}\lambda + \lambda A_1|\Omega| \\ &= c^p\left\{ \frac{\lambda^{-rp}}{p} - \lambda A_1 c^{q+1-p}\lambda^{-r(q+1)}\|\varphi\|_{q+1}^{q+1} \right\} + \lambda A_1|\Omega|. \end{aligned} \tag{2.6}$$

Substituting (2.5) into (2.6) yields

$$\begin{aligned} J_\lambda(s\varphi) &\leqslant c^p\left(\frac{\lambda^{-rp}}{p} - \frac{2}{p}\lambda^{1-r(q+1)} \right) + \lambda A_1|\Omega| \\ &= c^p\lambda^{-rp}\left(\frac{1}{p} - \frac{2}{p}\lambda^{1+rp-r(q+1)} \right) + \lambda A_1|\Omega| \\ &= -c^p\lambda^{-rp}\frac{1}{p} + \lambda A_1|\Omega|. \end{aligned} \tag{2.7}$$

Taking $\lambda_1 < \min\{1, (pA_1c^{-p}|\Omega|)^{-1/(1+pr)}\}$, the lemma is proven. □

Lemma 2.2. *There exist $\tau > 0$, $c_1 > 0$, and $\lambda_2 \in (0, 1)$ such that if $\|u\|_{1,p} = \tau\lambda^{-r}$, then $J_\lambda(u) \geqslant c_1(\tau\lambda^{-r})^p$ for all $\lambda \in (0, \lambda_2)$.*

Proof. By the Sobolev embedding theorem there exists $K_1 > 0$ such that if $u \in W_0^{1,p}(\Omega)$, then $\|u\|_{q+1} \leqslant K_1\|u\|_{1,p}$. Let

$$\tau = \min\{(2pK_1^{q+1}B_1)^{-r}, c\|\varphi\|_{1,p}\}. \tag{2.8}$$

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If $\|u\|_{W_0^{1,p}} = \tau\lambda^{-r}$, then

$$\begin{aligned} J_\lambda(u) &= \frac{(\tau\lambda^{-r})^p}{p} - \int_\Omega \lambda F(u) \\ &\geqslant \frac{(\tau\lambda^{-r})^p}{p} - \lambda \int_\Omega B_1 |u|^{q+1} - \lambda|\Omega| B_1 \\ &\geqslant \frac{(\tau\lambda^{-r})^p}{p} - \lambda B_1 K_1^{q+1} \|\nabla u\|_p^{q+1} - \lambda|\Omega| B_1 \\ &= \frac{(\tau\lambda^{-r})^p}{p} - \lambda B_1 K_1^{q+1} (\tau\lambda^{-r})^{q+1} - \lambda|\Omega| B_1 \\ &= \lambda^{-rp}\left[\frac{\tau^p}{2p} - \lambda^{1+rp}|\Omega| B_1\right] \\ &\geqslant \lambda^{-rp}\frac{\tau^p}{4p}, \end{aligned} \tag{2.9}$$

where we have used that $\tau \leqslant (2pK_1^{q+1}B_1)^{-r}$ (see (2.8)). Taking $c_1 = \tau^p/(4p)$ and $\lambda_2 = \tau^{p/(1+rp)}(4pB_1|\Omega|)^{-1/(1+rp)}$, the lemma is proven. □

Next, using the mountain pass theorem we prove that (1.1) has a solution $u_\lambda \in W_0^{1,p}(\Omega)$.

Lemma 2.3. *Let $\lambda_3 = \min\{\lambda_1, \lambda_2\}$. There exists $c_2 > 0$ such that, for each $\lambda \in (0, \lambda_3)$, the functional J_λ has a critical point u_λ of mountain pass type that satisfies $J_\lambda(u_\lambda) \leqslant c_2 \lambda^{-pr}$.*

Proof. First we show that J_λ satisfies the Palais–Smale condition.

Assume that $\{u_n\}_n$ is a sequence in $W_0^{1,p}(\Omega)$ such that $\{J_\lambda(u_n)\}_n$ is bounded and $J_\lambda'(u_n) \to 0$. Hence, there exists $\nu > 0$ such that $\langle J_\lambda'(u_n), u_n\rangle \leqslant \|\nabla u_n\|_p$ for $n \geqslant \nu$. Thus,

$$-\|\nabla u_n\|_p^p - \|\nabla u_n\|_p \leqslant -\lambda \int_\Omega f(u_n) u_n \,\mathrm{d}x \quad \text{for } n \geqslant \nu.$$

Let K be a constant such that $|J_\lambda(u_n)| \leqslant K$ for all $n = 1, 2, \dots$. From (1.3), we obtain

$$\frac{1}{p}\|\nabla u_n\|_p^p - \frac{\lambda}{\theta}\int_\Omega f(u_n) u_n \,\mathrm{d}x + \frac{\lambda}{\theta} M|\Omega| \leqslant \frac{1}{p}\|\nabla u_n\|_p^p - \lambda \int_\Omega F(u_n)\,\mathrm{d}x \leqslant K.$$

From the last two inequalities we have

$$\left(\frac{1}{p} - \frac{1}{\theta}\right)\|\nabla u_n\|_p^p - \frac{1}{\theta}\|\nabla u_n\|_p \leqslant K - \frac{\lambda}{\theta} M|\Omega|.$$

This proves that $\{u_n\}$ is a bounded sequence. Thus, without loss of generality, we may assume that $\{u_n\}$ converges weakly. Let $u \in W_0^{1,p}(\Omega)$ be its weak limit. Since $q < Np/(N-p)$, by the Sobolev embedding theorem we may assume that $\{u_n\}$ converges to u in $L^q(\Omega)$. These assumptions and Hölder's inequality imply

$$\int_\Omega \lambda f(u_n)(u_n - u) \to 0. \tag{2.10}$$

From (2.10) and $\lim_{n\to+\infty} J'_\lambda(u_n) = 0$ we have

$$\lim_{n\to+\infty} \int_\Omega |\nabla u_n|^{p-2}\nabla u_n(\nabla u_n - \nabla u)\,\mathrm{d}x = 0. \tag{2.11}$$

Using again that u is the weak limit of $\{u_n\}$ in $W_0^{1,p}(\Omega)$ we also have

$$\lim_{n\to+\infty} \int_\Omega |\nabla u|^{p-2}\nabla u(\nabla u_n - \nabla u)\,\mathrm{d}x = 0. \tag{2.12}$$

By Hölder's inequality,

$$\begin{aligned}\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)(\nabla u_n - \nabla u)\,\mathrm{d}x&\\ \geqslant \|\nabla u_n\|_p^p - \|\nabla u\|_p\|\nabla u_n\|_p^{p-1} - \|\nabla u_n\|_p\|\nabla u\|_p^{p-1} + \|\nabla u\|_p^p&\\ = (\|\nabla u_n\|_p^{p-1} - \|\nabla u\|_p^{p-1})(\|\nabla u_n\|_p - \|\nabla u\|_p)&\\ \geqslant 0.&\end{aligned} \tag{2.13}$$

From (2.11)–(2.13),

$$\lim_{n\to\infty} (\|\nabla u_n\|_p^{p-1} - \|\nabla u\|_p^{p-1})(\|\nabla u_n\|_p - \|\nabla u\|_p) = 0,$$

which implies that $\lim_{n\to\infty} \|\nabla u_n\|_p = \|\nabla u\|_p$. Since $u_n \rightharpoonup u$, $u_n \to u$ in $W_0^{1,p}$. This proves that J_λ satisfies the Palais–Smale condition.

From (2.6) we see that

$$\begin{aligned}\max\{J_\lambda(s\varphi); s \geqslant 0\} &\leqslant \frac{C^{1+pr}((q+1)^{r(q-p)} - p)}{D^{pr}p(q+1)^{r(q+1)}}\lambda^{-pr} + \lambda A_1|\Omega|\\ &:= c_2'\lambda^{-pr} + \lambda A_1|\Omega| \leqslant c_2'\lambda^{-pr} + A_1|\Omega|\lambda^{-pr}\\ &:= c_2\lambda^{-pr},\end{aligned} \tag{2.14}$$

where $C = \|\nabla\varphi\|_p^p$ and $D = A_1\|\varphi\|_{q+1}^{q+1}$.

With this estimate and lemma 2.2, the existence of $u_\lambda \in W_0^{1,p}(\Omega)$ such that $\nabla J_\lambda(u_\lambda) = 0$ and

$$c_1(\tau\lambda^{-r})^p \leqslant J_\lambda(u_\lambda) \leqslant c_2\lambda^{-pr} \tag{2.15}$$

follows by the mountain pass theorem. □

Remark 2.4. The solution $u_\lambda \in W_0^{1,p}(\Omega)$ is indeed in $C^{1,\alpha}(\bar{\Omega})$ (cf. [5]).

Lemma 2.5. *Let u_λ be as in lemma 2.3. Then there is a positive constant M_0 such that*

$$M_0\lambda^{-r} \leqslant \|u_\lambda\|_\infty. \tag{2.16}$$

Proof. We already know that there exists $c_1 > 0$ such that $J(u_\lambda) \geqslant c_1\lambda^{-rp}$. On the other hand, we have that $F(s) \geqslant \min F > -\infty$ and $f(s)s \leqslant B_1(|s|^{q+1} + |s|)$ for all

$s \in \mathbb{R}$. Then there is a constant $C_1 > 0$ such that

$$\begin{aligned}\lambda \int_\Omega f(u_\lambda)u_\lambda \,\mathrm{d}x &= \int_\Omega |\nabla u_\lambda|^p \,\mathrm{d}x \\ &= pJ(u_\lambda) + p\lambda \int_\Omega F(u_\lambda)\,\mathrm{d}x \\ &\geqslant pC_1\lambda^{-rp} + p|\Omega|\lambda \min F \\ &\geqslant C_1\lambda^{-rp}.\end{aligned}$$

Thus, $\lim_{\lambda\to 0} \|u_\lambda\|_\infty = +\infty$. On the other hand, by (2.3),

$$\begin{aligned}\lambda \int_\Omega f(u_\lambda)u_\lambda \,\mathrm{d}x &\leqslant B_1\lambda \int_\Omega (|u_\lambda|^{q+1} + |u_\lambda|)\,\mathrm{d}x \\ &\leqslant B_1\lambda \int_\Omega (\|u_\lambda\|_\infty^{q+1} + \|u_\lambda\|_\infty)\,\mathrm{d}x \\ &\leqslant 2B_1|\Omega|\lambda\|u_\lambda\|_\infty^{q+1},\end{aligned}$$

where we have used the fact that $0 < \lambda < 1$. Finally, taking $M_0 = C_1/2B_1|\Omega|$, the lemma is proven. □

LEMMA 2.6. *Let u_λ be as in lemma 2.3. Then there exists $c_3 > 0$ such that*

$$\|u_\lambda\|_{1,p}^p \leqslant c_3\lambda^{-pr} \tag{2.17}$$

for all $\lambda \in (0, \lambda_3)$.

Proof. By (1.3) and the definition of u_λ,

$$\begin{aligned}\lambda \int_\Omega \frac{\theta - p}{\theta} u_\lambda f(u_\lambda)\,\mathrm{d}x &\leqslant \lambda \int_\Omega (u_\lambda f(u_\lambda) - pF(u_\lambda))\,\mathrm{d}x - \frac{\lambda pM|\Omega|}{\theta} \\ &= \int_\Omega (|\nabla u_\lambda|^p - p\lambda F(u_\lambda))\,\mathrm{d}x - \frac{\lambda pM|\Omega|}{\theta} \\ &\leqslant c_2\lambda^{-rp} + \frac{\lambda pM|\Omega|}{\theta} \\ &\leqslant 2c_2\lambda^{-rp}, \end{aligned} \tag{2.18}$$

where we have used $0 < \lambda < 1$. Now the result follows from (2.18) and the fact that u_λ is a weak solution of (1.1). □

3. Proof of theorem 1.1

We prove theorem 1.1 by contradiction. Suppose there exists a sequence $\{\lambda_j\}_j$, $1 > \lambda_j > 0$ for all j, converging to 0 such that the measure $m(\{x \in \Omega;\ u_{\lambda_j}(x) \leqslant 0\}) > 0$.

Letting $w_j = u_{\lambda_j}/\|u_{\lambda_j}\|_\infty$, we see that

$$-\Delta_p(w_j) = \lambda_j f(u_{\lambda_j})\|u_{\lambda_j}\|_\infty^{1-p}. \tag{3.1}$$

From lemmas 2.5 and 2.6 there is a constant C_3 such that

$$\|w_j\|_{1,p} \leqslant C_3. \tag{3.2}$$

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By [4, proposition 3.7] the sequence w_j is uniformly bounded in $C^{1,\alpha}$ for some $\alpha \in (0,1)$. Hence, for any $\beta \in (0,\alpha)$, the sequence w_j has a subsequence that converges in $C_0^{1,\beta}$. Let us denote its limit by w.

Next, using comparison principles, we prove that $w(x) \geqslant 0$.

Let $v_0 \in W_0^{1,p}(\Omega)$ be the solution of

$$\left.\begin{aligned} -\Delta_p v_0 &= 1 \quad \text{in } \Omega, \\ v_0 &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.3}$$

Let $K_j := \lambda_j \min\{f(t);\ t \in \mathbb{R}\} \|u_{\lambda_j}\|_\infty^{1-p}$. Then the solution v_j of the equation

$$\left.\begin{aligned} -\Delta_p v_j &= K_j \quad \text{in } \Omega, \\ v &= 0 \qquad \text{on } \partial\Omega \end{aligned}\right\} \tag{3.4}$$

is given by $v_j = (-K_j)^{1/(p-1)} v_0$.

Since $\lambda_j f(u_{\lambda_j}) \|u_{\lambda_j}\|_\infty^{1-p} \geqslant K_j$, it follows by the comparison principle in [9] that $w_j \geqslant v_j$. Then the fact that $v_j(x) \to 0$ as $j \to 0$ implies that $w(x) \geqslant 0$ for all $x \in \Omega$.

Since, by hypothesis, $q > p-1$, we have $s = Npr/(N-p) > 1$. This result, together with the Sobolev embedding theorem, (1.2) and lemma 2.6, gives

$$\begin{aligned} \int_\Omega |f(u_{\lambda_j})|^s \|u_{\lambda_j}\|_\infty^{s(1-p)} \,\mathrm{d}x &\leqslant B^s 2^{s-1} \int_\Omega (|u_{\lambda_j}|^{(q+1-p)s} + 1) \,\mathrm{d}x \\ &\leqslant C(\|u_{\lambda_j}\|_{1,p}^{Np/(N-p)} + 1) \\ &\leqslant C(c_3 \lambda_j^{-rNp/(N-p)} + 1), \end{aligned} \tag{3.5}$$

where $C > 0$ is a constant independent of j and, without loss of generality, we have assumed $\|u_{\lambda_j}\|_\infty \geqslant 1$. From (3.5) and the fact that $rNp/(sN - sp) = 1$ we see that $\{\lambda_j f(u_{\lambda_j}) \|u_{\lambda_j}\|_\infty^{1-p}\}$ is bounded in $L^s(\Omega)$, so we may assume that it converges weakly. Let $z \in L^s(\Omega)$ be the weak limit of such a sequence. Since $\|u_{\lambda_j}\|_\infty^{1-p} \lambda_j \to 0$ as $j \to +\infty$ and f is bounded from below, $z \geqslant 0$. Now if $\phi \in C_0^\infty(\Omega)$, then

$$\begin{aligned} \int_\Omega \|\nabla w\|^{p-2} \langle \nabla w, \nabla \phi \rangle \,\mathrm{d}x &= \lim_{j\to\infty} \int_\Omega \|\nabla w_j\|^{p-2} \langle \nabla w_j, \nabla \phi \rangle \,\mathrm{d}x \\ &= \lim_{j\to\infty} \int_\Omega \|u_{\lambda_j}\|_\infty^{1-p} \|\nabla u_{\lambda_j}\|^{p-2} \langle \nabla u_{\lambda_j}, \nabla \phi \rangle \,\mathrm{d}x \\ &= \lim_{j\to\infty} \int_\Omega \|u_{\lambda_j}\|_\infty^{1-p} \lambda_j f(u_{\lambda_j}) \phi \,\mathrm{d}x \\ &= \int_\Omega z\phi \,\mathrm{d}x. \end{aligned} \tag{3.6}$$

Therefore, $-\Delta_p w = z$. Since $\|w_j\|_\infty = 1$, $w \neq 0$. By Hopf's maximum principle for the p-Laplacian operator (see [8, theorem 5.1]), $w > 0$ in Ω and

$$\frac{\partial w}{\partial \nu}(x) < 0 \quad \text{for all } x \in \partial\Omega.$$

Here $\partial/\partial n$ denotes the outward unit normal derivative. Therefore, since $\{w_j\}_j$ converges in $C^{1,a}$ to w, for sufficiently large j, $w_j(x) > 0$ for all $x \in \Omega$. Hence,

$u_{\lambda_j}(x) > 0$ for all $x \in \Omega$, which contradicts the assumption that

$$m(\{x; u_{\lambda_j}(x) < 0\}) > 0.$$

This contradiction proves theorem 1.1.

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