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The unified method for the heat equation: I. non-separable boundary conditions and non-local constraints in one dimension

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We use the heat equation as an illustrative example to show that the unified method introduced by one of the authors can be employed for constructing analytical solutions for linear evolution partial differential equations in one spatial dimension involving non-separable boundary conditions as well as non-local constraints. Furthermore, we show that for the particular case in which the boundary conditions become separable, the unified method provides an easier way for constructing the relevant classical spectral representations avoiding the classical spectral analysis approach. We note that the unified method always yields integral expressions which, in contrast to the series or integral expressions obtained by the standard transform methods, are uniformly convergent at the boundary. Thus, even for the cases that the standard transform methods can be implemented, the unified method provides alternative solution expressions which have advantages for both numerical and asymptotic considerations. The former advantage is illustrated by providing the numerical evaluation of typical boundary value problems.

Key words: initial-boundary value problems; heat equation; unified method; non-separable boundary conditions; non-local constraints

1 Introduction

Self-adjoint boundary value problems for linear partial differential equations (PDEs) formulated in a separable domain and involving separable boundary conditions can be solved via the standard transform methods. The unified method introduced by one of the authors [5, 6], although originally developed for integrable *nonlinear* PDEs, has had important implications for *linear* PDEs (see [7, 11]). In particular, it can be used to construct analytical solutions to certain linear boundary value problems which are either non-self-adjoint or involve non-separable boundary conditions; see [13–15] for applications of the unified method to the classical spectral theory of non-self-adjoint linear differential operators, and [2, 9] for recent reviews comparing the unified method with the standard approaches.

Even for the simple cases where the standard transform methods can be applied, the implementation of the classical techniques for constructing the relevant classical spectral representations is quite cumbersome. For example, Cohen [1] in order to solve a particular boundary value problem for the heat equation formulated on the half-line, had to construct the following spectral representation:

$$q(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos kx + k\sin kx}{k^2+1}\left[\int_0^\infty (\cos k\xi + k\sin k\xi)\, q(\xi)d\xi - q(0)\right] dk + 2e^{-x}\left[q(0) - \int_0^\infty e^{-\xi} q(\xi)d\xi\right]. \tag{1.1}$$

The derivation of (1.1) is rather elaborate, see [1] and also the relevant discussion in [8]. Analogously, the authors of [12], in order to study a particular boundary value problem for the heat equation formulated on the finite interval involving certain non-local constraints, had to perform a rather involved spectral analysis; in this case, the relevant eigenvalues, denoted by k, satisfy the *transcendental* equation

$$lk\sin(lk) + 2\cos(lk) - 2 = 0, \tag{1.2}$$

where l is the length of the interval.

The main advantages of the unified method are as follows: (i) It can be implemented in a straightforward way *without* the need to derive a spectral representation, and (ii) it yields an integral expression which is uniformly convergent at the boundary. Regarding (ii), we recall that the solution obtained via the standard transform methods has the serious disadvantage that it is *not* uniformly convergent at the boundary (unless the boundary conditions are homogeneous). Thus, since the unified method is easier to implement than the standard transform methods, and since it yields uniformly convergent solutions, it is clear that it is preferable to the standard transform methods, even in the case that the standard transform methods can indeed be applied. In particular, (a) it yields an effective approach for the numerical evaluation of typical boundary value problems (see [3,4]), and (b) it provides a straightforward approach to obtaining rigorous results.

Our goal in this paper is twofold: (1) Using as an example the heat equation in one dimension, we show that the unified method can be implemented in a straightforward way for problems involving non-separable boundary conditions as well as non-local constraints, and (2) we emphasise a new application of the unified method, namely that it can be used to construct classical spectral representations (such as (1.1) as well as the spectral representations associated with (1.2)) in an easier way than the standard approaches.

In Section 2 we solve the heat equation on the half-line with given initial data and with an oblique Robin boundary condition at $x = 0$,

$$u_t(0,t) + \alpha u_x(0,t) + \beta u(0,t) = \gamma(t), \quad t > 0, \tag{1.3}$$

where α, β are real constants and $\gamma(t)$ is a given function. The particular case of $\beta = \gamma = 0$ is the case considered by Cohen in [1]; using the general solution formula obtained by the unified method, we also derive in Section 2 equation (1.1) in a straightforward way.

In Section 3 we solve the heat equation on the finite interval $0 < x < l$ with given initial data with the boundary condition (1.3) at $x = 0$, and with an oblique Robin boundary condition at $x = l$,

$$u_t(l,t) + Au_x(l,t) + Bu(l,t) = \Gamma(t), \quad t > 0, \tag{1.4}$$

where A, B are real constants and $\Gamma(t)$ is a given function.

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In Section 4 we solve the heat equation on the finite interval $0 < x < l$ satisfying the non-local constraints

$$\int_0^l u(x,t)dx = \mathscr{F}(t), \quad \int_0^l (l-x)\,u(x,t)dx = \mathscr{R}(t), \tag{1.5}$$

where $\mathscr{F}(t)$ and $\mathscr{R}(t)$ are given functions. Furthermore, using the general solution formula obtained by the unified method, we derive in a straightforward way the classical spectral representation associated with equation (1.2).

In Section 5, using the technique introduced in [4] we show that the integral expressions obtained by the unified method can be evaluated numerically in a straightforward manner. Indeed, for typical initial and boundary data, this evaluation only requires a few lines of Mathematica.

The unified method. For the heat equation in one dimension, the unified method involves the following steps:

(1) Rewrite the heat equation

$$u_t = u_{xx} \tag{1.6}$$

in the divergence form

$$\left(e^{-ikx+k^2t}u\right)_t = \left[e^{-ikx+k^2t}\left(u_x + iku\right)\right]_x, \quad k \in \mathbb{C}. \tag{1.7}$$

Use Green's theorem in the plane in the given domain to obtain the so-called global relation, namely an equation coupling the Fourier transform of $u(x,t)$, denoted by $\hat{u}(k,t)$, with appropriate transforms of the initial data and the boundary values.

(2) Solve the global relation for $\hat{u}(k,t)$ and use the inverse Fourier transform to obtain $u(x,t)$ in terms of transforms of the initial data and the boundary values. Deform the contour of integration from the real line to an appropriate contour in the complex k-plane to obtain an integral representation of $u(x,t)$. This representation is not yet effective because it involves transforms of unknown boundary values.

(3) Use the global relation, as well as the equation obtained from the global relation under the transformation $k \mapsto -k$, to eliminate the transforms of the unknown boundary values to derive an effective solution expression (effective in the sense that it only involves transforms of the given data).

Classical spectral representations. For problems for which there exists a spectral representation, it is possible to deform the contours of integration from the complex plane back to the real line. The resulting expressions evaluated at $t = 0$ yields the classical spectral representation.

Numerical evaluations. Suppose the Fourier transform of the given data can be computed in closed form. Then the numerical evaluation of the solution involves only the computation of a single k-integral. By performing suitable contour deformations in the complex k-plane, it is possible to deform to paths in regions where the integrands decay exponentially for large k. This yields rapid convergence.

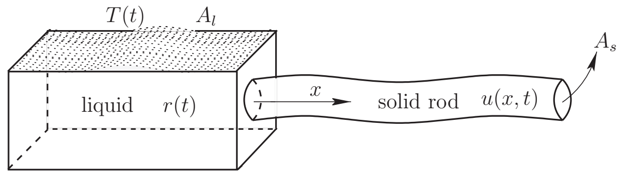

FIGURE 1. Heat conduction between the tank and the solid rod.

Rigorous considerations. An important advantage of the unified method is that it provides a straightforward approach to proving rigorous results. This is particularly simple for the case that the given data have 'sufficient smoothness and decay' (see Appendix). Analogous results for more restricted function spaces are given in [10].

2 The heat equation on the half-line with oblique Robin boundary conditions

Consider a tank filled with liquid, open at the top and with free surface area A_l. A horizontal semi-infinite solid rod is placed in direct contact with the tank on one of the vertical sides so that heat conduction takes place between the two. The liquid, of arbitrary initial temperature, absorbs heat from its surroundings which has a temperature $T(t)$, and is kept well stirred so that its temperature $r(t)$ is uniform throughout its volume. The lateral surface of the rod is insulated so that there is no heat flow to the surrounding medium. Moreover, since the length of the rod is infinite, the temperature u of the rod can also be regarded as uniform across the cross section A_s, i.e. we can assume that it is only a function of the distance x from the point of contact with the tank and of the time t (Figure 1).

Hence, $u(x,t)$ obeys the one-dimensional heat equation

$$u_t = u_{xx}, \quad 0 < x < \infty, \quad t > 0. \tag{2.1}$$

At time $t = 0$ we have

$$u(x,0) = u_0(x), \quad 0 < x < \infty. \tag{2.2}$$

Moreover, at the point of contact $x = 0$ we impose the boundary condition

$$u(0,t) = r(t), \tag{2.3}$$

where $r(t)$ is the unknown temperature of the liquid.

The rate of accumulation of heat in the liquid can be expressed in terms of the incoming heat from the surrounding medium and the heat which flows from the tank to the rod, i.e.

$$\sigma \frac{dr}{dt}(t) = -u_x(0,t) + \lambda \left[T(t) - r(t)\right], \tag{2.4}$$

where the real constants σ and λ depend on the specific heat, the mass and the conductivities of the rod and the liquid.

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Combining equations (2.3) and (2.4), we obtain the following boundary condition, known as an *oblique Robin* condition:

$$\sigma u_t(0,t) + u_x(0,t) + \lambda\,[u(0,t) - T(t)] = 0. \tag{2.5}$$

For simplicity of notation, we let

$$\alpha = \frac{1}{\sigma}, \quad \beta = \frac{\lambda}{\sigma}, \quad \gamma(t) = \frac{\lambda}{\sigma}\,T(t), \tag{2.6}$$

so that the boundary condition (2.5) becomes

$$u_t(0,t) + \alpha u_x(0,t) + \beta u(0,t) = \gamma(t). \tag{2.7}$$

Equations (2.2) and (2.7) define a well-posed initial-boundary value problem for the heat equation on the half-line. Let $\hat{u}(k,t)$ denote the Fourier transform with respect to x on the half-line,

$$\hat{u}(k,t) = \int_0^\infty e^{-ikx}u(x,t)dx, \quad k \in \mathbb{C}^-. \tag{2.8a}$$

The inverse Fourier transform is given by

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{u}(k,t)dk, \quad 0 < x < \infty. \tag{2.8b}$$

The symbols $\mathbb{C}^+$ and $\mathbb{C}^-$ denote the closure of the upper and the lower half of the complex plane respectively. Note that, as opposed to the Fourier transform along the real line $x \in \mathbb{R}$, the domain of definition of $\hat{u}(k,t)$ is now extended from the real line to the lower half of the complex k-plane (e^{-ikx} is analytic and bounded for all $k \in \mathbb{C}^-$).

In what follows, we apply steps 1–3 of the Introduction for the solution of the heat equation on the half-line with the oblique Robin boundary condition (2.7).

Step 1. Apply Green's theorem on the divergence form. Consider the formal adjoint of equation (2.1),

$$-\tilde{u}_t = \tilde{u}_{xx}, \quad \tilde{u} = \tilde{u}(x,t). \tag{2.9}$$

Then, combining equations (2.1) and (2.9), we find

$$(\tilde{u}u)_t = (\tilde{u}u_x - \tilde{u}_x u)_x\,. \tag{2.10}$$

Equation (2.9) admits the one-parameter family of solutions

$$\tilde{u}(x,t) = e^{-ikx+k^2t}, \quad k \in \mathbb{C}. \tag{2.11}$$

Thus, equation (2.10) yields the following divergence form for equation (2.1):

$$\left[e^{-ikx+k^2t}u(x,t)\right]_t = \left\{e^{-ikx+k^2t}\left[u_x(x,t) + ik\,u(x,t)\right]\right\}_x. \tag{2.12}$$

Let $\partial\Omega$ denote the positively oriented boundary of the domain $\Omega = \{0 < x < \infty,$

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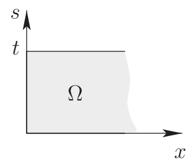

FIGURE 2. The domain Ω for Green's theorem.

$0 < s < t\}$, depicted in Figure 2. Green's theorem in Ω implies

$$\int_{\partial\Omega} e^{-ikx+k^2 s}\left\{u(x,s)dx + \left[u_x(x,s) + ik\,u(x,s)\right]ds\right\} = 0, \quad k \in \mathbb{C}^-. \tag{2.13}$$

Consequently, recalling the definition (2.8a) we obtain the *global relation*

$$e^{k^2 t}\hat{u}(k,t) = \hat{u}_0(k) - \left[g_1(k^2,t) + ik\,g_0(k^2,t)\right], \quad k \in \mathbb{C}^-, \tag{2.14}$$

where $\hat{u}_0(k)$ is the Fourier transform of the initial data $u_0(x)$ and the functions $g_j(k^2,t)$ are defined by

$$g_j(k^2,t) = \int_0^t e^{k^2 s}\partial_x^j u(0,s)ds, \quad j = 0,1. \tag{2.15}$$

The restriction $k \in \mathbb{C}^-$ is due to equation (2.8a).

Step 2. Invert the global relation. Using the inverse Fourier transform (2.8b), the global relation yields the following expression for $u(x,t)$:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2 t}\hat{u}_0(k)dk - \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2 t}\left[g_1(k^2,t) + ik\,g_0(k^2,t)\right]dk. \tag{2.16}$$

We now study the integrand of the second integral in the above expression. Using the definition (2.15) we find that this integrand involves the exponentials e^{ikx} and $e^{k^2(s-t)}$. Since $x \geqslant 0$ and $s - t \leqslant 0$, these exponentials are bounded as $|k| \to \infty$, respectively in $\mathbb{C}^+$ and in

$$\mathrm{Re}\left(k^2\right) \geqslant 0. \tag{2.17}$$

Define the regions D^+ and D^-, shown in Figure 3, as the regions of the upper half and the lower half of the complex k-plane where $\mathrm{Re}\left(k^2\right) < 0$:

$$D^+ = \{k \in \mathbb{C}^+ : (\mathrm{Im}\,k)^2 > (\mathrm{Re}\,k)^2\}, \quad D^- = \{k \in \mathbb{C}^- : (\mathrm{Im}\,k)^2 > (\mathrm{Re}\,k)^2\}. \tag{2.18}$$

Then according to equation (2.17) the second integral on the right-hand side of equation (2.16) is well defined not only for $k \in \mathbb{R}$ but actually for $k \in \mathbb{C}^+ \setminus D^+$. In fact, by using Cauchy's theorem and Jordan's lemma, we can deform the contour of integration from the real line to the contour ∂D^+ (see Figure 3) so that the following proposition is established.

Proposition 2.1 (*The integral representation*) *The heat equation* (2.1) *formulated on the half-line* $0 < x < \infty$ *with the initial data* (2.2) *admits the integral representation*

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2 t}\hat{u}_0(k)dk - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2 t}\left[g_1(k^2,t) + ik\,g_0(k^2,t)\right]dk, \tag{2.19}$$

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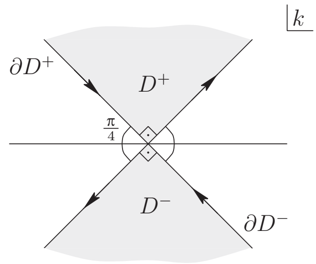

FIGURE 3. The regions D^+ and D^- and the associated contours ∂D^+ and ∂D^-.

where $\hat{u}_0(k)$ *is the Fourier transform* (2.8a) *of the initial data* $u_0(x)$*, the functions* $g_j(k^2, t)$ *are defined by equation* (2.15) *and the contour* ∂D^+ *is depicted in Figure* 3.

Step 3. Elimination of the unknown spectral functions. The integral representation (2.19) does not provide an effective solution expression, since both spectral functions g_0 and g_1 defined by equation (2.15) are unknown.

We observe that these functions depend on k only through k^2, and therefore they are *invariant* under the transformation $k \mapsto -k$. Under this transformation, which leaves the functions $g_j(k^2, t)$ invariant, the global relation (2.14) becomes

$$e^{k^2 t}\hat{u}(-k, t) = \hat{u}_0(-k) - \left[g_1(k^2, t) - ik\, g_0(k^2, t)\right], \quad k \in \mathbb{C}^+. \tag{2.20}$$

Multiplying the boundary condition (2.7) by $e^{k^2 s}$ and integrating with respect to s from 0 to t, we find,

$$\int_0^t e^{k^2 s} u_s(0, s)ds + \alpha \int_0^t e^{k^2 s} u_x(0, s)ds + \beta \int_0^t e^{k^2 s} u(0, s)ds = \int_0^t e^{k^2 s}\gamma(s)ds. \tag{2.21}$$

Integrating by parts we obtain the condition,

$$e^{k^2 t} u(0, t) - u_0(0) - k^2 g_0(k^2, t) + \alpha g_1(k^2, t) + \beta g_0(k^2, t) = \tilde{\gamma}(k^2, t), \tag{2.22}$$

where $\tilde{\gamma}$ is defined by

$$\tilde{\gamma}(k^2, t) = \int_0^t e^{k^2 s}\gamma(s)ds. \tag{2.23}$$

We solve equation (2.22) for $g_1(k^2, t)$ in terms of the unknown functions $g_0\left(k^2, t\right)$, $u(0, t)$ and the known functions $\gamma\left(k^2, t\right)$, $u_0(0)$, i.e.

$$g_1(k^2, t) = \frac{1}{\alpha}\left[\left(k^2 - \beta\right) g_0(k^2, t) - e^{k^2 t} u(0, t) + \tilde{\delta}(k^2, t)\right], \tag{2.24}$$

where $\tilde{\delta}(k^2, t)$ is defined by

$$\tilde{\delta}(k^2, t) = \tilde{\gamma}(k^2, t) + u_0(0). \tag{2.25}$$

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Then substituting for $g_1(k^2,t)$ in equation (2.20), we find

$$g_0(k^2,t) = \frac{1}{ik + \frac{\beta - k^2}{\alpha}} \left\{ e^{k^2 t}\hat{u}(-k,t) - \hat{u}_0(-k) + \frac{1}{\alpha}\left[\tilde{\delta}(k^2,t) - e^{k^2 t}u(0,t)\right]\right\}, \quad k \in \mathbb{C}^+. \tag{2.26}$$

Using (2.24) we obtain

$$\begin{aligned} g_1(k^2,t) + ik\, g_0(k^2,t) = &- \frac{k^2 + i\alpha k - \beta}{k^2 - i\alpha k - \beta}\left[e^{k^2 t}\hat{u}(-k,t) - \hat{u}_0(-k)\right] \\ &- \frac{2ik}{k^2 - i\alpha k - \beta}\left[\tilde{\delta}(k^2,t) - e^{k^2 t}u(0,t)\right], \quad k \in \mathbb{C}^+. \end{aligned} \tag{2.27}$$

Inserting (2.27) in the integral representation (2.19), we find

$$\begin{aligned} u(x,t) = &\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx - k^2 t}\hat{u}_0(k)dk \\ &+ \frac{1}{2\pi}\int_{\partial D^+} \frac{e^{ikx-k^2 t}}{k^2 - i\alpha k - \beta}\left\{\left(k^2 + i\alpha k - \beta\right)\left[e^{k^2 t}\hat{u}(-k,t) - \hat{u}_0(-k)\right]\right. \\ &\left. + 2i\, k\left[\tilde{\delta}(k^2,t) - e^{k^2 t}u(0,t)\right]\right\} dk. \end{aligned} \tag{2.28}$$

The representation (2.28) is not yet effective, since it involves the unknown functions $\hat{u}(-k,t)$ and $u(0,t)$. However, the contributions of the corresponding k-integrals can be computed explicitly. In this respect, we need to study the roots of the polynomial $k^2 - i\alpha k - \beta$, which are given by

$$k_{1,2} = \frac{1}{2}\left(i\alpha \pm \sqrt{-\alpha^2 + 4\beta}\right). \tag{2.29}$$

We consider the following two cases:

(1) *No roots in the region D^+.*

If $k_{1,2} \notin D^+$, then the integrals of equation (2.28) that involve $\hat{u}(-k,t)$ and $u(0,t)$ vanish by direct application of Cauchy's theorem and Jordan's lemma in the upper half complex k-plane.

(2) *One or more roots lie in the region D^+.*

The integrand of the second integral on the right-hand side of equation (2.28) involves the functions g_0 and g_1 (see equation (2.27)), which are entire functions. Thus, the singularities at k_1 and k_2 must be removable. Since it will be necessary to treat the terms $\{g_0\left(k^2,t\right),\ u(0,t)\}$ and $\{\gamma\left(k^2,t\right),\ u_0(0)\}$ separately, we first deform the contour ∂D^+ to avoid these singularities *before* splitting the integral. Let $\mathscr{L}^+$ denote the contour of the upper half k-plane such that the singularities $k_{1,2}$ lie outside the region enclosed by $\mathscr{L}^+$

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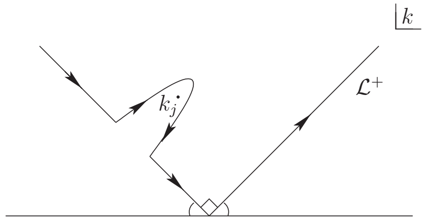

FIGURE 4. The contour $\mathscr{L}^+$.

(see Figure 4). Then equation (2.28) becomes

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{u}_0(k)dk + \frac{1}{2\pi}\int_{\mathscr{L}^+} \frac{e^{ikx-k^2t}}{k^2-i\alpha k-\beta}\left\{\left(k^2+i\alpha k-\beta\right)\left[e^{k^2t}\hat{u}(-k,t)-\hat{u}_0(-k)\right] + 2i\,k\left[\tilde{\delta}(k^2,t)-e^{k^2t}u(0,t)\right]\right\}dk. \tag{2.30}$$

By applying Cauchy's theorem and Jordan's lemma in the region enclosed by the contour $\mathscr{L}^+$, we find that the terms that involve $\hat{u}(-k,t)$ and $u(0,t)$ yield zero contributions, hence

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{u}_0(k)dk + \frac{1}{2\pi}\int_{\mathscr{L}^+} \frac{e^{ikx-k^2t}}{k^2-i\alpha k-\beta}\left[2ik\,\tilde{\delta}(k^2,t)-\left(k^2+i\alpha k-\beta\right)\hat{u}_0(-k)\right]dk. \tag{2.31}$$

Equation (2.31) provides an effective integral expression since it only involves transforms of known functions. An alternative integral expression can be obtained by replacing $\mathscr{L}^+$ with the contour ∂D^+ plus the residues due to the simple poles at $k_{1,2}$. Therefore, the following proposition is established.

Proposition 2.2 *Suppose that the roots $k_{1,2}$ of the polynomial $k^2-i\alpha k-\beta$, given by equation* (2.29)*, do not lie in the region D^+ depicted in Figure* 3*. Then the heat equation on the half-line with the initial data* (2.2) *and the boundary condition* (2.7) *admits the solution*

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{u}_0(k)dk + \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\,\frac{2ik\,\tilde{\delta}(k^2,t)-\left(k^2+i\alpha k-\beta\right)\hat{u}_0(-k)}{k^2-i\alpha k-\beta}\,dk, \tag{2.32a}$$

where $\hat{u}_0(k)$ denotes the Fourier transform (2.8a) *of the initial data, the function $\tilde{\delta}(k^2,t)$ is defined by equation* (2.25) *and the contour ∂D^+ is depicted in Figure* 3.

If $k_1 \in D^+$ and $k_2 \notin D^+$, then the solution is given by the expression (2.32a) *plus the function $u_1(x,t)$ defined by*

$$u_1(x,t) = -\frac{2k_1}{2k_1-i\alpha}\,e^{ik_1x-k_1^2t}\left[\alpha\,\hat{u}_0(-k_1)-\tilde{\delta}(k_1^2,t)\right]. \tag{2.32b}$$

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If $k_2 \in D^+$ *and* $k_1 \notin D^+$ *then the solution is given by the expression* (2.32a) *plus the function* $u_2(x,t)$ *defined by*

$$u_2(x,t) = -\frac{2k_2}{2k_2 - i\alpha}\ e^{ik_2x-k_2^2t}\left[\alpha\,\hat{u}_0(-k_2) - \tilde{\delta}(k_2^2,t)\right]. \tag{2.32c}$$

If both k_1 *and* k_2 *lie in the region* D^+*, then the solution is given by the expression* (2.32a) *plus the functions* $u_1(x,t)$ *and* $u_2(x,t)$.

Return to the real line and the classical spectral representation. The problem solved by Cohen [1] is the particular case of

$$\alpha = 1, \qquad \beta = \gamma = 0. \tag{2.33}$$

In this case, the boundary condition (2.7) becomes

$$u_t(0,t) + u_x(0,t) = 0. \tag{2.34}$$

Letting $\alpha = 1$ and $\beta = \gamma = 0$ in equation (2.32b), we find the solution expression

$$\begin{aligned} u(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{u}_0(k)dk \\ &+ \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\ \frac{2i\,u_0(0) - (k+i)\,\hat{u}_0(-k)}{k-i}dk \\ &- 2e^{-x+t}\left[\hat{u}_0(-i) - u_0(0)\right]. \end{aligned} \tag{2.35}$$

In order to obtain the representation of Cohen [1], we deform the contour of integration ∂D^+ back to the real line. Note that for general boundary value problems such a deformation is *not* possible; however, in this particular case the exponentials involved in equation (2.35) are bounded and analytic for $k \in \mathbb{C}^+ \setminus D^+$, thus ∂D^+ can indeed be deformed back to the real line:

$$\begin{aligned} u(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{u}_0(k)dk \\ &+ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\left[\frac{2i}{k-i}\,u_0(0) - \frac{k+i}{k-i}\,\hat{u}_0(-k)\right]dk \\ &+ 2e^{-x+t}\left[u_0(0) - \hat{u}_0(-i)\right]. \end{aligned} \tag{2.36}$$

Using the definition (2.8a) and rearranging, we find

$$\begin{aligned} u(x,t) =& \frac{1}{2\pi}\int_0^{\infty} e^{-k^2t}\left[e^{ikx}\int_0^{\infty} e^{-ik\xi}u_0(\xi)d\xi + e^{-ikx}\int_0^{\infty} e^{ik\xi}u_0(\xi)d\xi\right]dk \\ &- \frac{1}{2\pi}\int_0^{\infty} e^{-k^2t}\left[e^{ikx}\ \frac{k+i}{k-i}\int_0^{\infty} e^{ik\xi}u_0(\xi)d\xi + e^{-ikx}\ \frac{k-i}{k+i}\int_0^{\infty} e^{-ik\xi}u_0(\xi)d\xi\right]dk \\ &+ \frac{1}{i\pi}\int_0^{\infty} e^{-k^2t}\left[\frac{e^{-ikx}}{k+i} - \frac{e^{ikx}}{k-i}\right]u_0(0)dk + 2e^{-x+t}\left[u_0(0) - \int_0^{\infty} e^{-\xi}u_0(\xi)d\xi\right]. \end{aligned} \tag{2.37}$$

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Simplifying, we obtain the representation of Cohen [1]:

$$u(x,t) = \frac{2}{\pi}\int_0^{\infty} e^{-k^2 t}\,\frac{\cos kx + k\sin kx}{k^2+1}\left[\int_0^{\infty}(\cos k\xi + k\sin k\xi)\,u_0(\xi)d\xi - u_0(0)\right]dk$$
$$+ 2e^{-x+t}\left[u_0(0) - \int_0^{\infty} e^{-\xi}u_0(\xi)d\xi\right]. \quad (2.38)$$

For the boundary condition (2.34) the method of separation of variables yields a problem where the eigenvalue occurs in both the equation and the boundary condition. Cohen was able to construct the relevant spectral representation, which is given by equation (1.1), using a series of novel, ingenious steps. It is interesting to note that the spectral representation (1.1) can be obtained directly via the unified method in a much simpler way: Evaluating equation (2.38) at $t = 0$ and replacing $u(x,0)$ and $u_0(x)$ by $f(x)$, we find equation (1.1).

3 The heat equation on the finite interval with oblique Robin boundary conditions

Consider the heat equation (2.1) formulated on the finite interval $0 < x < l$. In addition to the initial data (2.2) and the boundary condition (2.7) at $x = 0$, we also prescribe an analogous boundary condition at $x = l$,

$$u_t(l,t) + Au_x(l,t) + Bu(l,t) = \Gamma(t). \quad (3.1)$$

The Fourier transform pair in this case is

$$\hat{u}(k,t) = \int_0^{l} e^{-ikx}u(x,t)dx, \quad k \in \mathbb{C}, \quad (3.2a)$$
$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{u}(k,t)dk, \quad 0 < x < l. \quad (3.2b)$$

Step 1. Apply Green's theorem on the divergence form. Green's theorem in the domain $\mathscr{E}$ depicted in Figure 5 yields the *global relation*

$$e^{k^2 t}\hat{u}(k,t) = \hat{u}_0(k) + e^{-ikl}\left[h_1(k^2,t) + ik\,h_0(k^2,t)\right] - \left[g_1(k^2,t) + ik\,g_0(k^2,t)\right], \quad k \in \mathbb{C}, \quad (3.3)$$

where the functions g_j, $j = 0,1$, are defined by equation (2.15), and the functions h_j, $j = 0,1$, are defined by

$$h_j(k^2,t) = \int_0^{t} e^{k^2 s}\partial_x^j u(l,s)ds, \quad j = 0,1. \quad (3.4)$$

The global relation is now valid for all complex values of the spectral variable k, since the Fourier transform with respect to x is now taken over a finite domain.

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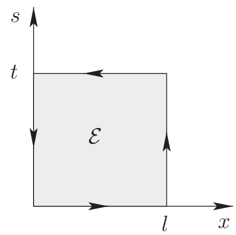

FIGURE 5. The domain $\mathscr{E}$ for Green's theorem.

Step 2. Invert the global relation. Inverting the global relation by means of equation (3.2b), we obtain the integral representation

$$\begin{aligned} u(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{u}_0(k)dk + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-l)-k^2t}\left[h_1(k^2,t)+ik\,h_0(k^2,t)\right]dk \\ &-\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\left[g_1(k^2,t)+ik\,g_0(k^2,t)\right]dk. \end{aligned} \tag{3.5}$$

Similarly to the half-line case, using Cauchy's theorem and Jordan's lemma in the regions $\mathbb{C}^+ \setminus D^+$ and $\mathbb{C}^- \setminus D^-$ of the complex k-plane, where the regions D^+ and D^- are defined by equation (2.18), we arrive at the following proposition.

Proposition 3.1 (*The integral representation*) *The heat equation* (2.1) *posed on the finite interval* $0 < x < l$ *with the initial data* (2.2) *admits the following integral representation:*

$$\begin{aligned} u(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{u}_0(k)dk - \frac{1}{2\pi}\int_{\partial D^-} e^{ik(x-l)-k^2t}\left[h_1(k^2,t)+ik\,h_0(k^2,t)\right]dk \\ &-\frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\left[g_1(k^2,t)+ik\,g_0(k^2,t)\right]dk, \end{aligned} \tag{3.6}$$

where $\hat{u}_0(k)$ *is the Fourier transform* (3.2a) *of the initial data* $u_0(x)$*, the functions* $g_j(k^2,t)$ *and* $h_j(k^2,t)$ *are defined by equations* (2.15) *and* (3.4) *respectively and the contours* ∂D^+ *and* ∂D^- *are shown in Figure* 3.

Step 3. Elimination of the unknown spectral functions. Equation (3.6) involves the unknown spectral functions g_j, h_j, $j = 0, 1$. Integrating the boundary condition (3.1), we find

$$e^{k^2t}u(l,t) - u_0(l) + A\,h_1(k^2,t) + \left(B-k^2\right)h_0(k^2,t) = \tilde{\Gamma}(k^2,t), \tag{3.7}$$

where

$$\tilde{\Gamma}(k^2,t) = \int_0^t e^{k^2s}\,\Gamma(s)ds. \tag{3.8}$$

Thus, combining equations (2.22) and (3.7), we can express g_1 and h_1 in terms of g_0 and h_0:

$$\alpha g_1(k^2,t) = \left(k^2-\beta\right)g_0(k^2,t) - e^{k^2t}u(0,t) + \tilde{\delta}(k^2,t), \tag{3.9a}$$

$$A h_1(k^2,t) = \left(k^2-B\right)h_0(k^2,t) - e^{k^2t}u(l,t) + \tilde{\Delta}(k^2,t), \tag{3.9b}$$

where $\tilde{\delta}(k^2,t)$ is defined by equation (2.25) and

$$\tilde{\Delta}(k^2,t) = \tilde{\Gamma}(k^2,t) + u_0(l). \tag{3.10}$$

Inserting these expressions in the global relation (3.3), we find

$$e^{k^2t}\hat{u}(k,t) = \hat{u}_0(k) + e^{-ikl}\left(\frac{k^2-B}{A}+ik\right)h_0(k^2,t) - \left(\frac{k^2-\beta}{\alpha}+ik\right)g_0(k^2,t) \\ + \frac{e^{-ikl}}{A}\left[\tilde{\Delta}(k^2,t) - e^{k^2t}u(l,t)\right] - \frac{1}{\alpha}\left[\tilde{\delta}(k^2,t) - e^{k^2t}u(0,t)\right], \qquad k \in \mathbb{C}. \tag{3.11a}$$

The symmetry $k \mapsto -k$ yields the additional identity

$$e^{k^2t}\hat{u}(-k,t) = \hat{u}_0(-k) + e^{ikl}\left(\frac{k^2-B}{A}-ik\right)h_0(k^2,t) - \left(\frac{k^2-\beta}{\alpha}-ik\right)g_0(k^2,t) \\ + \frac{e^{ikl}}{A}\left[\tilde{\Delta}(k^2,t) - e^{k^2t}u(l,t)\right] - \frac{1}{\alpha}\left[\tilde{\delta}(k^2,t) - e^{k^2t}u(0,t)\right], \qquad k \in \mathbb{C}. \tag{3.11b}$$

Equations (3.11) can be put in the form of the linear system,

$$\mathscr{A}(k)\begin{pmatrix} g_0(k^2,t) \\ h_0(k^2,t) \end{pmatrix} = \mathscr{N}(k,t), \tag{3.12}$$

where $\mathscr{A}(k)$ and $\mathscr{N}(k,t)$ are defined by

$$\mathscr{A}(k) = \begin{pmatrix} -\left(\frac{k^2-\beta}{\alpha}+ik\right) & e^{-ikl}\left(\frac{k^2-B}{A}+ik\right) \\ -\left(\frac{k^2-\beta}{\alpha}-ik\right) & e^{ikl}\left(\frac{k^2-B}{A}-ik\right) \end{pmatrix} \tag{3.13}$$

and

$$\mathscr{N}(k,t) = \begin{pmatrix} e^{k^2t}\hat{u}(k,t) - \hat{u}_0(k) - e^{-ikl}\vartheta(k^2,t) + \rho(k^2,t) \\ e^{k^2t}\hat{u}(-k,t) - \hat{u}_0(-k) - e^{ikl}\vartheta(k^2,t) + \rho(k^2,t) \end{pmatrix}, \tag{3.14}$$

with the functions ϑ and ρ defined by

$$\vartheta(k^2,t) = \frac{1}{A}\left[\tilde{\Delta}(k^2,t) - e^{k^2t}u(l,t)\right], \tag{3.15a}$$

$$\rho(k^2,t) = \frac{1}{\alpha}\left[\tilde{\delta}(k^2,t) - e^{k^2t}u(0,t)\right]. \tag{3.15b}$$

Let $P(k)$ denote the determinant of the matrix $\mathscr{A}(k)$, i.e.

$$P(k) = -\left(\frac{k^2-\beta}{\alpha}+ik\right)\left(\frac{k^2-B}{A}-ik\right)e^{ikl} + \left(\frac{k^2-\beta}{\alpha}-ik\right)\left(\frac{k^2-B}{A}+ik\right)e^{-ikl}. \tag{3.16}$$

The system (3.12) can be solved uniquely for g_0 and h_0 provided that the determinant P is non-zero along the contours ∂D^+ and ∂D^-. Actually, we are only interested in the large $|k|$ behaviour of this determinant, since we can deform the contours ∂D^+ and ∂D^- in order to avoid the *finite* zeros of $P(k)$.

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As $|k| \to \infty$, we find

$$P(k) \sim -\frac{1}{\alpha A} k^4 \left(e^{ikl} - e^{-ikl}\right). \tag{3.17}$$

Hence, the possible zeros of P for large $|k|$ lie on the real k-axis, and therefore solving equation (3.12) for $k \in \mathbb{C} \setminus \mathbb{R}$, we find the following expressions:

$$\begin{aligned} g_0(k^2,t) =& \frac{1}{P(k)} \left\{ e^{ikl} \left(\frac{k^2 - B}{A} - ik \right) \left[e^{k^2 t} \hat{u}(k,t) - \hat{u}_0(k) - e^{-ikl} \vartheta(k^2,t) + \rho(k^2,t) \right] \right. \\ & \left. - e^{-ikl} \left(\frac{k^2 - B}{A} + ik \right) \left[e^{k^2 t} \hat{u}(-k,t) - \hat{u}_0(-k) - e^{ikl} \vartheta(k^2,t) + \rho(k^2,t) \right] \right\}, \end{aligned} \tag{3.18a}$$

and

$$\begin{aligned} h_0(k^2,t) =& \frac{1}{P(k)} \left\{ \left(\frac{k^2 - \beta}{\alpha} - ik \right) \left[e^{k^2 t} \hat{u}(k,t) - \hat{u}_0(k) - e^{-ikl} \vartheta(k^2,t) + \rho(k^2,t) \right] \right. \\ & \left. - \left(\frac{k^2 - \beta}{\alpha} + ik \right) \left[e^{k^2 t} \hat{u}(-k,t) - \hat{u}_0(-k) - e^{ikl} \vartheta(k^2,t) + \rho(k^2,t) \right] \right\}. \end{aligned} \tag{3.18b}$$

Combining these two equations with equations (3.9), we can compute the integrands involved in the integral representation (3.5):

$$\begin{aligned} & h_1(k^2,t) + ik\, h_0(k^2,t) \\ & = \vartheta(k^2,t) + \frac{1}{P(k)} \left(\frac{k^2 - B}{A} + ik \right) \\ & \quad \times \left\{ \left(\frac{k^2 - \beta}{\alpha} - ik \right) \left[e^{k^2 t} \hat{u}(k,t) - \hat{u}_0(k) - e^{-ikl} \vartheta(k^2,t) + \rho(k^2,t) \right] \right. \\ & \quad \left. - \left(\frac{k^2 - \beta}{\alpha} + ik \right) \left[e^{k^2 t} \hat{u}(-k,t) - \hat{u}_0(-k) - e^{ikl} \vartheta(k^2,t) + \rho(k^2,t) \right] \right\}, \end{aligned} \tag{3.19a}$$

and

$$\begin{aligned} & g_1(k^2,t) + ik\, g_0(k^2,t) \\ & = \rho(k^2,t) + \frac{1}{P(k)} \left(\frac{k^2 - \beta}{\alpha} + ik \right) \\ & \quad \times \left\{ e^{ikl} \left(\frac{k^2 - B}{A} - ik \right) \left[e^{k^2 t} \hat{u}(k,t) - \hat{u}_0(k) - e^{-ikl} \vartheta(k^2,t) + \rho(k^2,t) \right] \right. \\ & \quad \left. - e^{-ikl} \left(\frac{k^2 - B}{A} + ik \right) \left[e^{k^2 t} \hat{u}(-k,t) - \hat{u}_0(-k) - e^{ikl} \vartheta(k^2,t) + \rho(k^2,t) \right] \right\}. \end{aligned} \tag{3.19b}$$

Analogous to the case of the half-line, the unknown functions $\hat{u}(k,t)$ and $\hat{u}(-k,t)$, as well as the unknowns $u(l,t)$ and $u(0,t)$ involved in the functions $\vartheta(k^2,t)$ and $\rho(k^2,t)$, correspond to terms of the integral representation (3.6) that can be handled with the help of Cauchy's theorem and Jordan's lemma. Indeed, the integrands of these terms are bounded and tend to zero uniformly as $|k| \to \infty$ in the regions D^+ and D^-. For example, consider the term

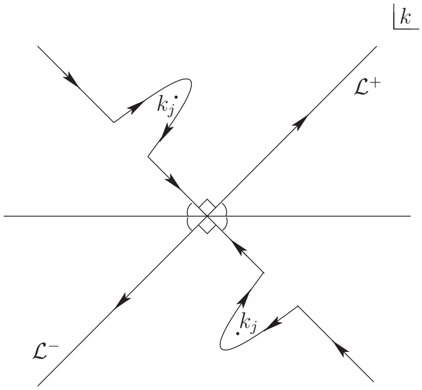

FIGURE 6. The contours $\mathscr{L}^+$ and $\mathscr{L}^-$.

corresponding to $\hat{u}(k,t)$ generated by (3.19b); this term is equal to

$$\int_{\partial D^+} e^{ikx} \frac{e^{ikl}}{P(k)} \left(\frac{k^2-\beta}{\alpha} + ik \right) \left(\frac{k^2-B}{A} - ik \right) \hat{u}(k,t)dk. \tag{3.20}$$

Since $k \in \mathbb{C}^+$,

$$P(k) \sim \frac{1}{\alpha A} k^4 e^{-ikl}, \quad |k| \to \infty, \tag{3.21}$$

hence, the behaviour of the integrand of (3.20) for large $|k|$ is governed by the exponential $e^{ik(x+2l-\xi)}$, which is analytic and tends to zero uniformly as $|k| \to \infty$.

To apply Cauchy's theorem, we need to consider the following cases:

(1) $P(k) \neq 0$ *for* $k \in D^+ \cup D^-$.

Then analyticity and uniform convergence to zero as $|k| \to \infty$ imply that the terms $\hat{u}(k,t)$, $\hat{u}(-k,t)$, $u(l,t)$ and $u(0,t)$ yield zero contributions. In this case the solution is obtained by inserting expressions (3.19a) and (3.19b) in the integral representation (3.6) after neglecting the aforementioned unknown functions.

(2) $P(k)$ *has zeros in the region* $D^+ \cup D^-$.

In this case we deform the contours ∂D^+ and ∂D^- to the contours $\mathscr{L}^+$ and $\mathscr{L}^-$ respectively so that the zeros of $P(k)$ lie outside the regions enclosed by $\mathscr{L}^+$ and $\mathscr{L}^-$ (see Figure 6). Then by applying Cauchy's theorem and Jordan's lemma we find that the integrals involving the unknown functions $\hat{u}(k,t)$, $\hat{u}(-k,t)$, $u(l,t)$ and $u(0,t)$ have zero contributions and therefore an effective integral expression is obtained. Furthermore, it is possible to replace the contours $\mathscr{L}^+$ and $\mathscr{L}^-$ with the contours ∂D^+ and ∂D^- respectively plus the residues associated with the zeros of the function $P(k)$. Therefore, the following proposition is established.

Proposition 3.2 *Suppose that the function* $P(k)$, *defined by equation* (3.16), *does not have any zeros for* $k \in D^+ \cup D^-$. *Then the heat equation* (2.1) *posed on the finite interval with the initial data* (2.2) *and the oblique Robin boundary conditions* (2.7) *and* (3.1) *admits the*

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solution

$$
\begin{aligned}
u(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{u}_0(k)dk \\
&+ \frac{1}{2\pi}\int_{\partial D^-} \frac{e^{ikx-k^2t}}{\alpha A P(k)} \left\{-2ik\left(k^2+i\alpha k-\beta\right)\tilde{\Delta}(k^2,t) + 2ik\left(k^2+iAk-B\right)e^{-ikl}\tilde{\delta}(k^2,t)\right. \\
&+ \left.\left(k^2-i\alpha k-\beta\right)\left(k^2+iAk-B\right)e^{-ikl}\hat{u}_0(k) - \left(k^2+i\alpha k-\beta\right)\left(k^2+iAk-B\right)e^{-ikl}\hat{u}_0(-k)\right\}dk \\
&+ \frac{1}{2\pi}\int_{\partial D^+} \frac{e^{ikx-k^2t}}{\alpha A P(k)} \left\{-2ik\left(k^2+i\alpha k-\beta\right)\tilde{\Delta}(k^2,t) + 2ik\left(k^2+iAk-B\right)e^{-ikl}\tilde{\delta}(k^2,t)\right. \\
&+ \left.\left(k^2+i\alpha k-\beta\right)\left(k^2-iAk-B\right)e^{ikl}\hat{u}_0(k) - \left(k^2+i\alpha k-\beta\right)\left(k^2+iAk-B\right)e^{-ikl}\hat{u}_0(-k)\right\}dk,
\end{aligned}
\tag{3.22a}
$$

where $\hat{u}_0(k)$ *denotes the Fourier transform* (3.2a) *of the initial data, the functions* $\tilde{\delta}(k^2,t)$, $\tilde{\Delta}(k^2,t)$ *and* $P(k)$ *are defined by equations* (2.25), (3.10) *and* (3.16) *respectively, and the contours* ∂D^+ *and* ∂D^- *are depicted in Figure* 3.

In the case that one or more zeros k_j *of* $P(k)$ *lie in the region* $D^+ \cup D^-$*, the solution is given by*

$$
\begin{aligned}
u(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{u}_0(k)dk \\
&+ \frac{1}{2\pi}\int_{\mathscr{L}^-} \frac{e^{ikx-k^2t}}{\alpha A P(k)} \left\{-2ik\left(k^2+i\alpha k-\beta\right)\tilde{\Delta}(k^2,t) + 2ik\left(k^2+iAk-B\right)e^{-ikl}\tilde{\delta}(k^2,t)\right. \\
&+ \left.\left(k^2-i\alpha k-\beta\right)\left(k^2+iAk-B\right)e^{-ikl}\hat{u}_0(k) - \left(k^2+i\alpha k-\beta\right)\left(k^2+iAk-B\right)e^{-ikl}\hat{u}_0(-k)\right\}dk \\
&+ \frac{1}{2\pi}\int_{\mathscr{L}^+} \frac{e^{ikx-k^2t}}{\alpha A P(k)} \left\{-2ik\left(k^2+i\alpha k-\beta\right)\tilde{\Delta}(k^2,t) + 2ik\left(k^2+iAk-B\right)e^{-ikl}\tilde{\delta}(k^2,t)\right. \\
&+ \left.\left(k^2+i\alpha k-\beta\right)\left(k^2-iAk-B\right)e^{ikl}\hat{u}_0(k) - \left(k^2+i\alpha k-\beta\right)\left(k^2+iAk-B\right)e^{-ikl}\hat{u}_0(-k)\right\}dk,
\end{aligned}
\tag{3.22b}
$$

where the contours $\mathscr{L}^+$ *and* $\mathscr{L}^-$ *are depicted in Figure* 6. *These contours can be deformed back to the contours* ∂D^+ *and* ∂D^- *respectively so that equation* (3.22b) *becomes equation* (3.22a) *plus the relevant residues at* $k = k_j$.

4 The heat equation on the finite interval with non-local constraints

Consider the heat equation for the function $u(x,t)$ formulated on the finite interval,

$$u_t = u_{xx}, \qquad x \in (0,l), \quad t > 0, \tag{4.1}$$

with the initial data

$$u(x,0) = u_0(x), \quad 0 < x < l. \tag{4.2}$$

We supplement equations (4.1) and (4.2) with the following non-local constraints:

$$\int_0^l u(x,t)dx = \mathscr{F}(t) \tag{4.3a}$$

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and

$$\int_0^l (l-x)\,u(x,t)dx = \mathscr{R}(t), \tag{4.3b}$$

where $\mathscr{F}$ and $\mathscr{R}$ are given functions.

Steps 1 and 2 of the unified method are the same as in Section 3 and yield the global relation (3.3) and the integral representation (3.6).

Step 3. Elimination of the unknown spectral functions. Letting

$$\frac{d\mathscr{F}}{dt}(t) = F(t), \quad \frac{d\mathscr{R}}{dt}(t) = R(t), \tag{4.4}$$

and using equation (4.1), the non-local constraints (4.3) imply the following boundary conditions:

$$\int_0^l u_t(x,t)dx = F(t) \Leftrightarrow \int_0^l u_{xx}(x,t)dx = F(t),$$

thus,

$$u_x(l,t) - u_x(0,t) = F(t) \tag{4.5a}$$

and

$$\int_0^l (l-x)\,u_t(x,t)dx = R(t) \Leftrightarrow \int_0^l (l-x)\,u_{xx}(x,t)dx = R(t),$$

hence

$$l\,u_x(0,t) = u(l,t) - u(0,t) - R(t). \tag{4.5b}$$

The conditions (4.5) imply the following relations for the spectral functions:

$$g_1\left(k^2,t\right) = \frac{1}{l}\left[h_0\left(k^2,t\right) - g_0\left(k^2,t\right) - r\left(k^2,t\right)\right], \tag{4.6a}$$

$$h_1\left(k^2,t\right) = h_0\left(k^2,t\right) - g_0\left(k^2,t\right) + f\left(k^2,t\right) - r\left(k^2,t\right), \tag{4.6b}$$

where

$$f\left(k^2,t\right) = \int_0^t e^{k^2 s}F(s)ds, \quad r\left(k^2,t\right) = \int_0^t e^{k^2 s}R(s)ds. \tag{4.7}$$

Combining equations (4.6) with the global relation (3.3), we find

$$e^{k^2t}\hat{u}(k,t) = \hat{u}_0(k) + \left[\left(\frac{1}{l} + ik\right)e^{-ikl} - \frac{1}{l}\right]h_0\left(k^2,t\right) + \left[\left(\frac{1}{l} - ik\right) - \frac{e^{-ikl}}{l}\right]g_0\left(k^2,t\right)$$
$$+ e^{-ikl}f\left(k^2,t\right) + \frac{1}{l}\left(1 - e^{-ikl}\right)r\left(k^2,t\right), \quad k \in \mathbb{C}. \tag{4.8a}$$

The transformation $k \mapsto -k$ implies the additional identity

$$e^{k^2t}\hat{u}(-k,t) = \hat{u}_0(-k) + \left[\left(\frac{1}{l} - ik\right)e^{ikl} - \frac{1}{l}\right]h_0\left(k^2,t\right) + \left[\left(\frac{1}{l} + ik\right) - \frac{e^{ikl}}{l}\right]g_0\left(k^2,t\right)$$
$$+ e^{ikl}f\left(k^2,t\right) + \frac{1}{l}\left(1 - e^{ikl}\right)r\left(k^2,t\right), \quad k \in \mathbb{C}. \tag{4.8b}$$

The spectral functions depend on k only through k^2, hence the transformation $k \mapsto -k$ leaves the spectral functions *invariant*.

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Equations (4.8) constitute a 2×2 linear system for the unknown functions $g_0(k^2, t)$ and $h_0(k^2, t)$. This system can be written in matrix form,

$$\mathscr{A}(k) \begin{pmatrix} g_0\left(k^2, t\right) \\ h_0\left(k^2, t\right) \end{pmatrix} = \mathscr{N}(k, t), \tag{4.9}$$

where $\mathscr{A}(k)$ and $\mathscr{N}(k, t)$ are defined by

$$\mathscr{A}(k) = \frac{1}{l} \begin{pmatrix} 1 - ilk - e^{-ikl} & (1 + ilk)\, e^{-ikl} - 1 \\ 1 + ilk - e^{ikl} & e^{ikl}\,(1 - ilk) - 1 \end{pmatrix} \tag{4.10}$$

and

$$\mathscr{N}(k, t) = \begin{pmatrix} e^{k^2 t}\hat{u}(k, t) - \hat{u}_0(k) - e^{-ikl} f\left(k^2, t\right) + \frac{1}{l}\left(e^{-ikl} - 1\right) r\left(k^2, t\right) \\ e^{k^2 t}\hat{u}(-k, t) - \hat{u}_0(-k) - e^{ikl} f\left(k^2, t\right) + \frac{1}{l}\left(e^{ikl} - 1\right) r\left(k^2, t\right) \end{pmatrix}. \tag{4.11}$$

The determinant of matrix $\mathscr{A}(k)$, denoted by $P(k)$, is equal to

$$P(k) = \frac{k}{l}\left[(lk - 2i)\, e^{-ikl} + 4i - (lk + 2i)\, e^{ikl}\right]. \tag{4.12}$$

We are interested in the roots of $P(k)$ when $|k| \to \infty$, since by using Cauchy's theorem we can deform the contours ∂D^+ and ∂D^- so that all the *finite* roots of $P(k)$ lie outside the regions D^+ and D^- (the theorem applies because the integrands are analytic and bounded for *finite* $|k|$). For $k \in \mathbb{C}^+ \setminus \mathbb{R}$,

$$P(k) \sim k^2 e^{-ikl} \to \infty, \quad |k| \to \infty, \tag{4.13a}$$

while for $k \in \mathbb{C}^- \setminus \mathbb{R}$,

$$P(k) \sim k^2\, e^{ikl} \to \infty, \quad |k| \to \infty. \tag{4.13b}$$

Hence, the only singularities in the case of large $|k|$ can occur along the real k-axis (for example, it is straightforward to see that $P(2n\pi/l) = 0$, $n \in \mathbb{Z}$).

Thus, inverting equation (4.9), we find

$$\begin{pmatrix} g_0\left(k^2, t\right) \\ h_0\left(k^2, t\right) \end{pmatrix} = \frac{1}{lP(k)} \begin{pmatrix} e^{ikl}\,(1 - ilk) - 1 & 1 - e^{-ikl}\,(1 + ilk) \\ e^{ikl} - 1 - ilk & 1 - ilk - e^{-ikl} \end{pmatrix} \mathscr{N}(k, t). \tag{4.14}$$

We note that the terms of $\mathscr{N}(k, t)$ involving $\hat{u}(k, t)$ and $\hat{u}(-k, t)$ yield a zero contribution in the representation (3.6) and therefore these can be neglected. Indeed, equations (4.6) yield

$$g_1\left(k^2, t\right) + ik\, g_0\left(k^2, t\right) = \frac{1}{l}\left[h_0\left(k^2, t\right) + (ilk - 1)\, g_0\left(k^2, t\right) - r\left(k^2, t\right)\right], \tag{4.15a}$$

and

$$h_1\left(k^2, t\right) + ik\, h_0\left(k^2, t\right) = (1 + ik)\, h_0\left(k^2, t\right) - g_0\left(k^2, t\right) + f\left(k^2, t\right) - r\left(k^2, t\right). \tag{4.15b}$$

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Hence, equation (4.14) implies that the terms involving $\hat{u}(k,t)$ and $\hat{u}(-k,t)$ in the representation (3.6) are equal to the following expressions:

$$-\frac{1}{2\pi}\int_{\partial D^-}\frac{e^{ik(x-l)-k^2t}}{lP(k)}\ \Big\{\big[(1+ik)\left(e^{ikl}-(1+ilk)\right)+\left(e^{ikl}\,(ilk-1)+1\right)\big]\,\hat{u}(k,t) + \big[(1+ilk)\left(1-ilk-e^{-ikl}\right)+\left(e^{-ikl}\,(1+ilk)-1\right)\big]\,\hat{u}(-k,t)\Big\}\ dk \tag{4.16a}$$

and

$$-\frac{1}{2\pi}\int_{\partial D^+}\frac{e^{ikx-k^2t}}{l^2P(k)}\ \Big\{\big[\left(e^{ikl}-1-ilk\right)+(ilk-1)\left(e^{ikl}\,(1-ilk)-1\right)\big]\,\hat{u}(k,t) + \big[\left(1-ilk-e^{-ikl}\right)+(ilk-1)\left(1-e^{-ikl}\,(1+ilk)\right)\big]\,\hat{u}(-k,t)\Big\}\ dk. \tag{4.16b}$$

Since $x\in(0,l)$, the integrands of the above two terms decay exponentially fast as $|k|\to\infty$. Hence, Cauchy's theorem and Jordan's lemma in the regions D^+ and D^- imply that the expressions (4.16a) and (4.16b) vanish.

Thus, equation (4.14) implies

$$g_0\left(k^2,t\right)\approx\frac{1}{lP(k)}\ \Big\{\left[e^{ikl}\,(ilk-1)+1\right]\hat{u}_0(k)+\left[e^{-ikl}\,(ilk+1)-1\right]\hat{u}_0(-k) + \left[2ilk+e^{-ikl}-e^{ikl}\right]f\left(k^2,t\right)+ik\left[e^{-ikl}+e^{ikl}-2\right]r\left(k^2,t\right)\Big\} \tag{4.17a}$$

and

$$h_0\left(k^2,t\right)\approx\frac{1}{lP(k)}\ \Big\{\left[1+ilk-e^{ikl}\right]\hat{u}_0(k)+\left[ilk-1+e^{-ikl}\right]\hat{u}_0(-k) + \left[(1+ilk)\,e^{-ikl}+(ilk-1)\,e^{ikl}\right]f\left(k^2,t\right)+ik\left[2-e^{-ikl}-e^{ikl}\right]r\left(k^2,t\right)\Big\}, \tag{4.17b}$$

where the symbol $\approx$ denotes that we have neglected the terms which yield zero contribution.

Consequently, we find the following expressions for the combinations of the spectral functions that appear in the representation (3.6):

$$g_1\left(k^2,t\right)+ik\ g_0\left(k^2,t\right)\approx\frac{k}{lP(k)}\ \Big\{\left[2i-(2i+lk)\,e^{ilk}\right]\hat{u}_0(k)-lk\,e^{-ikl}\hat{u}_0(-k) + 2i\left[e^{-ikl}+ilk-1\right]f\left(k^2,t\right)+2k\left(1-e^{-ikl}\right)r\left(k^2,t\right)\Big\} \tag{4.18a}$$

and

$$h_1\left(k^2,t\right)+ik\ h_0\left(k^2,t\right)\approx\frac{k}{lP(k)}\ \Big\{\left[2i-lk-2i\,e^{ikl}\right]\hat{u}_0(k)-lk\ \hat{u}_0(-k) + 2i\left[1+(ilk-1)\,e^{ikl}\right]f\left(k^2,t\right)+2k\left(e^{ikl}-1\right)r\left(k^2,t\right)\Big\}. \tag{4.18b}$$

Inserting these expressions in representation (3.6), we establish the following proposition.

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Proposition 4.1 *The heat equation* (4.1) *posed on the finite interval* $x \in (0, l)$ *with the initial data* (4.2) *and the non-local constraints* (4.3) *admits the following solution:*

$$
\begin{aligned}
u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{u}_0(k)dk \\
&- \frac{1}{2\pi} \int_{\partial D^-} e^{ik(x-l)-k^2t} \frac{k}{lP(k)} \left\{ \left[2i - lk - 2i\, e^{ikl}\right] \hat{u}_0(k) - lk\ \hat{u}_0(-k) \right. \\
&\left. + 2i \left[1 + (ilk-1)\, e^{ikl}\right] f\left(k^2,t\right) + 2k\left(e^{ikl}-1\right) r\left(k^2,t\right) \right\}\ dk \\
&- \frac{1}{2\pi} \int_{\partial D^+} e^{ikx-k^2t} \frac{k}{lP(k)} \left\{ \left[2i - (2i+lk)\, e^{ilk}\right] \hat{u}_0(k) - lk\, e^{-ikl}\hat{u}_0(-k) \right. \\
&\left. + 2i\left[e^{-ikl} + ilk - 1\right] f\left(k^2,t\right) + 2k\left(1-e^{-ikl}\right) r\left(k^2,t\right) \right\}\ dk,
\end{aligned} \tag{4.19}
$$

where the function $\hat{u}_0(k)$ *denotes the Fourier transform* (3.2a) *of the initial data* (4.2), *the contours* ∂D^+ *and* ∂D^- *are depicted in Figure* 3, *the determinant* $P(k)$ *is defined by equation* (4.12) *and the functions* $f\left(k^2,t\right)$ *and* $r\left(k^2,t\right)$ *are defined by equations* (4.7).

Return to the real line and the spectral representation. The integrands involved in equation (4.19) are bounded for $k \in \mathbb{C} \setminus \left(D^+ \cup D^-\right)$, hence by Cauchy's theorem and Jordan's lemma we can collapse the contours ∂D^+ and ∂D^- on the real line. In particular, the contour integrals of (4.19) are equal to the residues due to the roots of $P(k)$ in the region $\mathbb{C} \setminus \left(D^+ \cup D^-\right)$ plus the following principal value integrals:

$$
\begin{aligned}
&\frac{1}{2\pi} ⨍_{-\infty}^{\infty} e^{ik(x-l)-k^2t} \frac{k}{lP(k)} \left\{ \left[2i - lk - 2i\, e^{ikl}\right] \hat{u}_0(k) - lk\ \hat{u}_0(-k) \right. \\
&\quad \left. + 2i \left[1 + (ilk-1)\, e^{ikl}\right] f\left(k^2,t\right) + 2k\left(e^{ikl}-1\right) r\left(k^2,t\right) \right\}\ dk \\
&\quad - \frac{1}{2\pi} ⨍_{-\infty}^{\infty} e^{ikx-k^2t} \frac{k}{lP(k)} \left\{ \left[2i - (2i+lk)\, e^{ilk}\right] \hat{u}_0(k) - lk\, e^{-ikl}\hat{u}_0(-k) \right. \\
&\quad \left. + 2i\left[e^{-ikl} + ilk - 1\right] f\left(k^2,t\right) + 2k\left(1-e^{-ikl}\right) r\left(k^2,t\right) \right\}\ dk \\
&= -\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{u}_0(k)dk.
\end{aligned} \tag{4.20}
$$

Therefore, the principal value integrals cancel with the first term on the right-hand side of equation (4.19).

Regarding the residues, we first note that $k = 0$ is a removable singularity. Next, we recall our earlier remark that the contours ∂D^+ and ∂D^- can be deformed for *finite* k so that all *finite* roots of $P(k)$ lie outside the regions D^+ and D^-. Hence, when collapsing these contours on the real line, the roots with non-zero imaginary part contribute a full residue, while the roots on the real line contribute only *half* a residue.

However, by definition the roots of $P(k)$, denoted by k_m, satisfy the equation

$$
k_m\left[(lk_m - 2i)\, e^{-ik_ml} + 4i - (lk_m + 2i)\, e^{ik_ml}\right] = 0, \tag{4.21}
$$

therefore for $k_m \neq 0$,

$$2i - (2i + lk_m)\, e^{ik_m l} = (2i - lk_m)\, e^{-ik_m l} - 2i. \tag{4.22}$$

Thus, the integrands of the integrals along ∂D^+ and ∂D^- are equal at $k = k_m$ and hence the half residues at the real roots of $P(k)$ add up to a full residue.

Therefore,

$$u(x,t) = \sum_{\substack{k_m \\ k_m \neq 0}} e^{ik_m x - k_m^2 t} \frac{ik_m}{lP'(k_m)} \left\{ \left[(2i - lk_m)\, e^{-ik_m l} - 2i\right] \hat{u}_0(k_m) - lk_m\, e^{-ik_m l} \hat{u}_0(-k_m) \right.$$
$$\left. + 2i \left(ilk_m - 1 + e^{-ik_m l}\right) f(k_m^2) + 2k_m \left(1 - e^{-ik_m l}\right) r(k_m^2) \right\}, \tag{4.23}$$

where $P'(k)$ denotes the derivative of $P(k)$ with respect to k and k_m are the non-zero roots of $P(k)$ in the region $\mathbb{C} \setminus \left(D^+ \cup D^-\right)$ of the complex k-plane.

Evaluating the expression (4.23) at $t = 0$ and replacing $u(x,0)$ and $\hat{u}_0(k_m)$ by $q(x)$ and $\hat{q}(k)$ respectively, we obtain the following spectral representation:

$$q(x) = \sum_{\substack{k_m \\ k_m \neq 0}} e^{ik_m x} \frac{ik_m}{lP'(k_m)} \left\{ \left[(2i - lk_m)\, e^{-ik_m l} - 2i\right] \hat{q}(k_m) - lk_m\, e^{-ik_m l} \hat{q}(-k_m) \right\}, \tag{4.24a}$$

where

$$P(k) = \frac{k}{l} \left[(lk - 2i)\, e^{-ikl} + 4i - (lk + 2i)\, e^{ikl}\right] \tag{4.24b}$$

and k_m are the solutions of equation (1.2).

Remark 4.1 (Weyl's asymptotic law) The roots k_m, $m \in \mathbb{Z}$, of $P(k)$ appearing in the spectral representation (4.23) can be regarded as the eigenvalues of the one-dimensional Laplace operator, ∂_{xx}. It is straightforward to see that $P(2n\pi/l) = 0$, thus

$$k_{2n} = \frac{2n\pi}{l}, \quad \forall n \in \mathbb{Z}. \tag{4.25a}$$

Furthermore, it is shown in [12] that

$$k_{2n-1} = \frac{(2n-1)\pi}{l} + \mathcal{O}\left(\frac{1}{n}\right), \quad n \in \mathbb{Z}, \quad |n| \to \infty. \tag{4.25b}$$

An immediate consequence of equations (4.25) is the so-called Weyl's asymptotic law for the eigenvalues of the Laplace operator:

$$\lim_{|k_m| \to \infty} \frac{k_m^2\, l^2}{m^2\, \pi^2} = 1. \tag{4.26}$$

5 Numerical evaluations

We now consider particular initial and boundary conditions. In all three cases considered below, we have $u_0(0) = 0$ and hence, $\tilde{\gamma} \equiv \tilde{\delta}$.

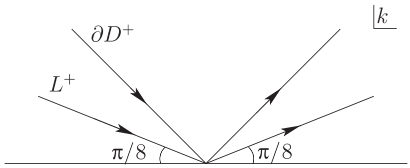

FIGURE 7. The steepest descent path L^+.

(1) Suppose that

$$u_0(x) = xe^{-a^2x}, \quad \gamma(t) = \sin bt, \quad a, b \in \mathbb{R}. \tag{5.1}$$

Then equations (2.8a) and (2.23) yield

$$\hat{u}_0(k) = \frac{1}{(ik + a^2)^2}, \quad \tilde{\gamma}(k^2, t) = \frac{1}{2i}\left[\frac{e^{(k^2+ib)t} - 1}{k^2 + ib} - \frac{e^{(k^2-ib)t} - 1}{k^2 - ib}\right]. \tag{5.2}$$

Thus, the solution formula (2.32a) becomes

$$\begin{aligned} u(x,t) = &\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\,\frac{1}{(ik+a^2)^2}\,dk \\ &- \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\,\frac{k^2 + i\alpha k - \beta}{(k^2 - i\alpha k - \beta)(-ik + a^2)^2}\,dk \\ &+ \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\left[\frac{e^{(k^2+ib)t} - 1}{k^2 + ib} - \frac{e^{(k^2-ib)t} - 1}{k^2 - ib}\right]\frac{k}{k^2 - i\alpha k - \beta}\,dk. \end{aligned} \tag{5.3}$$

Following the hybrid analytical-numerical method of Flyer and Fokas [4], we deform the contour of integration from ∂D^+ to the steepest descent path L^+, which forms an angle of $\pi/8$ with the real k-axis (see Figure 7). This deformation is very helpful regarding numerical computations since now both the x- and t-parts of the relevant exponentials decay as $|k| \to \infty$, providing the integrand with the maximal exponential decay. Then using the change of variables $k = i\sin(\pi/8 - i\theta)$, a few lines of code in Mathematica produce the graphical representations shown in Figures 8 and 9.

(2) Suppose that

$$u_0(x) = x^2e^{-a^2x}, \quad \gamma(t) = \sin bt, \quad a, b \in \mathbb{R}. \tag{5.4}$$

Then equations (2.8a) and (2.23) yield

$$\hat{u}_0(k) = \frac{2}{(ik + a^2)^3}, \quad \tilde{\gamma}(k^2, t) = \frac{1}{2i}\left[\frac{e^{(k^2+ib)t} - 1}{k^2 + ib} - \frac{e^{(k^2-ib)t} - 1}{k^2 - ib}\right]. \tag{5.5}$$

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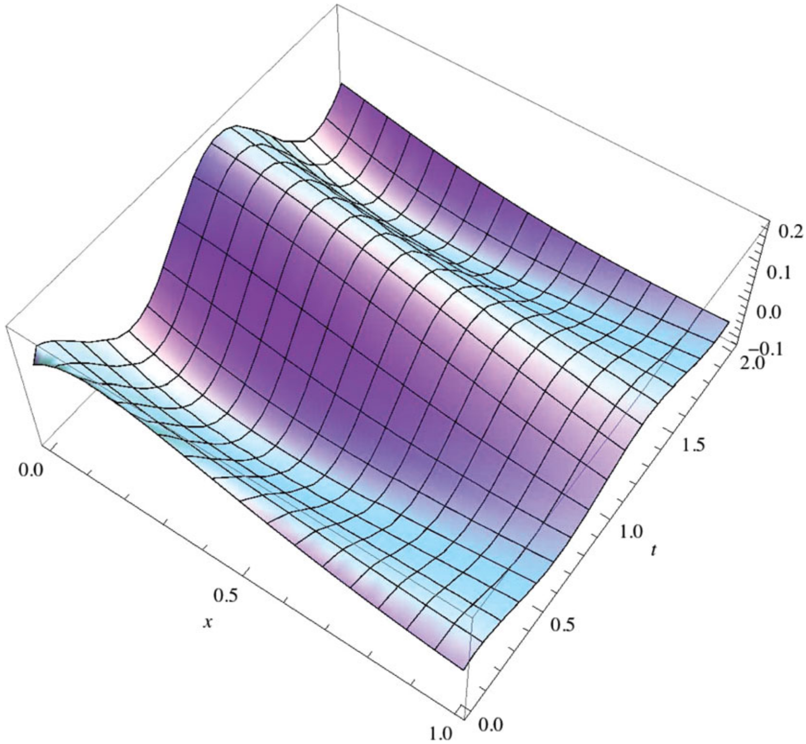

FIGURE 8. (Colour online) The solution (5.3) depicted over $(x, t) \in [0, 1] \times [0, 2]$ for $a = 2$, $b = 5$, $\alpha = 2$, $\beta = 1$.

Thus, the solution formula (2.32a) becomes

$$
\begin{aligned}
u(x,t) = &\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\,\frac{2}{(ik+a^2)^3}\,dk\\
&-\frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\,\frac{2(k^2+i\alpha k-\beta)}{(k^2-i\alpha k-\beta)(-ik+a^2)^3}\,dk\\
&+\frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\left[\frac{e^{(k^2+ib)t}-1}{k^2+ib}-\frac{e^{(k^2-ib)t}-1}{k^2-ib}\right]\frac{k}{k^2-i\alpha k-\beta}\,dk. \qquad (5.6)
\end{aligned}
$$

The same technique as in the previous case yields the following graphical representations (see Figures 10 and 11):

(3) Finally, choosing

$$u_0(x) = 0, \quad \gamma(t) = te^{-\frac{t}{2}} \qquad (5.7)$$

gives

$$\hat{u}_0(k) = 0, \quad \tilde{\gamma}(k^2, t) = \frac{1}{(k^2-\frac{1}{2})^2}\left\{e^{(k^2-\frac{1}{2})t}\left[t\left(k^2-\frac{1}{2}\right)-1\right]+1\right\}. \qquad (5.8)$$

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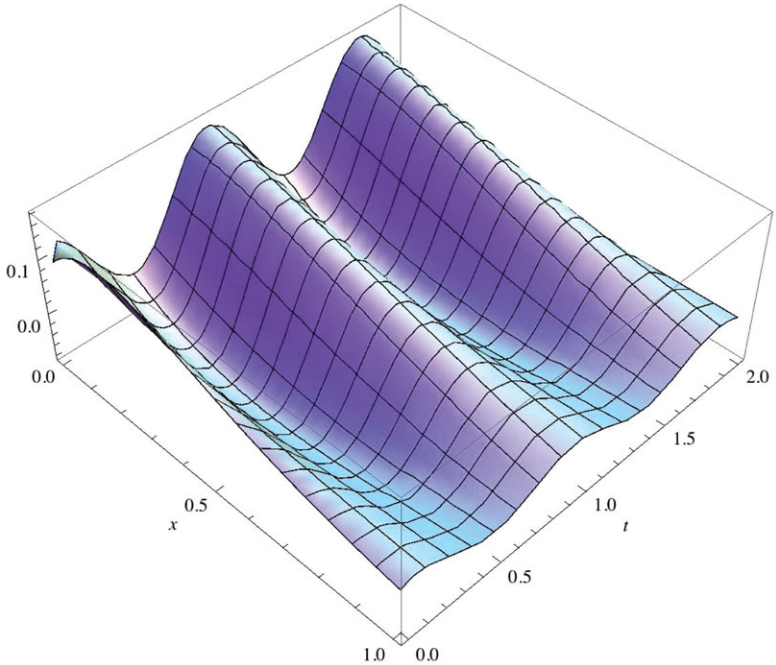

FIGURE 9. (Colour online) The solution (5.3) depicted over $(x,t) \in [0,1] \times [0,2]$ for $a = 2$, $b = 7$, $\alpha = 2$, $\beta = 1$.

The solution (2.32a) now becomes

$$u(x,t) = \frac{1}{2\pi} \int_{\partial D^+} \frac{e^{ikx-k^2t}}{k^2 - i\alpha k - \beta} \frac{2ik}{(k^2 - \frac{1}{2})^2} \left\{ e^{\left(k^2-\frac{1}{2}\right)t} \left[t\left(k^2 - \frac{1}{2}\right) - 1 \right] + 1 \right\} dk. \quad (5.9)$$

A graph of this solution in the case of $\alpha = 2$ and $\beta = 1$ is shown in Figure 12.

6 Conclusion

The integral expression (2.32) expresses the solution of the heat equation on the half-line with given initial data and the oblique Robin boundary condition (2.7). The particular case of the simplified boundary condition (2.34) was solved by Cohen in [1]. The integral expression (2.35) obtained by the unified method is different from the expression (2.38) obtained by Cohen. It is stated in [1] that the expression obtained via a Laplace transform in the temporal variable t is not particularly useful either for the physical interpretation of the solution or for numerical purposes, as opposed to (2.38). In this respect, it is remarkable that the expression (2.35) obtained via the unified method is even better than Cohen's representation (2.38). Indeed, the solution expression (2.35) is uniformly convergent at the boundary and this allows the *direct* evaluation of $u(x,t)$ and its derivatives at $x = 0$. Furthermore, it involves exponentially decaying integrands which make numerical computations very efficient. An illustration of typical numerical evaluations is presented in Section 5.

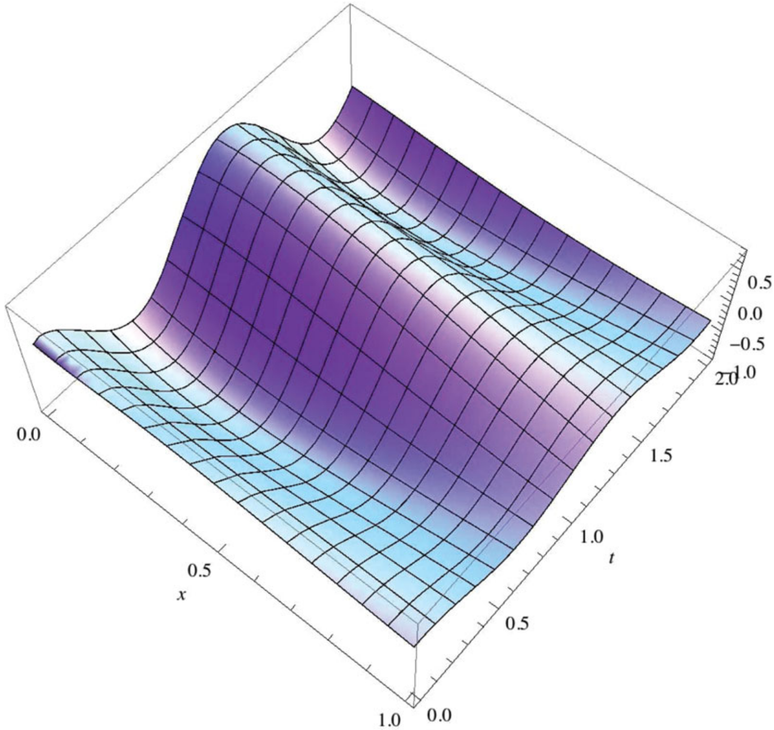

FIGURE 10. (Colour online) The solution (5.6) depicted over $(x, t) \in [0, 1] \times [0, 2]$ for $a = 2$, $b = 5$, $\alpha = 2$, $\beta = 1$.

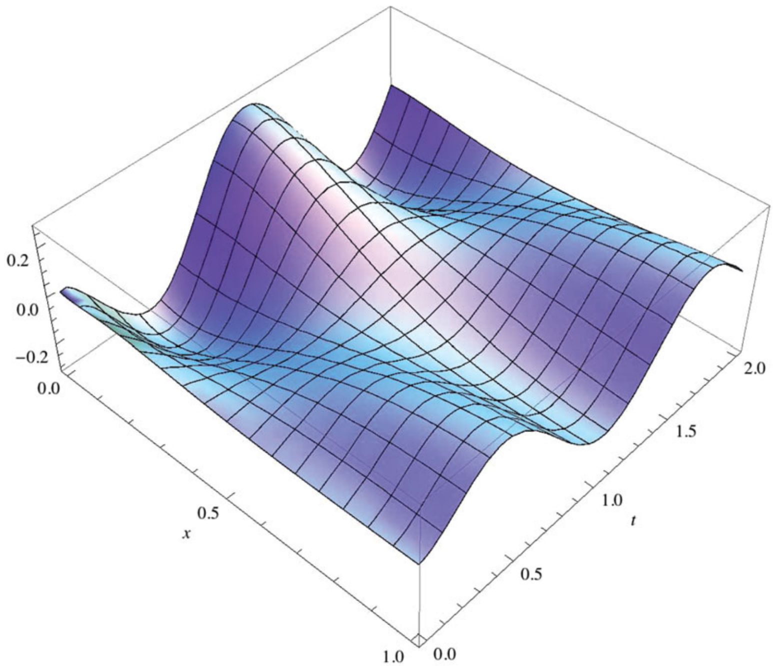

FIGURE 11. (Colour online) The solution (5.6) depicted over $(x, t) \in [0, 1] \times [0, 2]$ for $a = 3$, $b = 5$, $\alpha = 6$, $\beta = 9$.

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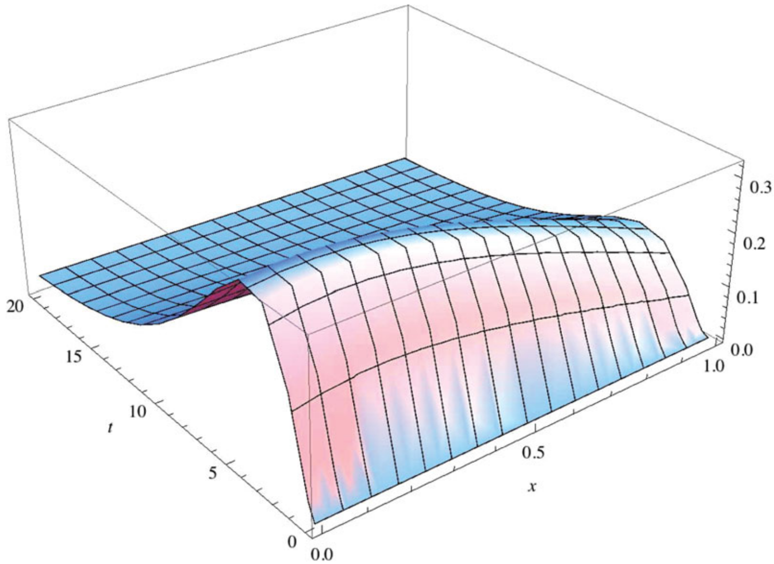

FIGURE 12. (Colour online) The solution (5.9) for $x \in [0,1]$ and $t \in [0,20]$ for $\alpha = 2$, $\beta = 1$.

The unified method can also be used for the derivation of the spectral representation (1.1). Actually the unified method yields the alternative 'spectral representation',

$$q(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{q}(k)dk + \frac{1}{2\pi}\int_{\partial D^+} e^{ikx}\,\frac{2i\,q(0) - (k+i)\,\hat{q}(-k)}{k-i}\,dk$$
$$+ 2e^{-x}\left[q(0) - \hat{q}(-i)\right]. \tag{6.1}$$

The spectral interpretation of the above expression remains an interesting open question. In this respect we emphasise that for non-self-adjoint problems there do *not* exist the analogues of equation (1.1) but there always exist the analogues of equation (6.1).

The integral expression (3.22) expresses the solution of the heat equation on the finite interval $0 < x < l$ with given initial data and the oblique Robin boundary conditions (2.7) and (3.1). This expression is uniformly convergent at the boundary and this allows the *direct* evaluation of $u(x,t)$ and its derivatives at $x = 0$ and $x = l$. Furthermore, (3.22) involves exponentially decaying integrands which make numerical computations very efficient.

The integral expression (4.19) expresses the solution of the heat equation on the finite interval $0 < x < l$ with given initial data and the non-local constraints (4.3). In this case, the classical spectral representation is given by equations (4.24), where the eigenvalues k_m satisfy equation (4.22) (see also equation (1.2) of [12]). The expression (4.19) is uniformly convergent at the boundary and this allows the *direct* evaluation of $u(x,t)$ and its derivatives at $x = 0$ and $x = l$. Moreover, it involves exponentially decaying integrands which make numerical computations very efficient.

Appendix Verification of the solution

In the applied mathematics literature, a PDE is solved under the *assumption* of existence. Indeed, since the solution is obtained by applying an appropriate transform, this procedure makes sense only if one assumes that the solution exists and has certain decay and smoothness properties. In order to eliminate this assumption one should prove *a posteriori* that the expression obtained by this approach satisfies the PDE and the given initial and boundary conditions (then one has to address independently the question of uniqueness). However, this is not done in the applied literature. Actually this verification is *not* straightforward because any representation obtained via the usual transform methods is *non-uniformly convergent* at the boundary. A major advantage of the unified method is that it constructs a solution which is uniformly convergent at the boundary. Thus, it is straightforward, at least formally, to verify that the solution satisfies the given PDE and the given data. The rigorous implementation of this verification for a large class of PDEs is discussed in [10].

Here we illustrate this feature by considering the integral expression (2.32a) derived for the heat equation on the half-line with given initial data and the oblique Robin boundary condition (2.7):

$$\begin{aligned} u(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{u}_0(k)dk \\ &+ \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\,\frac{2ik\,\tilde{\delta}(k^2,t)-\left(k^2+i\alpha k-\beta\right)\hat{u}_0(-k)}{k^2-i\alpha k-\beta}\,dk, \end{aligned} \tag{A.1}$$

where $\hat{u}_0(k)$ is the Fourier transform of the initial data $u_0(x)$, the function $\tilde{\delta}(k^2,t)$ is defined by equation (2.25) and the contour ∂D^+ is depicted in Figure 3.

First, we note that for *any* $T > t$ we have:

$$\int_{\partial D^+} e^{ikx-k^2t}\,\frac{2ik\,\tilde{\delta}(k^2,t)}{k^2-i\alpha k-\beta}\,dk = \int_{\partial D^+} e^{ikx-k^2t}\,\frac{2ik\,\tilde{\delta}(k^2,T)}{k^2-i\alpha k-\beta}\,dk. \tag{A.2}$$

The above equation is a direct consequence of Cauchy's theorem and Jordan's lemma applied in the region D^+. Consequently, the expression (A.1) can be written in the following *equivalent* form:

$$\begin{aligned} u(x,t) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{u}_0(k)dk \\ &+ \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-k^2t}\,\frac{2ik\,\tilde{\delta}(k^2,T)-\left(k^2+i\alpha k-\beta\right)\hat{u}_0(-k)}{k^2-i\alpha k-\beta}\,dk. \end{aligned} \tag{A.3}$$

The heat equation. The integral expression of $u(x,t)$ in the form (A.3) depends on x and t only through the exponential e^{ikx-k^2t}, which is a particular solution of the heat equation, thus the expression (A.3) satisfies the heat equation.

Initial condition. Evaluating the expression (A.1) at $t = 0$, we find

$$u(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{u}_0(k)dk + \frac{1}{2\pi}\int_{\partial D^+} e^{ikx}\,\frac{2ik\,u_0(0) - \left(k^2 + i\alpha k - \beta\right)\hat{u}_0(-k)}{k^2 - i\alpha k - \beta}\,dk. \tag{A.4}$$

The integrand of the integral along the contour ∂D^+ is bounded and analytic for all $k \in \mathbb{C}^+$, hence by Cauchy's theorem and Jordan's lemma this integral vanishes. Thus,

$$u(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{u}_0(k)dk = u_0(x), \tag{A.5}$$

where the last equality above follows by the definition (2.8b) of the inverse Fourier transform.

Boundary condition. We will compute the derivatives $u_t(0,t), u_x(0,t)$ as well as function $u(0,t)$ by using the expression (A.3):

$$u_t(0,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k^2 t}\left(-k^2\right)\hat{u}_0(k)dk + \frac{1}{2\pi}\int_{\partial D^+} e^{-k^2 t}\,\left(-k^2\right)\frac{2ik\,\tilde{\delta}(k^2,T) - \left(k^2 + i\alpha k - \beta\right)\hat{u}_0(-k)}{k^2 - i\alpha k - \beta}\,dk, \tag{A.6a}$$

$$u_x(0,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k^2 t}\,ik\,\hat{u}_0(k)dk + \frac{1}{2\pi}\int_{\partial D^+} e^{-k^2 t}\,ik\,\frac{2ik\,\tilde{\delta}(k^2,T) - \left(k^2 + i\alpha k - \beta\right)\hat{u}_0(-k)}{k^2 - i\alpha k - \beta}\,dk \tag{A.6b}$$

and

$$u(0,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k^2 t}\hat{u}_0(k)dk + \frac{1}{2\pi}\int_{\partial D^+} e^{-k^2 t}\,\frac{2ik\,\tilde{\delta}(k^2,T) - \left(k^2 + i\alpha k - \beta\right)\hat{u}_0(-k)}{k^2 - i\alpha k - \beta}\,dk. \tag{A.6c}$$

Taking the combination $u_t(0,t) + \alpha u_x(0,t) + \beta u(0,t)$, we find

$$u_t(0,t) + \alpha u_x(0,t) + \beta u(0,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k^2 t}\left(-k^2 + i\alpha k + \beta\right)\hat{u}_0(k)dk + \frac{1}{2\pi}\int_{\partial D^+} e^{-k^2 t}\,\left[\left(k^2 + i\alpha k - \beta\right)\hat{u}_0(-k) - 2ik\,\tilde{\delta}(k^2,T)\right]dk. \tag{A.7}$$

The change of variables $k \mapsto -k$ in the first integral on the right-hand side of the above equation yields

$$u_t(0,t) + \alpha u_x(0,t) + \beta u(0,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k^2 t}\left(-k^2 - i\alpha k + \beta\right)\hat{u}_0(-k)dk + \frac{1}{2\pi}\int_{\partial D^+} e^{-k^2 t}\,\left[\left(k^2 + i\alpha k - \beta\right)\hat{u}_0(-k) - 2ik\,\tilde{\delta}(k^2,T)\right]dk. \tag{A.8}$$

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Furthermore, Cauchy's theorem and Jordan's lemma for $k \in \mathbb{C}^+ \setminus D^+$ imply that we can deform the contour of the first integral from the real line to ∂D^+, i.e.

$$u_t(0,t) + \alpha u_x(0,t) + \beta u(0,t) = \frac{1}{2\pi}\int_{\partial D^+} e^{-k^2 t}\left(-k^2 - i\alpha k + \beta\right)\hat{u}_0(-k)dk + \frac{1}{2\pi}\int_{\partial D^+} e^{-k^2 t}\ \left[\left(k^2 + i\alpha k - \beta\right)\hat{u}_0(-k) - 2ik\,\tilde{\tilde{\delta}}(k^2, T)\right] dk. \quad \text{(A.9)}$$

Hence, recalling the definition (2.25) of $\tilde{\tilde{\delta}}(k^2, T)$, we find

$$u_t(0,t) + \alpha u_x(0,t) + \beta u(0,t) = -\frac{1}{2\pi}\int_{\partial D^+} e^{-k^2 t}\ 2ik\left[\int_0^T e^{k^2 s}\gamma(s)ds + u_0(0)\right] dk. \quad \text{(A.10)}$$

Letting $k^2 = il$, equation (A.10) becomes

$$u_t(0,t) + \alpha u_x(0,t) + \beta u(0,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ilt}\left[\int_0^T e^{ils}\gamma(s)ds + u_0(0)\right] dl = \gamma(t), \quad \text{(A.11)}$$

where the last equality follows by using Cauchy's theorem in the lower half complex l-plane and by the definition of Dirac's δ-function:

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{il(s-t)}dl. \quad \text{(A.12)}$$

Hence, the oblique Robin boundary condition (2.7) has been verified.

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